

\textbf{Z}_2^k\text{-actions fixing } \mathbb{RP}^2 \cup \mathbb{RP}^{\text{even}}

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\textbf{Abstract} This paper determines, up to equivariant cobordism, all manifolds with \( Z_2^k \)-action whose fixed point set is \( \mathbb{RP}^2 \cup \mathbb{RP}^n \), where \( n > 2 \) is even.

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\section{Introduction}

Suppose \( M \) is a smooth and closed manifold and \( T : M \to M \) is a smooth involution defined on \( M \). It is well known that the fixed point set of \( T, F \), is a finite and disjoint union of closed submanifolds of \( M \). For a given \( F \), a basic problem in this context is the classification, up to equivariant cobordism, of the pairs \((M,T)\) for which the fixed point set is \( F \). For related results, see for example [2], [3], [4], [10], [15], [6, Theorem 27.6], [1, Page 309], [16], [17] and [18]. Specifically, in [2], D. C. Royster studied this problem with \( F \) being the disjoint union of two real projective spaces, \( F = \mathbb{RP}^m \cup \mathbb{RP}^n \) (for \( F = \mathbb{RP}^n \), the classification was established in [6] and [16]), establishing the results via a case-by-case method depending on the parity of \( m \) and \( n \), with special arguments when one of the components is \( \mathbb{RP}^0 = \{\text{point}\} \), and leaving open the case in which \( m \) and \( n \) are even, with the exception of \((m,n) = (0,\text{even})\) (Royster remarked that the methods applied in [2] are not sufficient to handle the case \((m,n) = (\text{even,even})\) with \( m,n > 0 \)). If \( m = n = \text{even} \), one knows from [1] that \((M,T)\) is an equivariant boundary when \( \dim(M) \geq 2n \); this case was completed in [3], where it was shown that \((M,T)\) also is a boundary when \( n \leq \dim(M) < 2n \). To understand the case \((m,n) = (0,\text{even})\) and also to explain the goal of this paper, for any \( m \) and \( n \) consider the involution \((\mathbb{RP}^{m+n+1},T_{m,n})\) defined in homogeneous coordinates by

\[ T_{m,n}[x_0, x_1, \ldots, x_{m+n+1}] = [-x_0, -x_1, \ldots, -x_m, x_{m+1}, \ldots, x_{m+n+1}] \, . \]

The fixed set of \( T_{m,n} \) is \( \mathbb{RP}^m \cup \mathbb{RP}^n \). From \( T_{m,n} \), it may be possible to obtain other involutions fixing \( \mathbb{RP}^m \cup \mathbb{RP}^n \): in general, for a given involution \((W,T)\)
with fixed set $F$ and with $W$ being a boundary, the involution $\Gamma(W, T) = (S^1 \times W, \tau)$ is equivariantly cobordant to an involution fixing $F$; here, $S^1$ is the 1-sphere, $Id$ is the identity map and $\tau$ is the involution induced by $c \times Id$, where $c$ is the complex conjugation (see [7]). If $S^1 \times W - Id \times T$ is a boundary, we can repeat the process taking $\Gamma^2(W, T)$, and so on. If $F$ is nonbounding, this process finishes, that is, there exists the first natural number $r \geq 1$ for which the underlying manifold of $\Gamma^r(W, T)$ is nonbounding; this follows from the 5/2-theorem of J. Boardman of [5] and its strengthened version of [1]. In particular, if $m$ and $n$ are even and $m < n$, $RP^m \cup RP^n$ does not bound and $RP^{m+n+1}$ bounds, so this number makes sense for $(RP^{m+n+1}, T_{m, n})$ and we call it $h_{m,n}$.

In [2], Royster proved the following

**Theorem.** Let $(M, T)$ be an involution fixing $\{\text{point}\} \cup RP^n$, where $n$ is even. Then $(M, T)$ is equivariantly cobordant to $\Gamma^j(RP^{m+1}, T_{0,n})$ for some $0 \leq j \leq h_{0,n}$. Later, in [14], R. E. Stong and P. Pergher determined the value of $h_{0,n}$, thus answering the question posed by Royster in [2; page 271]: writing $n = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd, they showed that $h_{0,n} = 2$ if $p = 1$ and $h_{0,n} = 2^p - 1$ if $p > 1$.

In this paper, we contribute to this problem by solving the case $(m, n) = (2, \text{even})$. Specifically, we will prove the following

**Theorem 1.** Let $(M, T)$ be an involution fixing $RP^2 \cup RP^n$, where $M$ is connected and $n \geq 4$ is even. Then, if $n > 4$, $(M, T)$ is equivariantly cobordant to $\Gamma^j(RP^{n+3}, T_{2,n})$ for some $0 \leq j \leq h_{2,n}$; if $n = 4$, $(M, T)$ is either equivariantly cobordant to $\Gamma^j(RP^5, T_{2,4})$ for some $0 \leq j \leq h_{2,4}$, or equivariantly cobordant to $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$.

In addition, we generalize the result of Stong and Pergher of [14], calculating the general value of $h_{m,n}$ (which, in particular, makes numerically precise the statement of Theorem 1).

**Theorem 2.** For $m, n$ even, $0 \leq m < n$, write $n - m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd. Then $h_{m,n} = 2$ if $p = 1$ and $h_{m,n} = 2^p - 1$ if $p > 1$.

Finally, we also extend the results for $Z_k^k$-actions. This extension is automatic from the combination of the above results and the case $F = RP^{even}$ with a recent paper [12]. The details concerning this extension will be given in Section 4. Sections 2 and 3 will be devoted, respectively, to the proofs of Theorems 1 and 2.
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2 Involution fixing $RP^2 \cup RP^{even}$

We start with an involution $(M, T)$ fixing $RP^2 \cup RP^n$, where $M$ is connected and $n \geq 4$ is even, and first establish some notations. We will always use $\lambda_r \to RP^r$ to denote the canonical line bundle over $RP^r$. Denote by $\alpha \in H^1(RP^2, Z_2)$ and $\beta \in H^1(RP^n, Z_2)$ the generators of the 1-dimensional $Z_2$-cohomology. The model involution $(RP^{n+3}, T_{2,n})$ fixes $RP^2 \cup RP^n$ with normal bundles $(n+1)\lambda_2 \to RP^2$ and $3\lambda_n \to RP^n$. The total Stiefel-Whitney classes are $W((n+1)\lambda_2) = (1+\alpha)^{n+1}$, $W(3\lambda_n) = (1+\beta)^3$. Denote by $\eta \to RP^2$ and $\xi \to RP^n$ the normal bundles of $RP^2$ and $RP^n$ in $M$. To prove Theorem 1, it suffices to prove the following

Lemma. If $n > 4$, then $W(\eta) = (1+\alpha)^{n+1}$ and $W(\xi) = (1+\beta)^3$; if $n = 4$, then either $W(\eta) = (1+\alpha)^5$ and $W(\xi) = (1+\beta)^3$, or $W(\eta) = 1+\alpha$ and $W(\xi) = 1+\beta$.

In fact, suppose the lemma is true, and denote by $R$ the trivial one-dimensional vector bundle over any base space. Set $k = dim(\eta)$, $l = dim(\xi)$, that is, $k = dim(M) - 2$, $l = dim(M) - n \geq 1$. First consider $n > 4$. For $0 \leq j < h_{2,n}$, the involution $\Gamma^j(RP^{n+3}, T_{2,n})$ is equivariantly cobordant to an involution with fixed data $((n+1)\lambda_2 \oplus jR \to RP^2) \cup (3\lambda_n \oplus jR \to RP^n)$ (see [7]). Using the notations $W = 1+w_1+w_2+\ldots$ for Stiefel-Whitney classes and $\binom{\alpha}{3}$ for binomial coefficients mod 2, note that $w_3(\xi) = \binom{\alpha}{3} \beta^3 = \beta^3 \neq 0$ and thus $l \geq 3$. Then $\eta \cup \xi$ and $((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$ are cobordant because they have the same characteristic numbers. If $l \leq 3 + h_{2,n}$, one then has from [6] that $(M, T)$ and $\Gamma^{l-3}(RP^{n+3}, T_{2,n})$ are equivariantly cobordant, proving the result. By contradiction, suppose then $l > 3 + h_{2,n}$. Again from [6], $((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$ is the fixed data of an involution $(W, S)$, and by removing sections if necessary we can suppose, with no loss, that $dim(W) = n + h_{2,n} + 4$ (see [6, Theorem 26.4]). Let $(N, T\iota)$ be an involution cobordant to $\Gamma^{h_{2,n}}(RP^{n+3}, T_{2,n})$ and with fixed data $((n+1)\lambda_2 \oplus h_{2,n}R) \cup (3\lambda_n \oplus h_{2,n}R)$; one knows that $N$ is not a boundary. Then $\Gamma(N, T\iota) \cup (W, S)$ is cobordant to an involution with fixed data $R \to N$, and from [6] $R \to N$ then is a boundary, which is impossible. Now suppose $n = 4$. The case $W(\eta) = (1+\alpha)^5$ and $W(\xi) = (1+\beta)^3$ is included in the above approach, hence suppose $W(\eta) = 1+\alpha$.
and $W(\xi) = 1 + \beta$. Since $h_{0,2} = 2$, the involution $\Gamma^2(RP^3, T_{0,2})$ is cobordant to an involution with fixed data $(5R \to \{\text{point}\}) \cup (\lambda_2 \oplus 2R \to RP^2)$. Then the involution $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$ is cobordant to an involution $(W^5, T)$ with fixed data $(\lambda_2 \oplus 2R \to RP^2) \cup (\lambda_4 \to RP^4)$, and the total Stiefel-Whitney classes are $W(\lambda_2 \oplus 2R) = 1 + \alpha$, $W(\lambda_4) = 1 + \beta$. Because $h_{0,2} = 2$, the underlying manifold of $\Gamma^2(RP^3, T_{0,2})$ does not bound; since $RP^5$ bounds, $W^5$ does not bound. By contradiction, suppose $l \geq 2$. Using the hypothesis, [6] and removing sections if necessary, we can suppose with no loss that $(M, T)$ has fixed data $(\lambda_2 \oplus 3R \to RP^2) \cup (\lambda_4 \oplus R \to RP^4)$. Using the same above argument for $\Gamma(W^5, T) \cup (M, T)$, we conclude that $R \to W$ is a boundary, which is false. Then $l = 1$ and $(M, T)$ and $(W^5, T)$ (hence $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$) have fixed data with same characteristic numbers.

In order to prove the lemma, we will intensively use the following basic fact from [6]: the projective space bundles $RP(\eta)$ and $RP(\xi)$, with the standard line bundles $\lambda \to RP(\eta)$ and $\nu \to RP(\xi)$, are cobordant as elements of the bordism group $N_{k+1}(BO(1))$. Then any class of dimension $k + 1$, given by a product of the classes $w_i(RP(\eta))$ and $w_i(\lambda)$, evaluated on the fundamental homology class $[RP(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(RP(\xi))$ and $w_i(\nu)$, evaluated on $[RP(\xi)]$. To evaluate characteristic numbers, the following formula of Conner will be useful (see [9; Lemma 3.1]): if $\pi : \mu \to N$ is any $r$-dimensional vector bundle, $c$ is the first Stiefel-Whitney class of the standard line bundle over $RP(\mu)$, $W(\mu) = 1 + \pm_1(\mu) + \pm_2(\mu) + ...$ is the dual Stiefel-Whitney class defined by $W(\mu)W(\mu) = 1$ and $\alpha \in H^r(N, Z_2)$, then $c^j\pi^*([RP(\mu)]) = \pm_{j-r+1}(\mu)\alpha[N]$ when $j \geq r - 1$. In this context, our numerical arguments will always be considered modulo 2. Write $W(\lambda) = 1 + c$ and $W(\nu) = 1 + d$ for the Stiefel-Whitney classes of $\lambda$ and $\nu$. The structure of the Grothendieck ring of orthogonal bundles over real projective spaces says that $W(\eta) = (1 + \alpha)^p$ and $W(\xi) = (1 + \beta)^q$ for some $p, q \geq 0$. From [6, 23.3], one then has

$$W(RP(\eta)) = (1 + \alpha)^3(\sum_{i=0}^{2} (1 + c)^{k-i} {p \choose i} \alpha^i)$$

and

$$W(RP(\xi)) = (1 + \beta)^{n+1}(\sum_{i=0}^{l} (1 + d)^{l-i} {q \choose i} \beta^i),$$

where here we are suppressing bundle maps.

**Fact 1.** $p$ and $q$ are odd; in particular, $w_1(\eta) = \alpha$ and $w_1(\xi) = \beta$. 

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**Proof.** One has \(w_1(RP(\eta)) = \binom{k}{1} c + \alpha + \binom{q}{1} \alpha\) and \(w_1(RP(\xi)) = \binom{l}{1} d + \beta + \binom{q}{1} \beta\). Since \(k + 2 = l + n\) and \(n\) is even, \(\binom{k}{1} = \binom{1}{1}\), and thus \(w_1(RP(\eta)) + \binom{q}{1} c = (\binom{q}{1} + 1) \alpha\) and \(w_1(RP(\xi)) + \binom{q}{1} d = (\binom{q}{1} + 1) \beta\) are corresponding characteristic classes. Because \(n > 2\), it follows that

\[
0 = ((\binom{q}{1} + 1) \alpha^n c^{l-1}[RP(\eta)] + ((\binom{q}{1} + 1) \alpha^n d^{l-1}[RP(\xi)] = (\binom{q}{1} + 1) \beta^n [RP^m] = (\binom{q}{1} + 1),
\]

which gives that \(q\) is odd. Also \((\binom{q}{1} + 1) \alpha^2 c^{k-1}[RP(\eta)] = (\binom{q}{1} + 1) \beta^2 d^{k-1}[RP(\xi)] = 0\), and \(p\) is odd.

**Fact 2.** If \(l = 1\), then \(n = 4\), \(W(\eta) = 1 + \alpha\) and \(W(\xi) = 1 + \beta\).

**Proof.** Since \(l = 1\) and \(w_1(\xi) = \beta\), \(W(\xi) = 1 + \beta\). Then the involution \((M, T) \cup (RP^{n+1}, T_{0, n})\) is cobordant to an involution with fixed data \((\eta \to RP^2) \cup ((n+1) R \to \{point\})\). From [2] and the fact that \(h_{0, 2} = 2\), \(W(\eta) = 1 + \alpha\) and \(n = 4\).

Fact 2 reduces our lemma to the following assertion: if \(l > 1\), then \(W(\eta) = (1 + \alpha)^{n+1}\) and \(W(\xi) = (1 + \beta)^{l+1}\); so, we assume throughout the remainder of this section that \(l > 1\). Note that \((1 + \alpha)^{n+1} = (1 + \alpha)^3\) if \(\binom{n}{2} = 1\) and \((1 + \alpha)^{n+1} = 1 + \alpha\) if \(\binom{n}{2} = 0\). Denote by \(r\) the greatest power of 2 that appears in the 2-adic expansion of \(n\); that is, \(4 \leq 2^r \leq n < 2^{r+1}\). We can assume \(q < 2^{r+1}\) and \(p < 4\). Then Facts 3 and 4 below show that \(W(\eta) = (1 + \alpha)^{n+1}\):

**Fact 3.** If \(\binom{n}{2} = 1\), then \(p = 3\).

**Fact 4.** If \(\binom{n}{2} = 0\), then \(p = 1\).

Set \(p' = 4 - p\), \(q' = 2^{r+1} - q\). Then the dual Stiefel-Whitney classes of \(\eta\) and \(\xi\) are given by \(\tilde{W}(\eta) = (1 + \alpha)^{p'}\), \(\tilde{W}(\xi) = (1 + \beta)^{q'}\). Since \(p\) and \(q\) are odd, \(p'\) and \(q'\) are odd; further, \(\binom{p}{2} + \binom{p'}{2} = 1\) and \(\binom{q}{2} + \binom{q'}{2} = 1\) for each \(1 \leq u \leq r\). Now we prove Fact 3. We will use several times the fact that a binomial coefficient \(\binom{a}{b}\) is nonzero modulo 2 if and only if the 2-adic expansion of \(b\) is a subset of the 2-adic expansion of \(a\). We have \(n = 4j + 2\), with \(j \geq 1\), and want to show that \(p = 3\); since \(p < 4\) is odd, it suffices to show that \(\binom{n}{2} = 1\), or equivalently that \(\binom{p'}{2} = 0\). Suppose by contradiction that \(\binom{p'}{2} = 1\). By Conner’s formula, \(c^{k+1}[RP(\eta)] = \binom{k}{2} \alpha^2 [RP^2] = \binom{p'}{2} = d^{k+1}[RP(\xi)] = \binom{q'}{2}\). Then \(\binom{q'}{2} = 1\) and consequently \(\binom{q}{2} = 1\). We formally introduce the class (with \(l - 1 \geq 1\))

\[
\tilde{W}(RP(\xi)) = \frac{W(RP(\xi))}{(1 + c)^{l-1}}.
\]
Since $k = l + 4j$ and $p$ and $q$ are odd, on $RP^2$ this class is

$$
\widetilde{W}(RP(\eta)) = (1 + \alpha)^3(1 + c^4)^j(1 + c + \alpha + (1 + c)^{-1}(p)\alpha^2),
$$

and on $RP^n$ it is

$$
\widetilde{W}(RP(\xi)) = (1 + \beta)^{4j+3}(1 + d + \beta + (1 + d)^{-1}(q)\beta^2 + (1 + d)^{-2}(q/3)\beta^3 + ...).
$$

Then $\tilde{w}_3(RP(\eta)) = \alpha^2 c + (\eta/2)\alpha^2 c = (\eta/2)\alpha^2 c = \alpha^2 c$, and since $(\eta/2) + (\eta/2) = 0$ because $q$ is odd, $\tilde{w}_3(RP(\xi)) = (\eta/2)\beta^2 d = \beta^2 d$. Now we observe that, if $a$ and $b$ are one-dimensional cohomology classes, then by the Cartan formula one has $Sq^{2^n}(a^{2^n} b) = a^{2^n+1} b$, where $Sq$ is the Steenrod operation and $u \geq 1$. Also one has, by the Wu and Cartan formulae, that $Sq^j$ evaluated on a product of characteristic classes gives a polynomial in the characteristic classes. Then

$$
Sq^{2^r-1}(...(Sq^4(Sq^2(\alpha^2 c)))...) = \alpha^2 r c \quad \text{and} \quad Sq^{2^r-1}(...(Sq^4(Sq^2(\beta^2 d)))...) = \beta^2 r d
$$

are corresponding classes on $RP^2$ and $RP^n$. Using the Conner formula and the fact that $2^r \geq 4$, one then has

$$
0 = (\alpha^2 r c) c^{4j+1-2^r+l-1}[RP(\eta)] = (\beta^2 r d) d^{4j+1-2^r+l-1}[RP(\xi)] = \left(4j + 2 - 2^r\right).
$$

Since $(\eta/2)^j = 1$ and $2^r$ belongs to the 2-adic expansion of $4j + 2$, also $(\eta/2)^{j+2} = 1$, which is impossible. Hence Fact 3 is proved. To prove Fact 4, we can consider $n = 4j$ with $j \geq 1$; in this case, to show that $p = 1$, it suffices to show that $(\eta/2) = 1$, and again by contradiction we suppose $(\eta/2) = 0$. Then $(\eta/2)^j = 1$ and $k = l + 4j - 2$ gives

$$
\widetilde{W}(RP(\eta)) = (1 + \alpha)^3(1 + c)^{4j-1} + (1 + c)^{4j-2}\alpha + (1 + c)^{4j-3}\alpha^2)
$$

and $\tilde{w}_2(RP(\eta)) = c^2 + \alpha^2 + c\alpha$. Also

$$
\widetilde{W}(RP(\xi)) = (1 + \beta)^{4j+1}(1 + d + \beta + (1 + d)^{-1}(q/2)\beta^2 + (1 + d)^{-2}(q/3)\beta^3 + ...)
$$

and $\tilde{w}_2(RP(\xi)) = (\eta/2)^2 + \beta d + \beta^2$. Let $2^t$ be the lesser power of 2 of the 2-adic expansion of $n = 4j$ ($2^t \geq 4$). For $t \leq x \leq r$ and with the same preceding
tools, we then get

\[ S q^{2^{j+1}} \cdots (S q^4 (S q^2 (w_2 (RP(\eta))) \cdots ) c^{4j+1-2^r-2} [RP(\eta)] = \]

\[ (c^{2^r} + \alpha c + c^{2^r} \alpha) \cdots c^{4j+1-2^r-2} [RP(\eta)] = \binom{q}{2} \}

\[ S q^{2^{j+1}} \cdots (S q^4 (S q^2 (w_2 (RP(\xi))) \cdots ) d^{4j+1-2^r-2} [RP(\xi)] = \]

\[ (\binom{q}{2}) \beta^{2^r} d + \beta d^{2^r} + \beta^{2^r} d) d^{4j+1-2^r-2} [RP(\xi)] = \]

\[ (\binom{q}{2}) (q^{j-2r}) + (q^{j-1}) + (q^{j-2}) = (q^j) (q^{j-2r}) + (q^{j-1}) \]

\[ 0 = \binom{q}{2} = c^{k+1} [RP(\eta)] = d^{k+1} [RP(\xi)] = \binom{q}{2} \]

\[ \tilde{w}_2 (RP(\eta)) \beta^{2^r+1-3} [RP(\eta)] = \binom{q}{2} + 1 \]

\[ \tilde{w}_2 (RP(\xi)) d^{2^r+1-3} [RP(\xi)] = \binom{q}{2} (q^{j-2r}) + (q^{j-1}) + (q^{j-2}) = (q^j) (q^{j-2r}) + (q^{j-1}) \]

That is, we get the equations: i) 0 = \binom{q}{2}, ii) 0 = \binom{q}{2} (q^{j-2r}) + (q^{j-1}) and iii) 1 = \binom{q}{2} (q^{j-2r}) + (q^{j-1}). By using equations ii) and iii), we conclude that \( (\binom{q}{2}) = 1 \) and \((q^{j-2r}) \neq (q^{j-1}) \). Suppose \( t < r \). If \((q^{j-2r}) = 1 \), equation i) and the fact that \( 2^r \) belongs to the 2-adic expansion of \( 4j \) imply that \( 2^r \) is the only power of 2 of the 2-adic expansion of \( 4j \) that does not belong to the 2-adic expansion of \( q \). Hence \((q^{j-2r}) = 0 \), which is a contradiction. Then \((q^{j-2r}) = (q^{j-2r}) = 0 \). In this case, equation i) and \( (q^{j-2r}) = 1 \) give that \( 2^r \) is the only power of 2 of the 2-adic expansion of \( 4j \) that does not belong to the 2-adic expansion of \( q \), which gives the contradiction \((q^{j-2r}) = 1 \). Now suppose \( t = r \), that is, \( n = 4j + 2^r \). One has

\[ \tilde{w}_2 (RP(\eta)) c^{2^r+1-5} [RP(\eta)] = (q^j) + 1 = (\tilde{w}_2 (RP(\xi)) c^{2^r+1-5} [RP(\xi)] = \]

\[ (q^j) (q^{j-2r}) + (q^{j-4}) + (q^{j-2}) = (q^j) (q^{j-2r}) + (q^{j-4}) + (q^{j-2}) \]

Since \((q^j) = 1 \), \((q^{j-4}) = (q^{j-2}) \), which gives a contradiction. Thus Fact 4 is proved.

Now we prove that \( q = 3 \). To do this, first we prove

**Fact 5.** \( \binom{q}{2} = 1 \); in particular, \( q \geq 3 \).

**Proof.** As before, first consider \( n = 4j + 2 \), with \( j \geq 1 \). In this case, we know that \( 0 = \binom{q}{2} = (q^{j+2}) \), \( \tilde{w}_2 (RP(\eta)) = \binom{q}{2} \alpha^2 + \alpha c = \alpha^2 + \alpha c \) and \( \tilde{w}_2 (RP(\xi)) = \binom{q}{2} \beta^2 + \beta d \). Then

\[ (\tilde{w}_2 (RP(\eta))) c^{2^r+1-3} [RP(\eta)] = 1 = (\tilde{w}_2 (RP(\xi)) c^{2^r+1-3} [RP(\xi)] = \]

\[ (q^j) (q^{j-2r}) + (q^{j-1}) \]

Since \((q^j) = (q^{j-2r}) = 1 \) and 2 belongs to the 2-adic expansion of \( 4j - 2 \), one has that \((q^j) (q^{j-2r}) = 0 \), and thus \((q^{j-2r}) = 1 \). Now \((q^{j+2}) = 0 \) and \((q^{j-1}) = 1 \) imply that \((q^{j-2r}) = 0 \), and thus \((q^j) = 1 \). Since \( q \) is odd, this means that \( q \geq 3 \).
Now suppose \( n = 4j \), with \( j \geq 1 \). In this case, one has \( \binom{q}{2} = 1 \), \( \bar{w}_3(RP(\eta)) = c^3 + \binom{q}{2} \alpha^2c = c^3 + \alpha^2c \) and \( \bar{w}_3(RP(\xi)) = \binom{q}{2} \beta^2d \). Then

\[
S q^{2r-1}(... (S q^4 (S q^2 (w_3 (RP(\eta))))... )c^{4j+l-2r-2}[RP(\eta)] = \\
(2^c c + \alpha^2c) c^{4j+l-2r-2}[RP(\eta)] = \binom{q}{2} = 1 = \\
S q^{2r-1}(... (S q^4 (S q^2 ((\binom{q}{2} \beta^2d))))... )d^{4j+l-2r-2}[RP(\xi)] = \\
((\binom{q}{2} \beta^2d)d^{4j+l-2r-2}[RP(\xi)] = \binom{q}{2}\binom{q}{2}(4j-2r).
\]

Thus \( \binom{q}{2} = 1 \), and Fact 5 is proved.

To end our task, we will show that \( q \leq 3 \). The strategy will consist in finding nonzero characteristic numbers coming from characteristic classes involving \( \alpha^{q-1} \). To do this, we need the following

**Fact 6.** \( n + l - 1 > 2(q - 1) \).

**Proof.** First suppose \( n = 4j + 2 \), \( j \geq 1 \). From the proof of Fact 5, \( \binom{q}{2} = 1 \), and thus \( \binom{q}{2} = 1 \) and \( \binom{q}{2} = 0 \). Since \( q < 2r+1 \), \( q < 2r < 4j+2 \). In particular, \( w_q(\xi) = \alpha^q \neq 0 \) and \( q \leq l \). Then \( n + l - 1 = 4j + 2 + l - 1 > 2q - 1 > 2(q - 1) \).

Now suppose \( n = 4j \), \( j \geq 1 \). In this case, \( \binom{q}{2} = 1 = \binom{q}{2} \), so the argument is the same.

Fact 6 says that we can consider characteristic numbers coming from classes involving \( \bar{w}_2^{-1} \); in this direction, first consider \( n = 4j + 2 \), \( j \geq 1 \). In this case, \( \bar{w}_2(RP(\eta)) = \binom{q}{2} \alpha^2 + \alpha c = \alpha(\alpha + c) \) and \( \bar{w}_2(RP(\xi)) = \binom{q}{2} \beta^2 + \beta d = \beta(\beta + d) \). Thus

\[
(\alpha^{q-1}(\alpha + c)^{q-1}c^{4j+l-2q+3})[RP(\eta)] = \binom{q}{2}(\beta^{q-1}(\beta + d)^{q-1}d^{4j+l-2q+3})[RP(\xi)].
\]

By Conner’s formula, the last term is the coefficient of \( \beta^{4j+2} \) in \( \beta^{q-1}(1 + \beta)^{q-1}(1 + \beta)^{q-1} \). If \( n = 4j \), \( j \geq 1 \), similarly one has \( \bar{w}_2(RP(\eta)) + c^2 = (c^2 + \binom{q}{2} \alpha^2 + \alpha c) + c^2 = \alpha c, \bar{w}_2(RP(\xi)) + d^2 = \binom{q}{2}\beta^2 + \beta d + d^2 = (\beta + d)d, \)

\[
\left((\alpha^{q-1}\alpha^{q-1})\beta^{4j+l-2q+1}\right)[RP(\eta)] = \left(\beta + d\right)d^{q-1}d^{4j+l-2q+1}[RP(\xi)],
\]

and the last term is the coefficient of \( \beta^{4j} \) in \( (1 + \beta)^{q-1}(1 + \beta)^{q-1} \). Since \( (1 + \beta)^{q-1} \), these numbers have value 1, which means that \( \alpha^{q-1} \neq 0 \) and \( q - 1 \leq 2 \), thus ending the proof.

### 3 Calculation of \( h_{m,n} \)

Denote by \( W_r \) the underlying manifold of \( \Gamma^r(\mathcal{P}^{m+n+1}, T_{m,n}) \), and by \( \mathcal{P}_r \) the total space of the iterated fibration

\[
RP((m + 1)\mu_r \oplus (n + 1)R) \rightarrow RP(\lambda_1 \oplus (r - 1)R) \rightarrow RP^1,
\]

8
where $\mu_r$ is the standard line bundle over $RP(\lambda_1 \oplus (r - 1)R)$.

**Lemma** $\mathcal{W}_r$ is cobordant to $\mathcal{P}_r$.

**Proof.** If $(W, T)$ is a free involution and $\lambda \to W/T$ is the usual line bundle, the sphere bundle $S(\lambda \oplus R)$ with the antipodal involution in the fibers can be identified to the free involution $(\frac{W \times S^1}{T} \times c, \tau)$, where $c$ is the complex conjugation and $\tau$ is induced by $Id \times -Id$. Starting with $(S^1, -Id)$ and by iteratively applying this fact, we can see that $\mathcal{W}_r$ is diffeomorphic to the total space of the iterated fibration $RP((m + 1)\xi_r \oplus (n + 1)R) \to RP(\xi_{r-1} \oplus R) \to \ldots \to RP(\xi_2 \oplus R) \to RP(\xi_1 \oplus R) \to RP^1$; here, $\xi_1 = \lambda_1$ and $\xi_i$ is the standard line bundle over $RP(\xi_{i-1} \oplus R)$, for each $i > 1$. From [6], one knows that $\mathcal{N}_r(BO(1))$ is a free $\mathcal{N}_r$-module, where $\mathcal{N}_r$ is the unoriented cobordism ring, with one generator $X_j$ in each dimension $j \geq 0$; these generators are characterized by the fact that $c^j[V^j] = 1$, where $\lambda \to V^j$ is a representative of $X_j$ and $c$ is the first Whitney class of $\lambda$. Further, it was shown in [8, Theorem 24.5] that there is a unique basis $\{X_j\}_{j=0}^\infty$ for $\mathcal{N}_r(BO(1))$ which satisfies: i) $\triangle(X_j) = X_{j-1}$, $j \geq 1$, where $\triangle: \mathcal{N}_r(BO(1)) \to \mathcal{N}_{r-1}(BO(1))$ is the Smith homomorphism; ii) if $\lambda \to V^j$ is a representative of $X_j$ for $j \geq 1$, then $V^j$ bounds. Also it was shown in [8, Theorem 24.5] that $X_1 = [\xi_1 \to RP^1]$ and $X_j = [\xi_j \to RP(\xi_{j-1} \oplus R)]$ for $j \geq 2$. For $j \geq 1$, set $Y_j = [\mu_j \to RP(\lambda_1 \oplus (j-1)R)]$. One has $c^j[RP(\lambda_1 \oplus (j-1)R)] = w(T)(\lambda_1)[S^1] = 1$, $Y_1 = X_1$ and $\triangle([\mu_j \to RP(\lambda_1 \oplus (j-1)R)]) = [\mu_{j-1} \to RP(\lambda_1 \oplus (j-2)R)]$ for $j \geq 2$; further, every projective space bundle over $S^1$ bounds (see [7, Lemma 2.2]). By the uniqueness, $Y_j = X_j$ for $j \geq 1$, and the result follows. □

With the above lemma in hand, Theorem 2 can now be paraphrased as

**Theorem 2.** For $m, n$ even, $0 \leq m < n$, write $n = m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd. Then,

a) if $p = 1$, $\mathcal{P}_1$ bounds and $\mathcal{P}_2$ does not bound;

b) if $p > 1$, $\mathcal{P}_r$ bounds for each $1 \leq r \leq 2^p - 2$ and $\mathcal{P}_{2^p - 1}$ does not bound.

Denote by $\alpha \in H^1(RP^1, Z_2)$ the generator and by $\theta_r \to \mathcal{P}_r$ the standard line bundle; set $W(\mu_r) = 1 + c$ and $W(\theta_r) = 1 + d$. The following lemma, which follows from Conner’s formula, will be useful in our computations:

**Lemma.** i) For $f + g + h = m + n + 1 + r$, $c^f(c + d)^g d^h [\mathcal{P}_r]$ is the coefficient of $c^r$ in $\frac{c^f(1 + c)^g}{(1 + c)^{m+1}}$. 9
ii) For \( f + g + h = m + n + r \), \( \alpha c^j (c + d)^{g} d^{h} [P_r] \) is the coefficient of \( c^r \) in \( c^{f+1}(1+c)^{g} \over (1+c)^{m+1} \).

If \( M \) is a closed manifold and \((1 + t_1)(1 + t_2)\ldots(1 + t_i)\) is the factored form of \( W(M) \), one has the \( s \)-class \( s_j \) given by the polynomial in the classes of \( M \)
corresponding to the symmetric function
\[
\frac{1}{1 + t_1 + t_2 + \ldots + t_i}.
\]

Since
\[
W(P_r) = (1 + c + \alpha)(1 + c)^{r-1}(1 + c + d)^{m+1}(1 + d)^{n+1},
\]
c\( ^i \) = 0 if \( i > r \) and \( \alpha^i = 0 \) if \( i > 1 \), the \( s \)-class \( s_{m+n+1+r} \) of \( P_r \) then is
\[
s_{m+n+1+r} = (c + \alpha)^{m+n+1+r} + (r - 1)c^{m+n+1+r} + (m + 1)(c + d)^{m+n+1+r} + \]
\[
(n + 1)d^{m+n+1+r} = (c + d)^{m+n+1+r} + d^{m+n+1+r}.
\]

Using part i) of the above lemma and the fact that
\[
\frac{1}{1 + (1+c)^{m+1}} = 1 + \sum_{i=1}^{r} \binom{m+i}{i} c^i
\]
in \( H^*(P_r, \mathbb{Z}_2) \), one then has
\[
s_{m+n+1+r} [P_r] = \text{coefficient of } c^r \text{ in } (1+c)^{n+r} + \text{coefficient of } c^r \text{ in } \left( \frac{1}{1 + (1+c)^{m+1}} \right).
\]

Because \( n = 2^p q + m \) and \( q \) is odd, one then gets
\[
s_{m+n+1+2^p} [P_{2^p}] = \left( \binom{n+2^p}{2^p} \right) + \left( \binom{m+2^p}{2^p} \right) = 1.
\]

It follows that \( P_{2^p} \) does not bound; because \( P_1 \) is a projective space bundle over \( S^1 \) and hence a boundary, this in particular proves part a) of Theorem 2.

So we can assume from now that \( p > 1 \) and \( r < 2^p \). Using again \( n = 2^p q + m \), we rewrite \( W(P_r) \) as
\[
W(P_r) = (1 + c + \alpha)(1 + c)^{r-1}(1 + c + d(c + d))^{m+1}(1 + d^{2^p})^g.
\]

Then a general characteristic number of \( P_r \) is a sum of terms of the form \( \alpha^e c^j (d(c + d))^{g} d^{2^p h} [P_r] \), where \( e + f + 2g + 2^ph = m + n + 1 + r \) and either \( e = 0 \) or \( e = 1 \). Since by the above lemma \( \alpha^e c^j (d(c + d))^{g} d^{2^p h} [P_r] = c^{f+1}(d(c + d))^{g} d^{2^p h} [P_r] \), we can assume \( e = 0 \). Thus, to prove the first statement of part b) of Theorem 2, it suffices to show that \( c^j (d(c + d))^{g} d^{2^p h} [P_r] = 0 \) when \( f + 2g + 2^ph = m + n + 1 + r \) and \( r < 2^p - 1 \). Since \( c^j = 0 \) if \( f > r \), we assume
\( f \leq r \) and thus \( 0 \leq r - f < 2^p - 1 \). Take \( s > p \) with \( 2^s > m + 1 \); in particular, \( 2^s > 2^p > r \) and \( \frac{1}{(1 + c)^{m+1}} = (1 + c)^{2^s-m-1} \). Then

\[
c^f (d(c + d))^g d^{2p} h [P_r] = \text{coefficient of } c^r \text{ in } \frac{c^f (1 + c)^g}{(1 + c)^{m+1}} = \text{coefficient of } c^r \text{ in } c^f (1 + c)^g (1 + c)^{2^s - m - 1} = (2^s + g - m - 1 - \frac{r - f + 1}{2} - 1).
\]

Write \( r - f + 1 = 2^t a \), where \( a \) is odd. Since \( r - f + 1 = 2g + 2ph - m - n \) is even and \( r - f + 1 < 2^p \), one has \( 1 \leq t \leq p - 1 \). Then \( 2^t - 1 \) belongs to the \( 2 \)-adic expansion of \( r - f \) and does not belong to the \( 2 \)-adic expansion of \( 2^{p-1}(2^s-p+1+q-h)+\frac{r-f+1}{2} - 1 \), which means, as required, that the above number is zero.

Finally, we must to show that \( P_{2^p-1} \) does not bound. One has \( w_2(P_{2^p-1}) = \alpha + (m + 2^p + d(c + d)) \). We have seen above that \( c^f (d(c + d))^g d^{2p} h [P_r] = 0 \) for \( f + 2g + 2ph = m + n + 1 + r \) and \( 0 \leq r - f < 2^p - 1 \); in particular, this is true for \( r = 2^p - 1 \) and \( f > 0 \). In this way,

\[
w_2(P_{2^p-1}) = \frac{m + n + 2^p}{2} - \frac{m + n + 2^p}{2} = 0.
\]

Then

\[
\text{coefficient of } c^{2^p-1} \text{ in } \frac{(1 + c)^{m+1}}{(1 + c)^m + 2^p - 1} = (2^{p-1}q + 2^{p-1} - 1) = 1,
\]

and \( P_{2^p-1} \) does not bound.

### 4 \( \mathbb{Z}_2^k \)-actions fixing \( RP^2 \cup RP^{even} \)

Let \( F^n \) be a connected, smooth and closed \( n \)-dimensional manifold satisfying the following property, which we call property \( \mathcal{H} \) : if \( N^m \) is any smooth and closed \( m \)-dimensional manifold with \( m > n \) and \( T : N^m \to N^m \) is a smooth involution whose fixed point set is \( F^n \), then \( m = 2n \). From [1], this implies that \( (N^m, T) \) is cobordant to the \textit{twist involution} \( (F^n \times F^n, t) \), given by \( t(x, y) = (y, x) \). This concept was introduced and studied in [13], and it was inspired
in [6; 27.6] (or [8; 29.2]), where it was shown that $R^{\text{even}}$ has this property. In [12], we studied the equivariant cobordism classification of smooth actions $(M; \Phi)$ of the group $Z_2^k$ on closed and smooth manifolds $M$ for which the fixed point set $F$ of the action is the union $F = K \cup L$, where $K$ and $L$ are submanifolds of $M$ with property $\mathcal{H}$ and with $\dim(K) < \dim(L)$. We showed that, for this $F$, the $Z_2^k$-classification is completely determined by the corresponding $Z_2$-classification. Specifically, the equivariant cobordism classes of $Z_2^k$-actions fixing $K \cup L$ can be represented by a special set of $Z_2^k$-actions which are explicitly obtained from involutions fixing $K \cup L$, $K$ and $L$. Together with the results of Sections 2 and 3 and the case $F = R^{\text{even}}$, this gives a precise cobordism description of the $Z_2^k$-actions fixing $RP^2 \cup RP^n$, where $n > 2$ is even; next we give this description. Here, $Z_2^k$ is considered as the group generated by $k$ commuting involutions $T_1, T_2, ..., T_k$. The fixed data of a $Z_2^k$-action $(M; \Phi)$, $\Phi = (T_1, T_2, ..., T_k)$, is $\eta = \bigoplus_{\rho} \varepsilon_{\rho} \to F$, where $F = \{x \in M / T_i(x) = x \text{ for all } 1 \leq i \leq k\}$ is the fixed point set of $\Phi$ and $\eta = \bigoplus_{\rho} \varepsilon_{\rho}$ is the normal bundle of $F$ in $M$, decomposed into eigenbundles $\varepsilon_{\rho}$ with $\rho$ running through the $2k - 1$ nontrivial irreducible representations of $Z_2^k$. A collection of $Z_2^k$-actions fixing $F$ can be obtained from an involution fixing $F$ through the following procedure: let $(W, T)$ be any involution. For each $r$ with $1 \leq r < k$, consider the $Z_2^k$-action $\Gamma^r_k(W, T)$, defined on the cartesian product $W^{2r - 1} = W \times \cdots \times W$ ($2^{r-1}$ factors), and described in the following inductive way: first set $\Gamma^1_k(W, T) = (W, T)$. Taking $k \geq 1$ and supposing by inductive hypothesis one has constructed $\Gamma^{k-1}_{k-1}(W, T)$, define $\Gamma^k_k(W, T) = (W^{2k-1}; T_1, T_2, ..., T_k)$, where $(W^{2k-1}; T_1, T_2, ..., T_{k-1}) = (W^{2k-2} \times W^{2k-2}; T_1, T_2, ..., T_{k-1}) = \Gamma^{k-1}_{k-1}(W, T) \times \Gamma^{k-1}_{k-1}(W, T)$ and $T_k$ acts switching $W^{2k-2} \times W^{2k-2}$. This defines $\Gamma^k_r(W, T)$ for any $k \geq 1$. Next, define $\Gamma^k_k(W, T) = (W^{2r-1}; T_1, T_2, ..., T_k)$ setting $(W^{2r-1}; T_1, T_2, ..., T_{k}) = \Gamma^r_k(W, T)$ and letting $T_{r+1}, ..., T_k$ act trivially. If $(W, T)$ fixes $F$ and if $\eta \to F$ is the normal bundle of $F$ in $W$, then $\Gamma^k_r(W, T)$ fixes $F$ and its fixed data consists of $2r-1$ copies of $\eta$, $2^{r-1} - 1$ copies of the tangent bundle of $F$ and $2^k - 2r$ copies of the zero-dimensional bundle over $F$. In particular, for the twist involution $(F \times F, t)$, $\Gamma^k_k(F \times F, t) = (F^{2r}; T_1, T_2, ..., T_k)$, where $(T_1, T_2, ..., T_r)$ is the usual twist $Z_2^r$-action on $F^{2r}$ which interchanges factors and $T_{r+1}, ..., T_k$ act trivially, with the fixed data having in this case $2r - 1$ copies of the tangent bundle of $F$ and $2^k - 2r$ zero bundles. In this special case, we allow $r$ to be zero, setting $\Gamma^k_0(F \times F, t) = (F; T_1, T_2, ..., T_k)$, where each $T_i$ is the identity involution.

Now, from a given $Z_2^k$-action $(M; \Phi)$, $\Phi = (T_1, ..., T_k)$, we can obtain a collection of new $Z_2^k$-actions, described as follows: first, each automorphism $\sigma : Z_2^k \to Z_2^k$ yields a new action given by $(M; \sigma(T_1), ..., \sigma(T_k))$; we denote this
action by $\sigma(M; \Phi)$. The fixed data of $\sigma(M; \Phi)$ is obtained from the fixed data of $(M; \Phi)$ by a permutation of eigenbundles, obviously depending on $\sigma$. Next, it was shown in [11] that if $(M; \Phi)$ has fixed data $\bigoplus_\rho \varepsilon_\rho \to F$ and one of the eigenbundles $\varepsilon_\theta$ is isomorphic to $\varepsilon_\theta' \oplus R$, then there is an action $(N; \Psi)$ with fixed data $\bigoplus_\rho \mu_\rho \to F$, where $\mu_\rho = \varepsilon_\rho$ if $\rho \neq \theta$ and $\mu_\theta = \varepsilon_\theta'$. We say in this case that $(N; \Psi)$ is obtained from $(M; \Phi)$ by removing one section. Thus, the iterative process of removing sections may possibly enlarge the set \{\sigma(M; \Phi), \sigma \in \text{Aut}(Z^k_2)\}. Summarizing, from a given involution $(W; T)$ that fixes $F$, we obtain a collection of $Z^k_2$-actions fixing $F$ by applying the operations $\sigma \Gamma^k_r$ on $(W; T)$ and next by removing the (possible) sections from the resultant eigenbundles. The results of [12] say that when $F = K \cup L$, where $K$ and $L$ have property $\mathcal{H}$ and $\dim(K) < \dim(L)$, then, up to equivariant cobordism, all $Z^k_2$-actions fixing $F$ are obtained, with the above procedure, from involutions fixing $K \cup L$, $K$ and $L$. Together with the $Z_2$-classification obtained in Sections 2 and 3 and the case $F = \mathbb{R}^2 \cup \mathbb{R}^n$, we obtain a collection of $Z^k_2$-actions fixing $F$, where $\sigma \in \text{Aut}(Z^k_2)$.

**Theorem.** Let $(M; \Phi)$ be a $Z^k_2$-action fixing $\mathbb{R}^2 \cup \mathbb{R}^n$, where $n > 2$ is even. Then $(M; \Phi)$ is equivariantly cobordant to an action belonging to the set $A \cup B$, where the sets $A$ and $B$ are described below in terms of $n$:

i) $n - 2 = 2^p q$, with $q$ odd and $p > 1$: $A = \emptyset = \text{the empty set}; B = \text{the set obtained from } \{\sigma \Gamma^k_r \Gamma^{2p-1}(\mathbb{R}^{n+3}, T_{2,n}), \sigma \in \text{Aut}(Z^k_2), 1 \leq r \leq k\}$ by removing sections.

ii) $n - 2 = 2q$, with $q$ odd, and $n$ is not a power of 2: $A = \emptyset; B = \text{the set obtained from } \{\sigma \Gamma^k_r \Gamma^2(\mathbb{R}^{n+3}, T_{2,n}), \sigma \in \text{Aut}(Z^k_2), 1 \leq r \leq k\}$ by removing sections;

iii) $n = 2^t$ is a power of 2 with $t \geq 3$: $A = \{\sigma \Gamma^k_r(\mathbb{R}^2 \times \mathbb{R}^2, \text{twist}) \cup \sigma' \Gamma^k_{r-1}(\mathbb{R}^2 \times \mathbb{R}^2, \text{twist}), \sigma, \sigma' \in \text{Aut}(Z^k_2), t - 1 \leq r \leq k\}; B = \text{the set obtained from } \{\sigma \Gamma^k_r \Gamma^2(\mathbb{R}^{2^t+3}, T_{2,2^t}), \sigma \in \text{Aut}(Z^k_2), 1 \leq r \leq k\}$ by removing sections (by dimensional reasons, in this case $A = \emptyset$ if $t - 1 > k$);

iv) $n = 4$: $A = \{\sigma \Gamma^k_{r+1}(\mathbb{R}^2 \times \mathbb{R}^2, \text{twist}) \cup \sigma' \Gamma^k_r(\mathbb{R}^4 \times \mathbb{R}^4, \text{twist}), \sigma, \sigma' \in \text{Aut}(Z^k_2), 0 \leq r \leq k - 1\} \cup \{\sigma \Gamma^k_r(W^3, T), \sigma \in \text{Aut}(Z^k_2), 1 \leq r \leq k\}$, where $(W^5, T) = \Gamma^2(R^3, T_{0,2}) \cup (\mathbb{R}^5, T_{0,4}); B = \text{the set obtained from } \{\sigma \Gamma^k_r \Gamma^2(\mathbb{R}^7, T_{2,4}), \sigma \in \text{Aut}(Z^k_2), 1 \leq r \leq k\}$ by removing sections.
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