On some combinatorial aspects of Representation Theory

Doctoral Defense

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March 24, 2004

Overview

- Degree one nonsymmetric Pieri rule
- New recursions for characters and multiplicities
- Complexity of character formulas

Part 1

Degree 1 nonsymmetric Pieri rule

The classical Pieri rule

- The classical Pieri rule is a combinatorial description for the product of a Schur polynomial with a complete (or an elementary) symmetric polynomial:

$$h_m s_\mu = \sum_{\substack{\lambda - \mu \\ \text{hor. } m-\text{strip}}} s_\lambda .$$

- This is a special case of the well-known Littlewood-Richardson rule for the product of any two Schur polynomials.

Jack polynomials

- Jack polynomials $J_\lambda^{(\alpha)}$ generalize Schur polynomials.
- In fact, $J_\lambda^{(1)}$ is a scalar multiple of s_λ.
- The Pieri rule for these was found by Stanley in 1989:

$$J_m^{(\alpha)} J_\mu^{(\alpha)} = \sum_{\substack{\lambda - \mu \\ \text{hor. } m-\text{strip}}} c_{m,\mu}^\lambda(\alpha) J_\lambda^{(\alpha)*}.$$

- The Littlewood-Richardson rule is not known for the $J_\lambda^{(\alpha)}$.

The nonsymmetric Pieri rule

- Nonsymmetric analogs $F_\eta^{(\alpha)}$ where introduced in 1995 by Heckman and Opdam.
- They are indexed by compositions, i.e. $\eta \in \mathbb{Z}_+^n$.
- The Pieri rule for them is

$$F_\nu^{(\alpha)} F_\eta^{(\alpha)} = \sum_\lambda g_{\nu\eta}^\lambda(\alpha) F_\lambda^{(\alpha)*}$$

  where $\nu \in \{0,1\}^n$.
- This is not fully known yet, but we will give a complete answer for the case $\nu = \varepsilon_k$.

Nonsymmetric Jack polynomials

Up to a scalar multiple, F_η can be defined as a simultaneous polynomial eigenfunction for the commuting family of Cherednik operators

$$\xi_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{k<i} \frac{x_i}{x_i - x_k}(1 - s_{ik}) + \sum_{k>i} \frac{x_k}{x_i - x_k}(1 - s_{ik}) + 1 - i,$$

for $i = 1, \ldots, n$, with eigenvalues

$$\overline{\eta}_i = \eta_i \alpha - \#\{k < i \,|\, \eta_k \geq \eta_i\} - \#\{k > i \,|\, \eta_k > \eta_i\}.$$

Here s_{ik} is the transposition permuting x_i and x_k.

Main Results

Two main questions answered for the product $F_{\varepsilon_k} F_\eta$:

- Which polynomials occur in the decomposition?
- What are values of the coefficients?

To answer the first question we need a few definitions.

Composition diagrams

- The diagram of η is the set $\operatorname{diag}(\eta) = \{(i, j) | 1 \le i \le n, 1 \le j \le \eta_i\}$.

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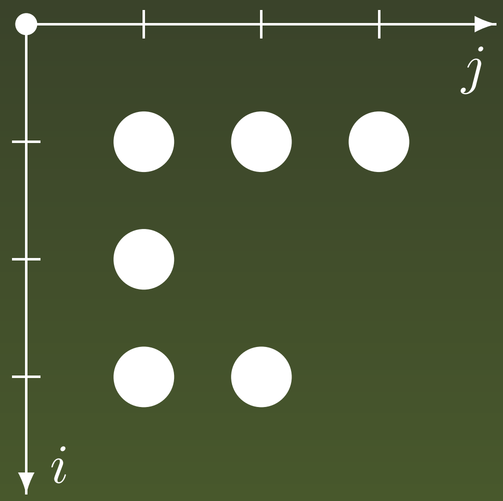
- Graphically, for $\eta = (312)$:



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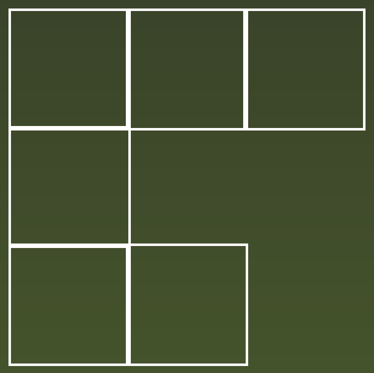

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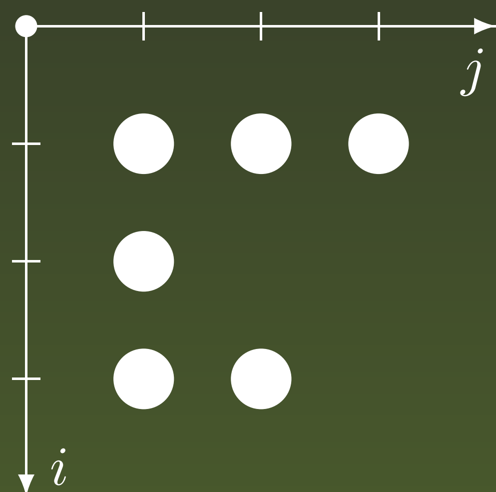

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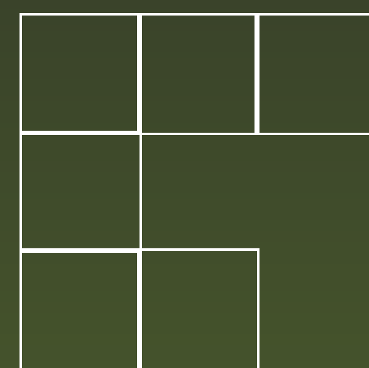


- Usually we identify $\operatorname{diag}(\eta)$ with η.

An ordering on compositions

Define an ordering on compositions by asking that $\nu \preceq \eta$ if there exists a permutation π such that

$$\begin{aligned} \nu_i < \eta_{\pi(i)}, \ \text{if } i < \pi(i), \\ \nu_i \leq \eta_{\pi(i)}, \ \text{if } i \geq \pi(i). \end{aligned}$$

This extends the usual ordering $\subseteq$ (inclusion of diagrams).

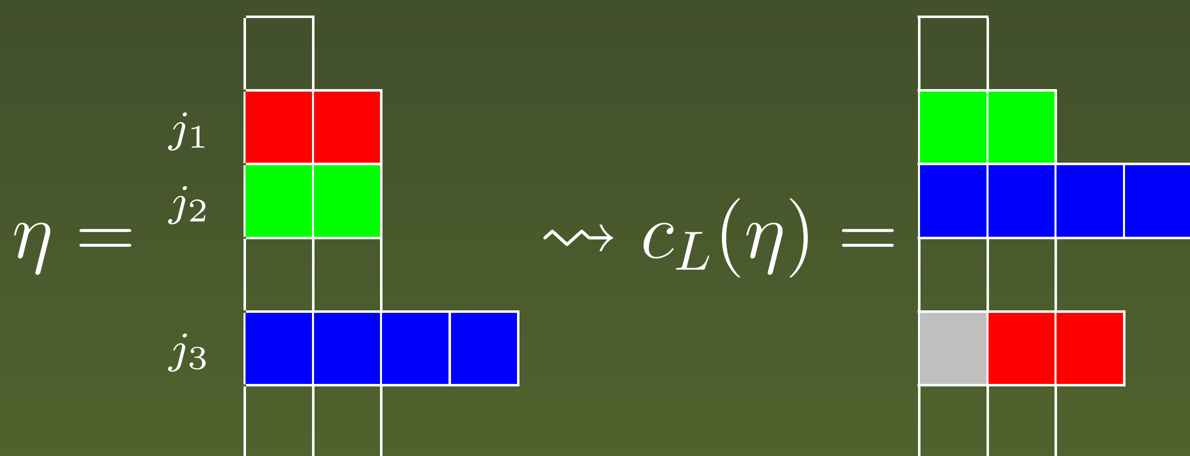
Minimal elements above η

For $L = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, n\}$, let

$$\eta = (\eta_1, \ldots, \eta_{j_1}, \ldots, \eta_{j_2}, \ldots, \eta_{j_\ell}, \ldots, \eta_n),$$

and define $\mu = C_L(\eta)$ by

$$\mu = (\eta_1, \ldots, \eta_{j_2}, \ldots, \eta_{j_3}, \ldots, \eta_{j_1} + 1, \ldots, \eta_n).$$



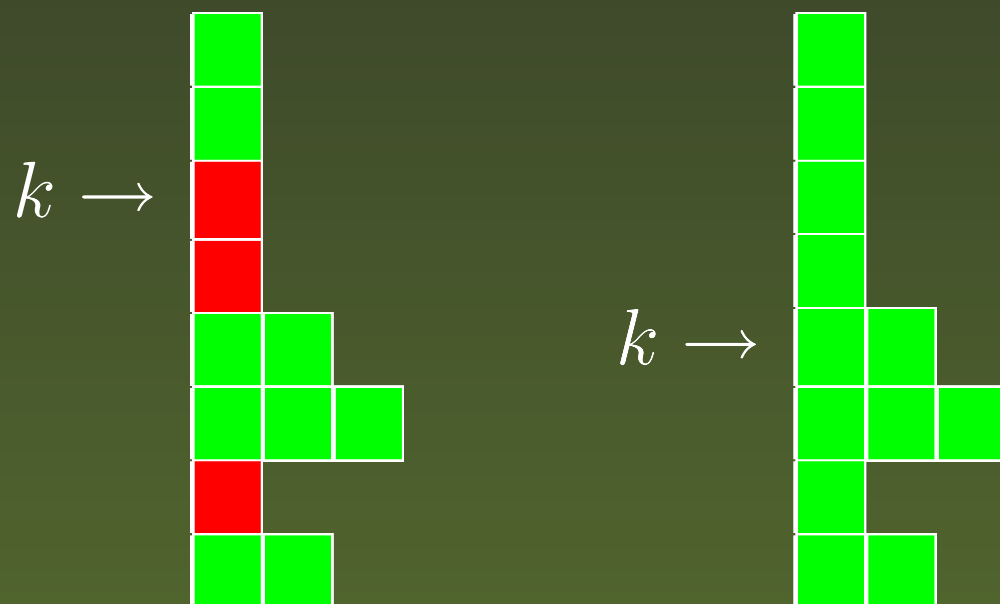
First main theorem

Theorem. $g^{\lambda}_{\varepsilon_k \eta} \neq 0$ if and only if:

(i) $\lambda = C_S(\eta)$ where $S = \{i_1, \ldots, i_s\} := \{i | \eta_i \neq \lambda_i\}$,

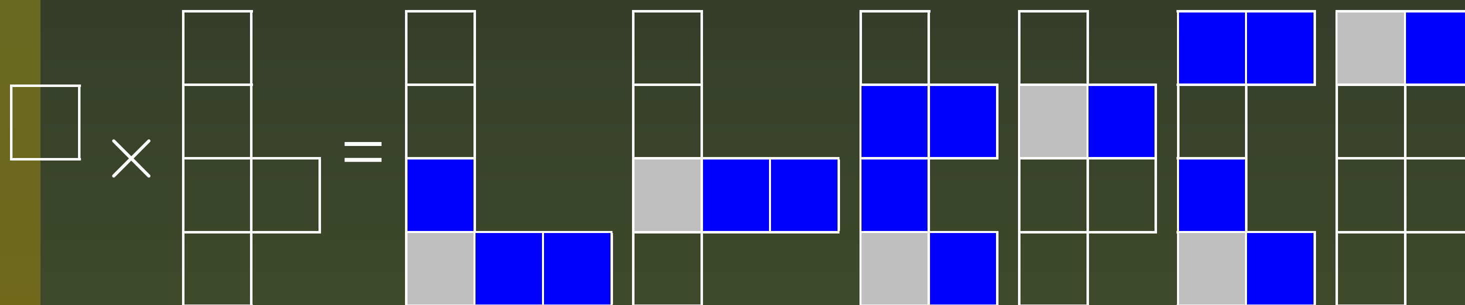
(ii) Either $i_s \geq k$ or $\#\{i \geq k | \eta_i = \eta_{i_1} + 1\} > 0$,

(iii) if $\eta_{i_1} = \eta_1 = \cdots = \eta_k$ then $i_1 < k$.

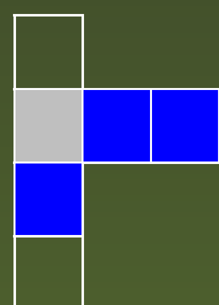


Example

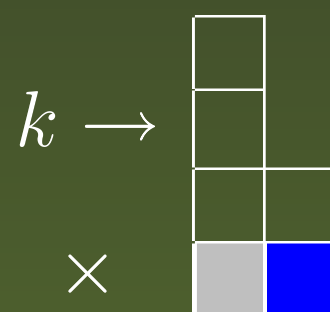
The following are all compositions occurring in the expansion of $F_{\varepsilon_2} F_{(1121)}$:



These are some compositions which **do not occur** for the indicated reason:



Not minimal



Wrong i_1

Remarks

- S need not be the only set to satisfy (i).
- In fact, as we will see below, there is always a (unique) "maximal" set L such that $\lambda = c_L(\eta)$.

Maximal sets

A subset $L = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, n\}$ is maximal w.r.t. η if:

- $\eta_i \neq \eta_{j_1}$, if $i < j_1$,
- $\eta_i \neq \eta_{j_t}$, if $j_{t-1} < i < j_t$,
- $\eta_i \neq \eta_{j_1} + 1$, if $i > j_\ell$.

One can show that if $\lambda = C_M(\eta) = C_L(\eta)$ and L is maximal, then $M \subset L$, hence $\lambda = C_L(\eta)$ determines L uniquely. In particular, $S \subset L$.

Remarks

- Let $S \subset L$. Then (ii) is equivalent to $j_\ell \geq k$.
- We are going to describe $g^{\lambda}_{\varepsilon_k \eta}$ combinatorially.
- We will need two auxiliary polynomials $b_{\eta\lambda}(\alpha)$ and $b^{(k)}_{\eta\lambda}(\alpha)$.
- They will be defined by playing a *jeu de flèches* (an arrow shooting game).
- First, we recall some fundamental concepts.

Composition statistics

For a box $s = (i, j)$ of η we define the *coarm* length and the *arm* lengths as

coarm $a'_\eta(s)$ arm $a_\eta(s)$

so

$$\#\square = a'_\eta(s) = j - 1, \qquad \#\square = a_\eta(s) = \eta_i - j.$$

Composition statistics

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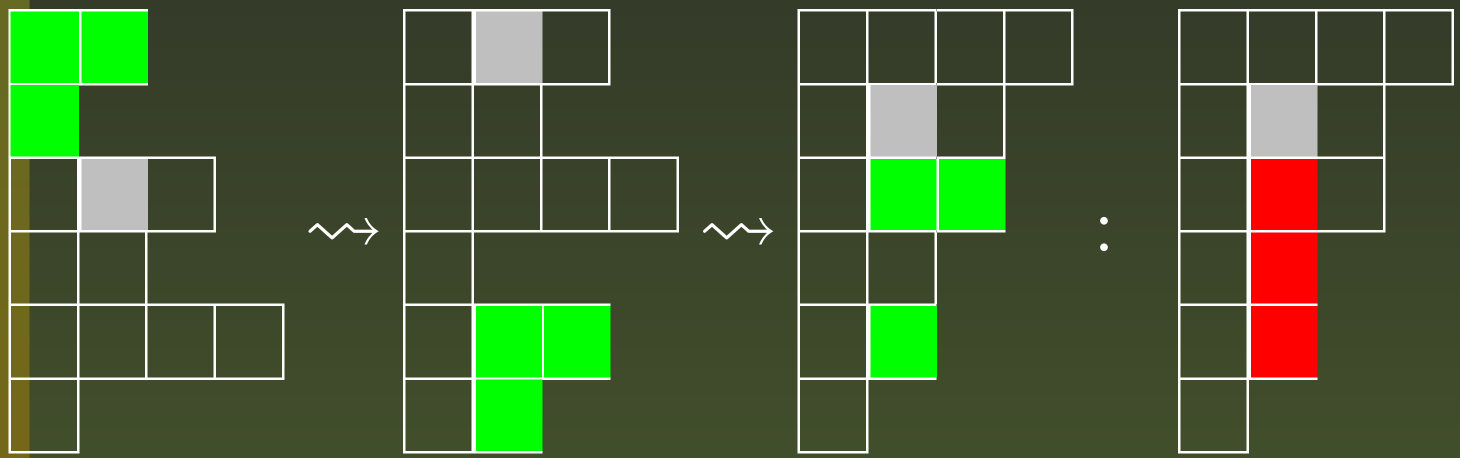

so

$$\#\blacksquare = l_\eta(s) = \#\{k < i \,|\, j \leq \eta_k + 1 \leq \eta_i\} + \#\{k > i \,|\, j \leq \eta_k \leq \eta_i\}.$$

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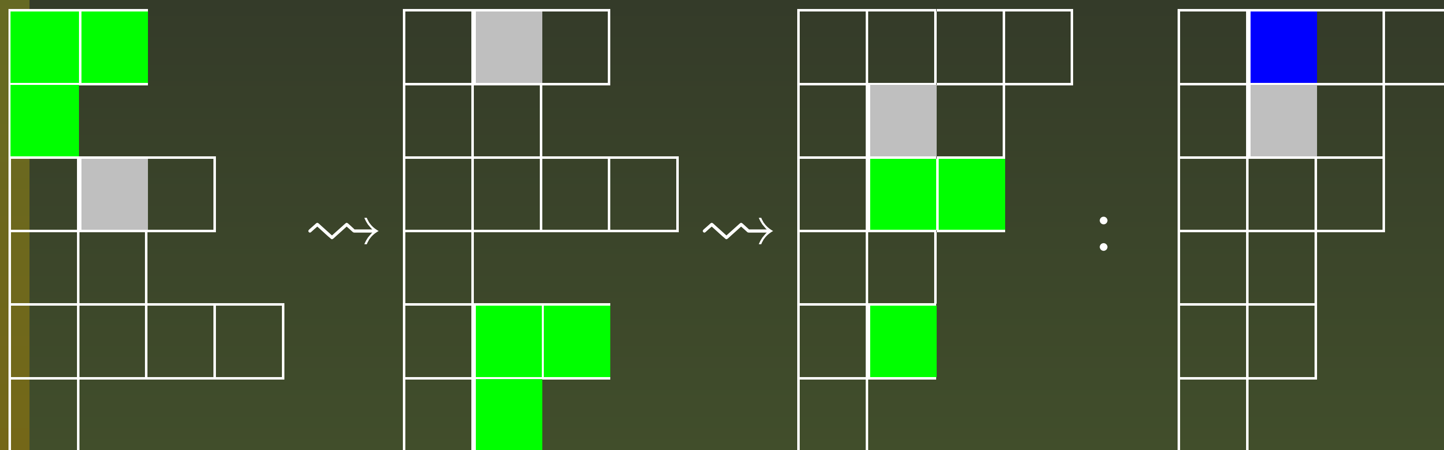


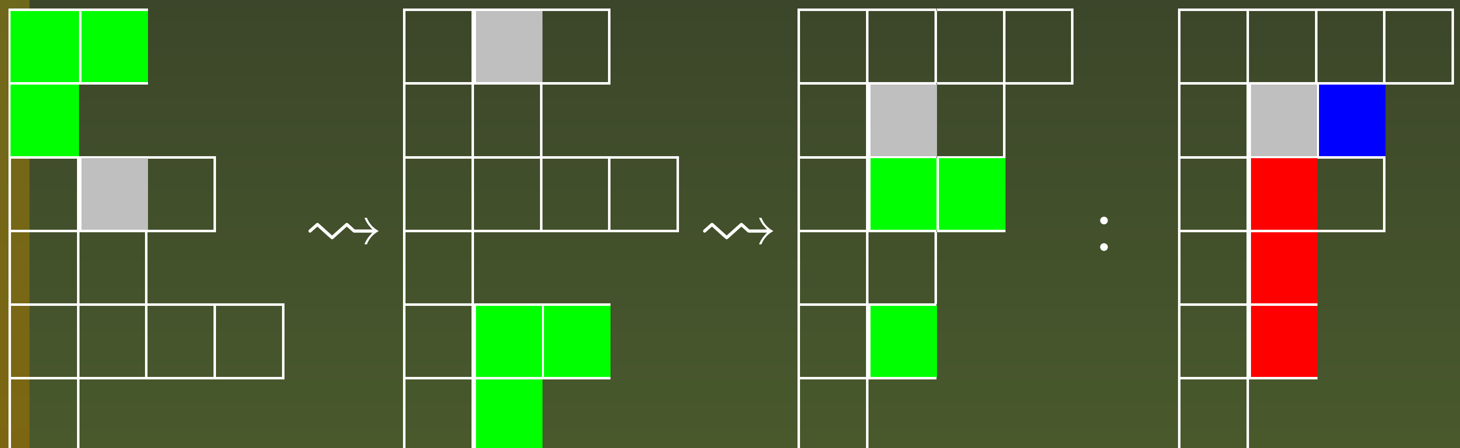
so

$$\#\blacksquare = l'_\eta(s) \;=\; \#\{k < i \mid \eta_k \geq \eta_i\} \;+\; \#\{k > i \mid \eta_k > \eta_i\}.$$

Composition statistics

For a box $s = (i, j)$ of η we define the *lower hook* length polynomial as

$$d'_\eta(s) = (a_\eta(s) + 1)\alpha + l_\eta(s).$$



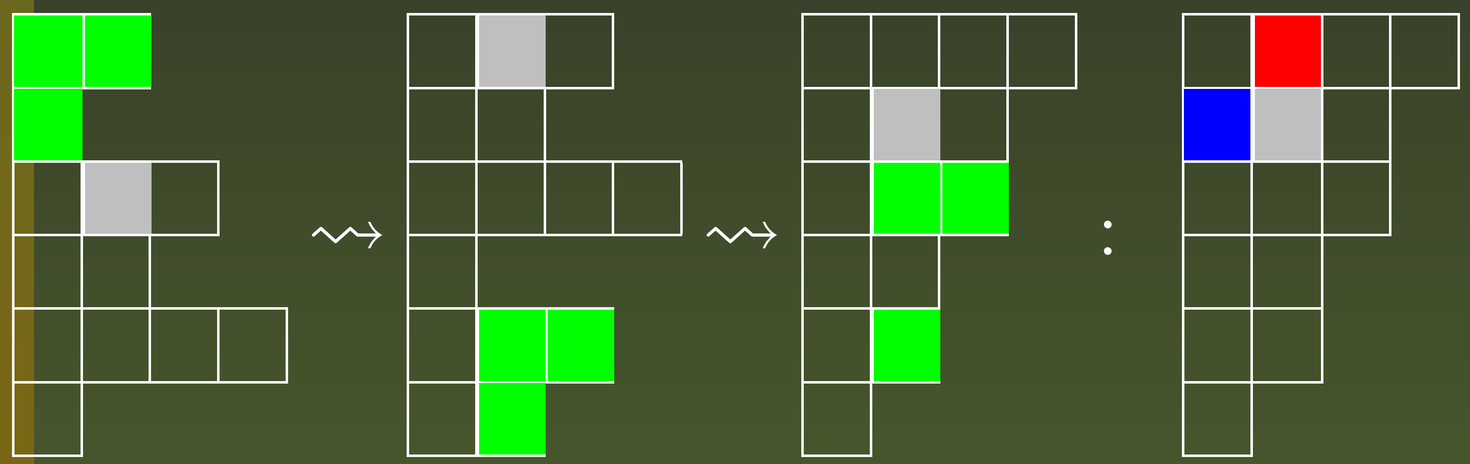
e.g. $d'_\eta(3, 2) = 2\alpha + 3$. We also define the *upper hook* polynomial as

$$d_\eta(s) = d'_\eta(s) + 1.$$

Composition statistics

The *content* of a box in η is defined as

$$c_\eta(s) = (a'_\eta(s) + 1)\alpha - l'_\eta(s).$$

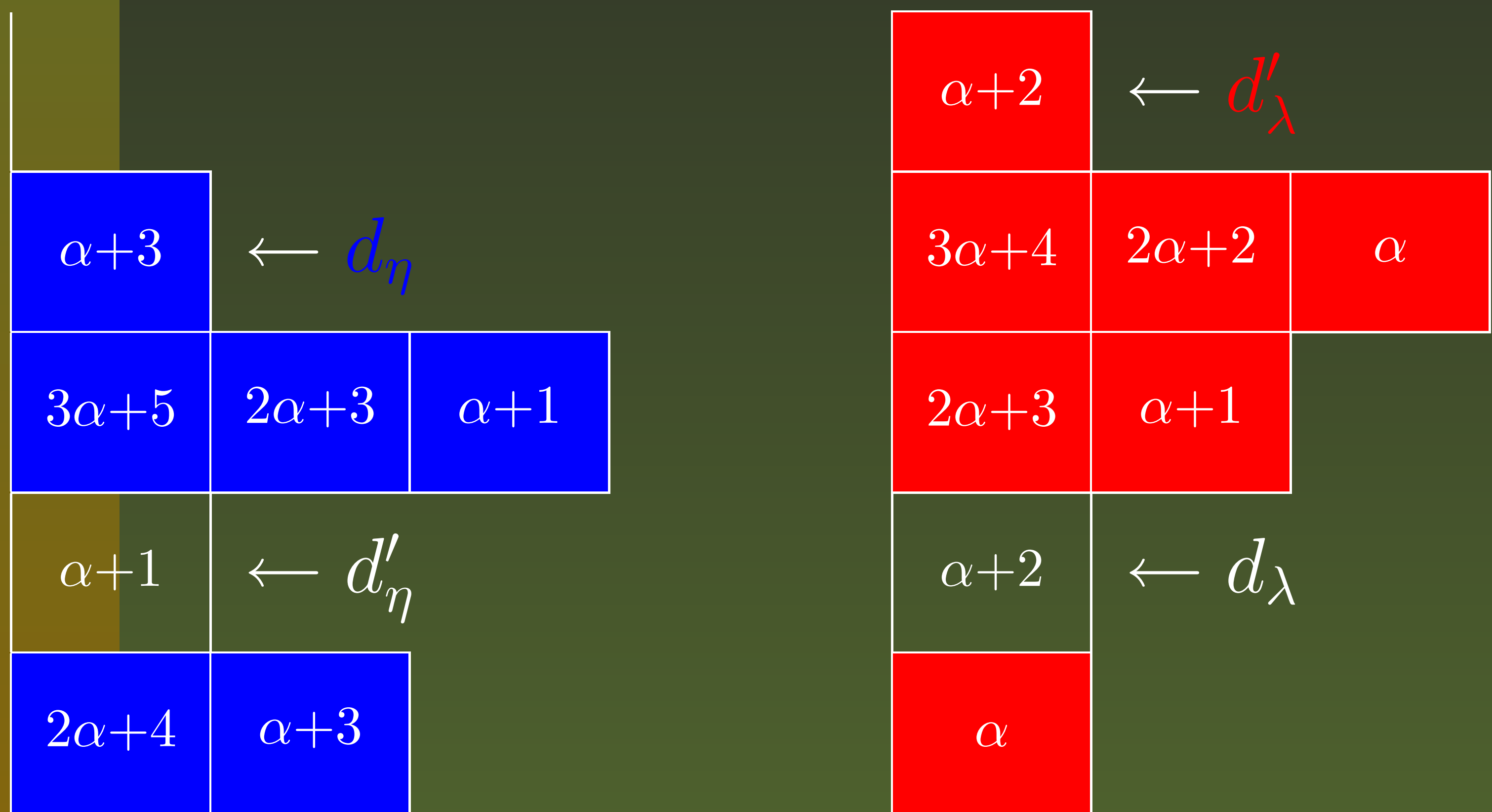


e.g. $c_\eta(3,2) = 2\alpha - 1.$

Remark: the eigenvalue $\overline{\eta}_i$ is just the content $c_\eta(i, \eta_i)$ of the rightmost box on row i of η.

Hook tableaux

Let $\eta = (01312)$, $\lambda = (13211)$ and $k = 1$. Then $L = \{1, 2, 3, 5\}$ is maximal w.r.t. η and $\lambda = C_L(\eta)$.



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For the moment, forget the non-colored rows and think of the colored ones as if they formed a cycle, namely regard the first row as if it came below the last one.

Place the arrows close to the right end of each (colored) row as follows.

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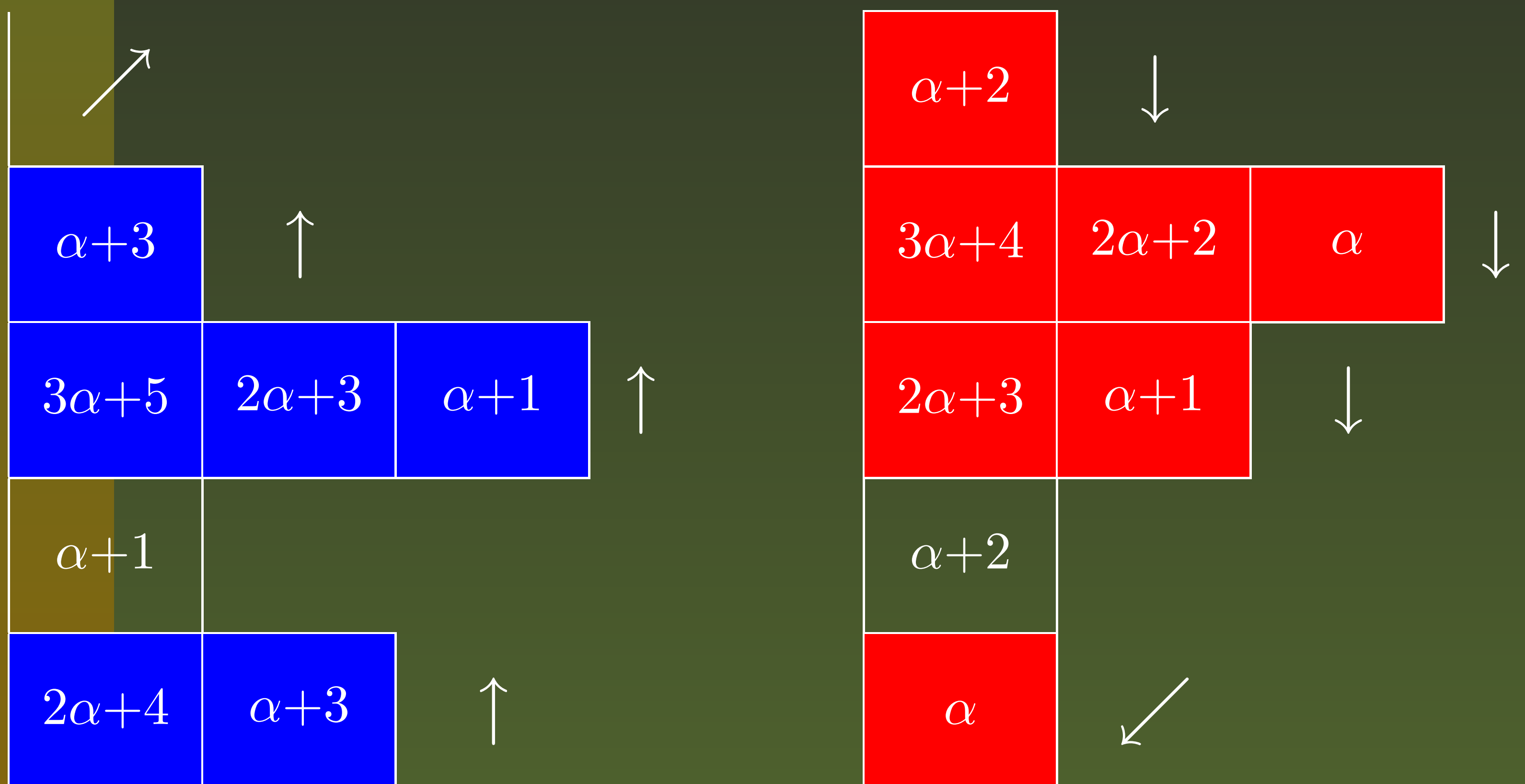

The arrows can only reach as far as the next row in cycle.

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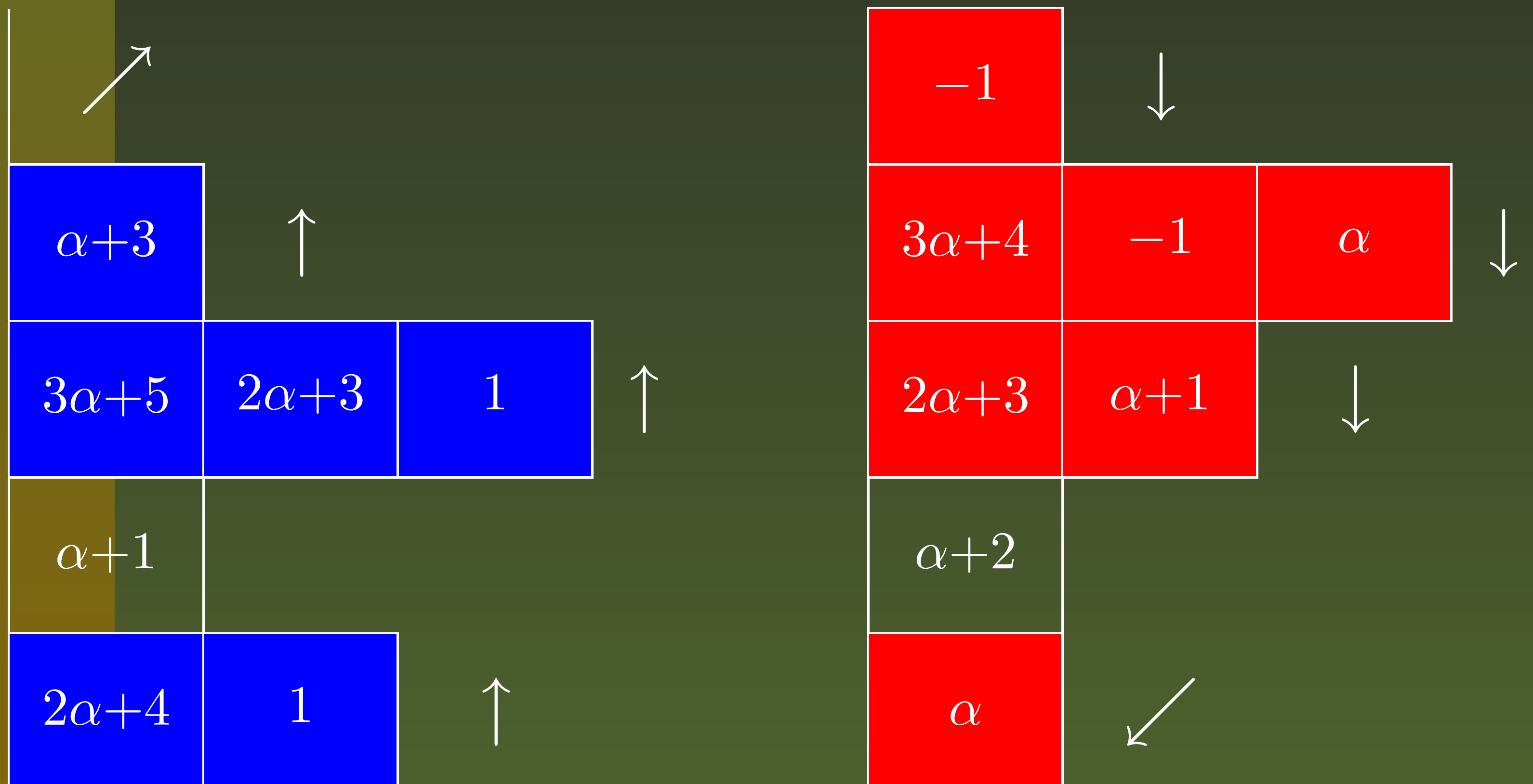

$$b_{\eta\lambda} := 2\,(\alpha+3)(3\,\alpha+5)(2\,\alpha+3)^2(\alpha+1)^2(\alpha+2)^2(3\,\alpha+4)\alpha^2$$

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Now we are going to take k into account.

Do so by placing a shield on row k protecting from the arrows.

Then repeat the jeu de flèches.

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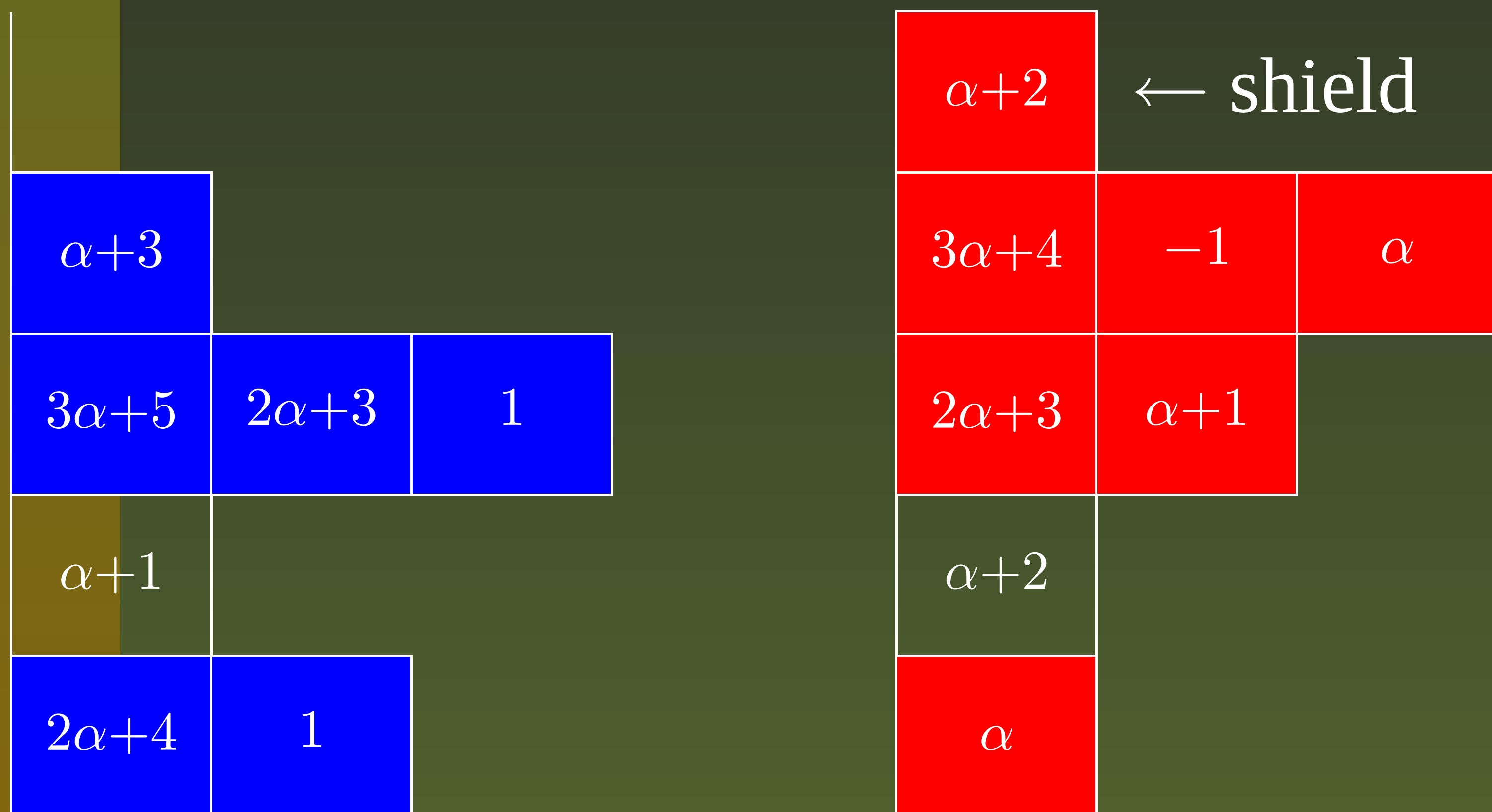

$$b^{(1)}_{\eta\lambda} := -2\,(\alpha+3)(3\,\alpha+5)(2\,\alpha+3)^2(\alpha+1)^2(\alpha+2)^3(3\,\alpha+4)\alpha^2$$

Second main theorem

Theorem. If $g^{\lambda}_{\varepsilon_k \eta} \neq 0$ then

$$g^{\lambda}_{\varepsilon_k \eta} = \begin{cases} (\alpha + k) b^{(k)}_{\eta\lambda} + (c_\lambda(j_p) - c_\lambda(j_\ell)) b_{\eta\lambda}, \\ \qquad\qquad\qquad\qquad \text{if } k = j_p \in L, \\ (c_\lambda(j_p) - c_\lambda(j_\ell)) b_{\eta\lambda}, \ \text{if } j_p < k < j_{p+1}, \\ -\alpha b_{\eta\lambda}, \ \text{if } k < j_1, \end{cases}$$

where

$$c_\lambda(i) := c_\lambda(i, \lambda_i) = \lambda_i \alpha - l'_\lambda(i)$$

is the *content* of the rightmost box on row i of λ.

Example

Using the aforementioned values of $b_{\eta\lambda}$ and $b^{(1)}_{\eta\lambda}$ for $\eta = (01312)$ and $\lambda = (13211)$, we find that

$$c_\lambda(1) = \alpha - 2, \qquad c_\lambda(5) = \alpha - 4,$$

and thus

$$\begin{aligned} g^{\lambda}_{1\eta} &= (\alpha+1)b^{(1)}_{\eta\lambda} + ((\alpha-2)-(\alpha-4))b_{\eta\lambda} \\ &= -2(\alpha+1)^2(\alpha+3)^2(3\alpha+5)(2\alpha+3)^2(\alpha+2)^2(3\alpha+4)\alpha^3 \end{aligned}$$

Example

For $\eta = (045)$, $\lambda = (451)$ and $k = 2$, we find that

$$\begin{aligned} g^{(451)}_{2,(045)} &= -24(\alpha+2)(2\alpha+1)^3(3\alpha+1)^2 \\ &\quad (\alpha+1)^2(5\alpha+3)(3\alpha+2) \\ &\quad (4\alpha+1)(5\alpha+2)(\alpha-1)\alpha^5. \end{aligned}$$

Thus it is not always possible to define a “global sign” for each coefficient.

Idea of the proof

- There are some recursions for the $g^{\lambda}_{\varepsilon_k \eta}$.
- There is an exact, non-combinatorial formula for $x_i F_\eta$ by Marshall.
- First guess which λ will occur and guess the value of $g^{\lambda}_{\varepsilon_k \eta}$.
- Then use either recursion or closed formula to show the guess is correct.

End of part 1

Part 2

New recursions for characters and multiplicities

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- $E = \mathbb{R}\langle \alpha_1, \ldots, \alpha_n \rangle$ with Killing form $\langle \, , \, \rangle$

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- $\rho = \omega_1 + \cdots + \omega_n = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

The character

The (formal) character of V^λ is

$$\chi_\lambda = \sum_{\mu \in P^\lambda} \dim(V_\lambda(\mu)) e^\mu,$$

where $V^\lambda(\mu) = \{v \in V^\lambda \,|\, h \cdot v = \mu(h)v, \forall h \in \mathfrak{h}\}$ is the weight space of V^λ of weight μ relative to $\mathfrak{h}$. This makes sense since $\mathfrak{h}$ is abelian and consists of semisimple elements.

The WCF

The Weyl Character Formula (WCF) states that

$$\delta\chi_\lambda = \sum_{w\in W} \varepsilon(w)e^{w(\lambda+\rho)}$$

where $\varepsilon(w) = (-1)^{l(w)}$ and δ is the Weyl denominator

$$\delta = \sum_{w\in W} \varepsilon(w)e^{w\rho} = \prod_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

The girdle

Is defined as

$$\theta_\lambda = \sum_{\mu \in P^\lambda} e^\mu.$$

Looks like the character, except the coefficients are all 1's.

A partition function

- Let $R'^{+} = R^{+} \backslash \Delta$ be the positive **non**-simple roots.
- For any $\phi \subset R'^{+}$, let $\langle \phi \rangle = \sum_{\beta \in R'^{+}} \beta$
- Then we can write

$$\prod_{\alpha \in R'^{+}} (1 - e^{-\alpha}) = 1 + \sum_{\langle \phi \rangle} c_{\langle \phi \rangle} e^{-\langle \phi \rangle},$$

where

$$c_\beta = \#\{\langle \phi \rangle = \beta \,|\, \#\phi \text{ is even}\} \\ - \#\{\langle \phi \rangle = \beta \,|\, \#\phi \text{ is odd}\}.$$

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- Let $\varepsilon_\mu = (-1)^{l(w_\mu)}$, if $\mu + \rho$ is regular, and zero otherwise.
- Define the *signed characters* by:

$$\chi_\mu := \varepsilon_\mu \cdot \chi_{\overline{\mu}}$$

Main Theorem

Theorem. For a dominant λ,

$$\Theta_\lambda = \chi_\lambda + \sum_{\langle\phi\rangle} c_{\langle\phi\rangle}\chi_{\lambda-\langle\phi\rangle}$$

is a recursion for χ_λ.

Idea of the proof

We show that both sides of

$$\Theta_\lambda = \chi_\lambda + \sum_{\langle\phi\rangle} c_{\langle\phi\rangle}\chi_{\lambda-\langle\phi\rangle}$$

are equal to

$$\sum_{w\in W} \varepsilon(w)\frac{e^{w\lambda}}{\prod_{\alpha\in R'^+}(1-e^{-w\alpha})}.$$

LHS

Proposition.

$$\Theta_\lambda = \sum_{w \in W} \varepsilon(w) \frac{e^{w\lambda}}{\prod_{\alpha \in R'+}(1 - e^{-w\alpha})}.$$

We give an elementary proof, but this also follows from the work of Brion.

RHS

Proposition.

$$\sum_{w\in W} \varepsilon(w) \frac{e^{w\lambda}}{\prod_{\alpha\in R'+}(1 - e^{-w\alpha})} = \chi_\lambda + \sum_{\langle\phi\rangle} c_{\langle\phi\rangle}\chi_{\lambda-\langle\phi\rangle}.$$

The proof is: just expand and use the WCF.

Example

- For $\mathfrak{g}$ of type A_2,

$$\chi_\lambda = \Theta_\lambda + \chi_{\lambda-\rho}.$$

- For $\mathfrak{g}$ of type B_2,

$$\chi_\lambda = \Theta_\lambda + \chi_{\lambda-\alpha_1-\alpha_2} + \chi_{\lambda-\alpha_1-2\alpha_2} - \chi_{\lambda-2\alpha_1-3\alpha_2}$$

Consequences

- **Corollary.**

$$m_\lambda(\mu) = 1 - \sum_{\langle\phi\rangle} c_{\langle\phi\rangle} m_{\lambda - \langle\phi\rangle}(\mu),$$

where $m_\lambda(\mu) = \varepsilon_\lambda \dim V^{\overline{\lambda}}(\mu)$.

Consequences

- **Corollary.**

$$m_\lambda(\mu) = 1 - \sum_{\langle\phi\rangle} c_{\langle\phi\rangle} m_{\lambda-\langle\phi\rangle}(\mu),$$

where $m_\lambda(\mu) = \varepsilon_\lambda \dim V^{\overline{\lambda}}(\mu)$.

- **Corollary.**

$$|P^\lambda| = \dim V^\lambda + \sum_{\langle\phi\rangle} \varepsilon_{\lambda-\langle\phi\rangle} c_{\langle\phi\rangle} \dim V^{\overline{\lambda-\langle\phi\rangle}}.$$

Consequences

Corollary.

$$|P^\lambda| = \sum_{w\in W} \frac{1}{\prod_{i=1}^n \langle \rho, w\alpha_i\rangle}$$
$$\times \sum_{j=0}^{n} \frac{\langle \rho, w\lambda\rangle^j}{j!} \mathcal{T}_{n-j}(\langle w\alpha_1\rangle, \ldots, \langle w\alpha_n\rangle),$$

where $\mathcal{T}_i$ is the i-th Todd polynomial defined by

$$\prod_{i\geq 1} \frac{tx_i}{1-e^{-tx_i}} = \sum_{i=0}^{\infty} \mathcal{T}(x_1, x_2, \ldots)t^i.$$

Example

For $\mathfrak{g}$ of type A_2, and $\lambda = (a, b)$, we get

$$|P^\lambda| = \binom{a+b+2}{2} + ab.$$

Example

- For $\mathfrak{g}$ of type B_2, $m_{(i,j)}(0,0) = 0$ unless j is even.
- So when $\lambda = (i, 2j)$ for $i, j \in \mathbb{Z}_+$ we have the recursion

$$m_\lambda(0,0)=1+m_{\lambda-\alpha_1-\alpha_2}(0,0)+m_{\lambda-\alpha_1-2\alpha_2}(0,0)-m_{\lambda-2\alpha_1-3\alpha_2}(0,0).$$

- This can be solved exactly:

$$m_{(i,2j)}(0,0) = \frac{1+(-1)^i}{4} + \frac{(i+1)(2j+1)}{2}.$$

End of Part 2

Part 3

The complexity of character formulas

Two character formulas

We studied two character formulas:

- One classic: Freudenthal's formula expressing the multiplicity of a weight in terms of multiplicity for "higher" weights.
- One new: Sahi's formula is a recursion for non-symmetric analogs P_λ of characters χ_λ.

Statement of the Problem

Given a dominant weight λ of length m, we want to compute the character table $\mathcal{T}_m$, containing the characters for all dominant weights μ whose length is $\leq m$.

The size of the Problem

- The size of χ_λ is precisely $|P^\lambda|$ and this is about $O(m^n)$.
- Thus the size of $\mathcal{T}_m$ is about $O(m^{2n})$.

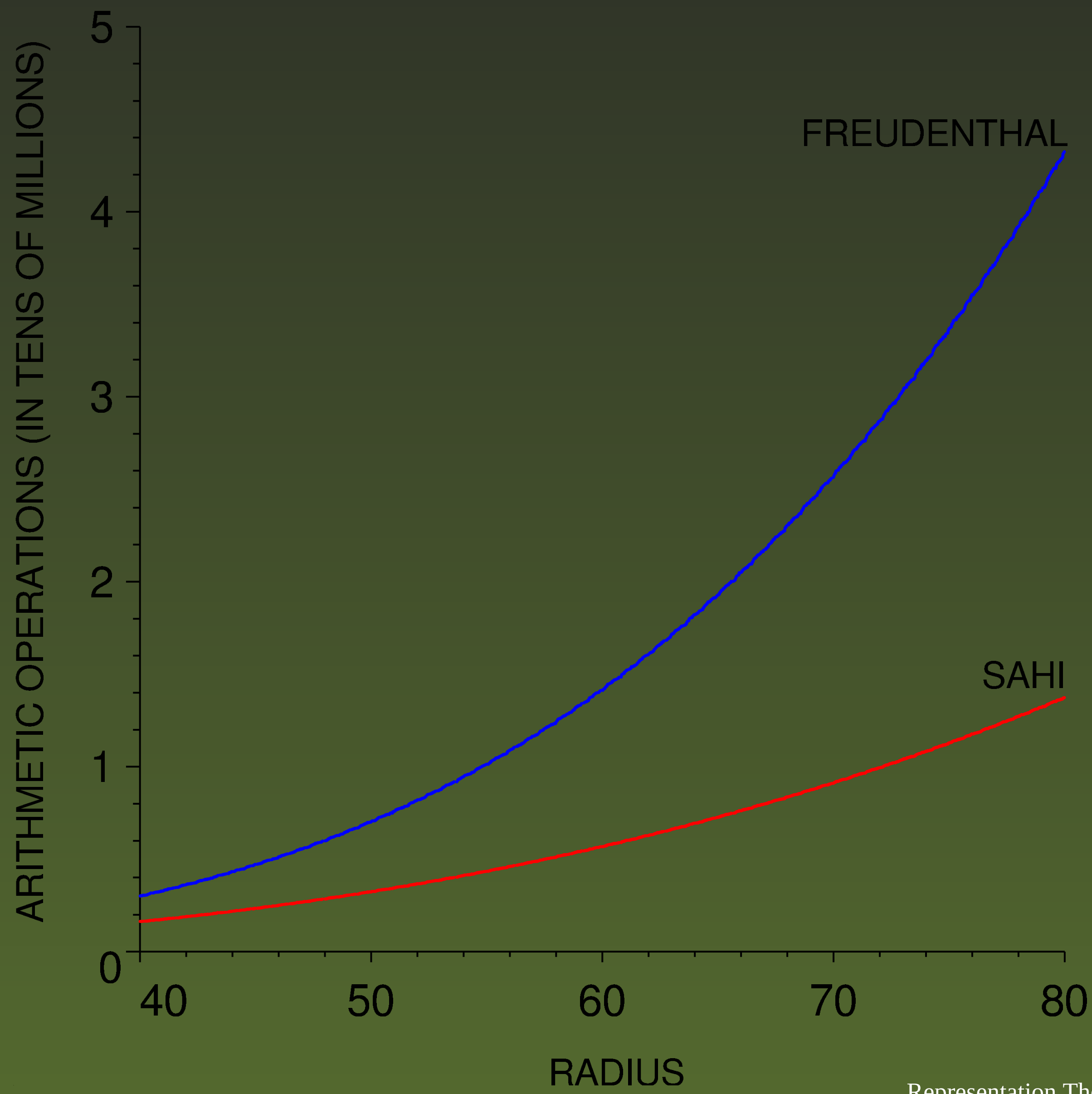
Computational model

- We adopted straight-line program model with uniform cost function, so that the time complexity depends only on the total number of arithmetic operations.
- Hence, an optimal method for computing $\mathcal{T}_m$, takes $O(m^{2n})$ time.

Main Theorem

Theorem. Let λ be a weight of length m. Then Freudenthal computes $\mathcal{T}_m$ in $O(m^{2n+1})$ time while Sahi computes it in $O(m^{2n})$ time, therefore Sahi is of optimal performance.

Computational experience



End of Part 3