THE REAL JACOBIAN CONJECTURE ON $\mathbb{R}^2$ IS TRUE WHEN
ONE OF THE COMPONENTS HAS DEGREE 3

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Abstract. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (p, q)$, be a polynomial mapping such that
det $DF$ never vanishes. In this paper it is shown that if either $p$ or $q$ has
degree less or equal 3, then $F$ is injective. The technique relates solvability of
appropriate vector fields with injectivity of the mapping.

1. Introduction

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (p, q)$, be a polynomial mapping. Suppose
$$\det DF(x) \neq 0, \forall x \in \mathbb{R}^2. \tag{1.1}$$
In 1994, Pinchuk presented, in [6], an example of such an $F$ that is not injective.
In Pinchuk’s example, $p$ has degree 10 and $q$ has degree 35. On the other hand, in
2001, Gwoździewicz showed, in [4], that if $p$ and $q$ are polynomials of degree less or
equal 3, then $F$ is injective. So a question arises: How far can the degree of $p$ or $q$
be increased in order to have Gwoździewicz’s conclusion? In this paper it is proved
the following:

Theorem 1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (p, q)$, be a polynomial mapping such that $p$
has degree less or equal 3. If (1.1) holds, then $F$ is one-to-one.

This Theorem is the main result of this paper and it will be proved in Section 4
as a consequence of results stated and proved earlier in the paper.

The ideas in this paper are based on the second named author’s paper [1]: Let
$H_p$ be the hamiltonian field of $p$, i. e.
$$H_p = \left( \frac{\partial p}{\partial x_2}, -\frac{\partial p}{\partial x_1} \right).$$
The results in [1] give that $F$ is one-to-one if $H_p$ (or $H_q$) is globally solvable (recall
that a vector field $X = (X_1, X_2)$ is said to be globally solvable if for all $C^\infty$
fundction $f$, there is a $C^\infty$ function $u$ such that $X u = X_1 \partial u / \partial x_1 + X_2 \partial u / \partial x_2 = f$).
So to show Theorem 1, one can concentrate only in the polynomial $p$ (and in the
associated hamiltonian vector field $H_p$).

The paper is organized as follows: In section 2, it will be proved that global
solvability of $H_f$, $f : \mathbb{R}^2 \to \mathbb{R}$ being a $C^\infty$ function, is equivalent to connectedness
of all level sets $f^{-1}((c))$, provided that $H_f$ never vanishes. Since the latter implies
the injectivity of $F = (f, g)$ when (1.1) holds, one will have an alternative proof
of Santos Filho’s result for the plane (Corollary 1). In section 3, by means of a
classification, it will be seen that if $H_p$ never vanishes (a necessary condition to

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2. Connected components and Global solvability

The following notation will be used throughout the paper: let \( X : \mathbb{R}^2 \to \mathbb{R}^2 \) be a vector field. Denote by \( \gamma_x \) the integral curve of \( X \) such that \( \gamma_x(0) = x \), and by \( I_x \) its maximal interval of definition.

In Theorem 3 below, global solvability of \( H_f \) is characterized. In order to prove that Theorem, some preliminaries are needed:

**Lemma 1.** Let \( X : \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^\infty \) vector field such that \( X(x) \neq 0 \), \( \forall x \in \mathbb{R}^2 \). Then

\[ \gamma_x^+ = \{ \gamma_x(t) \mid t \geq 0, t \in I_x \} \quad \text{and} \quad \gamma_x^- = \{ \gamma_x(t) \mid t \leq 0, t \in I_x \} \]

are unbounded sets.

**Proof.** It is a consequence of Poincaré-Bendixon Theorem.

Recall now that a vector field \( X : \mathbb{R}^2 \to \mathbb{R}^2 \) is said to have a Reeb component if there is a compact \( K \subset \mathbb{R}^2 \), such that for all compact \( K' \subset \mathbb{R}^2 \), there are an integral curve \( \gamma_x \) of \( X \), and \( s_1 < s_2 < s_3 \in I_x \) such that \( \gamma_x(s_i) \in K \), \( i = 1, 3 \), and \( \gamma_x(s_2) \notin K' \). From Theorem 6.4.2 of [2], one has:

**Theorem 2.** Let \( X : \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^\infty \) vector field. Then \( X \) is globally solvable if, and only if, the following holds:

1. \( X(x) \neq 0 \), \( \forall x \in \mathbb{R}^2 \);  
2. \( X \) has no Reeb component.

From now on let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^\infty \) function. First recall the following simple result:

**Lemma 2.** Every integral curve of \( H_f \) is contained in a connected component of a level set of \( f \).

If \( \nabla f(x) \neq 0 \), \( \forall x \in \mathbb{R}^2 \), then every connected component of a level set of \( f \) defines an integral curve of \( H_f \).

**Proof.** For a given \( x \in \mathbb{R}^2 \), let \( c = f(x) \) and \( C_x \) be the connected component of the level set \( f^{-1}\{\{c\}\} \) such that \( x \in C_x \). It is very simple to see that \( \gamma_x \subset C_x \). If \( \nabla f \neq 0 \), \( C_x \) is a (connected) \( C^\infty \) manifold of dimension 1 (by the Inverse Function Theorem), then, as \( H_f \neq 0 \), it follows from Lemma 1 that \( C_x = \gamma_x \).

Hence, one has:

**Theorem 3.** If \( \nabla f(x) \neq 0 \), \( \forall x \in \mathbb{R}^2 \), then:

\( H_f \) is globally solvable if, and only if, the level set \( f^{-1}\{\{c\}\} \) is connected for all \( c \in \mathbb{R} \).

**Proof.** Throughout this proof, \( \gamma_x \) will denote the integral curve of \( H_f \) such that \( \gamma_x(0) = x \). First it will be proved \((\Rightarrow)\). Suppose, by contradiction, that there is a \( c \in \mathbb{R} \) such that \( \Gamma_1 \) and \( \Gamma_2 \) are two distinct connected component of \( f^{-1}\{\{c\}\} \). One will construct a Reeb component of \( H_f \) and then reach in a contradiction with the hypothesis and Theorem 2.
Take \( a \in \Gamma_1 \) and \( b \in \Gamma_2 \). From Lemma 2, it follows that \( \Gamma_1 = \gamma_a \) and \( \Gamma_2 = \gamma_b \). From Lemma 1 one has that \( \Gamma_1 \) and \( \Gamma_2 \) separate the plane into three open unbounded connected regions. Call \( R \) the region whose boundary is \( \Gamma_1 \cup \Gamma_2 \). Let \( L : [0, 1] \to \mathbb{R}^2 \) be a \( C^\infty \) curve without self-intersections such that \( L(0) = a \), \( L(1) = b \) and \( L([0, 1]) \subset R \), and call \( K = L([0, 1]) \). Notice that \( K \) separates \( R \) into two open unbounded connected regions \( R_1 \) and \( R_2 \). Define the following sets:

\[ \mathcal{C} \equiv \{ \gamma : I \subset \mathbb{R} \to R \mid \forall r > 0, \exists t_1 < t_2 \in I : |\gamma(t)| > r, \forall t < t_1, \forall t > t_2 \in I \} , \]

\[ \mathcal{C}_1 \equiv \{ \gamma \in \mathcal{C} \mid \exists t' < t'' \in I : \gamma(t) \in R_j, \forall t < t', \forall t > t'' \in I \} , j = 1, 2, \]

and

\[ \mathcal{C}_2 \equiv \mathcal{C} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2) . \]

Notice that one has the disjoint union \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_2 \). Furthermore, observe that, from Lemma 1, Lemma 2 and from Tubular Flow Theorem, for all \( x \in R \), the integral curve \( \gamma_x \) of \( H_f \) is an element of \( \mathcal{C} \). Consider the function \( h(t) \equiv f(L(t)) \), \( t \in [0, 1] \). Since \( h(0) = h(1) \) and \( h \) is not constant (by Lemma 2), changing \( f \) by \( -f \) (if necessary), it can be assumed that \( h \) has a global maximum in a point \( t_0 \in (0, 1) \), with \( h(t_0) > h(0) \).

**Figure 1.** The integral curve \( \gamma_{L(t_0)} \) is crossing the curve \( L \).

Notice that \( \gamma_{L(t_0)} \subset R_1 \cup K \) or \( \gamma_{L(t_0)} \subset R_2 \cup K \). In particular, \( \gamma_{L(t_0)} \in \mathcal{C}_1 \) or \( \gamma_{L(t_0)} \in \mathcal{C}_2 \), respectively. In fact, if both of the alternatives were false, there would be \( s_1 \neq s_2 \in I_{L(t_0)} \) such that \( \gamma_{L(t_0)}(s_1) \in R_1 \) and \( \gamma_{L(t_0)}(s_2) \in R_2 \) (Figure 1). Then because \( \nabla f \neq 0 \), it would follow from continuous dependence on the initial data and from Lemma 2 that there would be \( t' \in (0, 1) \) such that

\[ h(t') = f(L(t')) > f(L(t_0)) = h(t_0) , \]

a contradiction with \( t_0 \) being a maximum point of \( h \).

Now suppose, without loss of generality, that

\[ t_0 = \inf \{ t \in (0, 1) \mid t \text{ is a global maximum of } h \} , \]

and that \( \gamma_{L(t_0)} \subset R_1 \cup L \) (so \( \gamma_{L(t_0)} \in \mathcal{C}_1 \)) (as in Figure 2). Defining

\[ t_1 = \inf \{ t \in (0, 1) \mid \gamma_{L(t)} \in \mathcal{C}_1 \} , \]

there are three possibilities for \( t_1 \):

1. \( t_1 = 0 \);  
2. \( t_1 = t_0 \);  
3. \( t_1 = t_2 \).
These three possibilities will be analyzed below. In each of them it will be possible to construct a Reeb component of \( H_f \), finishing the proof of \( \Rightarrow \).

If (1) occurs, then there is a sequence \( t_n \to 0 \) such that \( \gamma_{L(t_n)} \in C_1 \). For the compact set \( K \) given before, if \( K' \subset \mathbb{R}^2 \) is any compact set, there are \( y \in \Gamma_1 \) and \( \varepsilon > 0 \) such that \( B(y, \varepsilon) \cap (K \cup K') = \emptyset \) and \( B(y, \varepsilon) \cap R_2 \neq \emptyset \), where \( B(y, \varepsilon) = \{ x \in \mathbb{R}^2 \mid |x - y| < \varepsilon \} \) is the ball of center \( y \) and radius \( \varepsilon \). Let \( s_y \in I_a \) be such that \( \gamma_a(s_y) = y \). One has from continuous dependence on the initial data that there is \( \delta > 0 \) such that \( |x - a| < \delta \implies |\gamma_x(s_y) - y| < \varepsilon \). So, taking \( t_n \), such that \( |L(t_n) - a| < \delta \), it follows that \( \gamma_{L(t_n)}(s_y) \notin K' \). As \( \gamma_{L(t_n)} \in C_1 \), there is \( s' \in L(t_n) \) such that \( s_y \) is contained in the interval defined by 0 and \( s' \), and \( \gamma_{L(t_n)}(s') \in K \). Calling \( s_1 = 0, s_2 = s_y \) and \( s_3 = s' \) it follows that \( \gamma_{L(t_n)}(s_i) \in K, i = 1, 3 \), and \( \gamma_{L(t_n)}(s_2) \notin K' \). Thus one has a Reeb component of \( H_f \).

If (2) occurs, for all \( t \in (0, t_1) \), \( \gamma_{\Gamma(t)} \in C_2 \cup C_2' \). Again considering the compact set \( K \) given before, if \( K' \subset \mathbb{R}^2 \) is any compact set, there is \( s' < 0 < s'' \in I_{\Gamma(t)} \) and \( \varepsilon > 0 \) such that \( B(\gamma_{\Gamma(t)}(s'), \varepsilon) \cap (K \cup K') = \emptyset \) and \( B(\gamma_{\Gamma(t)}(s''), \varepsilon) \cap (K \cup K') = \emptyset \). Then one has that \( \delta > 0 \) such that \( |x - L(t)| < \delta \implies |\gamma_x(s') - \gamma_{\Gamma(t)}(s')| < \varepsilon \) and \( |\gamma_x(s'') - \gamma_{\Gamma(t)}(s'')| < \varepsilon \). Taking \( t \in (0, t_0) \) such that \( |L(t) - L(t_0)| < \delta \), it follows that \( \gamma_{\Gamma(t)}(s'), \gamma_{\Gamma(t)}(s'') \notin K' \). If \( \gamma_{\Gamma(t)} \in C_2' \), there are at least two points \( s_1 < s' < s'' < s_3 \in I_{\Gamma(t)} \) such that \( \gamma_{\Gamma(t)}(s_i) \in K, i = 1, 3 \). In this case, take \( s_2 = s' \). On the other hand, if \( \gamma_{\Gamma(t)} \notin C_2' \), there is at least one point \( s \in I_{\Gamma(t)} \), \( s < s', \) or \( s > s'' \) such that \( \gamma_{\Gamma(t)}(s) \in K \). In this case, take \( s_1 = s \) and \( s_3 = s \), such that \( s_1 < s_3 \), and take \( s_2 = s' \) or \( s_2 = s'' \), respectively. Then, in both cases above, one will have \( s_1 < s_2 < s_3 \) such that \( \gamma_{\Gamma(t)}(s_i) \in K, i = 1, 3 \) and \( \gamma_{\Gamma(t)}(s_2) \notin K' \). This gives a Reeb component of \( H_f \).

If (3) occurs, one has three cases: \( \gamma_{\Gamma(t_1)} \notin C_2' \), \( \gamma_{\Gamma(t_1)} \in C_1 \) or \( \gamma_{\Gamma(t_1)} \in C_2' \). In the first one, one has just to repeat the arguments of the proof of the case (1) above (changing \( \Gamma_1 = \gamma_0 \) for \( \gamma_{\Gamma(t_1)} \)), while in the second one, one can repeat the arguments of the proof of the case (2) above (changing \( \gamma_{\Gamma(t_0)} \) for \( \gamma_{\Gamma(t_1)} \)), to obtain a Reeb component of \( H_f \). Now the third one can not occur, because if it could, it would be possible to take \( s'' \in I_{\Gamma(t_1)} \) such that \( \gamma_{\Gamma(t_1)}(s) \in H_2, \) for all \( s \leq s'' \).
and $s \geq s''$, $s \in I_{L(t_i)}$, and $\varepsilon > 0$ such that $B(\gamma_{L(t_i)}(s'), \varepsilon), B(\gamma_{L(t_i)}(s''), \varepsilon) \subset R_2$, and $B(\gamma_{L(t_i)}(s'), \varepsilon) \cap B(\gamma_{L(t_i)}(s''), \varepsilon) = \emptyset$. For the definition of $t_1$, there would be $t > t_1$ such that

$$\gamma_{L(t)} \in \mathcal{E}_1 \quad \text{and} \quad \left\{ \begin{array}{l}
\gamma_{L(t)}(s') \in B(\gamma_{L(t_1)}(s'), \varepsilon) \\
\gamma_{L(t)}(s'') \in B(\gamma_{L(t_1)}(s''), \varepsilon).
\end{array} \right.$$ 

Since $\gamma_{L(t)} \in \mathcal{E}_1$, there would be $s_1 < s' < s'' < s_2$ such that $\gamma_{L(t)}(s_i) \in L_i$, $i = 1, 2$, and $\gamma_{L(t)}(s) \in R_1$ for all $s < s_1$, $s > s_2$, $s \in I_{L(t)}$. But then it would be a contradiction with the definition of $t_1$ (because $\gamma_{L(t)}(s_i) = L(t_2)$, with $t_2 < t_1$, for $i = 1$ or $i = 2$).

Now it will be proved ($\Leftarrow$). Suppose, by contradiction, that $H_f$ is not globally solvable. It will be found a level set of $f$ with two distinct connected component. Using Theorem 2 there is a compact set $K \subset \mathbb{R}^2$ such that for all $n \in \mathbb{N}$, there are $x_n \in K$ and $0 < s_n < t_n \in I_{x_n}$ such that $\gamma_{x_n}(t_n) \in K$ but $\gamma_{x_n}(s_n) \notin B(0, n)$. Taking a subsequence, one may assume that $x_n \to a \in K$ and $\gamma_{x_n}(t_n) \to b \in K$. By the continuity of $f$ and Lemma 2, it follows that $f(a) = f(b)$.

Notice that $\gamma_a$ separates the plane into two open unbounded connected regions $R_1$ and $R_2$. From the Tubular Flow Theorem, there is $r > 0$ such that all the pieces of integral curves inside of $B(a, r)$ have all the same orientation as $\gamma_a$. Taking another subsequence (if necessary), one can assume that $x_n \in R_1 \cap B(a, r)$, and as a consequence $\gamma_{x_n} \subset R_1$ for all $n \in \mathbb{N}$. Furthermore, there exists $n_0 \in \mathbb{N}$ such that $\gamma_{x_n}(s_n) \notin B(a, r)$ for all $n_0 \leq n \in \mathbb{N}$.

Notice that $\gamma_{x_n}(t_n) \notin B(a, r)$, $\forall n \geq n_0$. Indeed, if $\gamma_{x_n}(t_n) \in B(0, r)$, then in this point $\gamma_{x_n}$ would be orientated as $\gamma_a$, and it would be a contradiction with $\gamma_{x_n}^+$ and $\gamma_{x_n}$ being unbounded sets (observe the two possibilities in Figure 3).

![Figure 3](image-url)

**Figure 3.** The two possibilities for the integral curve $\gamma_{x_n}$ if $\gamma_{x_n}(t_n) \in B(a, r)$.

Thus $b \notin B(a, r/2)$ and one has two possibilities for $b$: $b \in \gamma_a$ or $b \in R_1$. If $b \in \gamma_a$, there is $0 < t_0 \in I_a$ such that $\gamma_a(t_0) = b$. Then one use the Tubular Flow Theorem to construct a bounded tubular neighborhood $T$ around the compact interval of curve $\{\gamma_a(t) \mid t \in [0, t_0]\}$, small enough to have its beginning part inside of $B(a, r)$ (this will ensure that each integral curve $\gamma_a$ will pass through the tube just once). But then $\gamma_{x_n}(s_n) \in T, \forall n \geq n_0$, a contradiction with $T$ being bounded. So $b \in R_1$, and $\gamma_a$ and $\gamma_b$ are two distinct connected component of the level set $f^{-1}(\{f(a)\})$. This finishes the proof. \(\square\)
Now, as a direct consequence of Theorem 3, one has Santos Filho’s result:

**Corollary 1.** Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be such that \( \det DF(x) \neq 0, \forall x \in \mathbb{R}^2 \). If \( H_{F_1} \) or \( H_{F_2} \) is globally solvable, then \( F \) is an one-to-one function.

**Proof.** To see this, suppose \( F(a) = F(b), a = \gamma(t_1) \) and \( b = \gamma(t_2), \gamma \) an integral curve of \( H_{F_1} \) (Lemma 2). If \( t_1 \neq t_2 (t_1 < t_2, \text{say}), \exists t_0 \in (t_1, t_2) \) such that \(- \det F(\gamma(t_0)) = (F_2 \circ \gamma)'(t_0) = 0, \) a contradiction. \( \square \)

### 3. Polynomials of degree 3

Hereafter a point in \( \mathbb{R}^2 \) will be denoted \((x, y), x, y \in \mathbb{R}\). The following result will be shown throughout this section:

**Theorem 4.** Let \( p : \mathbb{R}^2 \to \mathbb{R} \) be a polynomial of degree less or equal 3. If \( H_p \) does not vanish in any point of \( \mathbb{R}^2 \) and it is not globally solvable, then there are \( F \), an affine change of coordinates, and \( \alpha \neq 0 \) such that if \( \widetilde{p} = \alpha p \circ F^{-1} \), then

\[
\widetilde{p}(x, y) = \kappa + x(1 + xy),
\]

(3.1)

with \( \kappa \in \mathbb{R} \).

To the proof of this Theorem, consider some preliminary results:

**Lemma 3.** Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be an affine change of coordinates. Then up to multiplication by a scalar, \( F \) carries hamiltonian fields in hamiltonian fields. More specifically, if \( p : \mathbb{R}^2 \to \mathbb{R} \) is a differentiable function and \( \tilde{p} : \mathbb{R}^2 \to \mathbb{R} \) is defined by \( \tilde{p} = p \circ F^{-1} \), where \( F(x, y) = (ax + by + e_1, cx + dy + e_2) \), then

\[
H_{\tilde{p}} = \frac{1}{(ad - bc)} \tilde{X},
\]

where \( \tilde{X}(x_1, y_1) = DF(H_p(x, y)) \), with \( (x, y) = F^{-1}(x_1, y_1) \).

**Proof.** The Lemma follows by Chain Rule. \( \square \)

The following Lemma is a classification of all the quadratic vector fields. For a proof see Lemma 1 of [3].

**Lemma 4.** If \( X(x, y) = (P(x, y), Q(x, y)) \) is a vector field with \( P \) and \( Q \) polynomials of degree less or equal 2, then there are \( F \), an affine change of coordinates, and \( \alpha \neq 0 \) such that if \( \tilde{X} = (\tilde{P}, \tilde{Q}) = \alpha DF(X) \), then either

\[
\begin{align*}
(I) & \quad \tilde{P}(x, y) = 1 + xy, & (VI) & \quad \tilde{P}(x, y) = 1 + x^2, \\
(II) & \quad \tilde{P}(x, y) = xy, & (VII) & \quad \tilde{P}(x, y) = x^2, \\
(III) & \quad \tilde{P}(x, y) = y + x^2, & (VIII) & \quad \tilde{P}(x, y) = x, \\
(IV) & \quad \tilde{P}(x, y) = y, & (IX) & \quad \tilde{P}(x, y) = 1, \text{ or} \\
(V) & \quad \tilde{P}(x, y) = -1 + x^2, & (X) & \quad \tilde{P}(x, y) = 0,
\end{align*}
\]

and \( \tilde{Q}(x, y) = d + ax + by + tx^2 + mxy + ny^2 \).

Now using the two lemmas above, it will be given a classification of the polynomials of degree less or equal 3 up to an affine change of coordinates.
Lemma 5. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of degree less or equal 3. Then there are $F$, an affine change of coordinates, and $\alpha \neq 0$ such that $\alpha p \circ F^{-1}$ is equal to either

\[(I) \quad p_1(x, y) = y + \frac{1}{2}xy^2 + g(x), \quad (VI) \quad p_6(x, y) = y + x^2y + g(x),\]
\[(II) \quad p_2(x, y) = \frac{1}{2}xy^2 + g(x), \quad (VII) \quad p_7(x, y) = x^2y + g(x),\]
\[(III) \quad p_3(x, y) = \frac{1}{2}y^2 + x^2y + g(x), \quad (VIII) \quad p_8(x, y) = xy + g(x),\]
\[(IV) \quad p_4(x, y) = \frac{1}{2}y^2 + g(x), \quad (IX) \quad p_9(x, y) = y + g(x), \quad \text{or}\]
\[(V) \quad p_5(x, y) = -y + x^2y + g(x), \quad (X) \quad p_{10}(x, y) = g(x),\]

with $g(x) = \kappa + l_2x + l_1x^2 + l_0x^3$.

Proof. Use Lemma 4 to transform $H_\rho$ in $\tilde{X}$, one of the ten vector fields presented there, by an affine change of coordinates $F$ and a multiplication by a scalar. Then by Lemma 3 there is $0 \neq \tilde{\tau} \in \mathbb{R}$ such that, if $\tilde{\rho} = \tilde{\tau}p \circ F^{-1}$, then $H_{\tilde{\rho}} = \tilde{X}$. So one has just to integrate in $y$ each of the ten polynomial presented in Lemma 4, to obtain the polynomials presented here.

Now one begins with the proof of Theorem 4, analyzing each of the cases of Lemma 5. Notice that both the polynomials $p_7$, with $l_2 \neq 0$, and $p_1$, with $l_0 = l_1 = l_2 = 0$, satisfy the hypotheses of Theorem 4 (because in both the level set $p_7^{-1}(\{\kappa\})$ has three distinct connected component, so $H_\rho$ is not globally solvable by Theorem 3). The first one is transformed into the polynomial (3.1) by the affine change of coordinates $x_1 = l_2x, y_1 = l_2^2(l_1 + l_0x + y)$, and the second one is carried into the polynomial (3.1) by the change $x_1 = y, y_1 = x/2$.

It will be shown bellow that all the other cases of Lemma 5 do not satisfy the hypotheses of Theorem 4, and so the proof of it will be completed.

Simple calculations show that $H_{p_i}(x, y) = 0$ for some $(x, y) \in \mathbb{R}^2$, for $i = 3, 5$ and 8, and for $i = 7$, with $l_2 = 0$. On the other hand, for $i = 6$ and 9, $H_{p_i}$ never vanishes, but in both cases the equation $p_i(x, y) = c$ defines a single function $y = g(x), x \in \mathbb{R}$, for all $c \in \mathbb{R}$, thus all the level sets of $p_i$ are connected, and $H_{p_i}$ is globally solvable by Theorem 3. Further, if $H_{p_{10}}$ never vanishes, then $g(x) \neq 0, \forall x \in \mathbb{R}$, thus $p(x, y) = c$ defines a (single) vertical line for all $c \in \mathbb{R}$, and then $H_{p_{10}}$ is globally solvable (Theorem 3).

For $p_i, i = 1, 2$ and 4, it will be supposed that $H_{p_i}$ never vanishes. So, if $l_0 = 0$, it is easy to see that $l_1 = 0$ for $p_i, i = 1, 2$ and 4. With these hypotheses, in the case of $p_1$ it is necessary that $l_2 \geq 0$. The case $l_2 = 0$ was already considered above and in the case $l_2 > 0$, for all $c \in \mathbb{R}$ the equation $p_1(x, y) = c$ defines a (single) function $x = x(y), y \in \mathbb{R}$, and then $H_{p_1}$ is globally solvable. For $p_i, i = 2$ and 4, it is necessary that $l_2 > 0$ and $l_2 \neq 0$, respectively. So the equation $p_2(x, y) = c$ defines a function $x = x(y), y \in \mathbb{R}$, and it follows that $H_{p_i}$ is globally solvable for $i = 2$ and 4.

For $p_i, i = 1, 2$ and 4, when $l_0 \neq 0$, one will use the lemma bellow to conclude that if $H_{p_i}$ never vanishes, it will be globally solvable. One starts recalling an elementary result about a cubic equation. Consider the equation

$$x^3 + Ax^2 + Bx + C = 0,$$
and take \( P = B - A^2/3 \) and \( Q = C - AB/3 + 2A^3/27 \). The discriminant of the above equation is \( D = Q^2/4 + P^3/27 \). It is known that

- if \( D < 0 \), the equation has three distinct, real solutions,
- if \( D = 0 \), the equation has three real solutions, with two being equal,
- if \( D > 0 \), the equation has one real and two complex solutions.

As a consequence, one has

**Lemma 6.** Let \( p(x, y) = x^3 + A(y)x^2 + B(y)x + C(y) \) be such that \( \nabla p(x, y) \neq (0, 0), \forall (x, y) \in \mathbb{R}^2 \), where \( A, B \) and \( C \) are polynomials. If the discriminant \( D(y) = \sum_{i=0}^{6} a_i y^i \) of the equation \( p(x, y) = 0 \) is such that \( a_k > 0 \) and \( k \) is even, then \( p^{-1}\left((0)\right) \) is connected.

**Proof.** Let \( [a, b] \) be such that \( D(y) > 0, \forall y \notin [a, b] \). Then by the remarks just before the Lemma, one has that for each \( y \notin [a, b] \), there is only one real root \( x = x(y) \) of \( p(x, y) = 0 \). Observe that by Implicit Function Theorem,

\[
x(y) \text{ defines a real analytic function for } y \notin [a, b].
\]

On the other hand, defining \( \|f\|_{\infty} := \sup_{y \in [a, b]} |f(y)|, \forall f \in C^0([a, b]) \), consider

\[
R = 1 + 3\|A\| + \sqrt{3}\|B\| + \sqrt{3}\|C\|.
\]

If \( |x| > R \), one has that

\[
|p(x, y)| = |x|^3 \left| 1 + \frac{A(y)}{x} + \frac{B(y)}{x^2} + \frac{C(y)}{x^3} \right| > 0, \forall y \in [a, b].
\]

So for \( y \in [a, b] \),

\[
p(x, y) \neq 0, \forall x \notin [-R, R].
\]

Thus from Lemma 1 and Lemma 2, it follows that \( p^{-1}\left((0)\right) \) is a single real analytic curve, therefore a connected set. (For an illustration, see Figure 4.) \( \square \)

In order to apply Lemma 6 one will first put the polynomials in the hypotheses of it. Observe that if \( H_{p_1} \) never vanishes, then \( l_0 > 0 \) (to see this, take \( y = -1/x \) and notice that if \( l_0 < 0 \), \( H_{p_1}(x, y) = 0 \), for some \( x \)). Take the affine change of coordinates \( x_1 = \sqrt{l_0}x, y_1 = y \) to transform \( p_1 \) in the polynomial (with the same notation) \( p_1(x, y) = x^3 + l'_1x^2 + (l'_2 + l_3)y^2)x + y + \kappa \), with \( l_3 = 1/\left(2\sqrt{l_0}\right) > 0 \). Performing the calculations of \( D(y) \) (as in Lemma 6) to the polynomial \( p_1 - c \), one has that \( D(y) = \sum_{i=0}^{6} a_i y^i \), with \( a_6 = l_3^3/27 \). So it follows from Lemma 6 that all the level sets of \( p_1 \) are connected, and then \( H_{p_1} \) is globally solvable (if it never vanishes).

In the case of \( p_2 \), one has that \( l_0 > 0 \) if \( H_{p_2} \) never vanishes, and then the change of coordinates \( x_1 = \sqrt{l_0} x, y_1 = y \) carries \( p_2 \) into the polynomial (again with the same notation) \( p_2(x, y) = x^3 + l'_1x^2 + (l'_2 + y^2)x + \kappa \). Here \( D(y) \) for the polynomial \( p_2 - c \) is \( D(y) = \sum_{i=0}^{6} a_i y^i \), with \( a_6 = 1/27 \), so it follows that \( H_{p_2} \) is globally solvable (if it never vanishes).

Finally, for \( p_4 \), the change \( x_1 = \sqrt{l_0}x, y_1 = y \) carries \( p_4 \) into the polynomial \( p_4(x, y) = x^3 + l'_1x^2 + l'_2x + y^2/2 + \kappa \). As above, \( D(y) = \sum_{i=0}^{4} a_i y^i \), with \( a_4 = 1/16 \), and then \( H_{p_4} \) is globally solvable (if it never vanishes).
4. Proof of the main result

Throughout this section, $p$ will be the polynomial (3.1), that is,

$$p(x, y) = \kappa + x(1 + xy).$$

In Theorem 5 below, it will be given a necessary integral condition on $g$ in order to exist a solution of $H_pq = g$. In Proposition 1 below, it will be shown that Theorem 5 can be applied under the hypothesis of Theorem 1. Finally, in the end of the section, the proof of Theorem 1 will be detailed.

Since it will be dealt with level sets of $p$, it will be supposed, without loss of generality, that $\kappa = 0$. Consider the function $y(x) = -1/x$, for $x > 0$, and the horizontal curve $L(t) = (t, -1)$, $0 \leq t \leq 1$. From Lemma 2 one has that $y_c(x) = c/x^2 - 1/x$, $x > 0$, defines an integral curve of $H_p$ for all $c \in \mathbb{R}$. For $c > 0$, notice that these curves live between $x = 0$ and $y(x) = -1/x$, and they are oriented in the direction of crescent $x$ (because the first coordinate of $H_p(x, y)$ is $x^2 > 0$). Thus observe that if $\gamma(x_1,y_1)(t)$ is such an integral curve of $H_p$, with $0 < x_1 < 1$ and $y(x_1) = y_1 < -1$, then there are $s_1 < 0 < s_2 \in I(x_1,y_1)$ such that $\gamma(x_1,y_1)(s_i) \in L$, $i = 1, 2$ (to see this, just observe that $\gamma(x_1,y_1)$ is a curve $y_c(x)$ for some $c > 0$, and then $y_c(x) \to -\infty$ as $x \to 0$ (so it cuts $L$), and $y_c(x)$ cuts $L$ in another point $x > x_1$, because $y_c(x) > y(x)$ and approaches 0 when $x \to \infty$). So one has

**Lemma 7.** For all $n \in \mathbb{N}$, there are $c_n > 0$ and $x_n \in (0, 1)$ such that $y_{c_n}(x_n) = -n$. Furthermore, putting $z_n = (x_n, y_{c_n}(x_n))$, there are $s_{1n} < 0 < s_{2n} \in I_{z_n}$ such that $\gamma_{z_n}(s_{1n}), \gamma_{z_n}(s_{2n}) \in L$.

**Proof.** In fact, taking $x_n = 1/(n + 1)$ and $c_n = x_n^2$, one has that $y_{c_n}(x_n) = -n$, so the first part of the Lemma follows. The second one follows from the computations made just before the Lemma. \qed
Remark 1. It follows from Lemma 7 that
\[ B = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < 1 \text{ and } -\frac{1}{x} < y < -1 \right\} \]
is a Reeb component of \( H_p \). Furthermore, it follows that if
\[ B_{cn} = \left\{ (x,y) \in \mathbb{R}^2 \mid x_{1n} < x < x_{2n} \text{ and } c_n - \frac{1}{x} < y < -\frac{1}{x} \right\}, \]
where \( x_{1n} \) and \( x_{2n} \) are the first coordinates of \( \gamma_{z_n}(s_{1n}) \) and \( \gamma_{z_n}(s_{2n}) \), respectively, then \( B_{cn} \subseteq B_{cn+1} \subseteq B \) (notice that \( c_n > c_{n+1} \)), for all \( n \in \mathbb{N} \), and \( B = \bigcup B_{cn} \).
Because of these, it follows from Monotone Convergence Theorem that
\[ \lim_{n \to \infty} \int_{B_{cn}} g = \int_B g, \quad (4.1) \]
for all measurable function \( g : \mathbb{R}^2 \to [0, \infty) \). With these notations, one has the following:

**Theorem 5.** Let \( U \) be an open set such that \( B \subseteq U \). If \( g : U \to [0, \infty) \) is a continuous function such that \( \int_B g = \infty \), then there is not a \( C^1 \) function \( q : U \to \mathbb{R} \) such that \( H_p q = g \) in \( U \).

**Proof.** Suppose there is a function \( q \) satisfying \( H_p q = g \). Since
\[ \frac{\partial p}{\partial y} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial q}{\partial y} = \frac{\partial}{\partial x} \left( q \frac{\partial p}{\partial y} \right) - \frac{\partial}{\partial y} \left( q \frac{\partial p}{\partial x} \right), \]
it follows from Green’s formula that
\[ \int_{B_{cn}} g = \int_{s_{1n}}^{s_{2n+1}} q(\gamma(t)) \left( \frac{\partial p}{\partial x}(\gamma(t)) \gamma'_1(t) + \frac{\partial p}{\partial y}(\gamma(t)) \gamma'_2(t) \right) dt, \quad (4.2) \]
where \( \gamma(t) = \left\{ \begin{array}{ll} \gamma_{z_n}(t), & t \in (s_{1n}, s_{2n}) \\ (x_{2n} + (x_{1n} - x_{2n})(t - s_{2n}), -1), & t \in (s_{2n}, s_{2n+1}). \end{array} \right. \)
Since for \( t \in (s_{1n}, s_{2n}) \) \( \gamma(t) \) satisfies \( \gamma'(t) = H_p(\gamma(t)) \), it follows that
\[ \frac{\partial p}{\partial x}(\gamma(t)) \gamma'_1(t) + \frac{\partial p}{\partial y}(\gamma(t)) \gamma'_2(t) = 0, \quad \forall t \in (s_{1n}, s_{2n}), \]
so, taking
\[ M = \max \left\{ q(x,y) \frac{\partial p}{\partial x}(x,y) \mid (x,y) \in L \right\}, \]
and observing that for \( t \in (s_{2n}, s_{2n+1}) \), \( |\gamma'_1(t)| \leq 1 \) and \( \gamma'_2(t) = 0 \), it follows from (4.2) that
\[ \int_{B_{cn}} g \leq M. \]
Letting \( n \to \infty \), it follows from (4.1) that
\[ \int_B g \leq M, \]
in contradiction with the hypothesis. \( \square \)
Now it will be shown that if \( g \) is a positive polynomial, then Theorem 5 is always true, that is:

**Proposition 1.** If \( g : \mathbb{R}^2 \to (0, \infty) \) is a polynomial, then \( \int_{B} g = \infty \).

**Proof.** First one introduces a change of variables that linearizes \( H_p \) on \( B \). Consider

\[
A = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < 1 \text{ and } -\frac{1}{x} < y < 0 \right\}.
\]

Note that \( B \subset A \), and \( A \setminus B \) is a bounded set, so, to show the Proposition, it is sufficient to show that \( \int_{A} g = \infty \). Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by

\[
F(x, y) = \left( x(1 + xy), 1 + xy \right),
\]

and take \( a = x(1 + xy) \) and \( b = 1 + xy \). Observe that \( x = a/b \), if \( b \neq 0 \), and then

\[
\left\{ \begin{array}{l}
0 < x < 1 \\
-\frac{1}{x} < y < 0
\end{array} \right\} \iff \left\{ \begin{array}{l}
0 < a < b \\
0 < b < 1
\end{array} \right\},
\]

so

\[
F(A) = \left\{ (a, b) \in \mathbb{R}^2 \mid 0 < b < 1 \text{ and } 0 < a < b \right\}.
\]

It is easy to see that

\[
(a, b) \mapsto \left( \frac{a}{b}, \frac{b}{a} (b - 1) \right),
\]

defined in \( F(A) \), is the inverse of \( F = F|_A \). Furthermore, the expressions of \( F \) and \( F^{-1} \) show that \( F : A \to F(A) \) is a diffeomorphism. Since

\[
\det(DF^{-1}(a, b)) = \frac{1}{a},
\]

it follows that

\[
\int_{A} g(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{b} \frac{g \circ F^{-1}(a, b)}{a} \, da \, db,
\]

for all measurable function \( g : \mathbb{R}^2 \to (0, \infty) \). If \( g \) is the polynomial \( g(x, y) = \sum_{|\alpha| \leq M} c_{\alpha} x^{\alpha_1} y^{\alpha_2} \), one has that

\[
\overline{g}(a, b) = \frac{g \circ F^{-1}(a, b)}{a} = \sum_{|\alpha| \leq M} c_{\alpha} a^{\alpha_1 - \alpha_2 - 1} b^{\alpha_2 - \alpha_1} (b - 1)^{\alpha_2}.
\]

Defining

\[
\tau_g = \min_{|\alpha| \leq M} \{ \alpha_1 - \alpha_2 - 1 \mid c_{\alpha} \neq 0 \},
\]

it follows that

\[
\overline{g}(a, b) = a^\tau_g b^{-\tau_g - 1} \sum_{|\alpha| \leq M} c_{\alpha} (b - 1)^{\alpha_2} + r(a, b),
\]

where

\[
r(a, b) = \sum_{\alpha_1 - \alpha_2 - 1 = \tau_g} c_{\alpha} a^{\alpha_1 - \alpha_2 - 1} b^{\alpha_2 - \alpha_1} (b - 1)^{\alpha_2} = \sum_{i=\tau_g+1}^{m} u_i(b) a^i,
\]

for some \( m \in \mathbb{N} \), and \( u_i \) rational functions of \( b \). The proof will follow from the two Lemmas bellow.

**Lemma 8.** Suppose \( k_1 \leq k_2 \in \mathbb{Z} \), and \( h(t) = \sum_{i=k_1}^{k_2} A_i t^i \), with \( A_i \in \mathbb{R} \) and \( A_{k_i} \neq 0 \). If \( h(t) > 0 \) for all \( t \in (0, c) \), for some \( c > 0 \), then
(1) If $k_1 > 0$;
(2) If $k_1 < 0$, then $\int_0^c h(t)dt = \infty$.

Proof. Observe that
\[
\frac{h(t)}{t^{k_1}} = \sum_{i=0}^{k_2-k_1} A_{k_1+i}c^i.
\]
So, since $h(t) > 0$, it follows that $A_{k_1} > 0$, and (1) holds. If now $k_1 < 0$ and $\int_0^c h(t)dt < \infty$, then by Hölder’s Inequality, $\int_0^c t^{-k_1-1}h(t)dt < \infty$, and it follows that
\[
A_{k_1}\int_0^c \frac{1}{t}dt + \int_0^c \sum_{i=1}^{k_2-k_1} A_{k_1+i}c^{i-1}dt < \infty,
\]
a contradiction. So (2) holds, and the proof is completed. \hfill \Box

**Lemma 9.** If $g : \mathbb{R}^2 \to (0, \infty)$ is a polynomial, then $\int_A g = \infty$.

Proof. Observe that
\[
s(b) \doteq \sum_{\alpha_1 - \alpha_2 = 1 = \tau_g} c_{\alpha}(b-1)^{\alpha_2}
\]
is not identically 0. Because if it was, since it is a polynomial in the variable $b-1$, $c_{\alpha} = 0$, for all $\alpha$, $|\alpha| \leq M$, such that $\alpha_1 - \alpha_2 - 1 = \tau_g$, a contradiction with the definition of $\tau_g$ in (4.4). Thus, as $s(b)$ is not the 0 polynomial, there is $b_1 \in \mathbb{R}$, $1 > b_1 > 0$, such that $s(b)$ does not vanish in the interval $(0, b_1)$. Furthermore, since $g$ is positive, $c_{(0,0)} > 0$, and then $\tau_g \leq 0 - 0 - 1 = -1 < 0$. Therefore, from (4.5) and (2) of Lemma 8, one has that $\int_0^{b_1} \mathcal{g}(a,b)da = \infty$, for all $b \in (0, b_1)$ (because $\mathcal{g}$ is positive). Thus, as
\[
\int_0^1 \int_0^{b_1} \mathcal{g}(a,b)dadb \geq \int_0^{b_1} \int_0^{b_1} \mathcal{g}(a,b)dadb,
\]
it follows from (4.3) that $\int_A g = \infty$. \hfill \Box

**Proof of Theorem 1.** Observe that $\det DF = -H_p \mathcal{q}$, so $H_p$ never vanishes. Thus by Theorem 4, up to an affine change of coordinates and a multiplication by a scalar, either $H_p$ is globally solvable or $p$ is the polynomial (3.1). In the first case, it follows from Corollary 1 that $F$ is one-to-one. In the second case (changing $q$ by $-q$, if necessary) Theorem 5 combined with Proposition 1 show that there is not a polynomial $q$ such that $H_p q(x, y) \neq 0$, $\forall (x, y) \in \mathbb{R}^2$. This is a contradiction, as a change of variables and a scalar multiplication do not modify the hypothesis $\det DF \neq 0$. This finishes the proof. \hfill \Box

**Remark 2.** The use of Green’s formula in the proof of Theorem 5 was motivated by a similar use made in [5].

**Remark 3.** For a polynomial $p$ of degree greater than 3, it is not always true that the integral of a positive polynomial $g$ in a “minimal” Reeb component of $H_p$ is infinite. For an example, take $p(x, y) = x^2 y^2 - x$. Observe that the level set $p = 0$ is formed by the three curves $x = 0$, $y \in \mathbb{R}$; $y = 1/\sqrt{x}$, $x > 0$; and $y = -1/\sqrt{x}$, $x > 0$. It is easy to see that the region $A$ whose boundary is formed by these three curves for $x \leq 1$ is a “minimal” Reeb component of $H_p$. But $\int_A 1 = 4$. So in this case Proposition 1 is not true.
References


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