The Saddle Point Property. Hartman's Theorem.

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Outline.

- The Banach Fixed Point with Parameters.
- Motivation: A Simple Example.
- The Stable and the Unstable Subspaces for an autonomous linear system.
- The Saddle Point Property. The Local Stable and the Local Unstable Manifold for a nonlinear system.
- Hartman's Theorem.

The Banach Fixed Point with Parameters.

Definition. We say that $T : M \times \Lambda \to M$ given by $T(x, y)$ is an uniform contraction with respect to $y \in \Lambda$ if there exists $\rho \in [0, 1)$ such that

$$d(T(x_1, y), T(x_2, y)) \leqslant \rho \; d(x_1, x_2), \; \forall \; x_1, x_2 \in M, \; \forall \, y \in \Lambda.$$

The Banach-Cacciopoli Fixed Point Theorem.
Let M be a complete metric space, Λ a metric space and $T : M \times \Lambda \to M$ an uniform contraction with respect to $y \in \Lambda$. Then for each $y \in \Lambda$ there exists a unique fixed point in M, $x = x(y) \stackrel{\text{def}}{=} g(y)$. If for each x fixed in M the map $y \in \Lambda \mapsto T(x, y) \in M$ is continous then $g(y)$ é contínua em Λ.

Motivation: A simple Example. The Linear Equation

$$\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = -y_2 \end{cases} \tag{H}$$

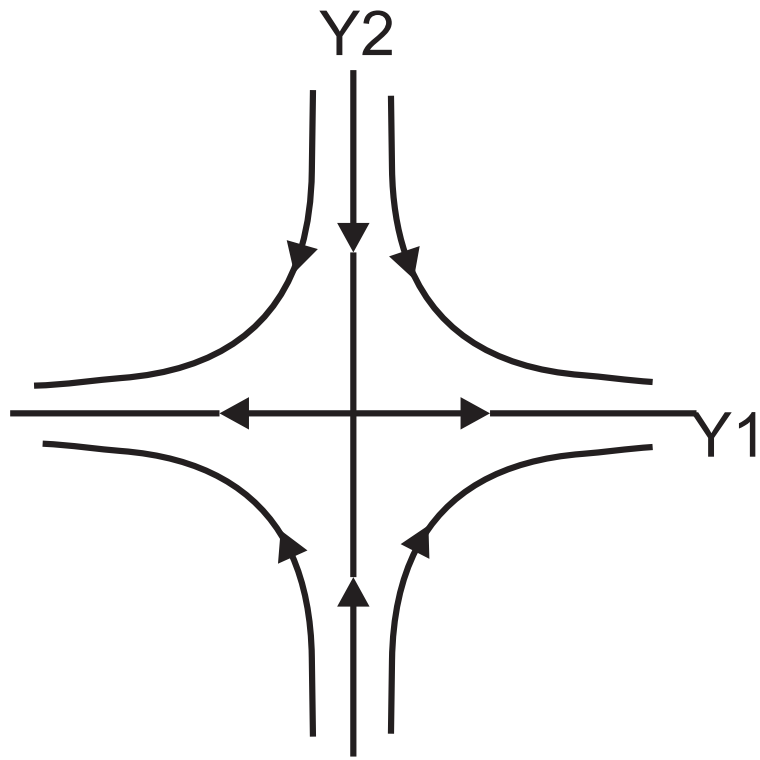

Figura: Retrato de fase de (H)

Motivation: A simple Example. The Nonlinear Equation

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 + x_1^3 \end{cases} \tag{P}$$

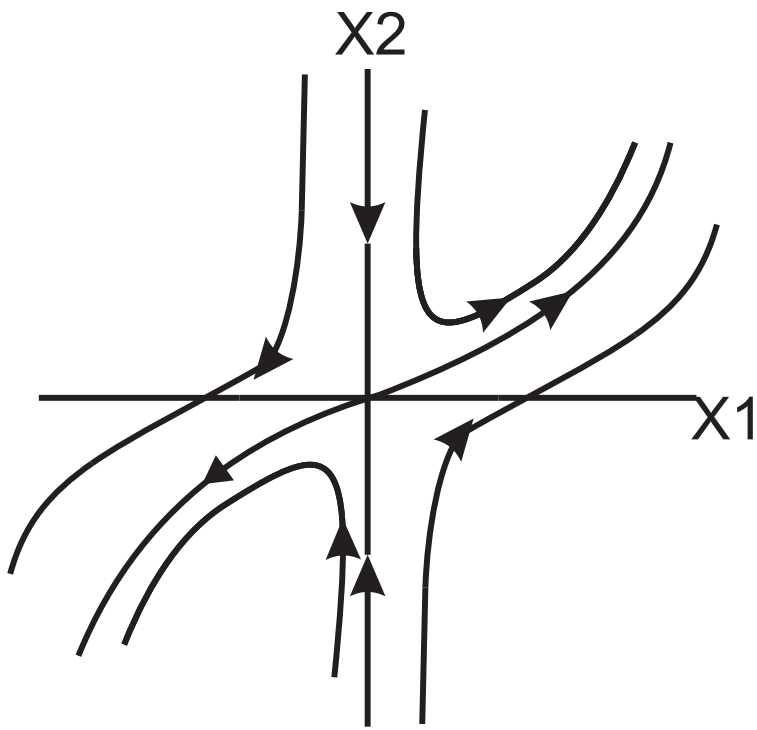

Figura: Retrato de fase de (P)

The Stable and the Unstable Manifolds.

The solution of (P) that for $t = 0$ holds $\begin{pmatrix} a \\ b \end{pmatrix}$ is given by

$$x_1(t) = e^t a$$

$$x_2(t) = e^{-t}(b - \frac{a^3}{4}) + \frac{a^3}{4} e^{3t}.$$

Therefore $x(t) \to 0,\ t \to \infty \Leftrightarrow a = 0$ and

$x(t) \to 0, t \to -\infty \Leftrightarrow b = \frac{a^3}{4}$.

If $b = \frac{a^3}{4}$ then $x_2(t) = \frac{x_1^3(t)}{4}$.

An Integral Inequality.

Let $\alpha, \gamma > 0$, $K, L, M \geqslant 0$ and $u(t)$ a nonnegative solution continuous and bounded for $t \geqslant 0$ of

$$u(t) \leqslant Ke^{-\alpha t} + L\int_0^t e^{-\alpha(t-s)}u(s)ds + M\int_t^\infty e^{\gamma(t-s)}u(s)ds. \quad (1)$$

Suppose that $\beta \stackrel{def}{=} \frac{L}{\alpha} + \frac{M}{\gamma} < 1$. Then

$$u(t) \leqslant \frac{1}{1-\beta}Ke^{-(\alpha - \frac{L}{1-\beta})t}, \ t \geqslant 0.$$

If $u(t)$ is a nonnegative solution, continuous and bounded for $t \leqslant 0$ of

$$u(t) \leqslant Ke^{\alpha t} + L\int_t^0 e^{\alpha(t-s)}u(s)ds + M\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds. \quad (2)$$

then

$$u(t) \leqslant \frac{1}{1-\beta}Ke^{(\alpha - \frac{L}{1-\beta})t}, \ t \leqslant 0.$$

Exponential dichotomy.

Let A $n \times n$ a complex matrix. We assume that there are $k \leq n$ eigenvalues (not necessarely distincts) with positive real parts and no eigenvalue with zero real part. Consider the system:

$$\dot{x} = Ax \qquad (H)$$

Then $\mathbb{C}^n$ can be decomposed as $\mathbb{C}^n = \mathbb{C}^n_+ \bigoplus \mathbb{C}^n_-$ where

$$\mathbb{C}^n_+ = \bigoplus_{\substack{\lambda \in \sigma(A) \\ \mathrm{Re}\lambda > 0}} \mathcal{M}_\lambda(A), \quad \mathbb{C}^n_- = \bigoplus_{\substack{\lambda \in \sigma(A) \\ \mathrm{Re}\lambda < 0}} \mathcal{M}_\lambda(A)$$

and $\sigma(A)$ indictes the spectrum of A and $\mathcal{M}_\lambda(A)$ indicates the generalized eigenspace associated to λ. We call $\mathbb{C}^n_+$ and $\mathbb{C}^n_-$, respectively, the unstable subspace(manifold) and the stable subspace(manifold) of (H). They are invariant with respect to A and $e^{A\,t}$. Let Π_+ and Π_- the projections define by the above decomposition. Then there exist constants $K, \alpha > 0$ such that:

$$\begin{aligned} |e^{A\,t}\Pi_+ x_0| &\leqslant Ke^{\alpha\,t}|x_0|, \ t \leqslant 0 \\ |e^{A\,t}\Pi_- x_0| &\leqslant Ke^{-\alpha\,t}|x_0|, \ t \geqslant 0. \end{aligned}$$

Geometrically.

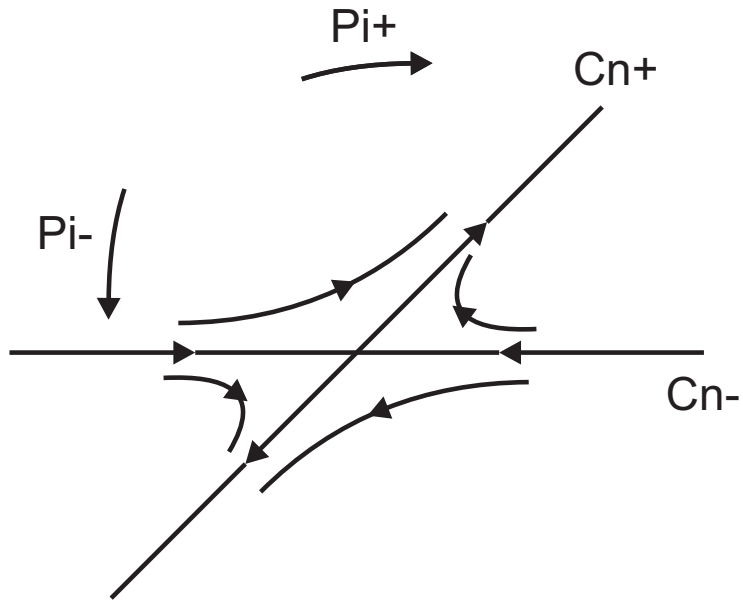

Figura:

The Nonlinear System.

Let $\eta : [0, \infty) \to [0, \infty)$ a continuous and increasing function such that $\eta(0) = 0$. We define
$\mathcal{L}ip(\eta) = \{f : \mathbb{C}^n \to: \mathbb{C}^n : f(0) = 0, |f(x) - f(y)| \leqslant$
$\eta(\sigma)|x - y|, \forall\ x, y$ such that $|x|, |y| \leqslant \sigma\}$.
Remark: Let $f :: \mathbb{C}^n \to: \mathbb{C}^n$ of class $\mathcal{C}^1$ such that
$f(0) = 0,\ f'(0) = 0$. Then there exist η such that $f \in \mathcal{L}ip(\eta)$.
For a function $f \in \mathcal{L}ip(\eta)$ we consider the peturbed system:

$$\dot{x} = Ax + f(x) \tag{3}$$

The Stable and the Unstable Manifolds of (3).

Theorem. Let A and f as above.
Then there exists $\delta > 0$ and a set S,

$$S = S_\delta = \{x_0 : |\Pi_- x_0| \leqslant \frac{\delta}{2K} \text{ e } |x(t, x_0)| \leqslant \delta,\ t \geqslant 0\}$$

where $x(t, x_0)$ is the solution of PVI

$$\begin{cases} \dot{x} = Ax + f(x) \\ x(0) = x_0. \end{cases} \qquad (P)$$

(4)

Moreover S is homeomorphic ($\Pi_-\big|S$ is the homeomorphism) to a ball of radius $\frac{\delta}{2K}$ of $\mathbb{C}^n_-$, S is tangent to $\mathbb{C}^n_-$ (that is, $\frac{|\Pi_+ x|}{|\Pi_- x|} \to 0$ as $x \to 0$, $x \in S$) and there are constants $M, \gamma > 0$ such that

$$|x(t, x_0)| \leqslant Me^{-\gamma t}|x_0|, x_0 \in S,\ t \geqslant 0.$$

A similar result holds for the unstable manifold.

The Picture.

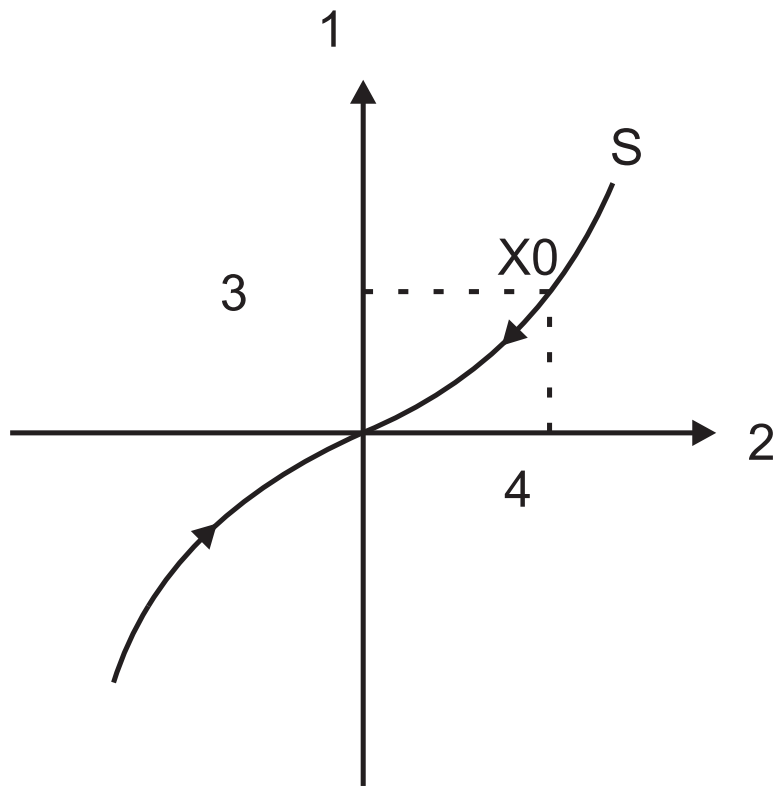

Figura: Decomposition of $\mathbb{C}^n$

The Proof.

Let L such that $|\Pi_{\pm}| \leqslant L$. It is possible to prove that if $x(t)$ is a solution of the perturbed system such that $x(t)$ is bounded for $t \geq 0$ then there exists $x_- \in \mathbb{C}^n_-$ such that the following holds:

$$x(t) = e^{A\,t}x_- + \int_0^t e^{A(t-s)}\Pi_- f(x(s))ds + \int_\infty^t e^{A(t-s)}\ \Pi_+ f(x(s))ds.$$

Therefore we will look for small solutions of the above integral equation.
For each $x_- \in \mathbb{C}^n_-$ let $\mathcal{T} = \mathcal{T}_{x_-}$ the operator defined by

$$(\mathcal{T}x)(t) \stackrel{\text{def}}{=} e^{At}x_- + \int_0^t e^{A(t-s)}\Pi_- f(x(s))ds + \int_\infty^t e^{A(t-s)}\ \Pi_+ f(x(s))ds.$$

The Banach Fixed Point Theorem for $\mathcal{T}$.

Let $\overline{B}_\delta$ the closed ball of radius $\delta > 0$ in the Banach space

$$\mathcal{BC}([0,\infty),\mathbb{C}^n) = \{\varphi : [0,\infty) \to \mathbb{C}^n : \varphi \text{ é contínua e limitada } \},$$

with the sup norm.
Using our assumptions and the exponential dichotomy one can prove that

$$(\mathcal{T}x)(t)| \leq K|x_-| + \frac{2}{\alpha}K\delta\eta(\delta).$$

If we take $\delta > 0$ in such a way that $\frac{2}{\alpha}K\eta(\delta) < \frac{1}{2}$ and $|x_-| \leqslant \frac{\delta}{2K}$ we can see that $\mathcal{T}$ let $\overline{B}_\delta$ invariant.
If $x, y \in \overline{B}_\delta$ then

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \leqslant \frac{2}{\alpha}K\eta(\delta)\sup_{t\geqslant 0}|x(t) - y(t)|$$

and so $\mathcal{T}$ is an uniform contraction with respect to $x_- \in \overline{B}_{\frac{\delta}{2K}}$ and so $\mathcal{T}$ has a unique fixed point in $\overline{B}_\delta$, $x^*(t) = x^*_{x_-}(t)$.

The Banach Fixed Point with Parameters.

Definition. We say that $T : M \times \Lambda \to M$ given by $T(x, y)$ is an uniform contraction with respect to $y \in \Lambda$ if there exists $\rho \in [0, 1)$ such that

$$d(T(x_1, y), T(x_2, y)) \leqslant \rho \; d(x_1, x_2), \; \forall \; x_1, x_2 \in M, \; \forall \, y \in \Lambda.$$

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The Stable Manifold.

$$x^*(t) = e^{A\,t}x_- + \int_0^t e^{A(t-s)}\Pi_- f(x^*(s))ds + \int_\infty^t e^{A(t-s)}\,\Pi_+ f(x^*(s))ds.$$

Consider its initial condition

$$x^*(0) = g(x_-) \stackrel{\text{def}}{=} x_- - \int_0^\infty e^{-As}\Pi_+ f(x^*_{x_-}(s))ds.$$

We have that $\Pi_- g(x_-) = x_-$, $g : \overline{B}_{\frac{\delta}{2K}} \subset \mathbb{C}^n_- \to \mathbb{C}^n$. Let $\mathcal{S} := g(\overline{B}_{\frac{\delta}{2K}})$.

One can show that $g : \overline{B}_{\frac{\delta}{2K}} \to g(\overline{B}_{\frac{\delta}{2K}})$ is a bijection and it defines the stable manifold $\mathcal{S}$.

An Integral Inequality.

Let $\alpha, \gamma > 0$, $K, L, M \geqslant 0$ and $u(t)$ a nonnegative solution continuous and bounded for $t \geqslant 0$ of

$$u(t) \leqslant Ke^{-\alpha t} + L\int_0^t e^{-\alpha(t-s)}u(s)ds + M\int_t^\infty e^{\gamma(t-s)}u(s)ds. \quad (5)$$

Suppose that $\beta \stackrel{def}{=} \frac{L}{\alpha} + \frac{M}{\gamma} < 1$. Then

$$u(t) \leqslant \frac{1}{1-\beta}Ke^{-(\alpha - \frac{L}{1-\beta})t}, \ t \geqslant 0.$$

If $u(t)$ is a nonnegative solution, continuous and bounded for $t \leqslant 0$ of

$$u(t) \leqslant Ke^{\alpha t} + L\int_t^0 e^{\alpha(t-s)}u(s)ds + M\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds. \quad (6)$$

then

$$u(t) \leqslant \frac{1}{1-\beta}Ke^{(\alpha - \frac{L}{1-\beta})t}, \ t \leqslant 0.$$

The Exponential Decay on $\mathcal{S}$.

From the above assumptions we obtain

$$|x(t,x_0)| \leqslant Ke^{-\alpha t}|x_-| + KL\eta(\delta)e^{-\alpha t}\int_0^t e^{\alpha s}|x(s,x_0)|ds +$$

$$+ KL\eta(\delta)e^{\alpha t}\int_t^\infty e^{-\alpha s}|x(s,x_0)|ds.$$

$$|x(t,x_0)| \leqslant \frac{K}{1-\frac{2}{\alpha}KL\eta(\delta)}e^{-\left[\alpha-\frac{KL\eta(\delta)}{1-\frac{2}{\alpha}KL\eta(\delta)}\right]t}|x_-|$$

para $t \geqslant 0$

Taking $\delta > 0$, sufficiently small we obtain the exponential decay on the stable manifold.

$$|x(t,x_0)| \leqslant 2Ke^{-\frac{\alpha}{2}t}|x_-| \leqslant 2KLe^{-\frac{\alpha}{2}t}|x_0|, \ t \geqslant 0 \qquad (+)$$

Some previous results.

The Hartman-Grobman Theorem.

C^0-Linearization for flows.

$$\begin{aligned} \dot{x} &= Ax + f(x) \quad (NL) \\ \dot{y} &= Ay \qquad\qquad\quad (H) \end{aligned}$$

L hyperbolic, $|f| << 1$, $Lip(f) << 1 \Rightarrow (NL)$ conjugate with (H). There exists a homeomorphism $H = I + \cdots$ such that if $\phi(t, x)$ is the solution of (NL), $\phi(0, x) = x$

$$H((t, H^{-1}x)) = e^{At}x$$

C^0-Linearization for maps.

$$\phi(1, x) = e^A x + \int_0^1 e^{A(1-s)} f((s, x))ds := Lx + \lambda(x) := \Lambda(x)$$

Find a homeomorphism h such that

$$h \circ \Lambda \circ h^{-1} = e^L$$

A Theorem of Hartman. Preliminaries.

Let E be a Banach space and let $L : E \to E$ be an isomorphism .
Definition. The isomorphism $L : E \to E$ is said to be hyperbolic if the space E has a decomposition in a direct sum $E = E^u \oplus E^s$ such that both E^u and E^s are invariant under L and the restriction $L_u = L|_{E^u}$ is an expansion while $L_s = L|_{E^s}$ is a contraction, that is, $\|L_u^{-1}\| < 1$ e $\|L_s\| < 1$.
We will assume that $a = \max\{\|L_u^{-1}, \|L_s\|\} < 1$ and that the norm in E is given by the norm $|x + y| := \max\{|x|, |y|\}$ para $x \in E^u$, $y \in E^s$. For $\mu > 0$ we define:

$$\mathcal{C}^0_*(E, E) = \{\text{bounded and uniformly continuous functions from } E \text{ to } E\}$$

$$\mathcal{L}_\mu(L) = \{\Lambda = L + \lambda : \lambda \in \mathcal{C}^0_*(E, E) \text{ is Lipschitzian, bounded by } \mu \text{ and has Lipschitz constant } \leq \mu\}$$

$$\mathcal{H} = \{h = I + g : g \in \mathcal{C}^0_*(E, E)\}, I \text{ is the Identity of E.}$$

If we consider the $\mathcal{C}^0$-uniform topology in $\mathcal{C}^0_*(E, E)$, this becomes a Banach space and $\mathcal{L}_\mu(L)$ and $\mathcal{H}$ become complete metric spaces.

Hartman's Theorem for mappings.

Theorem. If $\mu > 0$ is sufficiently small then for each $\Lambda \in \mathcal{L}_\mu(L)$, there is a unique $h = h_\Lambda \in \mathcal{H}$ such that $h\Lambda = Lh$. Furthermore this h_Λ is a homeomorphism depending continuously on $\Lambda \in \mathcal{L}_\mu(L)$.

Proof: To simplify things, we prove instead the strong looking assertion:
To each pair $\Lambda, \Lambda' \in \mathcal{L}_\mu(L)$ there corresponds a unique $h \in \mathcal{H}$ such that $h\Lambda = \Lambda' h$. h is a homeomorphism depending continuously on Λ, Λ'.
The equation $h\Lambda = \Lambda' h$ for $h \in \mathcal{H}$ is $\Lambda, \Lambda' \in \mathcal{L}_\mu(L)$ is

$$(I + g)(L + \lambda) = (L + \lambda')(I + g)$$

for $g \in \mathcal{C}^0_*(E, E)$. That is

$$g\Lambda - Lg = \lambda'(I + g) - \lambda. \tag{7}$$

(7) is equivalent to $g = [Lg + \lambda'(I + g) - \lambda]\,\Lambda^{-1}$, which expanded in the $E^u \times E^s$ coordinates, becomes

$$g_u = \left[L_u g_u + \lambda'_u(I+g) - \lambda_u\right]\Lambda^{-1} \quad (8)$$

$$g_s = \left[L_s g_s + \lambda'_s(I+g) - \lambda_s\right]\Lambda^{-1}. \quad (9)$$

On the other hand (7) is also equivalent to
$g = L^{-1}\left[g\Lambda + \lambda - \lambda'(I+g)\right]$ which expands to

$$g_u = L_u^{-1}\left[g_u\Lambda + \lambda_u - \lambda'_u(I+g)\right] \quad (10)$$

$$g_s = L_s^{-1}\left[g_s\Lambda + \lambda_s - \lambda'_s(I+g)\right]. \quad (11)$$

We will be looking to (9) e (10). For $\mu > 0$, (9) e (10) defines a contraction $K : \mathcal{C}^0_*(E,E) \to \mathcal{C}^0_*(E,E)$

$$g = (g_u, g_s) \mapsto \left(L_u^{-1}\left[g_u\Lambda + \lambda_u - \lambda'_u(I+g)\right], \left[L_s g_s + \lambda'_s(I+g) - \lambda_s\right]\Lambda^{-1}\right)$$

Since for all $g, g' \in \mathcal{C}^0_*(E,E)$

$$\begin{aligned}\left\|L_u^{-1}\right\|\left[|g_u\Lambda - g'_u\Lambda| + |\lambda_u - \lambda'_u| + |\lambda'_u(I+g) - \lambda'_u(I+g')|\right] \\ \leq a\left[|g_u - g'_u| + \mu|g - g'|\right] \leq (a+\mu)|g-g'|\end{aligned}$$

$$\begin{aligned}\|L_s\|\ |g_s\Lambda^{-1} - g'_s\Lambda^{-1}| + |(\lambda_s(I+g) - \lambda_s(I+g'))\Lambda^{-1}| \\ \leq a|g_s - g'_s| + \mu|g-g'| \leq (a+\mu)|g-g'|\end{aligned}$$

Therefore we can use the Banach Fixed Point depending on parameters to the unique solution of (9), (10), $g = g_{\Lambda,\Lambda'}$, depends continuously on $\Lambda, \Lambda' \in \mathcal{L}_\mu(L)$. Thus, $h_{\Lambda,\Lambda'} = I + g_{\Lambda,\Lambda'}$ also does. Clearly $h\Lambda = \Lambda' h$ is implied ty (9), (10).
h is a homeomorphism: the inverse of $h_{\Lambda,\Lambda'}$ is $h_{\Lambda',\Lambda}$. For

$$h_{\Lambda,\Lambda'} h_{\Lambda',\Lambda} \Lambda' = h_{\Lambda,\Lambda'} \Lambda h_{\Lambda',\Lambda} = \Lambda' h_{\Lambda,\Lambda'} h_{\Lambda',\Lambda}$$

so by the uniqueness of solutions of $h\Lambda' = \Lambda' h$, $h_{\Lambda,\Lambda'} h_{\Lambda',\Lambda} = I$.
This holds for all $\Lambda, \Lambda' \in \mathcal{L}_\mu(L)$ so it holds for Λ, Λ' reversed.
Hence $h_{\Lambda',\Lambda}$ is both a left and right inverse of $h_{\Lambda,\Lambda'}$.
This completes the proof.

Hartman's Theorem for flows.

Theorem If $\Lambda = L + \lambda$ is considered as a vector field on E, e^L is hyperbolc respecting $E = E^u \oplus E^s$ and $\nu > 0$ is small, then for each $\Lambda \in \mathcal{L}_\nu(L)$ there is a unique $H = H_\Lambda \in \mathcal{H}$ such that $H\phi_\Lambda(t,x) = \phi_L(t,Hx)$ for $x \in E$, $t \in \mathbb{R}$, where ϕ_L, ϕ_Λ are the L and Λ flows. H_Λ is a homeomorphism depending continuously on $\Lambda \in \mathcal{L}_\nu(L)$.

$$\dot{y} = Ly, \qquad \phi_L(t,y) = e^{Lt}y \tag{12}$$

$$\dot{x} = Lx + \lambda(x), \qquad \phi_\Lambda(t,x), \quad \Lambda = L + \lambda \tag{13}$$

Remark: From Variation of Constants Formula,

$$\phi_\Lambda(t,x) = e^{Lt}x + \int_0^t e^{L(t-s)}\lambda(\phi_\Lambda(s,x))ds.$$

$$\phi_\Lambda(1,x) = e^{L}x + \int_0^1 e^{L(1-s)}\lambda(\phi_\Lambda(s,x))ds.$$

Proof: Consider the time-one maps $\widetilde{\phi}_L = \phi_L(1,\cdot)$ and $\widetilde{\phi}_\Lambda = \phi_\Lambda(1,\cdot)$. The assignment $\Lambda \mapsto \widetilde{\phi}_\Lambda$ is easily seen to be continuous and so for some $\nu > 0$, $\Lambda \in \mathcal{L}_\nu(L) \Rightarrow \widetilde{\phi}_\Lambda \in \mathcal{L}_\mu(e^L)$ where μ is as in Hartman's Theorem for mappings. Accordingly there is a unique $h \in \mathcal{H}$ such that $\widetilde{\phi}_L h = h\widetilde{\phi}_\Lambda$.
We define

$$H(x) = \int_0^1 e^{-Ls} h(\phi_\Lambda(s,x))ds.$$

A calculation shows that

$$\phi_L(t, H(x)) = H(\phi_\Lambda(t,x)).$$

For $t = 1$ we obtain

$$\phi_L(1, H(x)) = H(\phi_\Lambda(1,x)),$$

or

$$\widetilde{\phi}_L H = H\widetilde{\phi}_\Lambda.$$

By uniqueness of solution we obtain $h = H$.

Localization.

In applications one is usually interested in the existence of a local equivalence homeomorphism, given a local flow or map. Let L be hyperbolic as above. For any closed disc D around the origin, let

$$\mathcal{L}_\mu^D(L) = \{L + \lambda : \lambda \in \mathcal{C}_*^0(D, E) \text{ é Lipschitz, limitada por } \mu \text{ and has Lipschitz constant } \leq \mu\}.$$

There are many continuous linear extensions operators $\mathcal{E} : \mathcal{L}_\mu^D(L) \to \mathcal{L}_{2\mu}(L)$. For instance $\Lambda = L + \lambda \in \mathcal{L}_\mu^D(L)$, let $\mathcal{E}\Lambda = L + \lambda$ where

$$\lambda' x = \begin{cases} \lambda x, & \text{se } x \in D \\ \lambda x', & \text{se } x \notin D \end{cases}$$

x' being a point of ∂D on the segment from 0 to x.
For flows the situation is similar. In both cases the uniqueness is totally lost. This is because the extension of Λ to E is not unique. it was quite useful to be dealing with Lipschitz maps instead of $\mathcal{C}^1$ because of the difficulty of extending Λ to be uniformly $\mathcal{C}^1$ near L on E, if E is a general Banach space.