A NOTE ON $S$-ASYMPTOTICALLY PERIODIC FUNCTIONS

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Abstract. In [1, Lemma 2.1] is established that an scalar $S$-asymptotically $\omega$-periodic function (that is, a continuous and bounded function $f : [0, \infty) \rightarrow \mathbb{R}$ such that $\lim_{t \rightarrow \infty} (f(t+\omega) - f(t)) = 0$) is asymptotically $\omega$-periodic. In this note we give two examples to show that this assertion is false.

1. Introduction

In [1, Lemma 2.1] is established that every scalar continuous and bounded function $f : [0, \infty) \rightarrow \mathbb{R}$ such that $\lim_{t \rightarrow \infty} (f(t+\omega) - f(t)) = 0$, for some real number $\omega > 0$, is asymptotically $\omega$-periodic. In this note, we provide two examples which shows that this assertion is false. For completeness, we recall some concepts and definitions.

In this note, $C_b([0, \infty), \mathbb{R})$ is the space of all continuous and bounded functions from $[0, \infty)$ into $\mathbb{R}$ endowed with the norm of the uniform convergence norm denoted by $| \cdot |_{\infty}$. Its subspaces, $C_b([0, \infty), \mathbb{R})$ and $C_{\omega}([0, \infty), \mathbb{R})$, $\omega > 0$, are defined by

$$C_0([0, \infty), \mathbb{R}) = \left\{ x \in C_b([0, \infty), \mathbb{R}) : \lim_{t \rightarrow \infty} |x(t)| = 0 \right\},$$

$$C_{\omega}([0, \infty), \mathbb{R}) = \left\{ x \in C_b([0, \infty), \mathbb{R}) : x \text{ is } \omega \text{-periodic} \right\}.$$

Definition 1.1. A function $f \in C(\mathbb{R}, \mathbb{R})$ is called almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{H}(\varepsilon, f)$ of $\mathbb{R}$ such that $|f(t+\xi) - f(t)| < \varepsilon$, for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, f)$.

Definition 1.2. A function $f \in C_b([0, \infty), \mathbb{R})$ is called asymptotically almost periodic if there exists an almost periodic function $g$ and $\varphi \in C_0([0, \infty), \mathbb{R})$ such that $f = g + \varphi$. If $g$ is periodic (resp. $\omega$-periodic) $f$ is said asymptotically periodic (resp. asymptotically $\omega$-periodic).

For additional facts on almost periodic and asymptotically almost periodic functions, we refer the reader to [2, 3].

Definition 1.3. A function $f \in C_b([0, \infty), \mathbb{R})$ is called $S$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} (f(t+\omega) - f(t)) = 0$. In this case, we say that $\omega$ is an asymptotic period of $f$ and that $f$ is $S$-asymptotically $\omega$-periodic.

2. Examples

The following examples are contrary to [1, Lemma 2.1].
Example 2.1. Let \((b_n)_n\) be a sequence of real numbers such that \(b_n \neq 0\), \(n = 0, 1, 2, \ldots\), \(b_n \to 0\) as \(n \to \infty\), and the sequence \((a_n)_n = (\sum_{i=0}^{n} b_i)_n\) is bounded and non-convergent. We note that under these conditions, \(a_n - a_{n-1} \to 0\) as \(n \to \infty\).

Let \(f : [0, \infty) \to \mathbb{R}\) be the function defined by \(f(n) = a_n\) and

\[
f(t) = a_{n+1} + (a_{n+1} - a_n)(t - n - 1), \quad t \in [n, n+1], \quad n = 0, 1, 2, \ldots
\]

That is, the graph of \(f\) consists of the line segments joining the points \((n, a_n)\), \(n = 0, 1, 2, \ldots\) Therefore, \(f\) is bounded. Moreover, since \(|f(t) - f(s)| \leq \max_{k \geq n} |a_{k+1} - a_k||t - s|\), for every \(t, s \in [n, \infty)\), it following that \(f\) is uniformly continuous and

\[
\lim_{t \to \infty} |f(t + \omega) - f(t)| = 0,
\]

for every \(\omega > 0\). Therefore, \(f\) is an \(S\)-asymptotically \(\omega\)-periodic function, for any \(\omega > 0\).

In particular, \(f\) is \(S\)-asymptotically \(1\)-periodic.

However, \(f\) is not asymptotically \(1\)-periodic. In fact, let us assume \(f = g + \alpha\), where \(g \in C_1([0, \infty), \mathbb{R})\) and \(\alpha \in C_0([0, \infty), \mathbb{R})\). In such case, \(a_n = f(n) = g(n) + \alpha(n) = g(0) + \alpha(n) \to g(0), \quad n \to \infty\), which is contrary to the construction of \((a_n)_n\).

From the above remarks, we have that \(f\) is \(S\)-asymptotically \(1\)-periodic but not asymptotically \(1\)-periodic.

The following elementary lemma is immediate and plays a role in the example below.

Lemma 2.1. If \(g \in C_\omega([0, \infty), \mathbb{R})\), then \(g([t, t + \omega]) = \mathcal{R}(g)\), the range of \(g\), for any \(t \in [0, \infty)\)

Example 2.2. Define \(f : [0, \infty) \to \mathbb{R}\) by \(f(t) = \sin \ln(t + 1), \quad t \in [0, \infty)\). Since \(f'(t) = (\cos \ln(t + 1))/(t + 1)\) we have \(\lim_{t \to \infty} f'(t) = 0\).

For any \(\omega > 0\), \(f(t + \omega) - f(t) = f'(t + \sigma \omega)\omega\), where \(0 < \sigma < 1\), which implies that \(\lim_{t \to \infty} |f(t + \omega) - f(t)| = 0\). That is, \(f\) is \(S\)-asymptotically \(\omega\)-periodic, for any \(\omega > 0\).

But \(f\) is not asymptotically \(\omega\)-periodic. In fact, given \(\omega > 0\), let \(k\) be sufficiently large such that \(e^{2k\pi + \omega} < e^{2k\pi + \frac{\pi}{2}}\). Note that \(f\) is increasing in \([e^{2k\pi - 1}, e^{2k\pi + \frac{\pi}{2}} - 1]\), with let \(0 < \varepsilon < 1/2\), and take \(k\) larger if necessary in such a way that \(t \geq e^{2k\pi - 1}\) implies \(|\varphi(t)| < \varepsilon\). Therefore, if \(g = f - \varphi\), one sees that \(g([e^{2k\pi - 1}, e^{2k\pi + \frac{\pi}{2}} - 1]) \subset (-\varepsilon, 1 + \varepsilon)\).

Since \(g(e^{2k\pi + \frac{\pi}{2}} - 1) = -1 + \varepsilon < -\varepsilon\) we have \(g(e^{2k\pi + \frac{\pi}{2}} - 1) < -1 + \varepsilon < -\varepsilon\) we have \(g(e^{2k\pi + \frac{\pi}{2}} - 1) \notin (-\varepsilon, 1 + \varepsilon)\) and, \(g([e^{2k\pi - 1}, e^{2k\pi + \frac{\pi}{2}} - 1]) \subset g([e^{2k\pi - 1}, e^{2k\pi + 1 + \omega}])\).

This means that for \(t = e^{2k\pi - 1}, \mathcal{R}(g) \setminus g([t, t + \omega]) \neq \emptyset\) and, according to Lemma 2.1, there is not \(\omega > 0\) such that \(g\) can be \(\omega\)-periodic. Consequently, \(f\) cannot be asymptotically \(\omega\)-periodic.

Remark 2.1. The examples above exploit the fact that the functions \(f\) under consideration are Lipschitz continuous, with the possibility of choosing the Lipschitz constant arbitrarily small by restricting \(f\) to an interval \([T, \infty)\), with \(T > 0\) sufficiently large. By consequence, the \(S\)-asymptotic period \(\omega > 0\) of \(f\) is arbitrary. If fact suggests the following question: Does lemma 2.1 of [1] hold if restricted to scalar \(S\)-asymptotically \(\omega\)-periodic functions that have a minimum \(S\)-asymptotic period \(\omega > 0\)? The answer is
negative, as one can see by taking for instance the function \( F \) given by \( F(t) = f(t) + \cos t, \) \( t \in \mathbb{R}, \) where \( f \) is the function given in example 2.2. Straightforward arguments show that \( F \) is \( S \)-asymptotically \( 2\pi \)-periodic, being \( 2\pi \) its minimum \( S \)-asymptotic period, but it is not asymptotically \( 2\pi \)-periodic.

References


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