CANTOR SINGULAR CONTINUOUS SPECTRUM FOR OPERATORS ALONG INTERVAL EXCHANGE TRANSFORMATIONS

M. COBO, C. GUTIERREZ, AND C. R. DE OLIVEIRA

Abstract. It is shown that Schrödinger operators, with potentials along the shift embedding of Lebesgue almost every interval exchange transformations, have Cantor spectrum of measure zero and pure singular continuous for Lebesgue almost all points of the interval.

1. Introduction and Main Results

In [7] the spectrum $\sigma(H_\omega)$ of the discrete Schrödinger operators $H_\omega : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$,

$$ (H_\omega \psi)_j = \psi_{j+1} + \psi_{j-1} + \omega_j \psi_j, $$

with $\omega = (\omega_j)_{j \in \mathbb{Z}}$ a sequence of real numbers (the so-called potential) modulated along the shift embedding of interval exchange transformations (jets) [10, 11] was investigated. There it was proved the presence of pure singular continuous spectrum for $H_\omega$, for the shift associated with a dense set of jets and for a.e. points of the interval (a.e. with no specification means almost everywhere with respect to Lebesgue measure). In this work we give a step further by showing that the above mentioned results hold for Lebesgue almost every jets. The proof of absence of eigenvalues involves a rather different argument, mainly the Rauzy induction map. As a by-product of our a.e. results, which are summarized in Theorem 1 ahead, there is a set of interesting results in the literature ready to be applied, which will lead to Cantor spectrum of zero Lebesgue measure. We next recall some notations and a description of the jets necessary to state and prove our results. Let $\pi$ be a permutation of the symbols $\{1, 2, \ldots, n\}$ and

$$ a = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\} $$

be a partition of the interval $[a, b]$. An jet $E : [a, b) \to [a, b)$ is associated to the pair $(\pi, a)$ by cutting the interval $[a, b)$ in $n$ semi-open intervals

$$ I_1 := [a_0, a_1), I_2 := [a_1, a_2), \ldots, I_n := [a_{n-1}, a_n) $$

(which are naturally ordered from left to right) and then exchanging their positions according to the permutation $\pi$ in such a way that the interval in the $i^{th}$ position, $I_i$, is translated to the $\pi(i)^{th}$ position (from left to right). In this way the transformation obtained is of the form

$$ E(x) = x + d_i, \quad x \in I_i, \quad i = 1, 2, \ldots, n, $$

Key words: Schrödinger operator; interval exchange transformation; singular continuous spectrum; Cantor spectrum.

2000 Mathematics Subject Classification. 47B36,47B37,37B05,37B10.
for some numbers \(d_1, \ldots, d_n\). We will consider the symbolic coding, that is, the map \(A_E : [a, b) \to \{1, 2, \ldots, n\}\) given by
\[
A_E(x) = i \quad \text{if and only if} \quad x \in I_i,
\]
and write simply \(A\) in situations where \(E\) is clear from the context. In this work we consider mostly interval exchanges of the form \(E : [0, b) \to [0, b), b > 0\). \(E\) is said to be normalized if \(b = 1\). Denote by \(P_n\) the set of all permutations of the symbols \(\{1, 2, \ldots, n\}\) and by \(G_n\) the set of irreducible permutations of \(P_n\), i.e., those permutations \(\pi\) for which \(\pi(\{1, 2, \ldots, k\}) \neq \{1, 2, \ldots, k\}\) unless \(k = n\). \(\Delta^{n-1}\) will denote the standard simplex of \(\mathbb{R}^n\), i.e.,
\[
\Delta^{n-1} := \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n_+ : \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.
\]
\(\Lambda_n\) will denote the set of all partitions of \([0, 1]\). In the standard simplex \(\Delta^{n-1}\) we will consider the \((n-1)\)-dimensional Lebesgue measure. There relations \(\lambda_i = a_i - a_{i-1}, 1 \leq i \leq n\) determine a one-to-one correspondence between partitions \(a \in \Lambda_n\) and vectors \(\lambda \in \Delta^{n-1}\).

Identify the product \(P_n \times \Delta^{n-1}\) with the set of all interval exchanges of \(n\) intervals. For a (fixed) irreducible permutation \(\pi \in G_n\), denote by \(E(\pi)\) the set of all sets \(E : [0, 1) \to [0, 1)\) with permutation \(\pi\). We identify the metric spaces \(\Delta^{n-1}\) and \(E(\pi)\) by the homeomorphism \(\Delta^{n-1} \ni \lambda \mapsto E_\lambda := (\pi, \lambda)\). Denote the \(E_\lambda\)-orbit of \(x \in [0, 1)\) by
\[
O_\lambda(x) = \{E^k_\lambda(x) : k \in \mathbb{Z}\}.
\]
Let \(\Sigma_n := \{1, 2, \ldots, n\}^\mathbb{Z}\). Associated to each orbit is \(\phi_\lambda(x) \in \Sigma_n\) given by \(\phi_\lambda(x) := A(O_\lambda(x))\); i.e., \(\phi_\lambda(x)\) is a natural coding of the \(E_\lambda\)-orbit of \(x\) by assigning to each entry of this orbit the number of the interval which contains it. The same concept is defined for intervals: take numbers \(j, k \in \mathbb{Z}\) with \(j \leq k\), suppose that \(I \subset [a, b)\) is a nonempty interval (which may be reduced to a point) such that, for all integer \(i \in [j, k]\), \(E^i|_I\) is continuous; then the sequence
\[
A(E^j(I)) A(E^{j+1}(I)) \cdots A(E^k(I))
\]
will be said to be the \(E\)-itinerary of \(I\) associated to \([j, k]\). Set
\[
\Omega_\lambda = \text{closure } \{\phi_\lambda([0, 1))\}
\]
in \(\Sigma_n\), so that \(\Omega_\lambda\) with the left shift dynamics is a subshift over the alphabet \(\{1, 2, \ldots, n\}\). Given \(\omega \in \Sigma_n\) and an injective map \(V : \{1, 2, \ldots, n\} \to \mathbb{R}\), consider the potential \(V(\omega) := (V(\omega_j))_{j \in \mathbb{Z}}\) and the operator \(H_V(\omega)\) as in \((1)\). \(E_\lambda\) is aperiodic if no sequence in \(\Omega_\lambda\) is periodic. A nonempty set in a metric space is a Cantor set if it is closed with empty interior and no isolated points.

**Theorem 1.** Fix \(\pi \in G_n\). Given \(V\) as above, there is a subset \(\mathcal{F} \subset E(\pi)\) of full Lebesgue measure so that:

(i) each \(E_\lambda \in \mathcal{F}\) is minimal, aperiodic and uniquely ergodic;

(ii) for each \(E_\lambda \in \mathcal{F}\), the spectrum of \(H_V(\omega)\) in \((1)\) is the same for all \(\omega \in \Omega_\lambda\) and it is a Cantor set of zero Lebesgue measure.

(iii) for each \(E_\lambda \in \mathcal{F}\) the corresponding Schrödinger operators \((1)\) with potentials \(V(\phi_\lambda(x))\) have pure singular continuous spectrum for a.e. \(x \in [0, 1]\).

We recall that for \(n = 2\) and \(\pi(1, 2) = (2, 1)\) there is only one discontinuity point \(a_1 \in [0, 1]\) and the system is reduced to rotations of the circle by the angle \((1 - a_1)\). In this case the potentials \(\Omega_E\) are the Sturmian sequences \([1, 5]\), which include
the well-known Fibonacci substitution sequence [13, 16]. Hence, the potentials generated by jets are natural generalizations of Sturmian potentials, one of the standard models of one-dimensional quasicrystals. We refer the reader to [7] for additional comments and to [6], as well as references therein, for related examples of Cantor zero measure spectrum. We close this section with an open question we have found interesting. Although almost every jets are minimal and uniquely ergodic [12, 17], there are cases of minimal jets with more than one ergodic component \((n/2)\) is an upper bound for the number of ergodic probability measures [11, 12]), so it is natural to ask if the characterization of the spectrum in Theorem 1 holds in such cases.

2. Proof of Theorem 1

The proof of this theorem will be reduced to the proof that, given an irreducible permutation \(\pi \in G_n\), for almost every jet the corresponding Schrödinger operator \(H_{V(\phi(x))}\) has no eigenvalues for Lebesgue almost every \(x \in [0, 1)\). In this section we clarify such statement and next sections are devoted to the related proof of absence of eigenvalues. Given a subshift \(\Omega\) over the finite alphabet \(\mathcal{B}\) (the dynamics is always given by the left shift), Boshernitzan [2, 3] introduced a condition which later was called condition (B) in [6]. The set of finite words in \(\Omega\) is

\[
W = W(\Omega) = \{\omega(j) \cdots \omega(j + n - 1) : j \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\},
\]

and for \(w \in W\) denote by \(|w|\) its length and by

\[
V_w = V_w(\Omega) = \{\omega \in \Omega : \omega(1) \cdots \omega(|w|) = w\}
\]

the cylinders. For each invariant probability measure \(\mu\) on \(\Omega\) set

\[
\eta_\mu(n) = \min\{\mu(V_w) : w \in W, |w| = n\}.
\]

The subshift \(\Omega\) satisfies the condition (B) if there exists an ergodic probability measure \(\mu\) on \(\Omega\) with

\[
\limsup_{n \to \infty} n \eta_\mu(n) > 0.
\]

This condition was shown by Boshernitzan [4] to imply unique ergodicity for minimal subshifts, and in the particular case of jets also by Veech [18]. Recall also the following basic result:

**Lemma 1.** [10] If \(E_\lambda\) is minimal, then \(\Omega_\lambda\) is a minimal subshift, i.e., every \(\omega \in \Omega_\lambda\) has dense orbit in \(\Omega_\lambda\).

With respect to the spectrum of discrete Schrödinger operators, the following important result was proved in [6]:

**Theorem 2.** Let \(\Omega\) be a minimal subshift which satisfies condition (B). If \(\Omega\) is aperiodic, then there exists a Cantor set \(\Sigma \subset \mathbb{R}\) of Lebesgue measure zero so that the spectrum \(\sigma(H_\omega) = \Sigma\) for every \(\omega \in \Omega\).

For jets Boshernitzan [4] has proved

**Theorem 3.** Let \(\pi \in G_n\). Then for Lebesgue almost every \(\lambda \in \Delta^{n-1}\) the subshift \(\Omega_\lambda\) satisfies condition (B).

Hence, by merging the above theorems one concludes part of Theorem 1. In fact, we have:
Theorem 4. Let $\pi \in G_n$ and $V$ as in Theorem 1. Then there is a subset $\mathcal{L} \subset E(\pi)$ of full Lebesgue measure so that:

(i) each $E_\lambda \in \mathcal{L}$ is minimal, aperiodic and uniquely ergodic;

(ii) for each $E_\lambda \in \mathcal{L}$ the spectrum of $H_{V(\omega)}$ in (1) is the same for all $\omega \in \Omega_\lambda$, and it is a Cantor set of zero Lebesgue measure.

(iii) for each $E_\lambda \in \mathcal{L}$ the corresponding Schrödinger operators (1) with potentials $V(\phi_\lambda(x))$ have no absolutely continuous spectrum for all $x \in [0,1)$.

Proof. (i) It is well known that the set of minimal ets has full Lebesgue measure, and each of them is necessarily aperiodic. By Theorem 3 the set of ets that satisfy condition (B) also has full measure. Define $\mathcal{L}$ as the intersection of such sets. Then $\mathcal{L}$ has full Lebesgue measure and each of them is uniquely ergodic [4]. (ii) By minimality of $E_\lambda \in \mathcal{L}$ the spectrum is the same set for all $\omega \in \Omega_\lambda$ and by Theorem 2 it is a Cantor set of zero Lebesgue measure. (iii) By (ii) the spectrum has zero Lebesgue measure and so no absolutely continuous component.

Now, in order to complete the proof of Theorem 1 it is enough to show that, given $\pi \in G_n$, there is a set $\mathcal{P} \subset E(\pi)$ of full Lebesgue measure so that for each $E_\lambda \in \mathcal{P}$ the corresponding Schrödinger operators $H_{V(\phi_\lambda(x))}$ have no eigenvalues for Lebesgue almost every $x \in [0,1)$; thus only singular continuous spectrum remains. Then the set $\mathcal{F}$ in Theorem 1 can be defined by the intersection of this set $\mathcal{P}$ with $\mathcal{L}$. Our final task is to exclude eigenvalues. An important tool to exclude eigenvalues for a given operator $H_\omega$, $\omega \in \Sigma_n$, is the Delyon-Petritis [8] version of an argument of Gordon [9], by means of suitable local word repetitions.

Theorem 5. [8] If for given $\omega \in \Sigma_n$ there exists a sequence $k_i \to \infty$ such that

$$\omega_{j-k_i} = \omega_j = \omega_{j+k_i},$$

for all $1 \leq j \leq k_i$, then the Schrödinger operator $H_\omega$ in (1) has no eigenvalues.

Given an irreducible permutation $\pi$, the idea is to show that, for almost all $\lambda \in \Delta^{n-1}$ and $E_\lambda$, Theorem 5 applies to $H_\omega$, $\omega = \phi_\lambda(x)$, with $x$ in a set of total Lebesgue measure over $[0,1)$. In other words, we have to prove that for almost all $x \in [0,1)$ there is a sequence $(r_k)$ of natural numbers, such that the itinerary of $x$ associated to $[-r_k, 2r_k]$ if of the form

$$w_0 w_1 \ldots w_{r_k} \quad w_0 w_1 \ldots w_{r_k} \quad w_0 w_1 \ldots w_{r_k}, \quad w_i \in \{1, 2, \ldots, n\}.$$

Then we will prove

Proposition 1. Fix an irreducible permutation $\pi \in G_n$. Then for almost all $\lambda \in \Delta^{n-1}$, the corresponding ets $E_\lambda$ is minimal and, for a.e. $x \in [0,1)$, the coding $\phi_\lambda(x)$ satisfies the hypotheses of Theorem 5 and so, the operator $H_{\phi_\lambda(x)}$ has empty point spectrum.

To prove this proposition we will use the well-known Rauzy Renormalization Operator in the space of interval exchange transformations. In the next section we will introduce the appropriate definitions and preliminary results; the end of the proof of this proposition is delayed until Section 5.

3. Rauzy's Renormalization

Let $E : [a, b) \to [a, b)$ be an ets and $J := [c, d)$ a proper subinterval of $[a, b)$. Let us denote by $E_J$ the Poincaré’s first return map of $E$ to the interval $J$, that is, for
If \( x \in J \), \( E_J(x) \) is given by the first point in the positive orbit of \( x \) (by \( E \)) that returns to the interval \( J \). \( E_J \) is again an iet, that is, there is a partition \( \mathcal{A}' = (a_0, a_1, \ldots, a_p) \) of \([c, d]\) and a permutation \( \pi' \in G_p \) such that \( E_J = (\pi', \mathcal{A}') \). In general the number of intervals of continuity increases: if \( E \) exchanges \( n \) intervals then \( p \geq n \), \( E_J \) will be called the induced map of \( E \) on the interval \( J \). Let \( I_1, I_2, \ldots, I_p \) be the intervals of continuity of \( E_J \). For each \( 1 \leq k \leq p \) there exists an integer \( r_k > 0 \) such that

\[
E(I_k), E^2(I_k), \ldots, E^{r_k}(I_k)
\]

are all intervals disjoints of \( J \) whereas \( E^{r_k+1}(I_k) \) is totally contained in \( J \). By definition \( E_J(I_k) = E^{r_k+1}(I_k) \). The number \( r_k \) is called the return time of \( I_k \) to \( J \).

We want to remark that, by the bijectivity of \( E \), the whole interval \([a, b]\) is given by the union

\[
[a, b) = \bigcup_{k=1}^{p} \bigcup_{j=1}^{r_k} E^j(I_k).
\]

**Rauzy’s map.** Take \((\pi, \lambda) \in G_n \times \Delta^{n-1}\) and let \( E \) be the \((\pi, \lambda)\)-interval exchange.

Consider \( \nu = \nu(\pi, \lambda) \) defined as the minimum between \( \lambda_n \) and \( \lambda_{\pi^{-1}(n)} \) provided that these numbers are different. If \( E_J \) is the induced map of \( E \) on the interval \( J := [0, 1 - \nu), \nu > 0 \), it is proved in [14, 15] that \( E_J \) is an iet of exactly \( n \) intervals associated to a new irreducible permutation \( \pi' \in G_n \) and partition \( \mathcal{B}' = (b_0, b_1, \ldots, b_n) \) of \( J \). Then, by normalizing \( E_J \), we obtain a pair \((\pi', \lambda') \in G_n \times \Delta^{n-1}\).

Indeed

\[
\lambda' := \frac{1}{1-\nu}(b_1, b_2 - b_1, \ldots, b_n - b_{n-1}) \in \Delta^{n-1}.
\]

Rauzy’s Renormalization map is the association \((\pi, \lambda) \xrightarrow{R} (\pi', \lambda')\). We remark that such association is not defined when \( \lambda_n = \lambda_{\pi^{-1}(n)} \). The domain of \( R \). It is proved in [14] that \( G_n \) is divided into several subsets called Rauzy Classes which are invariant by the process of induction just defined. Let us denote by \( \mathcal{C} \) one of the Rauzy classes of \( G_n \). Then if \( \pi \in \mathcal{C} \) and \( R(\pi, \lambda) = (\pi', \lambda') \) then \( \pi' \in \mathcal{C} \). Set

\[
\Delta_{\pi}^{n-1} := \{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta^{n-1} : \lambda_n \neq \lambda_{\pi^{-1}(n)}, \pi \in \mathcal{C} \}.
\]

Then the transformation \( R(\pi, \lambda) = (\pi', \lambda') \) is well defined in \( \mathcal{C} \times \Delta_{\pi}^{n-1} \). In \( \mathcal{C} \times \Delta^{n-1} \) there is a natural measure \( m \) which is the product of the counting measure in \( \mathcal{C} \) and the \((n-1)\)-dimensional Lebesgue measure in \( \Delta^{n-1} \). As the set \( \Delta_{\pi}^{n-1} \) have total measure in \( \Delta^{n-1} \) (regarding the \((n-1)\)-dimensional Lebesgue measure), the map \( R \) is defined \( m \)-almost everywhere in \( \mathcal{C} \times \Delta^{n-1} \) and by abuse of language it is usually written

\[
R : \mathcal{C} \times \Delta^{n-1} \to \mathcal{C} \times \Delta^{n-1}.
\]

The Rauzy’s transformation plays a central role in the ergodic theory of iets due to the following result [17, 12].

**Theorem 6.** The Rauzy’s operator is ergodic with respect to a measure absolute continuous with respect to \( m \) (the natural measure described above).

Another related result proved in [17, 12] is

**Theorem 7.** For a fixed permutation \( \pi \in \mathcal{C} \) and for almost every vector \( \lambda \in \Delta^{n-1} \), the \( R \)-orbit of \( E_{\lambda} \) is dense in \( \mathcal{C} \times \Delta^{n-1} \).
Let us show now a relation between the process of renormalization and the important property stated in Lemma 1. Let \( E : [0, 1) \rightarrow [0, 1) \) be an interval exchange and let \( E_J \) be the induced map of \( E \) on \( J = [a, b] \subset [0, 1) \). Suppose that \( I \subset J \) is an interval of continuity of \( E_J \) and that, for some \( x \in I \), it happens that \( E_J(x) \in I \) and \( E_J^{-1}(x) \in I \). If \( r \) is the return time of \( I \) to \( J \), the itinerary of every point in \( I \) associated to \([0, r]\) is given by the word

\[
 w_0 w_1 \ldots w_r := \mathcal{A}(I) \mathcal{A}(E(I)) \mathcal{A}(E^2(I)) \ldots \mathcal{A}(E^r(I)).
\]

As \( E_J(x) \in I \) and \( E_J^{-1}(x) \in I \), the itinerary of \( x \) associated to \([-r, 2r]\) will be given by the word

\[
 w_0 w_1 \ldots w_r \ w_0 w_1 \ldots w_r \ w_0 w_1 \ldots w_r.
\]

A point \( x \in [a, b) \) with this property will be called a candidate point in the interval \([a, b)\) for the length \( r \). \( C_N([a, b)) \) will be the set of candidate points in \([a, b)\) for the lengths \( r \leq N \). Note that if there is a nested sequence of intervals shrinking to a point \( x \),

\[
[a_1, b_1) \supset [a_2, b_2) \supset \ldots [a_k, b_k) \supset \ldots
\]

and such that \( x \) is a candidate point for each \([a_k, b_k)\) for the length \( r_k \), then necessarily \( r_k \rightarrow \infty \) as \( k \rightarrow \infty \), and the itinerary of \( x \) associated to its whole orbit, \( \phi_E(x) \), is given by a sequence that satisfies the hypotheses of Lemma 1. From now on, \( \pi_n \) we will denote the permutation in \( G_n \) given by

\[
\pi_n(j) = n - j + 1, \quad j = 1, 2, \ldots, n.
\]

Let us denote by \( |C| \) the Lebesgue measure of a measurable set \( C \subset \mathbb{R} \).

**Lemma 2.** Let \( E : [0, 1) \rightarrow [0, 1) \) be an iet and let \( E_J \) be the induced map of \( E \) on the interval \( J = [a, b) \subset [0, 1) \). Suppose that \( E_J \) is associated with the permutation \( \pi_n \) and that the first interval of continuity of \( E_J \) is of the form \( I_1 = [a, b - \delta) \), where \( 0 < \delta < (b - a)/4 \). Let \( r_1 \) be the return time of \( I_1 \) to \( J \). Then the Lebesgue measure of \( C_{r_1}([a, b)) \) satisfies

\[
\frac{|C_{r_1}([a, b))|}{b - a} \geq 1 - \frac{3\delta}{b - a}.
\]

**Proof.** \( E_J \) restricted to \( I_1 \) should be of the form \( E_J(x) = x + \delta \) (because \( \pi_n(1) = n \)). Observe that the interval \( L := [a + \delta, b - 2\delta) \) is made of candidate points of length \( r_1 \). Indeed \( K := [a + \delta, b) \) is the last interval of continuity of \( E_J^{-1} \) and \( E_J^{-1}(x) = x - \delta, x \in K \). Then

\[
E_J(L) = [a + 2\delta, b - \delta) \quad \text{and} \quad E_J^{-1}(L) = [a, b - 3\delta)
\]

are both intervals contained in \( I_1 \). In this way, the Lebesgue measure of \( C_{r_1}([a, b)) \) is greater than \( |L| \geq (b - a) - 3\delta \).

\( \square \)

4. A Key Example

In this section we present an example that will play a key role in the proof of Proposition 1. Let \( q > 1 \) be a natural number and consider the partitions
\( a = (a_0, a_1, \cdots, a_n) \) and \( b = (b_0, b_1, \cdots, b_n) \) of \([0, 1)\):

\[
\begin{align*}
    a_k &= \begin{cases} 
        1 - q^{-k}, & 0 \leq k < n, \\
        1, & k = n,
    \end{cases} \\
    b_k &= \begin{cases} 
        0, & k = 0, \\
        q^{-(n-k)}, & 0 < k < n, \\
        1, & k = n.
    \end{cases}
\end{align*}
\]

(5)

\( a := \left(0, 1 - q^{-1}, 1 - q^{-2}, \ldots, 1 - q^{-(n-1)}, 1\right) \),  
\( b = \left(0, q^{-(n-1)}, q^{-(n-2)}, \ldots, q^{-1}, 1\right) \).

\( F_q \) will be the interval exchange associated to the pair \((\pi_n, a)\). Then \( F_q^{-1} \) is associated to \((\pi_n, b)\). Let \( I_k := [a_{k-1}, a_k) \) and \( J_k := [b_{k-1}, b_k), k = 1, \ldots, n, \) be the intervals of continuity of \( F_q \) and \( F_q^{-1} \) respectively. We will see that \( F_q \) has the property that each of the induced maps on \( I_k \) satisfies the conditions in Lemma 2. Note that \( F_q \) and \( F_q^{-1} \) are explicitly given by the formulas

\[
\begin{align*}
    F_q(x) &= \begin{cases} 
        x - a_{k-1} + b_{n-k}, & x \in I_k, 1 < k < n, \\
        x - a_{n-1}, & x \in I_n,
    \end{cases} \\
    F_q^{-1}(x) &= \begin{cases} 
        x + a_{n-k} - b_{k-1}, & x \in J_k, 1 < k < n, \\
        x - a_1, & x \in J_n.
    \end{cases}
\end{align*}
\]

(6)

**Lemma 3.** Let \( F_{q,k} \) be the induced map of \( F_q \) on \( I_k \), \( k = 1, \ldots, n \). Then if \( C_k \) is the set of candidate points in \( I_k \),

\[
\frac{|C_k|}{|I_k|} \geq 1 - \frac{3}{q-1}.
\]

**Proof.** It is not hard to check the following facts: 1. The induced map of \( F_q \) on the interval \([a_k, 1)\), \( k = 1, 2, \ldots, n-1 \) is given by the permutation \( \pi_{n-k+1} \) and the partition of \([a_k, 1), \)

\[
( a_k, a_{k+1}, a_{k+2}, \ldots, a_{n-4}, a_n ).
\]

2. The induced map of \( E \) on \( I_k = [a_{k-1}, a_k) \), \( k = 1, 3, \ldots, n-1 \), is associated to the pair \((\pi_{n-k+1}, c_k)\) where \( c_k \) is the partition of \( I_k \):

\[
\mathbf{c}_k = \left( a_{k-1}, a_k - q^{-k}, a_k - q^{-(k+1)}, \ldots, a_k - q^{-(n-1)}, a_k \right).
\]

In particular, \( F_{q,k}(x) = x + q^{-k} \) for each \( x \) in the first interval of continuity \( I_{k,1} \); 3. The induced map of \( F_q \) on the first interval \( I_1 = [0, 1 - q^{-1}) = [a_0, a_1) \) is given by the pair \((\pi_n, \mathbf{c}_1)\) where \( \pi_n = \pi \) and

\[
\mathbf{c}_1 = \left( 0, a_1 - q^{-1}, a_1 - q^{-2}, \ldots, a_1 - q^{-(n-1)}, a_1 \right).
\]

4. The induced map of \( F_q \) on the last interval \( I_n = [a_{n-1}, a_n) \) is the identity map. In particular the last interval is all made of candidate points. 5. \( F_{q,k} \) restricted to the first interval of continuity

\[
I_{k,1} := \left[a_{k-1}, a_k - q^{-k}\right)
\]


is given by

$$F_{q,k}(x) = x + q^{-k}.$$  

Then Lemma 2 applies to each map $F_{q,k}$, $1 \leq k < n$ (with $\delta = q^{-k}$) and therefore if $C_k$ is the set of candidate points in $I_k$, then

$$\frac{|C_k|}{|I_k|} \geq 1 - \frac{3\delta}{|J_k|} \geq 1 - \frac{3q^{-k}}{(q-1)q^{-k}} = 1 - \frac{3}{q-1}.$$  

Indeed, in this way the interval

$$L_k := \left[ a_{k-1} + \frac{1}{q^k}, a_k - \frac{2}{q^k} \right]$$  

which is contained in $I_{k,1}$, is made of candidate points and the length of $L_k$ is $|I_k| - 3q^{-k}$.  

We say that two ietrs $E_1 : [a, b) \to [a, b)$ and $E_2 : [c, d) \to [c, d)$ are equivalent if they are conjugated by a translation of the real line, more precisely

$$E_2(x + c - a) = E_1(x) + c - a, \text{ for all } x \in [a, b).$$  

In particular we should have $b - a = d - c$. In many situations we identify an interval exchange transformation with its equivalent model defined on $[0, b - a)$.

**Lemma 4.** Let $E : [a, b) \to [a, b)$ be an iet and $J := [c, d) \subset [a, b)$. If $E(J)$ is also an interval, say $K$, then the induced maps of $E$ on $J$ and $K$ are equivalent.

**Proof.** As $E(J)$ is an interval, there is a number $d$ such that, for all $x \in J$, $E(x) = x + d$. Put $K = [E(c), E(d))$ and $1$ Let $J_k \subset J$ be an interval of continuity of $E_J$ and let $r_k$ be the return time of $J_k$ to $J$, this means that

$$E(J_k), E^2(J_k), \ldots, E^{r_k}(J_k)$$  

are all intervals disjoints of $J$ and $E^{r_k+1}(J_k)$ is completely contained in $J$. By definition $E_J(J_k) = E^{r_k+1}(J_k)$. Observe that

$$E^{r_k+2}(J_k) = E(E^{r_k+1}(J_k)) = E_J(J_k) + d$$  

is contained in $K$. Let $K_k := E(J_k) = J_k + d \subset K$. The successive iterates of $K_k$ by $E$ are (the same as)

$$E^2(J_k), E^3(J_k), \ldots, E^{r_k}(J_k), E^{r_k+1}(J_k)$$  

and this intervals are disjoint of $K$ whereas

$$E^{r_k+2}(J_k) = E_J(J_k) + d \subset K,$$

that is, $K_k$ is an interval of continuity of $E_K$, with the same return time and more over,

$$E_K(K_k) = E^{r_k+2}(J_k) = E_J(K_k - d) + d.$$

As this happens for all the intervals of continuity of $E_J$ and $E_K$, this maps are equivalent.  

5. Proof of Proposition 1

Let an irreducible permutation $\pi$ and let $0 < \epsilon < 1$. Let $(q_m)_{m \geq 1}$ be an increasing sequence of natural numbers such that

$$\frac{3}{q_m - 1} < \frac{\epsilon}{2^m}, \ m \geq 1.$$  

Consider the sets $F_{q_m}$ as defined in Section 4 and let 

$$a_m = (a_{m}^{0}, a_{1}^{m}, \ldots, a_{n}^{m})$$  

be the correspondent partition of $[0, 1)$ associated to $F_{q_m}$ as in (5). Observe that any set which is sufficiently close to $F_{q_m}$ enjoys the property given in Lemma 3, that is, there exist a number $\delta_m > 0$ such that if $E_{\epsilon}$, the set associated to $(\pi_n, c)$ and the partition $\mathbf{c} = (c_0, c_1, \ldots, c_n)$ in $\Delta^{n-1}$ is such that

$$(7) \quad \max \{|c_i - a_i^{m}| : 1 \leq i \leq n\} < \delta_m,$$  

then the set $C_{\mathbf{c}, k}$ of candidate points in each $I_k := [c_{k-1}, c_{k}), 1 \leq k \leq n$ for $E_{\epsilon}$ verify

$$\frac{|C_{\mathbf{c}, k}|}{|I_k|} \geq 1 - \frac{3}{q_m - 1} \geq 1 - \frac{\epsilon}{2^m}.\tag{8}$$  

Let $\pi \in G_n$ be an irreducible permutation. By Theorem 7, for almost all $\lambda \in \Delta^{n-1}$, the Rauzy orbit of $E_{\lambda} = (\pi, \lambda)$ is dense in $C \times \Delta^{n-1}$. In this way there is a nested sequence of intervals

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \ldots$$

given by the Rauzy process of induction, and an increasing sequence $(n_m)_{m \geq 1}$ of natural numbers such that the induced map $E_{m} := R^{n_m}(E)$ of $(\pi, \lambda)$ on $J_m$ satisfies (7). So, if $I_1^{m}, I_2^{m}, \ldots, I_{n}^{m}$ are the intervals of continuity of $E_{m}$ in $J_m$ and $r_k^{m}$ is the return time of $I_k^{m}$ to $J_m$, $1 \leq k \leq n$, then the set of candidate points in $I_k^{m}$ for the length $r_k^{m}$ has Lebesgue measure greater than $1 - \epsilon \cdot 2^{-m}$ times the measure of $I_k^{m}$. Let $E_{m, k}$ be the induced map of $E_{m}$ on $I_k^{m}$, $1 \leq k \leq n$. Therefore the set of candidate points in $I_k^{m}$ for the length $N_m$ with

$$N_m = \max\{r_1^{m}, \ldots, r_n^{m}\},$$

a set named here as $C_{N_m}(J_m)$, satisfies

$$\frac{|C_{N_m}(J_m)|}{|J_m|} \geq \left(1 - \frac{\epsilon}{2^m}\right) \cdot |J_m|.\tag{9}$$

By Lemma 4, the induce map of $(\pi, \lambda)$ on each interval $E^{j}(I_k^{m})$, $1 \leq j \leq r_k^{m}$ is equivalent to $E_{k, m}$ and this implies that the set of candidate point for the length $r_k^{m}$ in the union

$$\bigcup_{j=1}^{r_k^{m}} E^{j}(I_k^{m})$$

is greater than $1 - \epsilon \cdot 2^{-m}$ times the measure of this union. Using the fact that

$$[0, 1) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{r_k^{m}} E^{j}(I_k^{m}),$$

$$\rho_r(\pi, \lambda) \leq \frac{1}{r \cdot 2^m} \geq (\frac{1}{r} - \epsilon) \cdot 2^{-m}.$$  

This completes the proof of Proposition 1.
(see relation (2)) we conclude that the set of candidate points in $[0, 1)$ for the lengths $r \leq N_m$ with

$$N_m = \max\{r_1^m, \ldots, r_n^m\}$$

(a set named here as $C_{N_m}([0, 1]))$, satisfies

$$|C_{N_m}([0, 1])| \geq 1 - \frac{\epsilon}{2^m}.$$  \hspace{1cm} (10)

Now it follows that the Lebesgue measure of the intersection

$$C_\epsilon := \bigcap_{m \geq 1} C_{N_m}([0, 1])$$

is greater than $1 - \epsilon \sum_{m \geq 1} 2^{-m} = 1 - \epsilon$. Observe that the sequence of integers $(n_m)$ can be chosen in such a way that $(N_m)$ is a strictly increasing sequence and then all the points in $C_\epsilon$ satisfy the condition of Proposition 1. As the number $\epsilon > 0$ is arbitrary, Proposition 1 is proved.

**Acknowledgments.** CRdE and O was partially supported by CNPq. CG thanks the partial support by FAPESP Grant 03/03107-9 and by CNPq Grant 306992/2003-5, Brazil.

**References**


SCHRÖDINGER OPERATORS ALONG IET

Department of Mathematics, UFES, Av. F. Ferrari 514, Vitória, ES, 19075-910 Brazil
E-mail address: miltonc@cce.ufes.br

Department of Mathematics, ICMC/USP, CxP 668, São Carlos, SP, 13560-970 Brazil
E-mail address: gutp@icmc.usp.br

Department of Mathematics, UFSCar, São Carlos, SP, 13560-970 Brazil
E-mail address: oliveira@dm.ufscar.br