Z$^2_2$-ACTIONS WITH N-DIMENSIONAL FIXED POINT SET

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Abstract. We describe the equivariant cobordism classification of smooth actions $(M^n, \Phi)$ of the group $G = Z^2_2$, considered as the group generated by two commuting involutions, on closed smooth $m$-dimensional manifolds $M^m$, for which the fixed point set of the action is a connected manifold of dimension $n$ and $m = 4n - 1$ or $4n - 2$. For $m \geq 4n$, the classification is known.

1. Introduction

In [7], R. E. Stong determined all possible equivariant cobordism classes of smooth involutions $T : M \to M$, defined on closed smooth $m$-dimensional manifolds $M$, for which the fixed point set $F$ has dimension $n$ and $m = 2n - 1$; specifically, he showed that any such involution pair $(M, T)$ is equivariantly cobordant to an union of involutions of the following two types:

i) Let $\pi : N \to S^1$ be a smooth fibering, where $N$ is a closed $n$-dimensional manifold and $S^1$ is the 1-sphere, and let $M_1 = \{(x, y) \in N \times N / \pi(x) = \pi(y)\}$, with involution $T_1(x, y) = (y, x)$.

ii) Let $P$ and $Q$ be closed manifolds with dimensions $p$ and $q$, respectively, where $p + q = n - 1$, and let $M_2(P, Q) = S^1 \times P \times P \times Q \times Q \sim \{x, p, p', q, q'\} \sim \{-x, p', p, q, q'\}$ with the involution $T_2[x, p, p', q, q'] = [x, p', p, q', q]$.

In general, for a given $(M, T)$, several summands of the second type, with different $p$ values, may be needed; every such union of involutions is equivariantly cobordant to an involution with $(n$-dimensional) connected fixed set.

1991 Mathematics Subject Classification. (2.000 Revision) Primary 57R85; Secondary 57R75.
Key words and phrases. $Z^2_2$-action, fixed data, equivariant cobordism class, characteristic number, projective space bundle, Stiefel-Whitney class.
The author was partially supported by CNPq and FAPESP.

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Note. The equivariant cobordism classification for $\dim(F) = n$ and $\dim(M) \geq 2n$ was obtained in [1].

The purpose of this note is to extend the above result for $\mathbb{Z}_2^2$-actions; here, $G = \mathbb{Z}_2^2$ is understood as the group generated by two commuting involutions $T_1, T_2$. Specifically, we determine all possible equivariant cobordism classes of $G$ or 4 in [3]. To state the results, we need to describe certain $G$-connected and $G$-given a fixed set $F$ bundle of $F$ then the same is true for ($\mathbb{R}$ this is realized by new $G$ can form the $W, T$ fixed data. This will be combined with the fact that, from an involution ($\varepsilon$ equivalent to $\varepsilon$ $M$ [4]: suppose ($\mathbb{R}$ has a section, that is, there exists a vector bundle of $\varepsilon$ ($F \Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3$), where $\varepsilon_i$ is the normal bundle of $F$ in $F_{T_i}$, $i = 1, 2, 3$. If ($F; \varepsilon_i, \varepsilon_j, \varepsilon_k$) is the fixed data of some $G$-action, then the same is true for ($F; \varepsilon_i, \varepsilon_j, \varepsilon_k$), where $(i, j, k)$ is any permutation of $(1, 2, 3)$; this is realized by new $G$-actions obtained from permutations of $(T_1, T_2, T_3)$. Denote by $R \to X$ the trivial one-dimensional vector bundle over any base space $X$ and by $\tau(F) \to F$ the tangent bundle of $F$, where $F$ is a manifold. In order to obtain the above mentioned $G$-actions, the crucial point is the following section theorem of [4]: suppose $(M^m, \Phi)$ a $G$-action with fixed data $(F \Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, and suppose that $\varepsilon_1$ has a section, that is, there exists a vector bundle $\varepsilon_1'$ over $F \Phi$ so that $\varepsilon_1' \oplus R$ is equivalent to $\varepsilon_1$. Then there exists a $G$-action $(N^{n-1}, \Psi)$ having $(F \Phi; \varepsilon_1', \varepsilon_2, \varepsilon_3)$ as fixed data. This will be combined with the fact that, from an involution $(W, T)$, we can form the $G$-action $(W \times W, \Phi), \Phi = (T_1, T_2)$, where $T_1(x, y) = (T(x), T(y))$ and $T_2(x, y) = (y, x)$; we denote this $G$-action by $\Gamma(W, T)$. The fixed data of $\Gamma(W, T)$ is $(F_T; \tau(F_T), \eta, \eta)$, where $\eta \to F_T$ is the normal bundle of $F_T$ in $W$.

A) $G$-actions $(M^m, \Phi)$ with $m = 4n - 1$: let $F^n$ be a connected and closed $n$-dimensional manifold such that $\tau(F^n)$ is cobordant, as a bundle, to a vector bundle $\mu^n \to F^n$ which has a section, say $\mu^n \cong \nu^{n-1} \oplus R$. From [2], one knows that there is an involution $(W, T)$, equivariantly cobordant to the twist involution on
$F^n \times F^n$, with fixed set $F^n$ and with $\mu^n$ being the normal bundle of $F^n$ in $W$. The fixed data of $\Gamma(W, T)$ then is $(F^n; \tau(F^n), \mu^n, \mu^n)$, and by removing one section one obtains a $G$-action $(N^{4n-1}, \Psi)$ with fixed data $(F^n; \tau(F^n), \mu^n, \nu^{n-1})$. Evidently, this includes the case in which $\tau(F^n)$ has itself a section; for example, when $F^n$ is any closed manifold with $n$ odd. In this case, these actions are obtained by removing one section from the fixed data of the action $(F^n \times F^n \times F^n \times F^n, T_1, T_2)$, $T_1(x, y, z, w) = (y, x, w, z), T_2(x, y, z, w) = (z, w, x, y)$.

B) $G$-actions $(M^m, \Phi)$ with $m = 4n - 2$: we describe two types of such actions. First, take $F^n$ as in A), and suppose that also $\nu^{n-1}$ has a section, $\nu^{n-1} \cong \theta^{n-2} \oplus R$. By removing this additional section, one obtains a $G$-action $(Q^{4n-2}, \varphi)$ with fixed data $(F^n; \tau(F^n), \mu^n, \theta^{n-2})$. Second, one takes the $G$-actions $\Gamma(W, T)$, where $(W, T)$ is any involution with $\dim(W) = 2n - 1$ and with $F_T$ connected and $n$-dimensional, that is, equivariantly cobordant to the ones described by Stong. This includes the case where, taking $F^n$ as in A), we remove two sections to get a $G$-action with fixed data $(F^n; \tau(F^n), \nu^{n-1}, \nu^{n-1})$.

The desired classification is given by the following

**Theorem.** Let $(M^m, \Phi)$ be a $Z_2^2$-action whose fixed point set $F_\Phi$ is connected and $n$-dimensional, and where either $m = 4n - 1$ or $m = 4n - 2$. Then, either $(M^m, \Phi)$ bounds equivariantly, or it is, up to permutations, equivariantly cobordant to an action of type A) if $m = 4n - 1$ and of type B) if $m = 4n - 2$.

**Note.** For involutions, the connectedness of the fixed point set is redundant, since any involution is equivariantly cobordant to an involution with the property that the $p$-dimensional part of the fixed set is connected. However, this is not true for $Z_2^2$-actions, since in this case the fixed data over different components of the $p$-dimensional part of the fixed set may have different lists of dimensions.

2. **Proofs of the results**

In order to prove the stated result, we need some preliminaries about $G$-actions. Let $(M, \Phi), \Phi = (T_1, T_2)$, be a $G$-action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Each
s-dimensional component of \((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\) can be considered as an element of 
\(\mathcal{N}_s(BO(n_1) \times BO(n_2) \times BO(n_3))\), the bordism of s-dimensional manifolds with 
a map into \(BO(n_1) \times BO(n_2) \times BO(n_3)\), where \(n_i\) is the dimension of \(\varepsilon_i\) over the 
component and \(BO(n_i)\) is the classifying space for \(n_i\)-dimensional vector bundles 
(this is the simultaneous cobordism between lists of bundles). According to [6], two 
\(G\)-actions are equivariantly cobordant if and only if they have fixed data simulta-
neously cobordant. Also, if \((M, \Phi)\) has fixed data \((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\) and 
\((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\) is simultaneously cobordant to \((F; \mu_1, \mu_2, \mu_3)\), then there exists a 
\(G\)-action \((N, \Psi)\) with fixed data \((F; \mu_1, \mu_2, \mu_3)\), hence equivariantly cobordant to \((M, \Phi)\). The fol-
lowing result is the lemma found in [3; Section 3; page 108], particularized to 
\(\mathbb{Z}_2\)-actions.

**Lemma 2.1.** Let \((M, \Phi)\) be a \(G\)-action with fixed data \((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\), where \(F\) is 
connected and \(n\)-dimensional. Then,

a) if \(\dim(\varepsilon_1) > n\), \((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\) bounds simultaneously;

b) if \(\dim(\varepsilon_1) = n\), \((F; \varepsilon_1, \varepsilon_2, \varepsilon_3)\) is simultaneously cobordant to \((F; \tau(F; \varepsilon_1, \varepsilon_2, \varepsilon_3))\).

Now consider a \(n\)-dimensional vector bundle \(\varepsilon \to F\), where \(F\) is a connected, 
closed and \(n\)-dimensional manifold, and let \(RP(\varepsilon) \to F\) be the real projective 
space bundle associated to \(\varepsilon\). Denote by \(\lambda \to RP(\varepsilon)\) the line bundle of the double 
cover \(S(\varepsilon) \to RP(\varepsilon), S(\varepsilon)\) the sphere bundle of \(\varepsilon\). Suppose that \(\mu\) and \(\nu\) are two 
additional vector bundles over \(F\), with \(\dim(\mu) = p, \dim(\nu) = q\) and \(p \geq q\). The 
crucial point of the argument to be used is the following

**Lemma 2.2.** Suppose that the list \((RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))\) bounds as an element of 
\(\mathcal{N}_{2n-1}(BO(1) \times BO(p + q))\). Then the list \((F; \varepsilon, \mu, \nu)\) is simultaneously cobordant 
to the list \((F; \varepsilon, \nu \oplus (p - q)R, \nu)\). Here, \((p - q)R\) means the Whitney sum of \(p - q\) 
copies of \(R\), and the bundles \(\mu\) and \(\nu\) are considered over \(RP(\varepsilon)\) via pullbacks by 
the projection.

**Proof.** One lets \(W(F) = 1 + w_1 + \cdots + w_n, W(\varepsilon) = 1 + v_1 + \cdots + v_n, W(\mu) = 
1 + u_1 + \cdots + u_p, W(\nu) = 1 + \theta_1 + \cdots + \theta_q\) and \(W(\lambda) = 1 + c\) be the Stiefel-Whitney
classes of $F$, $\varepsilon$, $\mu$, $\nu$ and $\lambda$. One knows that the Stiefel-Whitney class of $RP(\varepsilon)$ is

$$W(RP(\varepsilon)) = (1 + w_1 + \ldots + w_n).\{(1 + c)^n + v_1(1 + c)^{n-1} + \ldots + v_{n-1}(1 + c) + v_n\}$$

and the Stiefel-Whitney class of the bundle $\mu \oplus (\nu \otimes \lambda)$ is

$$W(\mu \oplus (\nu \otimes \lambda)) = (1 + u_1 + \ldots + u_p).\{(1 + c)^q + \theta_1(1 + c)^{q-1} + \ldots + \theta_q(1 + c) + \theta_{q+1}\}.$$ 

Because the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ is a simultaneous boundary, any class of dimension $2n - 1$ given by a product of classes $w_i(RP(\varepsilon))$, $c$, $w_j(\mu \oplus (\nu \otimes \lambda))$, evaluated on the fundamental homology class $[RP(\varepsilon)]$, gives a zero characteristic number. For any $r$, one lets

$$W[r] = \frac{W(RP(\varepsilon))}{(1 + c)^{n-r}} \quad \text{and} \quad V[r] = \frac{W(\mu \oplus (\nu \otimes \lambda))}{(1 + c)^{q-r}}.$$ 

That is,

$$W[r] = (1 + w_1 + \ldots + w_n)\{(1 + c)^r + v_1(1 + c)^{r-1} + \ldots + v_r + \frac{v_{r+1}}{1 + c} + \ldots + \frac{v_n}{(1 + c)^{n-r}}\}$$

and

$$V[r] = (1 + u_1 + \ldots + u_p)\{(1 + c)^r + \theta_1(1 + c)^{r-1} + \ldots + \theta_r + \frac{\theta_{r+1}}{1 + c} + \ldots + \frac{\theta_q}{(1 + c)^{q-r}}\}.$$ 

The classes $W[r]_i$ and $V[r]_j$ are polynomials in $w_i(RP(\varepsilon))$, $c$, $w_j(\mu \oplus (\nu \otimes \lambda))$, hence they can be used to give characteristic numbers; also, for these classes, one has the following special properties (see [5]):

$W[r]_i = w_i c^r + \text{terms with smaller } c \text{ powers},$

$W[r]_{i+r+1} = (w_{i+r+1} + v_{r+1})c^r + \text{terms with smaller } c \text{ powers},$

$W[r]_{i+r+2} = v_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers},$

and in the same way,

$V[r]_i = u_i c^r + \text{terms with smaller } c \text{ powers},$

$V[r]_{i+r+1} = (u_{i+r+1} + \theta_{r+1})c^r + \text{terms with smaller } c \text{ powers},$

$V[r]_{i+r+2} = \theta_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers}.$

For a sequence $\omega = (i_1, \ldots, i_s)$ of natural numbers, one lets $|\omega| = i_1 + \ldots + i_s$, and for $w = 1 + w_1 + \ldots + w_p$, one lets $w_\omega = w_{i_1} \ldots w_{i_s}$ be the product of the classes $w_i$. Then given sequences $\omega = (i_1, \ldots, i_s)$, $\omega' = (j_1, \ldots, j_t)$, $\beta = (a_1, \ldots, a_k)$ and $\beta' = (b_1, \ldots, b_l)$, and a natural number $1 \leq r \leq p$ with $|\omega| + |\omega'| + |\beta| + |\beta'| + r = n$, one may form the class

$$X = \left(\prod_{i \in \omega} W[i]_{2i}\right) \cdot \left(\prod_{i \in \omega'} W[i - 1]_{2i}\right) \cdot \left(\prod_{i \in \beta} V[i]_{2i}\right) \cdot \left(\prod_{i \in \beta'} V[i - 1]_{2i}\right) \cdot V[r - 1]_{2r-1}.$$
Since $X$ has dimension $2n - 1$, it gives the zero characteristic number $X[RP(\varepsilon)]$.

From the properties above listed, one has
\[
\begin{align*}
\prod_{i \in \omega} W[i]_{2i} &= W(F)_{\omega}, c|\omega| + \text{terms with smaller } c \text{ powers}, \\
\prod_{i \in \omega} W[i - 1]_{2i} &= W(\varepsilon)_{\omega}', c|\omega'| + \text{terms with smaller } c \text{ powers}, \\
\prod_{i \in \beta} V[i]_{2i} &= W(\mu)_{\beta}, c|\beta| + \text{terms with smaller } c \text{ powers}, \\
\prod_{i \in \beta'} V[i - 1]_{2i} &= W(\nu)_{\beta'}, c|\beta'| + \text{terms with smaller } c \text{ powers and} \\
V[r - 1]_{2r - 1} &= (u_r + \theta_r)\cdot c^r + \text{terms with smaller } c \text{ powers}.
\end{align*}
\]

It follows that $X$ has the form
\[
X = W(F)_{\omega}, W(\varepsilon)_{\omega}', W(\mu)_{\beta}, W(\nu)_{\beta'}, (u_r + \theta_r)\cdot c^{n-1} + \text{terms with smaller } c \text{ powers}.
\]

Now if a term of dimension $2n - 1$ involves a power of $c$ less than $n - 1$, it necessarily has a factor of dimension greater than $n$ coming from the cohomology of $F$, which is zero. Also one knows that $H^*(RP(\varepsilon); Z_2)$ is the free $H^*(F; Z_2)$ module on $1, c, c^2, ..., c^{n-1}$. Therefore
\[
0 = X[RP(\varepsilon)] = W(F)_{\omega}, W(\varepsilon)_{\omega}', W(\mu)_{\beta}, W(\nu)_{\beta'}, (u_r + \theta_r)\cdot c^{n-1}[RP(\varepsilon)] = W(F)_{\omega}, W(\varepsilon)_{\omega}', W(\mu)_{\beta}, W(\nu)_{\beta'}, (u_r + \theta_r)[F]
\]
and thus
\[
W(F)_{\omega}, W(\varepsilon)_{\omega}', W(\mu)_{\beta}, W(\nu)_{\beta'}, u_r[F] = W(F)_{\omega}, W(\varepsilon)_{\omega}', W(\mu)_{\beta}, W(\nu)_{\beta'}, \theta_r[F].
\]

This says that any class $u_r$ in a characteristic number of $(F; \varepsilon, \mu, \nu)$ can be replaced by $\theta_r$ without changing the value of the characteristic number. In particular, if $q < r \leq p$, the class $u_r$ can be replaced by the zero class, and thus $(F; \varepsilon, \mu, \nu)$ and $(F; \varepsilon, \nu \oplus (p - q)R, \nu)$ have the same characteristic numbers, which gives the result.

Lemma 2.3. Let $(M, \Phi)$ be a $G$-action with fixed data $(F; \varepsilon, \mu, \nu)$, where $F$ is connected, $\dim(F) = \dim(\varepsilon) = n$, $\dim(\mu) = p$ and $\dim(\nu) = q$, with $p \geq q$. Then $(M, \Phi)$ is equivariantly cobordant to an action obtained by removing $p - q$ sections from the third bundle of the fixed data of the action $\Gamma(W, T)$, where $(W, T)$ is an involution fixing $F$ and with the normal bundle of $F$ in $W$ being $\nu \oplus (p - q)R$ (if $p = q$, $(M, \Phi)$ is equivariantly cobordant to the action $\Gamma(W, T)$, where $(W, T)$ is an involution with fixed set and normal bundle $\nu \to F$).

Proof. From the argument outlined in [3; Section 2; pages 107 and 108], particularized to $Z^2_3$-actions, one has that the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \oplus \lambda))$ bounds as an
element of $N_{2n-1}(BO(1) \times BO(p+q))$. First using Lemma 2.2 for $(F; \varepsilon, \mu, \nu)$, and
next using part b) of Lemma 2.1 for $(F; \varepsilon, \nu \oplus (p-q)R, \nu)$, one concludes that
$(M, \Phi)$ is equivariantly cobordant to a $G$-action $(N, \Psi)$, $\Psi = (T_1, T_2)$, with fixed
data $(F; \tau(F), \nu \oplus (p-q)R, \nu)$. Let $W \subset N$ be the component of $F_{T_2}$ that contains
$F$. Then the involution $(W, T_1)$ fixes $F$ with normal bundle $\nu \oplus (p-q)R \rightarrow F$, and
thus $(N, \Psi)$ (hence $(M, \Phi)$) is equivariantly cobordant to a $G$-action obtained by
removing $p-q$ sections from the third bundle of the fixed data of $\Gamma(W, T_1)$. □

Now we prove the result. Let $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ denote the fixed data of the $G$-
action $(M^m, \Phi)$. If some $\varepsilon_i$ has dimension greater than $n$, then part a) of Lemma
2.1, used up to permutation, says that $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously, and
thus $(M^m, \Phi)$ bounds equivariantly. Therefore we can suppose that $\dim(\varepsilon_i) \leq n$ for $i = 1, 2, 3$. By making permutations if necessary, we can suppose that
$(\dim(\varepsilon_1), \dim(\varepsilon_2), \dim(\varepsilon_3))$ then is $(n, n, n-1)$ if $m = 4n-1$, and either $(n, n, n-2)$
or $(n, n-1, n-1)$ if $m = 4n-2$. In the case $(n, n, n-1)$ ($(n, n, n-2)$), Lemma
2.3 says that $(M^m, \Phi)$ is equivariantly cobordant to an action obtained by
removing one section (two sections) from the third bundle of the fixed data of the
action $\Gamma(W, T)$, where $(W, T)$ is an involution with fixed set and normal bundle
$\varepsilon_3 \oplus R \rightarrow F_\Phi$ ($\varepsilon_3 \oplus 2R \rightarrow F_\Phi$). Since $\dim(F_\Phi) = n$ and $\dim(W) = 2n$, one has from
[1] that $(W, T)$ is equivariantly cobordant to $(F_\Phi \times F_\Phi, twist)$, and thus $\varepsilon_3 \oplus R \rightarrow F_\Phi$
($\varepsilon_3 \oplus 2R \rightarrow F_\Phi$) is cobordant to $\tau(F_\Phi)$. This means that $(M^m, \Phi)$ is equivariantly cobordant to an action of type A) (of type B)).

In the case $(n, n-1, n-1)$, Lemma 2.3 says that $(M^m, \Phi)$ is equivariantly cobordant to the action $\Gamma(W, T)$, where $(W, T)$ is an involution with fixed set and
normal bundle $\varepsilon_3 \rightarrow F_\Phi$. Since $\dim(F_\Phi) = n$ and $\dim(W) = 2n-1$, $(W, T)$ is of
the Stong’s type, and thus $(M^m, \Phi)$ is equivariantly cobordant to an action of type
B). This ends the proof.

References
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