Global Solvability for First Order Real Linear Partial Differential Operators

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Abstract

F. Treves, in [17], using a notion of convexity of sets with respect to operators due to B. Malgrange and a theorem of C. Harvey, characterized globally solvable linear partial differential operators on $C^\infty(X)$, for an open subset $X$ of $\mathbb{R}^n$.

Let $P = L + c$ be a linear partial differential operator with real coefficients on a $C^\infty$ manifold $X$, where $L$ is a vector field and $c$ is a function. If $L$ has no critical points, J. Duistermaat and L. Hörmander, in [2], proved five equivalent conditions for global solvability of $P$ on $C^\infty(X)$.

Based on Harvey-Treves’s result we prove sufficient conditions for the global solvability of $P$ on $C^\infty(X)$, in the spirit of geometrical Duistermaat-Hörmander’s characterizations, when $L$ is zero at precisely one point. For this case, additional non-resonance type conditions on the value of $c$ at the equilibrium point are necessary.

\textit{Key words:} First Order Partial Differential Operators, Global Solvability.

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1. Introduction

Let $X$ be a $C^\infty$ manifold Hausdorff with a countable basis of open sets and $P : C^\infty(X) \to C^\infty(X)$ a linear partial differential operator. $P$ is said to be \textbf{globally solvable}, or \textbf{solvable}, on $C^\infty(X)$ when $P(C^\infty(X)) = C^\infty(X)$.
B. Malgrange ([9] p. 295) in 1955 introduced the notion of $P$–convexity and showed it to be equivalent to the global solvability of $P$ on $C^\infty (X)$, when $P$ has constant coefficients and $X$ is an open subset of $\mathbb{R}^n$. When $P$ has variable coefficients, he showed that $P$–convexity is a necessary condition for the global solvability of $P$ on $C^\infty (X)$.

Let $X$ be an $n$–dimensional $C^\infty$ manifold Hausdorff space with countable basis. Take $\mathcal{F}$ to be a local coordinate system $(X_\kappa, \kappa)$ for $X$. The space of distributions $\mathcal{D}'(X)$ is defined in the following way (see [7], p. 144), for every $\kappa$ consider a distribution $u_\kappa \in \mathcal{D}'(\kappa (X_\kappa))$ such that

$$u_\kappa' = u_\kappa \circ (\kappa \circ \kappa'^{-1}) \text{ in } \kappa' (X_\kappa \cap X_{\kappa'}) ,$$

in this case, $(u_\kappa)$ is called a distribution on $X$. The set of all distributions in $X$ is denoted by $\mathcal{D}'(X)$. Similarly we define the space of compact support distribution $\mathcal{E}'(X)$.

Denote $M \subset \subset X$ if $M$ is a compact subset of $X$ and $^tP$ the formal transpose of $P$. In this article supp $(u)$ denotes the support and $\text{singsupp} (u)$ denotes the singular support of the distribution $u$. We say that $X$ is $P$–convex for supports if $\forall K \subset \subset X, \exists K' \subset \subset X$ such that

$$u \in \mathcal{D}'(X), \text{ supp } (^tPu) \subset K \Rightarrow \text{ supp } (u) \subset K'.$$

In a similar way we define the $P$–convexity for singular supports.

In 1967, F. Treves ([17] p. 60) and C. Harvey ([5] p. 700) using the $P$–convexity for supports, gave a general characterization of globally solvable linear partial differential operators on $C^\infty (X)$.

Unless otherwise mentioned, from now on $P = L + c$ will be a linear partial differential operator with real coefficients in $C^\infty (X)$, where $L$ is a vector field and $c$ is a function. In 1972, when $L$ has no critical points, J. Duistermaat and L. Hörmander (see [2] p. 212) gave five equivalent conditions for global solvability of $P$ on $C^\infty (X)$. They used the notions of global transversal of $L$ on $X$ and of convexity of $X$ with respect to the trajectories of $L$. In [6], J. Hounie extended one of these characterizations for $L$ complex.

In order to state our main theorem we recall some definitions and results. We say that $X$ is convex with respect to the trajectories of $L$ if $\forall K \subset \subset X, \exists K' \subset \subset X$ such that any compact interval of trajectory of $L$ with endpoints in $K$, is contained in $K'$ (see [2], p. 208).

If $L$ has a critical point at the origin and $c \in \mathbb{C}$, V. Guillemin and D. Schaeffer ([3] p. 175) gave, in 1977, sufficient conditions for the equation $Pu = f$ to
have a $C^\infty$ solution in a neighborhood of zero, for an arbitrary $f \in C^\infty(\mathbb{R}^n)$ flat at the origin. We remark that in [3] and [11] results on propagation of singularities for operators of type $P = L + c$ are presented.

Suppose that $x_0$ is a critical point of $L$. Let $\lambda_1, \lambda_2, ..., \lambda_{n'}, \lambda_{n'+1}, ..., \lambda_n$ be the eigenvalues of $DL(x_0)$, where $\lambda_1, \lambda_2, ..., \lambda_{n'}$ are the real eigenvalues and $\lambda_{n'+1}, ..., \lambda_n$ are non-real eigenvalues. For $c = 0$, from S. Sternberg ([15] p. 629), see also E. Nelson ([10] p. 50) and V. Guillemin and D. Schaeffer ([3] p. 175), we have: If

$$\lambda_j \neq \sum_{k=1}^{n} m_k \lambda_k, \quad j = 1, 2, ..., n', m_1, ..., m_n \in \mathbb{N}, \quad \sum_{k=1}^{n} m_k \geq 2. \quad (\text{NRC 1})$$

then given $f \in C^\infty(\mathbb{R}^n)$ flat at $x_0$, $\exists u \in C^\infty(\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of $x_0$.

Observe that the condition (NRC 1) implies that every eigenvalue of $DL(x_0)$ has nonzero real part, that is, $x_0$ is a hyperbolic critical point for $L$.

If $c(x_0) = 0$ then, since $Lu(x_0) = 0$, we have $Pu(x_0) = 0$ hence the operator $P$ is not $C^\infty$-solvable at any neighborhood of $x_0$. Therefore we consider the following non-resonance condition

$$-c(x_0) \neq \sum_{j=1}^{n} m_j \Re \lambda_j, \quad \forall m_1, ..., m_n \in \mathbb{N}, \forall m_{n'+1}, ..., m_n \in 2\mathbb{N}. \quad (\text{NRC 2})$$

Our main result is:

**Theorem 1.** Let $P = L + c$ be a first order differential operator with coefficients in $C^\infty(X, \mathbb{R})$ with a critical point at $x_0$. If
(a) (NRC 1) and (NRC 2) are valid,
(b) no orbit of $L$ on $X \setminus \{x_0\}$ is relatively compact in $X$
and
(c) $X$ is convex with respect to the trajectories of $L$
then

$$P \text{ is solvable on } C^\infty(X).$$

Also in this paper we consider the relationship between $P$-convexity and convexity with respect to the trajectories of $L$ for $P = L + c$, see Proposition 1.
This paper is organized in the following way. In Section 2 we present results concerning the relationship between $P-$convexity for supports, $P-$convexity for singular supports and convexity with respect to the trajectories of $L$ when $L$ is a real vector field. In Section 3 we prove Theorem 1.

2. $L-$convexity for supports, $L-$convexity for singular supports and convexity with respect to the trajectories

In this section we use propagation of singularities and of supports to characterize, in geometrical terms, the $L-$convexity for supports and singular supports. From these characterizations, we obtain in our setting the equivalence between those conditions.

The main result of this section is:

**Proposition 1.** Let $L$ be a real vector field on $X$. The following conditions are equivalent:

(a) $X$ is $L-$convex for singular supports.

(b) (b.1) $\exists \tilde{K} \subset X$ such that no orbit of $L \mid_{X \setminus \tilde{K}}$ is relatively compact and

(b.2) $X$ is convex with respect to the trajectories of $L$.

Let $L$ be a non-singular real vector field on $X$. If one of the following conditions holds:

(i) $X$ is any open set of $\mathbb{R}^n$ and $L$ has constant coefficients

or

(ii) $X$ is a simply connected open subset of $\mathbb{R}^2$,

then condition (b.1) holds with $\tilde{K} = \emptyset$, because the orbits are lines in case (i) and because of the Poincaré-Bendixson theorem in case (ii). Therefore, under conditions (i) or (ii) above, from Proposition 1 we have (a) $\iff$ (b.2).

Observe that if $L \equiv 0$ then every manifold $X$ is convex with respect to the trajectories of $L$ but $X$ is not $L-$convex for singular supports. If $X \subset \mathbb{R}^2$ is not simply connected then (b.2) $\not\Rightarrow$ (a), for example take $X = \mathbb{R}^2 \setminus \{0\}$ and $L = x_2 \partial_1 - x_1 \partial_2$.

In [14], H. Seifert proposed the following question, which is known as Seifert’s Conjecture: Does every smooth vector field on the 3–dimensional sphere have a periodic orbit? This conjecture was proved to be false for $C^1$ vector fields by P. A. Schweitzer (see [13]) and latter in the $C^\infty$ case.
by K. Kuperberg (see [8]). In contrast with (ii), the second author in [16]
starting from an example for which the statement of the conjecture is true,
constructed a real non-singular vector field on $\mathbb{R}^3$ such that $(b2) \not\Rightarrow (a)$.

2.1. Proof of Proposition 1

We will introduce some definitions concerning vector fields. Let $L$ be
a real vector field on a manifold $X$ and $\gamma$ the associated flow. For each
$x \in X$, we denote the maximal interval of definition of the orbit passing
through $x$ by $I_x = (\omega_-(x), \omega_+(x))$ and the orbit (or trajectory) of $x$ by
$\Gamma_x = \{ \gamma(t, x) ; t \in I_x \}$. Also denote $\Gamma_x^+ = \{ \gamma(t, x) ; 0 \leq t < \omega_+(x) \}$ and $\Gamma_x^- = \{ \gamma(t, x) ; \omega_-(x) < t \leq 0 \}$.

When $\omega_+(x) = +\infty$ (resp. $\omega_-(x) = -\infty$) we define
$\omega(x) = \{ y \in X, \gamma(t_j, x) \to y \text{ for some sequence } t_j \to +\infty \}$
(resp. $\alpha(x) = \{ y \in X, \gamma(t_j, x) \to y \text{ for some sequence } t_j \to -\infty \}$).

We say that $\{ x_0 \} \subset X$ is a local attractor of $L$ when there exist a neigh-
borhood $U$ of $x_0$ such that $\lim_{t \to \omega_(x)} \gamma(t, x) = x_0$, $\forall x \in U$. In this case, the basin of attraction of $\{ x_0 \}$ is defined by $B(x_0) = \{ x \in X ; \lim_{t \to \omega_(x)} \gamma(t, x) = x_0 \}$.

When $B(x_0) = X$ we say that $\{ x_0 \}$ is a global attractor.

To prove Proposition 1 we will need some preliminary results, namely
Lemma 1 to Lemma 3. Choose a sequence $\{ K_j \}_{j=1}^\infty$ of compact subsets of $X$ such that
$\bigcup K_j = X$, $K_j \subset K_{j+1}$, $j = 1, 2, \ldots$ and
$\forall K \subset X, \exists j_0 \in \mathbb{N}$ such that $K \subset K_{j_0}$.

Here $A^o$ denotes the interior of the subset $A \subset X$.

If $K$ is a compact subset of $X$ then we denote by $C^\infty(K)$ the quotient of
$C^\infty(X)$ by the space consisting of elements vanishing of infinite order on $K$.
Then $C^\infty(K)$ is a Fréchet Space and the family of seminorms given by

$$ p_j \left( \hat{\phi} \right) = \inf_{\phi \in \hat{\phi}} \frac{1}{\sum_{|\alpha| \leq j} |\partial^\alpha \phi|}, \hat{\phi} \in C^\infty(K), j = 0, 1, 2, \ldots $$

is a basis of continuous seminorms of $C^\infty(K)$. Here $\hat{\phi}$ denotes the class of
$\phi \in C^\infty(X)$ in $C^\infty(K)$. Denote $B_{p_j} = \{ \phi \in C^\infty(K) ; p_j \left( \hat{\phi} \right) < 1 \}$. Then
$\forall j \in \mathbb{N}, \exists C > 0$ such that

$$ L \left( \frac{1}{C} B_{p_{j+1}} \right) \subset B_{p_j}. $$
This implies the continuity of $L$ on $C^\infty (K)$.

We use the identification $(C^\infty (K))' = \mathcal{E}' (K)$, where $\mathcal{E}' (K)$ denotes the space of distributions on $X$ with compact support contained in $K$. Using this identification we prove the following result, see Theorem 6.4.1 of [2].

**Lemma 1.** If $K \subset \subset X$ and $L (C^\infty (K)) = C^\infty (K)$ then $\exists \phi \in C^\infty (X)$ such that $L^2 \phi > 0$ on $K$.

**Proof.** Choose $j \in \mathbb{N}$ such that $K \subset K_j$ and consider $\phi_1 \in C^\infty (X)$ satisfying $\phi_1 = 1$ on $K$. From the hypothesis it follows that there exist $\phi_2, \phi \in C^\infty (K)$ such that

$$L \phi_2 - \phi_1 \in \frac{1}{4} B_{p_j},$$

and $L \phi - \phi_2 \in \frac{1}{4C} B_{p_{j+1}}$ (here $C > 0$ is given by (2)). From (2) we obtain

$$L (L \phi - \phi_2) \in \frac{1}{4} B_{p_j}.$$  \hfill (4)

Since $L^2 \phi - \phi_1 = L \left( L \phi - \phi_2 \right) + L \phi_2 - \phi_1$, from (3) and (4) we obtain

$L^2 \phi - \phi_1 \in \frac{1}{2} B_{p_j}$. Hence $\exists \psi \in L^2 \phi - \phi_1$ such that $\sum_{|\alpha| \leq j} \sup_{K_j} |\partial^\alpha \psi| \leq \frac{3}{4}$, in particular $\sup_{K_j} |\psi| \leq \frac{3}{4}$.

But $K \subset K_j$ and $L^2 \phi - \phi_1 = \psi$ on $K$, therefore $\sup_{K} |L^2 \phi - \phi_1| \leq \frac{3}{4}$. Since $\phi_1 = 1$ on $K$ it follows that $L^2 \phi > \frac{1}{4}$ on $K$.

Denote $\mathcal{D}' (X)$ the space of the distributions on $X$.

**Remark 1.** Let $L$ be a real non-singular vector field on $X$ and $c \in C^\infty (X)$. If $u \in \mathcal{D}' (X)$ and $(L + c) u = 0$ by the Flow Box theorem it follows that $\text{supp} (u)$ is invariant under the flow of $L$.

**Lemma 2.** If $\Gamma$ is a relatively compact orbit of the real vector field $L$ then

(i) $\exists u \in \mathcal{E}' (X)$ such that $t^L u = 0$ and $\text{supp} (u) = \overline{\Gamma}$. So $\text{singsupp} (u) = \Gamma$, if $\Gamma$ is a periodic orbit.

(ii) For each orbit $\Lambda$ satisfying $\Lambda \cap \partial \Gamma \neq \emptyset$, $\exists u \in \mathcal{E}' (X)$ such that $t^L u = 0$ and $\text{supp} (u) = \text{singsupp} (u) = \overline{\Lambda} \subset \overline{\Gamma}$.
Proof. We will divide the proof in four steps. From steps 1 and 2 we will have (i) and, from steps 3 and 4 will follow (ii).

**Step 1.** If $\Gamma$ is a periodic orbit then $\exists u \in \mathcal{E}'(X)$ such that $t^*Lu = 0$ and $\text{supp} (u) = \text{singsupp} (u) = \Gamma$.

In fact, if $\Gamma$ is a critical point then we may take $u$ to be Dirac distribution. If $\Gamma$ is a periodic orbit define $u (\phi) = \int_a^b \phi \circ \gamma (s) \, ds, \phi \in C^\infty (X)$,

\[
(i)
\]

where $a \neq b, \gamma (a) = \gamma (b)$ and $\gamma$ is the integral curve whose image is $\Gamma$. It is easy to see that $\text{supp} (u) = \Gamma$.

Since $WF (u) = \{ (x, \xi) \in T^* (X) ; x \in \Gamma, \xi \neq 0 \text{ and } L (x, \xi) = 0 \}$ (see Example 8.2.5 of [7]) we have $\text{singsupp} (u) = \Gamma$.

**Step 2.** If $\Gamma$ is a non-periodic orbit then $\exists u \in \mathcal{E}'(X)$ such that $t^*Lu = 0$ and $\text{supp} (u) = \Gamma$.

In fact, from Lemma 1 and a result concerning solvability on compact subsets due to Duistermaat-H"ormander (see Theorem 6.4.1 of [2]) we have $L (C^\infty (\Gamma)) \neq C^\infty (\Gamma)$. The Hahn-Banach theorem implies that there exists $0 \neq u \in \mathcal{E}' (\Gamma)$ such that $u = 0$ on $L (C^\infty (X))$. Since $t^*Lu = 0$ and $L$ is non-singular in a neighborhood of $\Gamma$, using Remark 1 we obtain $\text{supp} (u) = \Gamma$.

**Step 3.** If $\Lambda$ is a non-periodic orbit then (ii) holds.

In fact, using the invariance of the sets $\alpha (x)$ and $\omega (x)$ under the flow and the hypothesis $\Lambda \cap \partial \Gamma \neq \emptyset$ we obtain $\overline{\Lambda} \subset \overline{\Gamma}$. From (i) it follows that $\exists u \in \mathcal{E}' (X)$ such that $t^*Lu = 0$ and $\text{supp} (u) = \overline{\Lambda}$. We will prove that $\text{singsupp} (u) = \overline{\Lambda}$. From propagation of singularities (see Theorem 6.1.1 of [2]) it is sufficient to prove that

\[
\Lambda \cap \text{singsupp} (u) \neq \emptyset.
\]

Let $\lambda : \mathbb{R} \to X$ be the integral curve whose the image is $\Lambda$ and $\psi \in C^\infty (X)$ such that $-t^*L = L + \psi$. For each bounded interval $I \subset \mathbb{R}$, from Flow Box theorem $\exists \phi \in C^\infty (X)$ such that $L \phi = \psi$ in a neighborhood of $\lambda (I)$.

If $\Lambda \cap \text{singsupp} (u) = \emptyset$ then $u$ is a continuous function on $\Lambda$. Since $\text{supp} (u) = \overline{\Lambda} \subset \overline{\Gamma}$ it follows that $u = 0$ on $\partial \Gamma$.

\[
(7)
\]
Moreover, since $u$ is a $C^\infty$–function in a neighborhood of $\lambda(I)$ we have
\[
((e^\phi u) \circ \lambda)'(s) = L(e^\phi u) \circ \lambda(s) = (e^\phi (L\phi) u + e^\phi Lu) \circ \lambda(s), \forall s \in I.
\]
But $L\phi = \psi$ in a neighborhood of $\lambda(I)$ and $tLu = 0$, then
\[
((e^\phi u) \circ \lambda)'(s) = 0, \forall s \in I.
\]
We proved that for any bounded interval $I \subset \mathbb{R}$, $\exists \phi \in C^\infty(X)$ such that $e^\phi u$ is a constant function on $\lambda(I)$. Since $\text{supp}(u) = \ov{\Lambda}$ we obtain $u \neq 0$ on $\Lambda$. This is a contradiction with (7), since $\Lambda \cap \partial\Gamma \neq \emptyset$. The proof of (6) is finished.

**Step 4.** If $\Lambda$ is a periodic orbit then (ii) holds.

In fact, if $\Lambda$ is a critical point then the result follows from **Step 1.** Otherwise, consider $a < b$ such that $\lambda(a) = \lambda(b)$. In this case, take $I = (a - \epsilon, b + \epsilon)$, where $\epsilon > 0$ is sufficiently small. The proof follows in the same way as the proof of **Step 3.**

We say that $\Gamma := \gamma([a,b])$ is a non-periodic interval of trajectory of $L$ when $\Gamma$ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$.

**Lemma 3.** If $\Gamma = \gamma([a,b])$ is a non-periodic interval of trajectory of $L$ then there exists $u \in E'(X)$ such that
\[
\text{supp}(u) = \text{singsupp}(u) = \Gamma
\]
and
\[
\text{supp} \left( tPu \right) = \text{singsupp} \left( tPu \right) = \{ \gamma(a), \gamma(b) \}.
\]

**Proof.** As in (5) define
\[
v(\phi) = \int_a^b \phi \circ \gamma(s) \, ds, \phi \in C^\infty(X),
\]
It is easy to see that $\text{supp}(v) = \text{singsupp}(v) = \Gamma$ and
\[
tLv = \delta_{\gamma(b)} - \delta_{\gamma(a)}.
\]
Here $\delta_{\gamma(a)}$, $\delta_{\gamma(b)}$ are the Dirac distributions supported on $\gamma(a)$ and $\gamma(b)$, respectively. Since $\gamma(a) \neq \gamma(b)$ we obtain
\[
\text{supp} \left( tLv \right) = \{ \gamma(a), \gamma(b) \}.
\]
From the Flow Box theorem, it follows that \( \exists \phi \in C^\infty (X) \) such that \( L\phi = c \) in a neighborhood \( \Gamma \). Defining \( u = e^{\phi}v \) we obtain \( ^tPu = e^{\phi} \cdot ^tLv + e^\phi (c - L\phi) v \). Since \( c = L\phi \) in a neighborhood \( \Gamma \) and \( \text{supp}(v) = \Gamma \) we have \( ^tPu = e^{\phi} \cdot ^tLv \). From (8) we obtain the result.

\[ \square \]

**Proof of Proposition 1.** For each \( K \subset X \) define

\[ C_K = \{ \Gamma; \Gamma \text{ is a compact interval of trajectory with endpoints in } K \} \]. (9)

Let \( \{ K_j \} \) be a sequence of compact subsets of \( X \) with the properties (1).

**Proof of (a) \( \Rightarrow \) (b.1).** By taking \( K = \emptyset \) in the definition of the \( P \)-convexity for singular supports we have that \( \exists K' \subset X \) with the following property:

\[ u \in \mathcal{E}'(X), \; ^tLu = 0 \Rightarrow \text{singsupp}(u) \subset K'. \] (10)

We will prove that (b.1) holds with \( \tilde{K} = K' \). In fact, suppose that there exists an orbit \( \Gamma \) such that \( \tilde{\Gamma} \subset X \setminus K' \). If \( \Gamma \) is a periodic orbit then from Lemma 2-(i) there exists \( u \in \mathcal{E}'(X) \) such that \( ^tLu = 0 \) and \( \text{singsupp}(u) = \Gamma \). This contradicts (10). In case \( \Gamma \) is a non-periodic orbit then we have a contradiction with (10) because of the Lemma 2-(ii).

**Proof of (a) \( \Rightarrow \) (b.2).** If (b.2) is false then \( \exists K \subset X \) and a sequence of integral curves \( \gamma_j : [a_j, b_j] \rightarrow X \) such that \( \Gamma_j := \gamma_j ([a_j, b_j]) \in C_K \) but \( \Gamma_j \not\subset K_j, \forall j \in \mathbb{N} \).

Choose an open subset \( V_K \) of \( X \) such that \( K \subset V_K \) and \( \overline{V_K} \subset X \). Consider \( j_0 \in \mathbb{N} \) such that \( j \geq j_0 \Rightarrow \overline{V_K} \subset K_{j_0} \). Observe that \( \Gamma_j \) is not a critical point of \( L \) when \( j \geq j_0 \).

Suppose that \( j \geq j_0 \) and \( \Gamma_j \) is a periodic orbit of \( L \). Since \( V_K \) is an open subset of \( X \), \( \exists c_j \in (a_j, b_j) \) such that \( \gamma_j ([a_j, c_j]) \) is a non-periodic interval of trajectory, \( \gamma_j ([a_j, c_j]) \not\subset K_j \) and \( \gamma_j (a_j), \gamma_j (c_j) \in V_K \).

For each \( j \geq j_0 \) define \( \Gamma'_j = \Gamma_j \) if \( \Gamma_j \) is a non-periodic interval of trajectory and \( \Gamma'_j = \gamma_j ([a_j, c_j]) \), otherwise. From Lemma 3, \( \exists u_j \in \mathcal{E}'(X) \) such that \( \text{singsupp}(^tLu_j) \subset V_K \) and \( \text{singsupp}(u_j) = \Gamma'_j \not\subset K_j \). Hence \( X \) is not convex for singular supports.

**Proof of (b) \( \Rightarrow \) (a).** If \( X \) is not convex for singular supports then \( \exists K \subset X \) with the following property:

\[ \forall K' \subset X, \exists u \in \mathcal{E}'(X) \text{ such that } \text{singsupp}(^tLu) \subset K \text{ but } \text{singsupp}(u) \not\subset K'. \] (11)
Let $\tilde{K}$ be as in (b.1) and choose an open subset $V_{\tilde{K}}$ of $X$ such that $\tilde{K} \subset V_{\tilde{K}}$ and $\overline{V_{\tilde{K}}} \subset X$. Define $K_0 = K \cup \overline{V_{\tilde{K}}}$. From (b.2) we have that $\exists K'_0 \subset X$ such that

$$\Gamma \in C_{K_0} \Rightarrow \Gamma \subset K'_0. \tag{12}$$

Property (11) implies there exist $u_0 \in \mathcal{E}'(X)$ and $x \in X$ such that

$$\text{singsupp} (\mathsf{t} L u_0) \subset K \tag{13}$$

and $x \in \text{singsupp} (u_0) \setminus K'_0$. Hence $\Gamma^+_x \cap K_0 = \emptyset$ or $\Gamma^-_x \cap K_0 = \emptyset$. In fact, if $\Gamma^+_x \cap K_0 \neq \emptyset$ and $\Gamma^-_x \cap K_0 \neq \emptyset$ then, from (12), we have $x \in K'_0$. This is a contradiction. Then we may suppose that $K_0 \cap \Gamma^+_x = \emptyset$. Since $K \subset K_0$ we obtain $K \cap \Gamma^+_x = \emptyset$. Using (13) and propagation of singularities we obtain $\Gamma^+_x \subset \text{singsupp} (u_0)$. Hence $\overline{\Gamma^+_x} \subset X$. But using (b.1) we have that $\Gamma^+_x$ is not relatively compact.

Using the ideas of the proof of Proposition 1 we prove that the $L$-convexity for supports is equivalent to condition (b) of Proposition 1, when $L$ is a real vector field. Then we have:

**Remark 2.** Let $L$ be a real vector field on $X$. Then $X$ is $L$-convex for supports if, and only if, $X$ is $L$-convex for singular supports.

The proof of the following remark is analogous to the case $c \equiv 0$ proved in Proposition 1.

**Remark 3.** Let $L$ be a real vector field on $X$ and $c \in C^\infty (X)$. Define $P = L + c$. Consider the condition (b) of Proposition 1 and the following condition: (a') $X$ is $P$-convex for singular supports. Then (b) $\Rightarrow$ (a') and (a') $\Rightarrow$ (b.2). Moreover, if $c \in C^\infty_0 (X)$ then (a') $\Rightarrow$ (b.1).

### 3. Proof of Theorem 1

First we remark that any hyperbolic linear vector field on $\mathbb{R}^n$ satisfies the hypotheses (b) and (c) of Theorem 1. Since condition (NRC 1) implies that $x_0$ is a hyperbolic critical point of $L$, the following results imply Theorem 1.

**Lemma 4.** With $X = \mathbb{R}^n$, suppose (a) holds. Then $\forall f \in C^\infty (\mathbb{R}^n), \exists u \in C^\infty (\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of zero.
Theorem 2. Suppose that \( x_0 \) is a hyperbolic critical point. If (b) and (c) are true then \( \forall f \in C^\infty (X) \) such that \( f = 0 \) in a neighborhood of \( x_0, \exists u \in C^\infty (X) \), with \( u = 0 \) in a neighborhood of \( x_0 \), such that \( Pu = f \).

Observe that Theorem 2 holds for any smooth complex function \( c \) defined on \( X \).

3.1. Proof of Lemma 4

Before the proof of Lemma 4 we will prove the following preliminary result:

Lemma 5. Suppose that \( X = \mathbb{R}^n \) and \( x_0 = 0 \). Condition (NRC 2) is equivalent to the property: \( \forall f \in C^\infty (\mathbb{R}^n), \exists u \in C^\infty (\mathbb{R}^n) \) such that \( Pu - f \) is flat at the origin.

Proof. We denote by \( Pu \sim f \) when \( Pu - f \) is flat at the origin. Write \( L = \sum_{j=1}^{n} a_j \partial_j \) and consider formal Taylor expansions of \( u, a_j \) and \( c \) at \( x = 0 \):

\[
\sum_{\alpha} \frac{\partial^n u (0)}{\alpha!} x^\alpha,
\sum_{\alpha} \frac{\partial^n a_j (0)}{\alpha!} x^\alpha, j = 1, 2, \ldots, n,
\sum_{\alpha} \frac{\partial^n c (0)}{\alpha!} x^\alpha,
\]

respectively. Then \( Pu \sim f \) is equivalent to

\[
\sum_{j,k} \alpha_k \partial_k a_j (0) \partial^{\alpha + e_j - e_k} u (0) + c (0) \partial^n u (0) + R_\alpha = \partial^n f (0), \forall \alpha \in \mathbb{N}^n, \quad (14)
\]

where \( e_j \) is the unit vector of \( \mathbb{R}^n \) with 1 in the \( j \)th position. The term \( R_\alpha \) depends only on the derivatives of \( u \) of order \( \leq 1 \) evaluated at the origin and has the following property: if \( \partial^\beta u (0) = 0, \forall \beta \in \mathbb{N}^n \) such that \( |\beta| \leq |\alpha| - 1 \), then \( R_\alpha = 0 \), where \( |\alpha| = \sum_{j=1}^{n} \alpha_j, \forall \alpha \in \mathbb{N}^n \).

\( Pu \sim f \) is equivalent to a sequence of linear systems

\[
(B^m + c (0) I) u^m = f^m + v^{m-1}, m \in \mathbb{N}.
\]

Consider \( \Lambda^m_n = \{ \alpha \in \mathbb{N}^n; |\alpha| = m \} \) and \( M = \sharp \Lambda^m_n \). For each \( m \in \mathbb{N}, B^m \) is a real matrix \( M \times M \) which depends on \( DL (0) \) and on the choice of an ordering
of $\Lambda^m_n$. The components of $u^m \in \mathbb{C}^M$ (resp. $f^m \in \mathbb{C}^M$) are the derivatives of $u$ (resp. $f$) of order $m$ evaluated at the origin. If $m \geq 1$ then the vector $v^{m-1} \in \mathbb{C}^M$ corresponds to the term $R_\alpha$ of (14). Define $v^0 = 0 \in \mathbb{R}$. The vector $v^{m-1}$ depends only on the derivatives of $u$ of order $\leq m - 1$ and this vector has the following property:

$$\partial^\alpha u(0) = 0, \forall \alpha \in \mathbb{N}^n \text{ satisfying } |\alpha| \leq m - 1 \Rightarrow v^{m-1} = 0.$$  \hspace{1cm} (16)

Using the real Jordan form for a choice of ordering of $\Lambda^m_n$ we prove that

$$\text{Spec } B^m \cap \mathbb{R} = \left\{ \sum_{j=1}^n m_j \text{ Re } \lambda_j; m_1, m_2, \ldots, m_n' \in \mathbb{N} \text{ and } m_{n'+1}, m_{n'+2}, \ldots, m_n \in 2\mathbb{N} \right\}. \hspace{1cm} (17)$$

Here Spec $A$ denotes the set of the eigenvalues of the matrix $A$. Using (16) and (17) we conclude that the systems (15) can be solved recursively for $u^0, u^1, \ldots$ if, and only if, (NRC 2) holds.

**Proof of Lemma 4.** In view of Lemma 5 it is sufficient to prove that $\forall f \in C^\infty(\mathbb{R}^n)$ with $f$ flat at the origin, $\exists u \in C^\infty(\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of the origin.

From (NRC 2) we obtain $c(0) \neq 0$. Define $P_1 = \frac{1}{c}P$ in a neighborhood of the origin. Then $P_1 = L_1 + 1$, where $L_1 = \frac{1}{c}L$. Since $L(0) = 0$ we have

$$DL_1(0) = \frac{1}{c(0)}DL(0).$$

Then (NRC 1) holds for $L_1$. From Sternberg’s result there exists a change of coordinates which carries $P_1$ into $P_2$ corresponding to

$$\frac{1}{c(0)}DL(0) + 1.$$ 

From Guillemin-Schaeffer’s result we conclude the proof of Lemma 4. \hfill \blacksquare

**3.2. Preliminaries for Theorem 2**

Here, we will prove some preliminary results. Let $L$ be a real vector field on $\mathbb{R}^2$. Suppose that the origin is a local attractor of $L$ and $\{0\}$ is the unique critical point of $L$. Under these conditions, from Proposition 1 and since, for
the case, convexity with respect of supports and singular support are the same, the result of dos Santos Filho ([12], p. 263) can be written as, the origin is a global attractor of $L$ if, and only if, $\mathbb{R}^2 \setminus \{0\}$ is convex with respect to the trajectories of $L$. We begin this section with a version of this result for an arbitrary manifold.

**Lemma 6.** Suppose that $X$ is a connected manifold and that $\{x_0\}$ is a local attractor of $L$. If

(i) $\Gamma^+_x \subset \subset X \Rightarrow \omega (x) = \{x_0\}$

and

(ii) $X$ is convex with respect to the trajectories of $L$

then

$\{x_0\}$ is a global attractor of $L$.

**Proof.** We will see that the boundary $\partial B (x_0)$ of the basin of attraction $B (x_0)$ is empty. Suppose there exists $x \in \partial B (x_0)$. Since $\{x_0\}$ is a local attractor of $L$, $B (x_0)$ is an open subset of $X$. Hence $\Gamma^+_x \cap B (x_0) = \emptyset$ then $x_0 \notin \Gamma^+_x$. From (i) it follows that $\Gamma^+_x$ is not relatively compact orbit of $L$.

Consider neighborhoods $U_x$ of $x$ and $U_{x_0}$ of $x_0$ such that $U_x \cup U_{x_0} \subset \subset X$. Take $K = U_{x_0} \cup U_x$. It is easy to see that for such $K$ there is no compact $K'$ satisfying the condition for convexity with respect to the trajectories of $L$, so (ii) is not true.

If $x_0$ is a hyperbolic critical point local attractor for $L$, then the conditions (i) and (ii) of Lemma 6 are necessary for $\{x_0\}$ to be a global attractor of $L$.

**Definition 1.** A global transversal of $L$ on $X$ is a codimension one immersed submanifold $\Sigma$ of $X$ such that for all $x \in X$ there exists a unique $t \in \mathbb{R}$ such that $y = \gamma (t, x) \in \Sigma$ and $T_y (\Sigma) \oplus L (y) = T_y (X)$.

Here $T_x (M)$ denotes the tangent space of the manifold $M$ at the point $x \in M$. The Definition 1 is similar to the definition used in [1] p. 15. Now, we state some simple remarks regarding this notion.

**Remark 4.** Let $\Sigma$ be a global transversal of $L$ on $X$.

(i) Let $\tau : X \to \mathbb{R}$ given by: for each $x \in X$, $\tau (x)$ is such that $\gamma (\tau (x), x) \in \Sigma$. Then $\tau \in C^\infty (X, \mathbb{R})$.

(ii) $M = \{(t, y) ; y \in \Sigma, t \in I_y\}$ is an open subset of $\mathbb{R} \times \Sigma$. $h : M \to X$ defined by $h (t, y) = \gamma (t, y)$ is a $C^\infty$-diffeomorphism which carries $\partial \tau / \partial t$ into $L$.  

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From Remark 4-(ii) and Duistermaat-Hörmander’s theorem (see Theorem 6.4.2 of [2]) we get that the existence of a global transversal of $L$ on $X$ is equivalent to the global solvability of $L$ on $C^\infty(X)$. The next remark follows from Hartman’s theorem (see Theorem 7.1 of [4]).

**Remark 5.** Let $x_0$ be a hyperbolic critical point of $L$. If $\{x_0\}$ is a global attractor of $L$ then any global transversal of $L$ on $X \setminus \{x_0\}$ is a compact subset of $X \setminus \{x_0\}$.

Sketch of the proof: Take a “sphere $S$ centered at $x_0$” and contained at the neighborhood of $x_0$ preluded in Hartman’s theorem. Then, we define the mapping $T$ from $S$ to $\Sigma$ which takes any point of $S$ to the unique point of $\Sigma$ that belongs to the trajectory of $L$ that passes through $x_0$. By continuous dependence, the injective mapping $T$ is continuous. Therefore $T(S) \subset \Sigma$ is compact. But by the hypothesis of $x_0$ being a global attractor we have that, for any point $y$ of $\Sigma$, the trajectory starting at $y$ must go into the Hartman’s neighborhood therefore must intercept $S$. Then $T$ is onto, hence $\Sigma = T(S)$ is compact.

In the lemma below we construct a global transversal in the attractor case.

**Lemma 7.** Let $x_0$ be a hyperbolic critical point of $L$. If $\{x_0\}$ is a global attractor of $L$ then for all neighborhood $V$ of $x_0$, there exists a global transversal $\Sigma$ of $L$ on $X \setminus \{x_0\}$ such that $\Sigma \subset V \setminus \{x_0\}$.

**Proof.** Since $\{x_0\}$ is a global attractor, it follows that $\{x_0\}$ is the unique relatively compact orbit of $L$. From Hartman’s theorem it follows that there exists a neighborhood $U$ of $x_0$ such that $U \setminus \{x_0\}$ is convex with respect to the trajectories of $L$ and $U \subset V$. Now, Duistermaat-Hörmander’s theorem implies that exists a global transversal $\Sigma$ of $L$ on $U \setminus \{x_0\}$. Since $\{x_0\}$ is a global attractor of $L$ then $\Sigma$ is a global transversal of $L$ on $X \setminus \{x_0\}$. ■

The next result shows that an appropriated perturbation of a global transversal is still a global transversal.

**Lemma 8.** Let $\Sigma$ be a global transversal of $L$ on $X$ and $\chi \in C^\infty(\Sigma, \mathbb{R})$ such that $\omega_- (y) < \chi (y) < \omega_+ (y)$, $\forall y \in \Sigma$. The image of the mapping $\sigma: \Sigma \rightarrow X$ given by $\sigma(y) = \gamma(\chi(y), y)$ is a global transversal of $L$ on $X$.

**Proof.** From Remark 4-(ii) we may suppose that $X = M$ and $L = \frac{\partial}{\partial t}$. The result holds easily for this case. ■
3.3. Proof of Theorem 2

Let \( s \) be the number of the eigenvalues of \( DL(x_0) \) with negative real part. To prove Theorem 2 we consider two cases:

- **Case A:** \( s \in \{0, n\} \) (attractor or repellent case).
- **Case B:** \( s \notin \{0, n\} \) (saddle point case).

### 3.3.1. Proof of Case A

Suppose \( s = n \) (the case \( s = 0 \) is analogous). From Lemma 6 it follows that \( \{x_0\} \) is a global attractor of \( L \). Let \( U \) be a neighborhood of \( x_0 \) such that \( f = 0 \) on \( U \) and

\[
x \in U \Rightarrow \Gamma^+_x \subset U. \tag{18}
\]

Choose a neighborhood \( V \) of \( x_0 \) such that \( V \subset U \) and \( \theta \in \mathcal{C}^\infty(X) \) such that

\[
\theta = 0 \text{ on } V \text{ and } \theta = 1 \text{ on } \mathcal{C}U. \tag{19}
\]

From Remark 5 and Lemma 7 there exists a compact global transversal \( \Sigma \) of \( L \) on \( X \setminus \{x_0\} \) contained in \( V \setminus \{x_0\} \). From the Method of Characteristics it follows that \( \exists \psi \in \mathcal{C}^\infty(X \setminus \{x_0\}) \) such that \( L\psi = c\theta \) on \( X \setminus \{x_0\} \) and \( \psi = 0 \) in a neighborhood of \( x_0 \). Then we may suppose \( \psi \in \mathcal{C}^\infty(X) \) and \( L\psi = c\theta \) on \( X \).

In the same way, using (18) we obtain \( \phi \in \mathcal{C}^\infty(X) \) such that \( L\phi = e^\psi f \) on \( X \) and

\[
\phi = 0 \text{ on } U. \tag{20}
\]

Hence

\[
P \left( \phi e^{-\psi} \right) = f + ce^{-\psi} \phi (1 - \theta).
\]

From (19) and (20) it follows \( \phi (1 - \theta) = 0 \). Therefore, by taking \( u = \phi e^{-\psi} \) we have \( Pu = f \).

### 3.3.2. Preliminaries for Case B

We define the **stable** (resp. **unstable**) manifold of \( L \) at \( x_0 \) by

\[
W^s(x_0) = \left\{ x \in X; \lim_{t \to \omega_+(x)} \gamma(t,x) = x_0 \right\}
\]

(resp. \( W^u(x_0) = \left\{ x \in X; \lim_{t \to \omega_-(x)} \gamma(t,x) = x_0 \right\} \)), which is a \( \mathcal{C}^\infty \) immersed submanifold of \( X \). Take \( X^s = X \setminus W^s(x_0) \) and \( X^u = X \setminus W^u(x_0) \).
If \( \Sigma^s \) (resp. \( \Sigma^u \)) is a global transversal of \( L \) on \( X^s \) (resp. \( X^u \)), we denote 
\[ X^\pm_s(\Sigma^s) = \{ \gamma(t,y); y \in \Sigma^s, \pm t > 0 \} \quad (\text{resp.} \quad X^\pm_u(\Sigma^u) = \{ \gamma(t,y); y \in \Sigma^u, \pm t > 0 \}) \] 
subsets of \( X^s \) (resp. \( X^u \)).

The main result of this section is:

**Proposition 2.** Let \( U_1 \) be a neighborhood of \( \{x_0\} \). There exists a neighborhood \( U \) of \( \{x_0\} \), with \( U \subset U_1 \), global transversal \( \Sigma^s_1 \) and \( \Sigma^u_1 \) of \( L \) on \( X^s \), and 
global transversal \( \Sigma^s_2 \) and \( \Sigma^u_2 \) of \( L \) on \( X^u \) such that:

(i) \( \Sigma^u_2 \subset X^u_+(\Sigma^u_1) \) and \( \Sigma^s_1 \subset X^s_+(\Sigma^s_2) \),

(ii) \( X^u_+(\Sigma^u_1) \cup W^u(x_0) \subset X^s_+(\Sigma^s_1) \cup U \)

and

(iii) \( \forall f \in C^\infty(X) \) such that \( f = 0 \) on \( X^s_+(\Sigma^s_2) \cup W^s(x_0) \cup U \) (resp. \( X^u_+(\Sigma^u_1) \cup W^u(x_0) \)), \( \exists u \in C^\infty(X) \) such that \( Lu = f \) and \( u = 0 \) on \( U \) (resp. \( u = 0 \) on \( X^u_+(\Sigma^u_1) \cup W^u(0) \)).

For the proof of Proposition 2, we do not use that \( T_x(\Sigma^s_1) \oplus L_x = T_x(X) \), \( \forall x \in \Sigma^s_1 \), similarly for \( \Sigma^u_1 \).

In order to prove Proposition 2 we will use some preliminary results, here Lemma 9 to Lemma 13.

**Lemma 9.**

(i) \( W^s(x_0) \cap W^u(x_0) = \{x_0\} \).

(ii) \( W^s(x_0) \) (resp. \( W^u(x_0) \)) is a closed subset of \( X \).

**Proof.** (i) If \( x \in W^s(x_0) \cap W^u(x_0) \) then \( \alpha(x) = \omega(x) = \{x_0\} \). Hence \( \Gamma_x \subset \subset X \). From (b) it follows that \( x = x_0 \).

(ii) If \( W^s(x_0) \) is not closed in \( X \) then there exists a sequence \( \{x_j\} \subset W^s(x_0) \) converging to some \( x \in X \setminus W^s(x_0) \). Hence \( x_0 \in \omega(x) \). Since \( \omega(x) \) is invariant under the flow, from (b) it follows that \( \Gamma_x \) is not relatively compact. Using the same arguments of the proof of Lemma 6 we obtain the result.

From Lemma 9-(ii) we have:

**Remark 6.** \( X^s \) (resp. \( X^u \)) is an open subset of \( X \). Therefore \( X^+_s(\Sigma^s) \) and \( X^-s(\Sigma^s) \) (resp. \( X^+_u(\Sigma^u) \) and \( X^-u(\Sigma^u) \)) are open subsets of \( X \).

Moreover:

**Lemma 10.** \( X^s \) (resp. \( X^u \)) is convex with respect to the trajectories of \( L \).
Proof. Suppose that $X^s$ is not convex with respect to the trajectories of $L$, then there exist $K \subset X$, a sequence $\{\Gamma_j\}$ of compact intervals of trajectories of $L$ with endpoints in $K$ and a sequence $\{x_j\}$ such that

$$x_j \in \Gamma_j \setminus K_j, \forall j \in \mathbb{N}$$  \hspace{0.5cm} (21)

here $\{K_j\}$ is a sequence of compact subsets of $X^s$ satisfying the properties (1). From hypothesis (c) of Theorem 2 it follows that $\exists K' \subset \mathbb{R}^n$ such that $\{x_j\} \subset K'$. Hence there exist $x \in X$ and a subsequence $\{x_{j_k}\} \subset \{x_j\}$ such that $x_{j_k} \to x$. Without loss of generality, we may suppose that $x_j \to x$. Observe that from (21) we have

$$x \in W^s(x_0).$$ \hspace{0.5cm} (22)

We will divide the rest of the proof in two cases.

Case $x \neq x_0$.

In this case take a sequence $\{C_k\}$ of compact subsets of $X$ satisfying the properties (1). Since $x \neq x_0$, from (b) it follows that $\forall k \in \mathbb{N}, \exists y_k \in \Gamma_x \setminus C_k$. Using (22) we obtain $[x, y_k] \cap K = \emptyset$, then from Flow Box theorem there exists a neighborhood $V_k$ of $[x, y_k]$ such that $L|_{V_k}$ is conjugated to $\partial_1$ and $V_k \cap K = \emptyset$. Since $x_j \to x$ follows that $\exists j_k \in \mathbb{N}$ with the following property: $\forall j > j_k, \exists z_j \in \Gamma_j \cap C_k$. Then (c) fails.

Case $x = x_0$.

From the proof of the previous case it is sufficient to prove that there exist $w \in W^s(x_0)$, with $w \neq x_0$, and a sequence $w_j \to w$ such that $w_j \in \Gamma_j, \forall j \in \mathbb{N}$.

Since $K \cap W^s(x_0) = \emptyset$ and $x_0 \in W^s(x_0)$ there exists a neighborhood $V$ of $x_0$ satisfying $K \cap V = \emptyset$.

From Hartman’s theorem we have there exists an open subset $U$ of $X$ such that $x_0 \in U \subset V$ and $U \setminus W^s(x_0)$ is convex with respect to the trajectories of $L$.

Consider a neighborhood $W$ of $x_0$ such that $W \subset U$ and $\partial W$ is homeomorphic to the sphere $S^{n-1}$. Choose $j_0 \in \mathbb{N}$ such that $j > j_0 \Rightarrow x_j \in W$. Since the endpoints of $\Gamma_j$ are contained in $K$, from the continuity of $\Gamma_j$ it follows that there exist $w_j, w'_j \in \Gamma_j \cap \partial W$ such that $x_j \in [w_j, w'_j]$. From a compactness argument there exist subsequences $\{w_{j_k}\} \subset \{w_j\}$ and $\{w'_{j_k}\} \subset \{w'_j\}$ such that $w_{j_k} \to w$ and $w'_{j_k} \to w'$. It is sufficient to prove that

$$w \in W^s(x_0) \text{ or } w' \in W^s(x_0).$$ \hspace{0.5cm} (23)
If \( w \notin W^s(x_0) \) and \( w' \notin W^s(x_0) \) then the sequences \( \{w_j\} \) and \( \{w'_j\} \) are contained in a compact subset of \( \partial W \setminus W^s(x_0) \). Hence \( U \setminus W^s(x_0) \) is not convex with respect to the trajectories of \( L \).

Using Lemma 10 we obtain:

**Remark 7.** \( X_+^s(\Sigma^s) \) and \( X_-^s(\Sigma^s) \) (resp. \( X_+^u(\Sigma^u) \) and \( X_-^u(\Sigma^u) \)) are convex with respect to the trajectories of \( L \).

Let \( \Sigma^s \) be a global transversal of \( L \) on \( X^s \). Observe that \( W^u(x_0) \) and \( \Sigma^s \) are immersed submanifold of \( X \) and \( \Sigma^s \) is transversal to \( W^u(x_0) \). Then we have:

**Remark 8.** If \( \Sigma^s \) is a global transversal of \( L \) X^s (resp. \( \Sigma^u \) is a global transversal of \( L \) on \( X^u \)) then \( K := \Sigma^s \cap W^u(x_0) \) (resp. \( K := \Sigma^u \cap W^s(x_0) \)) is a global transversal of \( L|_{W^u(x_0)} \) on \( W^u(x_0) \setminus \{x_0\} \) (resp. of \( L|_{W^s(x_0)} \) on \( W^s(x_0) \setminus \{x_0\} \)), furthermore \( K \subset X \).

Hartman’s theorem is used to prove:

**Lemma 11.** If \( \Sigma^s \) (resp. \( \Sigma^u \)) is a global transversal of \( L \) on \( X^s \) (resp. on \( X^u \)) then \( X^s(\Sigma^s) \cup W^s(0) \) (resp. \( X^u(\Sigma^u) \cup W^u(0) \)) is an open subset of \( X \).

**Proof.** From Remark 6 is sufficient to prove that \( \forall x \in W^s(x_0) \) there exists a neighborhood \( V_x \) of \( x \) such that \( V_x \subset X^s(\Sigma^s) \cup W^s(x_0) \). In the other hand from the continuity of \( \gamma \) it is sufficient to prove that there exists a neighborhood \( V_0 \) of \( x_0 \) such that

\[
V_0 \subset X^s(\Sigma^s) \cup W^s(x_0).
\]  

(24)

Consider the function \( \tau : X^s \to \mathbb{R} \) given by the Remark 4-(i) and take \( K = \Sigma^s \cap W^u(x_0) \). We will divide the rest of the proof in two steps.

**Step 1.** There exists an open subset \( U_0 \) of \( X \) such that \( x_0 \in U_0 \) and \( U_0 \cap W^u(x_0) \setminus \{x_0\} \subset X^s(\Sigma^s) \).

In fact, since \( K \subset X \) (see Remark 8), there exists an open subset \( U_0 \) of \( X \) such that \( x_0 \in U_0 \), \( U_0 \cap K = \emptyset \), \( U_0 \) satisfies the conclusion of Hartman’s theorem and \( U_0 \) is convex with respect to the trajectories of \( L \).

It is enough to prove that \( \tau(y) > 0, \forall y \in U_0 \cap W^u(x_0) \setminus \{x_0\} \). From \( U_0 \cap K = \emptyset \) we have \( \tau(y) \neq 0 \). Suppose that \( \tau(y) < 0 \). Since \( x_0 \) is a hyperbolic
critical point of $L$ and $x_0$ is a global attractor of $-L$ on $W^u(x_0)$, there exists an open subset $A$ of $W^u(x_0)$, with $x_0 \in A \subset U_0 \cap W^u(x_0)$, such that

$$t \leq 0, z \in A \Rightarrow \gamma(t, z) \in A. \tag{25}$$

Choose $t_0 < 0$ such that $\gamma(t_0, y) \in A$. If $\tau(y) \leq t_0$, from (25) it follows that $\gamma(\tau(y), y) \in U_0$. This is a contradiction, because $U_0 \cap K = \emptyset$. Hence $t_0 < \tau(y) < 0$. Since $U_0$ is convex with respect to the trajectories of $L$, these inequalities imply $\gamma(\tau(y), y) \in U_0$ and this is a contradiction with $K \cap U_0 = \emptyset$. Hence $t_0 < \tau(y) < 0$.

**Step 2.** There exists a neighborhood $V_0$ of $x_0$ with the property (24).

In fact, from Hartman’s theorem there exists a subset $\Sigma'$ of $X$ such that $\Sigma' \subset U_0 \setminus \{x_0\}$ and $\Sigma'$ is homeomorphic to $S^{n-1}$. Define $\Delta = \Sigma' \cap W^u(x_0)$. From Lemma 9-(ii) we have $\Delta \subset \subset X$. From **Step 1** it follows that there exists a neighborhood $V_\Delta$ of $\Delta$ such that

$$V_\Delta \subset X^s(\Sigma') \cap U_0. \tag{26}$$

Using (26), Hartman’s theorem and the compactness of $\Delta$ we prove that there exists a neighborhood $V_0$ of $x_0$ such that $V_0 \setminus W^s(x_0) \subset X^s(\Sigma^s)$. This inclusion implies the statement of **Step 2**.

From these lemmas we will construct global transversal of $L$ on $X^s$ with special properties. Denote $[x, y]$ the interval of trajectory of $L$ with endpoints $x$ and $y$.

**Lemma 12.** Let $U_1$ be a neighborhood of $\{x_0\}$. Then there exists an open set $U$, with $x_0 \in U \subset U_1$, satisfying the conclusion of the Hartman’s theorem with $U$ convex with respect to the trajectories of $L$, and global transversal $\Sigma^s_1$ and $\Sigma^s_2$ of $L$ on $X^s$ such that:

(i) $\Sigma^s_1 \cap W^u(x_0) \subset U$,

(ii) $\Sigma^s_1 \subset X^s(\Sigma^s_2)$

and

(iii) $x \in \Sigma^s_2, y \in \Gamma^+_x \cap U \Rightarrow [x, y] \subset U$.

**Proof.** From the hypothesis (b) of Theorem 2, Lemma 10 and Duisteramaat-Hörmander’s theorem it follows that there exists a global transversal $\Sigma^s_0$ of $L$ on $X^s$. From Lemma 11 there exists an open subset $U$ of $X$, with $x_0 \in U \subset U_1$ such that: $U \subset X^s(\Sigma^s_0) \cup W^s(x_0)$, $U$ satisfies the conclusion of Hartman’s
theorem and $U$ is convex with respect to the trajectories of $L$. Observe that $U$ has the additional property:

$$y \in \Sigma^s_0, \gamma(t, y) \in U \Rightarrow t < 0.$$  \hfill (27)

We will divide the rest of the proof in four steps.

**Step 1.** There exist $T \in \mathbb{R}$ and an open subset $W_0$ of $\Sigma^s_0$, with $K \subset W_0$, such that

$$y \in W_0 \Rightarrow \omega_-(y) < T < 0$$  \hfill (28)

and

$$y \in W_0 \Rightarrow \gamma(T, y), \gamma(T/2, y) \in U.$$  \hfill (29)

In fact, consider an open subset $V$ of $\Sigma^s_0$ such that $W_u(x_0) \subset V$ and $\omega_-(y) = -\infty, \forall y \in V$. Take $K = \Sigma^s_0 \cap W_u(x_0)$. For each $y \in K$ take $t_y < 0$ such that $\gamma(t, y) \in U, \forall t \leq t_y$. From compactness of $K$ there exists $T < 0$ such that $t \leq T \Rightarrow \gamma(t, y) \in U, \forall y \in K$. By continuity of $\gamma$ it follows that there exists an open subset $V_0$ of $X$ such that $K \subset V_0 \subset V$ and $\gamma(T, y), \gamma(T/2, y) \in U, \forall y \in V_0$. Set $W_0 = V_0 \cap \Sigma^s_0$.

**Step 2.** There exist a sequence $\{t_j\}_{j=1}^\infty \subset \mathbb{R}$ and a locally finite cover $\{W_j\}_{j=1}^\infty$ of $\Sigma^s_0$ such that

$$y \in W_j \Rightarrow 0 < t_j < \omega_+(y).$$  \hfill (30)

In fact, for each $y \in \Sigma^s_0$ choose $t_y \in \mathbb{R}$ and a neighborhood $V_y$ of $y$ such that $0 < t_y < \omega_+(y), \forall y \in V_y$. Consider a locally finite refinement $\{W_j\}_{j=1}^\infty$ of the cover $\{V_y \cap \Sigma^s_0\}_{y \in \Sigma^s_0}$. For each $j \geq 1$ choose $V_y$ such that $W_j \subset V_y \cap \Sigma^s_0$ and define $t_j = t_y$. Hence **Step 2** follows.

Consider $\mu_0 \in C^\infty(\Sigma^s_0, \mathbb{R})$ such that $0 \leq \mu_0 \leq 1, \mu_0 = 1$ in a neighborhood of $K$ and $\text{supp}(\mu_0) \subset W_0$. Let $\{\mu_j\}_{j=1}^\infty$ be a partition of unity subordinated to the cover $\{W_j\}_{j=1}^\infty$. Consider the functions $\chi_1, \chi_2 \in C^\infty(\Sigma^s_0, \mathbb{R})$ given by

$$\chi_1 = \frac{T}{2} \mu_0 + (1 - \mu_0) \sum_{j=1}^\infty t_j \mu_j \text{ and } \chi_2 = T \mu_0.$$  

Then we have the following result:

**Step 3.** For each $j = 1, 2$, the image $\Sigma^s_j$ of the function

$$\sigma_j : \Sigma^s_0 \rightarrow X^*$$

$$y \mapsto \gamma(\chi_j(y), y)$$
is a global transversal of $L$ on $X^s$.

In fact, from (28) it follows that $\omega_-(y) < \chi_2(y) < \omega_+(y), y \in \Sigma_0^s$. In the same way, from (28) and (30) we have $\omega_-(y) < \chi_1(y) < \omega_+(y), y \in \Sigma_0^s$. From Lemma 8 it follows that $\Sigma_1^u$ and $\Sigma_2^u$ are global transversal of $L$ on $X^s$.

**Step 4.** The statements (i), (ii) and (iii) hold, if $\Sigma_1^s$ and $\Sigma_2^s$ are given as in **Step 3**.

In fact, to prove (i), observe that for each $x \in \Sigma_1^s \cap W^u(x_0), \exists y \in K$ such that $x = \gamma(\chi_1(y), y)$ because $\Sigma_0^s$ is a global transversal of $L$ on $X^s$ and $W^u(x_0)$ is invariant under the flow. Since $\mu_0(y) = 1$ and from (29) it follows that $x \in U$. So proof of (i) is concluded. Observe that (ii) follows from $\chi_2 < \chi_1$.

For (iii), first we observe that for each $x \in \Sigma_2^s$ and $y \in \Gamma^+ \cap U$, we can take $t \geq 0$ such that $\gamma(t, x) = y$. Since $U$ is convex with respect to the trajectories of $L$, it is sufficient to prove that $x \in U$. Choose $z \in \Sigma_0^s$ such that $\gamma(\chi_2(z), z) = x$. We will prove that $z \in W_0$. If $z \notin W_0$ then $\chi_2(z) = 0$. But $y = \gamma(t, \chi_2(z), z)$ have $y = \gamma(t, z)$. Therefore from (27) it follows that $t < 0$. This is a contradiction. Then we have $z \in W_0$.

Since $T \leq \chi_2(z) \leq t + \chi_2(z)$ and $U$ is convex with respect to the trajectories $L$, from (29) and $y \in U$ we have $x \in U$.  

Also we have:

**Lemma 13.** Let $U$ be the neighborhood of $x_0$ and $\Sigma_0^s$ the global transversal of $L$ on $X^s$ given by Lemma 12. There exist global transversal $\Sigma_1^u$ and $\Sigma_2^u$ of $L$ on $X^u$ such that:

(i) $\Sigma_1^u \cap W^s(x_0) \subset U,$

(ii) $\Sigma_2^u \subset X^u(\Sigma_1^u)$

and

(iii) $\Sigma_1^u = \Sigma_1^s$ on $\overline{U}$.

**Proof.** In the same way as the proof of Lemma 12 we have that there exists a global transversal $\Sigma_0^u$ of $L$ on $X^u$ such that $K := \Sigma_0^u \cap W^s(x_0) \subset U$. Consider the function $\tau : X^s \to \mathbb{R}$ given by $\gamma(\tau(y), y) \in \Sigma_1^s$. We will divide the rest of the proof in three steps.

**Step 1.** There exists an open subset $W_0$ of $\Sigma_0^s$ such that $K \subset W_0 \subset U$ and

$$y \in W_0 \Rightarrow \gamma(\tau(y), y) \in U.$$

(31)

In fact, consider a subset $\Sigma'$ of $U \setminus \{0\}$ homeomorphic to $S^{n-1}$. Here the homeomorphism is given by Hartman’s theorem. Take $\Delta = \Sigma' \cap W^u(x_0)$.
Using Lemma 12-(i) it follows that there exists a neighborhood \( V_\Delta \) of \( \Delta \) such that
\[
y \in V \Rightarrow \gamma (\tau (y), y) \in U. \tag{32}
\]
Moreover, using the compactness of \( \Delta \) and Hartman’s theorem we prove that there exists a neighborhood \( V_0 \) of \( x_0 \) with the following property:
\[
y \in V_0 \setminus W^s (0) \Rightarrow \exists t \in \mathbb{R} \text{ such that } \gamma (t, y) \in V. \tag{33}
\]
From (32), (33) and from the continuity of \( \gamma \) Step 1 follows.

Consider \( \mu \in C^\infty (\Sigma_0^u, \mathbb{R}) \) such that \( 0 \leq \mu \leq 1 \), \( \mu = 1 \) in a neighborhood of \( K \) and \( \text{supp} (\mu) \subset W_0 \). Since \( \Sigma_0^u \) is an immersed submanifold of \( X \), we have \( \tau |_{\Sigma_0^u \setminus K} \in C^\infty (\Sigma_0^u \setminus K) \). Let \( \chi_1 : \Sigma_0^u \rightarrow X^u \) be the function given by \( \chi_1 = (1 - \mu) \tau |_{\Sigma_0^u \setminus K} \). Then we have that \( \chi_1 \in C^\infty (\Sigma_0^u, \mathbb{R}) \).

Step 2. The image \( \Sigma_1^u \) of the function
\[
\sigma_1 : \Sigma_0^u \rightarrow X^u \quad y \mapsto \gamma (\chi_1 (y), y)
\]
is a global transversal of \( L \) on \( X^u \) which satisfies (i).

In fact, from Lemma 8, \( \Sigma_1^u \) is a global transversal of \( L \) on \( X^u \), since \( \mu = 1 \) on \( K \) we have \( \Sigma_1^u \cap W^s (0) = K \), hence \( \Sigma_1^u \cap W^s (0) \subset U \). Then Step 2 follows.

The existence of \( \Sigma_2^u \) with the property is proved in the same way as in the proof of Lemma 12-(iii).

Step 3. The statement (iii) holds.

In fact, we will prove that
\[
\Sigma_1^u \cap \mathcal{C} U \subset \Sigma_1^s \tag{34}
\]
and
\[
\Sigma_1^s \cap \mathcal{C} U \subset \Sigma_1^u. \tag{35}
\]
To prove (34), take \( x \in \Sigma_1^u \cap \mathcal{C} U \) and choose \( y \in \Sigma_0^s \) such that \( \gamma (\chi_1 (y), y) = x \). If \( y \in W_0 \) then from (31) and \( |\chi_1 (y)| \leq |\tau (y)| \) result \( x \in U \). This is a contradiction. From \( y \notin W_0 \) it follows that \( \chi_1 (y) = \tau (y) \). Hence \( x \in \Sigma_1^s \) and the proof of (34) is finished. In the same way we prove (35).

Proof of Proposition 2.

Proof of (i). Use Lemma 12-(ii) and Lemma 13-(ii), respectively.

Proof of (ii). From Lemma 12-(i) it follows that \( W^u (x_0) \subset X^+_s (\Sigma_1^u) \cup U \), and Lemma 13-(iii) implies \( X^+_u (\Sigma_1^u) \subset X^+_s (\Sigma_1^s) \cup U \).

3.3.3. Proof of Case B

Let $U_1$ be a neighborhood of $x_0$ such that $f = 0$ on $U_1$. With the notation of the Proposition 2, we will prove Case B in two steps.

**Step 1.** \( \forall f \in C^\infty (X) \text{ such that } f = 0 \text{ on } U \exists u_1 \in C^\infty (X) \text{ such that } P u_1 = f \text{ on } U \cup X_+^s(\Sigma_1^s). \)

In fact, from Proposition 2-(i) and Lemma 10 choose $\theta_1 \in C^\infty (X)$ such that

\[
\theta_1 = 0 \text{ on } X^s(\Sigma_2^u) \cup W^s(x_0) \text{ and } \theta_1 = 1 \text{ on } X_+^s(\Sigma_1^s). \tag{36}
\]

By the Method of Characteristics and Lemma 11, \( \exists \psi_1 C^\infty (X) \) such that \( L \psi_1 = c \theta_1 \). From Proposition 2-(iii), \( \exists \phi_1 C^\infty (X) \) such that \( L \phi_1 = \theta_1 f e^{\psi_1} \) and \( L \phi_1 = \theta_1 f e^{\psi_1} \) and

\[
\phi_1 = 0 \text{ on } U. \tag{37}
\]

Hence

\[
P (\phi_1 e^{-\psi_1}) = \theta_1 f + c e^{-\psi_1} \phi_1 (1 - \theta_1). \]

Since $f = 0$ on $U$, from (36) and (37) it follows that on $X^s_+ (\Sigma_1^s) \cup U$ we have

\[
\phi_1 (1 - \theta_1) = 0 \text{ and } \theta_1 f = f.
\]

Therefore, by taking $u_1 = \phi_1 e^{-\psi_1}$ Step 1 follows.

**Step 2.** \( \forall f \in C^\infty (X) \text{ such that } f = 0 \text{ on } U \cup X_+^s(\Sigma_1^s), \exists u \in C^\infty (X) \text{ such that } P u = f \text{ on } X. \)

In fact, from Proposition 2-(i) and Lemma 11, choose $\theta_2 \in C^\infty (X)$ such that

\[
\theta_2 = 0 \text{ on } X^u_+ (\Sigma_2^u) \cup W^u(x_0) \text{ and } \theta_2 = 1 \text{ on } X^u_- (\Sigma_1^u).
\]

Therefore, \( \exists \psi_2 \in C^\infty (X) \) such that \( L \psi_2 = c \theta_2 \). Since $f = 0$ on $U \cup X_+^s(\Sigma_1^s)$, from Proposition 2-(ii)-(iii) it follows that \( \exists \phi_2 \in C^\infty (X) \) such that \( L \phi_2 = f e^{\psi_2} \) and \( \phi_2 = 0 \text{ on } U \cup X_+^s(\Sigma_1^s). \)

Hence

\[
P (\phi_2 e^{-\psi_2}) = f + c e^{-\psi_2} \phi_2 (1 - \theta_2),
\]

and

\[
\phi_2 (1 - \theta_2) = 0 \text{ on } X^s_+ (\Sigma_1^u) \cup U \cup X^u_- (\Sigma_1^u).
\]

Therefore, taking $u = \phi_2 e^{-\psi_2}$ Step 2 follows.

**Remark 9.** The hypotheses (NRC 2) and (c) are necessary for global solvability of $P$ on $C^\infty (X)$ from Lemma 5; and Remark 2, Theorem 4 of [9], respectively.
When $L$ is a linear vector field on $\mathbb{R}^n$, it is easy to see that (b) and (c) of Theorem 1 are verified. In this case, the hypothesis of linearization (NRC 1) is dropped and we have that $P = L + c$ is globally solvable on $C^\infty(\mathbb{R}^n)$ if, and only if, (NRC 2) holds. In particular, the condition (NRC 1) is not necessary for global solvability.

Now, we present a family of operators for which the condition (b) is necessary for global solvability. Take $p(x) = \sum_{j=0}^{n} a_j x^j$, be a real polynomial. Let $L$ be the vector field on $\mathbb{R}^2$ given by

$$L = x_1(1 - x_1)\partial_1 + x_2g(x_1, x_2)\partial_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $g \in C^\infty(\mathbb{R}^2)$. Notice that $(0, 0), (1, 0)$ are critical points and $(0, 1) \times \{0\}$ is a relatively compact orbit of $L$. Take the operator $P = L + c$ with $c \in C^\infty(\mathbb{R}^2)$ satisfying

$$c(x_1, 0) = p(x_1), x_1 \in \mathbb{R}.$$

Under these hypotheses we have (see [16] p. 59)

If

$$a_0 \notin \mathbb{Z} \quad \text{and} \quad a_j \notin \{1, 2, ..., n\}, j = 1, 2, ..., n,$$

then $\exists u \in \mathcal{E}' \in (\mathbb{R}^2)$ such that $Pu = 0$ and $\text{supp}(u) = [0, 1] \times \{0\}$. Hence $P$ is not globally solvable on $C^\infty(\mathbb{R}^2)$.

**References**


