INVOLUTIONS FIXING $F^n \cup F^2$

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Abstract. Let $M^m$ be a closed and smooth manifold with an involution having fixed point set of the form $F^n \cup F^2$, where $F^n$ and $F^2$ are submanifolds with dimensions $n$ and 2, respectively, and where $2 < n < m$ and $F^n \cup F^2$ does not bound. The main result of this paper is to establish the upper bound for $m$, for each $n$. The existence of these bounds is guaranteed by the famous $5/2$-theorem of J. Boardman, which establishes that, under the above hypotheses, $m \leq 5/2n$.

1. Introduction

Suppose $M^m$ is a smooth and closed $m$-dimensional manifold and $T : M^m \mapsto M^m$ is a smooth involution defined on $M^m$. The fixed point set of $T$, $F$, is a disjoint union of closed submanifolds of $M^m$, $F = \bigcup_{j=0}^{n} F^j$, where $F^j$ denotes the union of those components of $F$ having dimension $j$. It is well known, from equivariant bordism theory, that if $(M^m, T)$ is nonbounding then $F$ cannot be too low dimensional. This fact was evidenced from an old result of P. Conner and E. E. Floyd (Theorem 27.1 of [4]), which stated: for each natural number $n$, there exists a number $\varphi(n)$ with the property that, if $(M^m, T)$ is an involution fixing $F = \bigcup_{j=0}^{n} F^j$ and if $m > \varphi(n)$, then $(M^m, T)$ bounds equivariantly. Later this was explicitly confirmed by the famous $5/2$-Theorem of J. Boardman of [3]: if $(M^m, T)$ fixes $F = \bigcup_{j=0}^{n} F^j$ and $M^m$ is nonbounding, then $m \leq \frac{5}{2} n$. A strengthened version of this fact was obtained by R.E. Stong and C. Kosniowski

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in [2]: if \((M^m, T)\) is a nonbounding involution fixing \(F = \bigcup_{j=0}^{n} F^j\), then \(m \leq \frac{5}{2} n\).

In particular, if \(F = \bigcup_{j=0}^{n} F^j\) is nonbounding (which means that at least one \(F^j\)
is nonbounding) and \((M^m, T)\) fixes \(F\), then \(m \leq \frac{5}{2} n\); this follows from the fact
that the equivariant cobordism class of \((M^m, T)\) is determined by the cobordism
class of its fixed data. The generality of this last result allows the possibility
that fixed components of all dimensions \(j, 0 \leq j \leq n\), occur; in this way, it
is natural to ask whether there exists a better upper bound for \(m\) when we
omit some components of \(F\). This is inspired by the following result of Stong
and Kosniwoski of [2]: if \((M^m, T)\) is an involution whose fixed set has constant
dimension \(n\), and if \(m > 2n\), then \((M^m, T)\) bounds equivariantly. In particular,
if \(F = F^n\) with constant dimension \(n\) is nonbounding, and if \((M^m, T)\) fixes
\(F\), then \(m \leq 2n\). This bound is best possible, as can be seen by taking the
involution \((F^n \times F^n, T)\), where \(F^n\) is any nonbounding \(n\)-dimensional manifold
(with the exception of \(n = 1\) and \(n = 3\)) and \(T\) switches coordinates. Thus
one has a concrete improvement of the Boardman’s bound when we omit all
\(j\)-dimensional components with \(j < n\).

The above considerations can be placed in the following general setting: for
each natural number \(n\) and each subset \(X \subset \{0, 1, 2, \ldots, n-1\}\) (we allow \(X\) to
be empty), we define \(m(n; X)\) as being the number

\[
m(n; X) = \text{maximum } \{m \mid \text{there exists an involution } (M^m, T) \text{ fixing } F \text{ such that } F \text{ does not bound, } n \text{ is the dimension of the non-empty component of } F \text{ of largest dimension, and if } F^j \text{ is a non-empty } j \text{-dimensional component of } F \text{ with } j < n, \text{ then } j \in X\}.
\]

As it was seen above, this number always exists (but it is not defined if we allow
\(F\) to be a boundary, since in this case one has involutions fixing \(F\) with any
codimension); further, if \(j \in X\), the number of \(j\)-dimensional components of \(F\)
has no influence in the value of \(m(n; X)\), since any involution is equivariantly
cobordant to an involution with the property that the \(j\)-dimensional part of the
fixed set is connected.
Under this setting, the Boardman’s bound is stated as “for every $n$ and every $X \subset \{0, 1, 2, \ldots, n-1\}$, $m(n; X) \leq \frac{5}{2}n$”, and the Strong-Kosniowski’s bound is stated as “for $n \neq 1$ and 3, and $X = \emptyset$, $m(n; X) = 2n$”.

Once the case $X = \emptyset$ is established, the next natural step is to consider $X$ containing a single element, which means to consider fixed sets of the form $F = F^n \cup F^j$, $j < n$. For $j = 0$, $F = F^n \cup F^0$ reduces to $F = F^n \cup \{\text{point}\}$. Concerning this case, recently Stong and Pergher proved the following result [5]: for each natural number $n$, write $n = 2^p q$, where $p \geq 0$ and $q$ is odd, and set
\[
m(n) = \begin{cases} 
(2^{p+1} - 1)q + p + 1 = 2n + p - q + 1, & \text{if } p \leq q + 1 \\
(2^{p+1} - 2^{p-q})q + 2^{p-q}(q + 1) = 2n + 2^{p-q}, & \text{if } p \geq q.
\end{cases}
\]

Then, if $(M^m, T)$ is an involution whose fixed set has the form $F = F^n \cup \{\text{point}\}$, $m \leq m(n)$; further, there are involutions with $m = m(n)$ fixing a point and some $F^n$.

Together with the case $X = \emptyset$, this result says that
\[
m(n; \{0\}) = \text{ maximum } \{m(n), 2n\} \text{ if } n \neq 3, \text{ and } m(3; \{0\}) = 4.
\]

The objective of this paper is to calculate $m(n; \{2\})$. Specifically, we shall prove that $m(n; \{2\}) = \text{ maximum } \{m(n - 2) + 4, 2n\}$ when $n \geq 3$.

Concerning $m(n; \{1\})$, in her doctoral thesis [6] (and in [7]), S. Kelton studied bounds for involutions $(M^m, T)$ whose fixed set has the form $F = F^n \cup \mathbb{R}P^j$, where $\mathbb{R}P^j$ is the $j$-dimensional real projective space. Among the results, one finds: suppose $(M^m, T)$ is an involution whose fixed set has the form $F = F^n \cup \mathbb{R}P^1$ and the normal bundle of $\mathbb{R}P^1$ in $M^m$ is nonbounding. Then, if $n$ is odd, $m \leq m(n - 1) + 1$, and if $n$ is even, $m \leq m(n - 1) + 2$; further, these bounds are best possible. Since $F^n \cup F^1$ reduces to $F^n \cup \mathbb{R}P^1$, these results give (for $n > 1$):
\[
m(n; \{1\}) = \begin{cases} 
\text{ maximum } \{m(n - 1) + 1, 2n\}, & \text{if } n \text{ is odd;} \\
\text{ maximum } \{m(n - 1) + 2, 2n\}, & \text{if } n \text{ is even.}
\end{cases}
\]
We remark that, in the cases $F = F^n \cup F^0$ and $F = F^n \cup F^1$, one has an unique nonbounding stable cobordism class of bundles over $F^j$, $j = 0$ or $1$ (the trivial bundle when $j = 0$, and the stable cobordism class of the canonical line bundle over $\mathbb{RP}^1$ when $j = 1$). As we will see, the technical difficulty in the calculation of $m(n; \{2\})$ lies in the fact that one has a lot of possible stable cobordism classes of bundles over $F^2$.

2. Computation of $m(n; \{2\})$

In this section we will show that $m(n; \{2\}) = \max\{m(n - 2) + 4, 2n\}$, where $n \geq 3$. By the definition of $m(n; X)$, one needs to consider involutions $(M^m, T)$ for which the fixed set $F$ does not bound and has the form $F = F^n$ or $F = F^n \cup F^2$, and one knows that $F^n$ and $F^2$ can be assumed to be connected. The first thing to do is to exhibit, for each $n \geq 3$, involutions $(M^m, T)$ with $m = 2n$ and $m = m(n - 2) + 4$, and with $F$ having the form described above. As already remarked, taking any $n$-dimensional nonbounding manifold $F^n$, the twist involution on $F^n \times F^n$ provides an example with $m = 2n$.

On the other hand, and as remarked in the previous section, in [5] Stong and Pergher constructed, for each $n \geq 1$, a special involution $(M^{m(n)}, T_n)$ for which the fixed set has the form $F^n \cup \{\text{point}\}$. Given $n \geq 3$, consider the involution $(M^{m(n-2)} \times \mathbb{RP}^2 \times \mathbb{RP}^2, T)$, where $T(x, y, z) = (T_{n-2}(x), z, y)$. The fixed set of $T$ has the form

$$(F^{n-2} \cup \{\text{point}\}) \times \mathbb{RP}^2 = F^{n-2} \times \mathbb{RP}^2 \cup \mathbb{RP}^2,$$

and since $\mathbb{RP}^2$ does not bound, this provides an example with $m = m(n - 2) + 4$.

Since $m(3 - 2) + 4 = 6 = 2 \cdot 3$, this approach causes no problem when $n = 3$.

With these examples on hand and taking into account the Stong-Kosniowski’s bound for connected fixed sets, all that remains is to show the following fact: if $(M^m, T)$ is an involution whose fixed set $F$ does not bound and has the form $F = F^n \cup F^2$, then either $m \leq 2n$ or $m \leq m(n - 2) + 4$. Let $\eta \mapsto F^n$, $\mu \mapsto F^2$ denote the normal bundles of $F^n$ and $F^2$ in $M^m$. If $\mu \mapsto F^2$ bounds, it can be
equivariantly removed to give an involution \((N^m, T')\), equivariantly cobordant to \((M^m, T)\), and with fixed data \(\eta \mapsto F^n\). Since \(F^2\) bounds, \(F^n\) does not bound and so \(m \leq 2n\). Thus the computation of \(m(n; \{2\})\) is reduced to the following

**Theorem 2.1.** Suppose that \((M^m, T)\) is an involution having fixed set \(F\) which does not bound and has the form \(F = F^n \cup F^2\). If the normal bundle over the component \(F^2\) does not bound, then \(m \leq m(n - 2) + 4\).

**Remark.** As we will see, the hypothesis “\(F\) does not bound” is really not necessary to the proof.

As above, denote by \((\eta \mapsto F^n) \cup (\mu \mapsto F^2)\) the fixed data of \((M^m, T)\). If \(\mu \mapsto F^2\) is cobordant to \(\mu' \mapsto F^{2'}\), then there exists an involution \((N^m, T')\), cobordant to \((M^m, T)\) and with fixed data \(\eta \cup \mu'.\) Thus, since we will be working with characteristic numbers, our first task will be to describe a complete list of explicit representatives for the possible nonbounding cobordism classes of bundles over 2-dimensional closed manifolds. We need some notations: if \(\xi\) is a vector bundle and \(n\) is a natural number, \(n\xi\) will denote the Whitney sum of \(n\) copies of \(\xi\). We will use \(\varepsilon^r\) to denote the trivial \(r\)-dimensional vector bundle over any base space. For any vector bundle over a closed 2-dimensional manifold, \(\mu \mapsto F^2\), one lets \(W(F^2) = 1 + w_1 + w_2\) be the Stiefel-Whitney class of \(F^2\) and \(W(\mu) = 1 + v_1 + v_2\) be the Stiefel-Whitney class of \(\mu\).

**Lemma 2.2.** For vector bundles as above, one has \(w_1^2 = w_2\) and \(v_1^2 = w_1 v_1\).

**Proof.** \(F^2\) is either a boundary or cobordant to \(\mathbb{R}P^2\). Since \(\mathbb{R}P^2\) and any manifold which bounds satisfy \(w_1^2 = w_2\), this is also true for \(F^2\). Now let \(U = 1 + u\) be the Wu class of \(F^2\); one knows that \(u = w_1\). Then \(Sq^1(v_1) = w_1 = w_1 v_1\), where \(Sq\) is the Steenrod operation; but also \(Sq^1(v_1) = v_1^2\), and the result follows. \(\square\)

The cobordism class of \(\mu \mapsto F^2\) is determined by its characteristic numbers. By the above lemma, these numbers are reduced to the ones obtained from \(w_1^2 (= w_2), v_2\) and \(v_1^2 (= w_1 v_1)\). This gives at most seven possibilities for nonbounding classes. Next we describe examples realizing each one of these
possibilities. Denote by $\xi \mapsto \mathbb{R}P^2$ the canonical line bundle. Then one has the bundles:

1) the 0-dimensional bundle $0 \mapsto \mathbb{R}P^2$, with $w_1^2 \neq 0$, $v_2 = 0$ and $v_1^2 = 0$;
2) $\xi \mapsto \mathbb{R}P^2$, with $w_1^2 \neq 0$, $v_2 = 0$ and $v_1^2 \neq 0$;
3) $2\xi \mapsto \mathbb{R}P^2$, with $w_1^2 \neq 0$, $v_2 \neq 0$ and $v_1^2 = 0$;
4) $3\xi \mapsto \mathbb{R}P^2$, with $w_1^2 \neq 0$, $v_2 \neq 0$ and $v_1^2 \neq 0$.

Now consider $\xi \oplus \varepsilon^1 \mapsto \mathbb{R}P^1$, where again $\xi$ denotes the canonical line bundle.

Consider $\mathbb{R}P(\xi \oplus \varepsilon^1) \mapsto \mathbb{R}P^1$ the real projective space bundle associated to $\xi \oplus \varepsilon^1$, and denote by $\lambda \mapsto \mathbb{R}P(\xi \oplus \varepsilon^1)$ the line bundle of the double cover $S(\xi \oplus \varepsilon^1) \mapsto \mathbb{R}P(\xi \oplus \varepsilon^1)$, $S(\xi \oplus \varepsilon^1)$ the sphere bundle of $\xi \oplus \varepsilon^1$. Note that $K^2 = \mathbb{R}P(\xi \oplus \varepsilon^1)$ is a closed 2-dimensional manifold, and one has the following examples with its respective characteristic numbers, obtained from standard computations in the cohomology of $K^2$:

5) $\lambda \mapsto K^2$, with $w_1^2 = 0$, $v_2 = 0$ and $v_1^2 \neq 0$;
6) $2\lambda \mapsto K^2$, with $w_1^2 = 0$, $v_2 \neq 0$ and $v_1^2 = 0$;
7) $3\lambda \mapsto K^2$, with $w_1^2 = 0$, $v_2 \neq 0$ and $v_1^2 \neq 0$.

We denote by $\beta_i$, $1 \leq i \leq 7$, the stable cobordism classes corresponding to these examples. The following lemma will be crucial to our purposes:

**Lemma 2.3.** If $m > m(n-2) + 4$, then $w_1^2 = v_1^2$ and $v_2 = 0$.

Note that the unique $\beta_i$ satisfying $w_1^2 = v_1^2$ and $v_2 = 0$ is $\beta_2$. Thus this lemma will reduce our task to the following

**Theorem 2.4.** Let $(M^m, T)$ be an involution having fixed set $F$ of the form $F = F^m \cup F^2$. If the normal bundle $\mu \mapsto F^2$ represents $\beta_2$, then $m \leq m(n-2)+4$.

The following basic fact from [4] will be needed for the proof of Lemma 2.3: the projective space bundles $\mathbb{R}P(\eta)$ and $\mathbb{R}P(\mu)$, with its standard line bundles $\lambda \mapsto \mathbb{R}P(\eta)$ and $\nu \mapsto \mathbb{R}P(\mu)$, are cobordant as elements of the bordism group $\mathcal{N}_{m-1}(BO(1))$. Then any class of dimension $m-1$, given by a product of the classes $w_i(\mathbb{R}P(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class
$[\mathbb{R}P(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(\mathbb{R}P(\mu))$ and $w_1(\nu)$, evaluated on $[\mathbb{R}P(\mu)]$. We will apply this using some very special classes. Set $k = m - n$, and write

\[
\mathbb{W}(F^n) = 1 + \theta_1 + \cdots + \theta_n, \\
\mathbb{W}(\eta) = 1 + u_1 + \cdots + u_k \\
\mathbb{W}(\lambda) = 1 + c
\]

for the Stiefel-Whitney classes of $F^n$, $\eta$ and $\lambda$, respectively. From [1] one knows that

\[
\mathbb{W}(\mathbb{R}P(\eta)) = (1 + \theta_1 + \cdots + \theta_n)(1 + c) - (1 + c)^{-1}u_1 + \cdots + (1 + c)^{k-1}u_k + u_k,
\]

where here we are suppressing bundle maps. For any integer $r$, one lets

\[
W[r] = \mathbb{W}(\mathbb{R}P(\eta)) / (1 + c)^{k-r}
\]

Note that each class $W[r]_j$ is a polynomial in the classes $w_i(\mathbb{R}P(\eta))$ and $c$. Further, these classes satisfy the following special properties (see [5], Section 2):

\[
W[r]_{2r} = \theta_r c^r + \text{terms with smaller c powers}, \\
W[r]_{2r+1} = (\theta_{r+1} + u_{r+1}) c^r + \text{terms with smaller c powers}.
\]

For $n \geq 3$, write $n - 2 = 2^p q$, where $p \geq 0$ and $q$ is odd, and suppose first that $p < q + 1$. Consider the list of integers $r_1, r_2, \ldots, r_p$, where $r_i = 2^p - 2^{p-i}$, and take the class

\[
X = W[2^p - 1]_{2^p+1} \cdot W[r_1]_{2r_1} \cdot W[r_2]_{2r_2} \cdots W[r_p]_{2r_p}
\]

(if $p = 0$, this class reduces to $X = W[0]^{q+1}$). The dimension of $X$ is

\[
(q + 1 - p)(2^{p+1} - 1) + 2 \sum_{i=1}^{p} (2^p - 2^{p-i}) = (2^{p+1} - 1)q + p + 1 = m(n - 2).
\]

From the properties above listed, one has

\[
X = ((\theta_{2^p} + u_{2^p}) c^{2^p-1} + \text{terms with smaller c powers })^{q+1-p}. \\
\cdots \\
(\theta_{r_p} c^{r_p} + \text{terms with smaller c powers }) = \\
= ((\theta_{2^p} + u_{2^p})^{q+1-p} \cdot \theta_{r_1} \cdot \theta_{r_2} \cdots \theta_{r_p}) c^{(q+1-p)(2^p-1) + \sum_{i=1}^{p} r_i + \text{terms with smaller c powers}}.
\]
Note that
\[(q + 1 - p) \cdot 2^p + \sum_{i=1}^{p} r_i = (q + 1 - p)2^p + p2^p - 2^p + 1 = 2^p q + 1 = n - 1.\]

Thus \(X\) has the form
\[X = A_{n-1} \cdot c^{m(n-2)-n+1} + \text{terms with smaller } c \text{ powers,}\]
where \(A_{n-1}\) is a class of dimension \(n - 1\) coming from the cohomology of \(F^n\).

Now suppose \(p \geq q + 1\), and consider the list \(r_1, r_2, \ldots, r_{q+1}\), where again \(r_i = 2^p - 2^{p-i}\). In this case, take
\[X = W[r_1]_2 r_1 \cdot W[r_2]_2 r_2 \cdots W[r_{q+1}]_2 r_{q+1}.\]

The dimension of \(X\) is
\[\sum_{i=1}^{q+1} r_i = \sum_{i=1}^{q+1} (2^{p+1} - 2^{p-i+1}) = (q + 1)2^{p+1} - 2^{p+1} + 2^{p-q} q 2^{p+1} + 2^{p-q} = (2^{p+1} - 2^{p-q})q + 2^{p-q}(q + 1) = m(n - 2)\]
and
\[X = \theta_{r_1} \cdot \theta_{r_2} \cdots \theta_{r_{q+1}} \cdot c^{r_1 + \cdots + r_{q+1}} + \text{terms with smaller } c \text{ powers.}\]

Note that
\[\sum_{i=1}^{q+1} r_i = \sum_{i=1}^{q+1} (2^p - 2^{p-i}) = (q + 1)2^p - 2^p + 2^{p-q-1} = 2^p q + 2^{p-q-1} = n - 2 + 2^{p-q-1} \geq n - 1.\]

Thus, for every \(n \geq 3\), \(X\) is a class of dimension \(m(n - 2)\) which has the form
\[X = A_l \cdot c^{m(n-2)-l} + \text{terms with smaller } c \text{ powers,}\]
where \(A_l\) has dimension \(l \geq n - 1\) and comes from the cohomology of \(F^n\).

Next we shall introduce some special classes of dimension 4 associated to line bundles \(\lambda \mapsto B^s\), where \(B^s\) is a closed \(s\)-dimensional manifold. Using the splitting principle, write
\[\mathbb{W}(B^s) = (1 + x_1) \cdot (1 + x_2) \cdots (1 + x_s)\]
and \( W(\lambda) = 1 + c \). Consider the symmetric polynomials in the variables
\( x_1, x_2, \ldots, x_s, c \), of degree 4, given by
\[
f_{\omega_1} = \sum_{i<j} x_i(x_i + c)x_j(x_j + c)
\]
and
\[
f_{\omega_2} = \sum_i x_i^2(x_i + c)^2
\]
Then \( f_{\omega_1} \) and \( f_{\omega_2} \) determine polynomials of dimension 4 in the classes \( w_i(B^s) \) and \( w_1(\lambda) = c \). Returning to \( \lambda \mapsto \mathbb{R}P(\eta) \), write
\[
W(F^n) = (1 + x_1) \cdot (1 + x_2) \cdots (1 + x_n) \quad \text{and} \quad W(\eta) = (1 + y_1) \cdot (1 + y_2) \cdots (1 + y_k).
\]
Then
\[
W(\mathbb{R}P(\eta)) = (1 + x_1) \cdots (1 + x_n)(1 + c + y_1) \cdots (1 + c + y_k).
\]
It follows that
\[
f_{\omega_1}(\lambda \mapsto \mathbb{R}P(\eta)) = \sum_{i<j} x_i(x_i + c)x_j(x_j + c) + \\
+ \sum_{t<l} y_t(y_t + c)y_l(y_l + c) + \\
+ \sum_{i,t} x_i(x_i + c)y_t(y_t + c) = \\
= \left( \sum_{i<j} x_ix_j + \sum_{t<l} y_ty_l + \sum_{i,t} x_iy_t \right) \cdot c^2 + \\
+ \text{terms with smaller } c \text{ powers},
\]
and
\[
f_{\omega_2}(\lambda \mapsto \mathbb{R}P(\eta)) = \sum_i x_i^2(x_i + c)^2 + \sum_t y_t^2(y_t + c)^2 = \\
= \left( \sum_i x_i^2 + \sum_t y_t^2 \right) \cdot c^2 + \sum_i x_i^4 + \sum_t y_t^4.
\]
Therefore every term of \( f_{\omega_1} \) and \( f_{\omega_2} \) has a factor of dimension at least 2 from the cohomology of \( F^n \). We have seen that each term of our previous class \( X \) has a factor of dimension at least \( n - 1 \) from the cohomology of \( F^n \), which means that, for \( i = 1, 2 \), \( f_{\omega_i} \cdot X \) is a class in \( H^{m(n-2)+4}(\mathbb{R}P(\eta), Z_2) \) with each one of its terms having a factor of dimension at least \( n + 1 \) from \( F^n \). Thus \( f_{\omega_i} \cdot X = 0 \).
Since $m > m(n-2) + 4$, one can form the class $f_\omega \cdot X \cdot c^{m-1-(m(n-2)+4)}$, which yields the zero characteristic number $f_\omega \cdot X \cdot c^{m-1-(m(n-2)+4)}[\mathbb{R}P(\eta)]$.

Our next task is to analyse the class associated to $\nu \mapsto \mathbb{R}P(\mu)$ which corresponds to $f_\omega \cdot X \cdot c^{m-1-(m(n-2)+4)}$. Setting $\mathbb{W}(\nu) = 1 + d$, this class is

$$f_\omega(\nu \mapsto \mathbb{R}P(\mu)) \cdot Y \cdot d^{m-1-(m(n-2)+4)},$$

where $Y$ is obtained from $X$ by replacing each $W[r]_i$ by $W[n + r - 2]_i$. The Stiefel-Whitney class of $\mathbb{R}P(\mu)$ is

$$\mathbb{W}(\mathbb{R}P(\mu)) = (1 + w_1 + w_2)((1 + d)^{n+k-2} + (1 + d)^{n+k-3}v_1 + (1 + d)^{n+k-4}v_2).$$

Writing $\mathbb{W}(F^2) = (1 + x_1)(1 + x_2)$ and $\mathbb{W}(\mu) = (1 + y_1)(1 + y_2)$, one has

$$\mathbb{W}(\mathbb{R}P(\mu)) = (1 + d)^{n+k-4}((1 + w_1 + w_2)((1 + d)^2 + (1 + d)v_1 + v_2)) = (1 + d)^{n+k-4}(1 + x_1)(1 + x_2)(1 + d + y_1)(1 + d + y_2).$$

Noting that the part $(1 + d)^{n+k-4}$ does not contribute to $f_\omega$, we get

$$f_\omega(\nu \mapsto \mathbb{R}P(\mu)) = x_1(x_1 + d)x_2(x_2 + d) + y_1(y_1 + d)y_2(y_2 + d) + \sum_{i,j} x_i(x_i + d)y_j(y_j + d) = (x_1x_2 + y_1y_2 + \sum_{i,j} x_iy_j)d^2 +$$

$$+ \text{terms with smaller } c \text{ powers},$$

and

$$f_{\omega_2}(\nu \mapsto \mathbb{R}P(\mu)) = x_1^2(x_1 + d)^2 + x_2^2(x_2 + d)^2 + y_1^2(y_1 + d)^2 + y_2^2(y_2 + d)^2 = (x_1 + x_2 + y_1 + y_2)^2d^2 + (x_1 + x_2 + y_1 + y_2)^4.$$
from $F^2$ and with positive dimension, one has that $f_{\omega_i} \cdot A = 0$ for each $A \in \mathcal{I}$. Thus, in the computation of $Y$, one needs to consider only that

$$W(\mathbb{R}P(\mu)) \equiv (1 + d)^{n+k-2} \mod \mathcal{I}$$

and for each integer $l$

$$W[l] \equiv (1 + d)^l \mod \mathcal{I}.$$ 

For $r_i = 2^p - 2^p - i$, $i = 1, 2, \ldots, p$, set $l_i = n + r_i - 2 = 2^p q + 2^p - 2^p - i - 2 = 2^p q + 2^p - 2^p - i$. Then

$$W[l_i]_{2r_i} \equiv \left(\frac{2^p q + 2^p - 2^p - i}{2^p + 1 - 2^p - i + 1}\right) d^{2r_i} \mod \mathcal{I}.$$ 

Also, if $r = 2^p - 1$, $l = n + r - 2 = 2^p q + 2^p - 1$ and

$$W[l]_{2r+1} \equiv \left(\frac{2^p q + 2^p - 1}{2^p + 1 - 1}\right) d^{2r+1} \mod \mathcal{I}.$$ 

The lesser term of the 2-adic expansion of $2^p q + 2^p$ is $2^p q + 1$. Using the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of $b$ is a subset of the 2-adic expansion of $a$, we conclude that the above binomial coefficients are nonzero modulo 2. It follows that all classes $W[r]$ occurring in $Y$ satisfy $W[r] \equiv d^l \mod \mathcal{I}$, which implies that $Y \equiv d^{n(n-2)} \mod \mathcal{I}$. Since $H^*(\mathbb{R}P(\mu), \mathbb{Z}_2)$ is the free $H^*(F^2, \mathbb{Z}_2)$-module on $1, d, d^2, \ldots, d^{n+k-3}$, we get

$$f_{\omega_1}(\nu) \cdot Y \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}P(\nu)] = d^{m-3} V_2[\mathbb{R}P(\nu)] = V_2[F^2]$$

and

$$f_{\omega_2}(\nu) \cdot Y \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}P(\nu)] = V_1[F^2].$$

Putting together with the previous calculations on $F^n$, we conclude that $V_2 = 0$ and $V_1^2 = 0$. Since $V_1 = v_1 + w_1$, we get $v_1 = w_1$, and since

$$V_2 = v_1 w_1 + v_2 + w_2 = S q^1(v_1) + v_2 + w_2 = v_1^2 + v_2 + w_2 = w_1^2 + v_2 + w_1^2 = v_2,$$

we get $v_2 = 0$. Thus Lemma 2.3 is proved.

Now we prove Theorem 2.4. One is considering an involution $(M^m, T)$ with fixed set $F$ of the form $F = F^n \cup F^2$, where the normal bundle $\mu \mapsto F^2$ represents $\beta_2$, and wants to show that $m \leq m(n-2) + 4$. We maintain the previous notations for the characteristic classes referring to the component $F^n$,
and we can suppose with no loss that \( \mu \mapsto F^2 = \xi \oplus \varepsilon^{m-3} \mapsto \mathbb{R}P^2 \). We repeat
the notations \( \nu \mapsto \mathbb{R}P(\mu) \) and \( W(\nu) = 1 + d \) for the standard line bundle over
\( \mathbb{R}P(\mu) \) and its characteristic class. Let \( \alpha \in H^1(F^2, \mathbb{Z}_2) \) be the generator. Since
\( H^*(\mathbb{R}P(\mu), \mathbb{Z}_2) \) is the free \( H^*(F^2, \mathbb{Z}_2) \)-module on \( 1, d, d^2, \ldots, d^{m-3} \) subject to
the relation \( d^{m-2} + d^{m-3} \alpha = 0 \), one has that \( d^{m-1} = d^{m-2} \alpha = d^{m-3} \alpha^2 \) is the
generator (top) of \( H^{m-1}(\mathbb{R}P(\mu), \mathbb{Z}_2) \). Our strategy will consist
in showing that, if \( m > m(n - 2) + 4 \), then it is possible to find polynomials in the characteristic
classes so that the corresponding characteristic numbers are zero on \( F^n \) and
nonzero on \( F^2 \). First consider \( n \) odd. In this case, we will obtain a stronger
result, noting that \( m(n - 2) + 4 = n + 3 \).

**Lemma 2.5** If \((M^m, T)\) is an involution fixing \( F = F^n \cup F^2 \), where \( n \) is odd
and \( \mu \mapsto F^2 = \xi \oplus \varepsilon^{m-3} \mapsto \mathbb{R}P^2 \), then \( m \leq n + 1 \) (hence \( m = n + 1 \)).

**Proof.** On \( F^n \) one has
\[
W[0] = (1 + \theta_1 + \theta_2 + \cdots + \theta_n)\left\{ \frac{u_1}{1 + c} + \cdots + \frac{u_k}{(1 + c)^k} \right\}.
\]
If \( m > n + 1 \), one can form the class \( W[0]_1 = (\theta_1 + u_1)^{n+1} \) of dimension \( m - 1 \).
Since \( W[0]_1 = (\theta_1 + u_1)^{n+1} \) comes from \( F^n \), this gives a zero characteristic
number. The class over \( F^2 \) corresponding to \( W[0] \) is \( W[n - 2] \). Now
\[
W(\mathbb{R}P(\mu)) = (1 + \alpha + \alpha^2)\left\{ (1 + d)^{m-2} + (1 + d)^{m-3} \alpha \right\}
\]
and
\[
W[n - 2] = (1 + \alpha + \alpha^2)\left\{ (1 + d)^{n-2} + (1 + d)^{n-3} \alpha \right\}.
\]
Since \( n \) is odd,
\[
W[n - 2]_1 = \binom{n - 2}{1} d + \alpha + \alpha = d,
\]
which gives the nonzero characteristic number
\[
W[n - 2]_1 d^{m-1-(n+1)}[\mathbb{R}P(\mu)] = d^{m-1}[\mathbb{R}P(\mu)].
\]

\[\square\]

Now we consider \( n \) even, which means in particular that \( n \geq 4 \). Write
\( n - 2 = 2^p q \), where \( p, q \geq 1 \). Over \( F^n \) one takes the same class \( X \) considered
before; that is , \( X \in H^{m(n-2)}(\mathbb{R}P(\eta), \mathbb{Z}_2) \) and each term of \( X \) has a factor
of dimension at least \(n - 1\) from the cohomology of \(F^n\). Note that, on \(F^n\),
\[
W[0]_2 = \theta_2 + \theta_1 u_1 + u_1 c + u_2.
\]
Hence every term of \(W[0]_2^2 = \theta_2^2 + \theta_1^2 u_1^2 + u_1^2 c^2 + u_2^2\) has a factor of dimension at least 2 from \(F^n\). If \(m > m(n - 2) + 4\), one then has the zero characteristic number
\[
X \cdot W[0]_2^2 \cdot c^{m-1-(m(n-2)+4)}[\mathbb{R}P(\eta)].
\]
Our next and final task will be to show that, over \(F^2\), the corresponding characteristic number
\[
Y \cdot W[n-2]_2^2 \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}P(\mu)]
\]
is nonzero. First note that a general element of \(H^t(\mathbb{R}P(\mu), \mathbb{Z}_2)\) is of the form
\[
a_0 d^t + a_1 \alpha d^{t-1} + a_2 \alpha^2 d^{t-2},
\]
where \(a_i = 0\) or 1. In particular, for the top-generator of \(H^{m-1}(\mathbb{R}P(\mu), \mathbb{Z}_2)\), the number of 1’s in \(\{a_0, a_1, a_2\}\) is 1 or 3. From
\[
W(\mathbb{R}P(\mu)) = (1 + \alpha + \alpha^2)\{(1 + d)^{m-2} + (1 + d)^{m-3}\alpha\}
\]
we get
\[
W[l] = (1 + \alpha + \alpha^2)\{(1 + d)^l + (1 + d)^{l-1}\alpha\}
\]
and
\[
W[l]_t = \binom{l}{t} d^t + \left\{\binom{l-1}{t-1} + \binom{l}{t-1}\right\} \alpha d^{t-1} + \left\{\binom{l-1}{t-2} + \binom{l}{t-2}\right\} \alpha^2 d^{t-2}.
\]
To compute \(Y\), now write \(r_i = 2^p - 2^i, \ i = 0, 1, \ldots, p - 1,\) and set as before
\(l_i = n + r_i - 2 = 2^p q + 2^p - 2^i\). Then
\[
W[l_i]_{2r_i} = \left(\frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1}}\right) d^{2r_i} +
\]
\[
+ \left\{\frac{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 1} + \frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 1}\right\} \alpha d^{2r_i-1} +
\]
\[
+ \left\{\frac{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 2} + \frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 2}\right\} \alpha^2 d^{2r_i-2}.
\]
By inspection of 2-adic expansions, one gets the following values for the above binomial coefficients:
i) \( \left( \frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1}} \right) \equiv 1 \mod 2, \)

ii) \( \left( \frac{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 1} \right) \equiv 0 \mod 2, \)

iii) \( \left( \frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 1} \right) \equiv \begin{cases} 1 \mod 2, & \text{if } i = 0, \\ 0 \mod 2, & \text{if } i \geq 1, \end{cases} \)

iv) \( \left( \frac{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 2} \right) \equiv \begin{cases} 1 \mod 2, & \text{if } i = 0, \\ 0 \mod 2, & \text{if } i \geq 1, \end{cases} \)

and

v) \( \left( \frac{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 2} \right) \equiv \begin{cases} 1 \mod 2, & \text{if } i = 0 \text{ or } 1, \\ 0 \mod 2, & \text{if } i \geq 2. \end{cases} \)

It follows that

\[
W[l_i]_{2r_i} \equiv \begin{cases} d^{2r_i} + \alpha d^{2r_i-1}, & \text{if } i = 0, \\ d^{2r_i} + \alpha^2 d^{2r_i-2}, & \text{if } i = 1, \\ d^{2r_i}, & \text{if } i \geq 2. \end{cases}
\]

For \( r = 2^p - 1, \ l = n + r - 2 = 2^p q + 2^p - 1 \) and

\[
W[l]_{2r+1} = \left( \frac{2^p q + 2^p - 1}{2^{p+1} - 1} \right) d^{2r+1} + \\
+ \left\{ \left( \frac{2^p q + 2^p - 2}{2^{p+1} - 2} \right) + \left( \frac{2^p q + 2^p - 1}{2^{p+1} - 2} \right) \right\} \alpha d^{2r} + \\
+ \left\{ \left( \frac{2^p q + 2^p - 2}{2^{p+1} - 3} \right) + \left( \frac{2^p q + 2^p - 1}{2^{p+1} - 3} \right) \right\} \alpha^2 d^{2r-1}. 
\]

In the above expression, the unique binomial coefficient which is zero is \( \left( \frac{2^p q + 2^p - 2}{2^{p+1} - 3} \right). \)

Thus \( W[l]_{2r+1} = d^{2r+1} + \alpha^2 d^{2r-1}. \) With these \( l_i's \) and \( l \), and for \( p \leq q + 1 \), one then has that

\[
Y = (W[l])^{q+1-p} \cdot \prod_{i=0}^{p-1} W[l_i]_{2r_i} = \\
= (d^{2r+1} + \alpha^2 d^{2r-1})^{q+1-p} \cdot (d^{2r_0} + \alpha d^{2r_0-1}) \cdot (d^{2r_1} + \alpha^2 d^{2r_1-2}) \cdot d^{2(r_2 + \cdots + r_{p-1})}. 
\]
Because of the rule
\[
(d^t + \alpha^2 d^{t-1})^s = \begin{cases} 
  d^{ts}, & \text{if } s \text{ is even}, \\
  d^{ts} + \alpha^2 d^{ts-2}, & \text{if } s \text{ is odd},
\end{cases}
\]
and the fact that \( q+1 - p \equiv p \mod 2 \), we get that
\[
Y = \begin{cases} 
  d^{m(n-2)} + \alpha d^{m(n-2)-1} + \alpha^2 d^{m(n-2)-2}, & \text{if } p \text{ is even}, \text{ and} \\
  d^{m(n-2)} + \alpha d^{m(n-2)-1}, & \text{if } p \text{ is odd}.
\end{cases}
\]
For \( p > q + 1 \), one has
\[
Y = \prod_{i=p-(q+1)}^{p-1} W[l_i]_{2r_i} = \begin{cases} 
  d^{m(n-2)} + \alpha^2 d^{m(n-2)-2}, & \text{if } p - (q + 1) = 1, \text{ and} \\
  d^{m(n-2)}, & \text{if } p - (q + 1) > 1.
\end{cases}
\]
With the values of \( Y \) on hand, the final step is the calculation of \( W[n-2]_2^2 \) on \( F^2 \). One has
\[
W[n-2] = (1 + \alpha + \alpha^2) \{(1 + d)^n - (1 + d)^{n-3} \alpha\}
\]
and
\[
W[n-2]_2^2 = \left( \binom{n-2}{2} d^2 + \binom{n-2}{1} + \binom{n-3}{1} \right) \alpha d = \left( \frac{2p^2 q}{2} \right) d^4 + \alpha^2 d^2 = \begin{cases} 
  \alpha^2 d^2, & \text{if } p > 1, \\
  d^4 + \alpha^2 d^2, & \text{if } p = 1.
\end{cases}
\]
Since \( Y \) has the form \( d^t, d^t + \alpha d^{t-1}, d^t + \alpha^2 d^{t-2} \) or \( d^t + \alpha d^{t-1} + \alpha^2 d^{t-2} \), for \( p > 1 \) one has \( Y \cdot W[n-2]_2^2 = \alpha^2 d^{m(n-2)+2} \). If \( p = 1 \), \( Y = d^{m(n-2)} + \alpha d^{m(n-2)-1} \) and
\[
Y \cdot W[n-2]_2^2 = (d^{m(n-2)} + \alpha d^{m(n-2)-1}) \cdot (d^4 + \alpha^2 d^2) = \begin{cases} 
  d^{m(n-2)+4} + \alpha d^{m(n-2)+3} + \alpha^2 d^{m(n-2)+2}, & \text{if } p = 1,
\end{cases}
\]
In any case, \( Y \cdot W[n-2]_2^2 \cdot d^{m-1-(m(n-2)+4)} [\mathbb{R} P(\mu)] \) is a nonzero characteristic number, and our task is ended.

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