Linearization in Infinite Dimensions. Some Previous and some New Results.

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Outline.

- Some previous and auxiliary results.

- An example for a resonant saddle (Hartman) and for a resonant contraction (Sternberg).

- An example of an analytic contraction that is not $C^1$-linearizable in an infinite dimensional Hilbert space. This contraction is the time one map of an analytic ODE that is asymptotic stable.

- A Hartman-Grobman Theorem with parameters.

- On the relationship between norms of bounded linear operators and their spectral radius.
Some previous results.

The Hartman-Grobman Theorem.

$C^0$-Linearization for flows.

\[ \dot{y} = Ly + f(y) \quad (NL) \]
\[ \dot{z} = Lz \quad (H) \]

$L$ hyperbolic, $|f| \ll 1$, $\text{Lip}(f) \ll 1 \Rightarrow (NL)$ conjugate with $(H)$. There exists a homeomorphism $H = I + \cdots$ such that if $\phi(t, x)$ is the solution of $(NL)$, $\phi(0, x) = x$

\[ H(\phi(t, H^{-1}x)) = e^{Lt}x \]

$C^0$-Linearization for maps.

\[ \phi(1, x) = e^Lx + \int_0^1 e^{L(1-s)}f(\phi(s, x))ds := Ax + g(x) := F(x) \]

Find local homeomorphism $h$ such that

\[ h \circ F \circ h^{-1} = A \]
On the Hartman-Grobman Theorem with Parameters.

$X$ = Banach space

$\Theta$ = metric space. $\Theta \ni \theta \mapsto L(\theta) \in \mathcal{L}(X)$ continuous

$f(\theta) : X \to X$.

**Problem:** Find $R(\theta)$ such that

$\theta \in \Theta$, $R(\theta) := I + g(\theta)$ is a homeomorphism

$R(\theta)(L(\theta) + f(\theta)) = L(\theta)R(\theta)$. 
A Hartman-Grobman Theorem with Parameters.

Suppose

\[
\begin{cases}
|L(\theta)^k P(\theta)| \leq Mr^k, & 0 < r < 1 \\
|L(\theta)^{-k}(I - P(\theta))| \leq Mr^k \\
L(\theta)P(\theta) = P(\theta)L(\theta) \\
|L(\theta)^{-1}| \leq N
\end{cases}
\]

$X=$Banach space, $\Theta =$metric space. $\Theta \ni \theta \mapsto L(\theta) \in \mathcal{L}(X)$ continuous $0 \notin \sigma(L(\theta))$

$Lip(f(\theta)) \leq \mu, \ |f(\theta)| \leq \mu$

If $\mu$ is sufficiently small then there exists a unique map $\Theta \ni \theta \mapsto g(\theta) \in BUC(X, X)$

$\theta \in \Theta, \ R(\theta) := I + g(\theta)$ is a homeomorphism

\[R(\theta)(L(\theta) + f(\theta)) = L(\theta)R(\theta).\]

Furthermore $\Theta \ni \theta \mapsto R(\theta)$ is continuous.
A Banach-Caccioppoli Theorem with parameters and variable distances.

\((\mathbb{M}, d)\) = complete metric space, \(\Theta\) = metric space.

\(T : (x, \theta) \in \mathbb{M} \times \Theta \mapsto T(x, \theta) \in \mathbb{M},\)

\(d_\theta\) a metric in \(\mathbb{M}\)

\[q \ d(x, y) \leq d_\theta(x, y) \leq Q \ d(x, y),\]

\[d_\theta(T(x, \theta), T(y, \theta)) \leq \rho \ d_\theta(x, y), \quad \rho \in (0, 1)\]

Let \(x = g(\theta)\) be the unique fixed point of \(T(\cdot, \theta)\).

Suppose that for each \(\theta_0 \in \Theta\) the function:
\(\Theta \ni \theta \mapsto T(g(\theta_0), \theta) \in \mathbb{M}\)

is continuous in \(\theta = \theta_0\).

Then the function \(g(\theta)\) is continuous in \(\Theta\).
To simplify things, as as before we prove instead the stronger-looking assertion: to each pair of continuous maps

$$\theta \mapsto f(\theta) \in \mathcal{F}_{\mu,0}(X), \; \theta \mapsto f'(\theta) \in \mathcal{F}_{\mu,0}(X)$$

there corresponds a unique map $$\theta \mapsto g(\theta) \in BUC(X,X)$$ such that

$$(I + g(\theta))(L(\theta) + f(\theta)) = (L(\theta) + f'(\theta))(I + g(\theta)) \quad (1)$$

This map $$g(\theta)$$ is unique for each $$\theta$$, and $$I + g(\theta)$$ is a homeomorphism. Furthermore it depends continuously on $$\theta$$ in the norm of $$BUC(X,X)$$.

Projecting equation (1) by $$P(\theta)$$ and $$I - P(\theta)$$ we see that it is equivalent to the following system, with $$g = g_s + g_u$$ and $$g_s = P(\theta)g$$:

$$g_s(\theta) = T_s(\theta, g) := [L_s(\theta)g_s(\theta) + f'_s(\theta)(I + g(\theta)) - f_s(\theta)](L(\theta) + f(\theta))^{-1}$$

$$g_u(\theta) = T_u(\theta, g) := L_u(\theta)^{-1}[g_u(\theta)(L(\theta) + f(\theta)) + f_u(\theta) - f'_u(\theta)(I + g(\theta))] \quad (2)$$
There exists a norm $| \cdot |_{\theta}$ on $\mathbb{X}$ and constant $K > 0$ such that
\[
\frac{1}{2} |x| \leq |x|_{\theta} \leq K |x|, \quad \forall \ x \in \mathbb{X}.
\]
This norm induces a norm on $L(\mathbb{X})$ that we indicate also by $| \cdot |_{\theta}$
and there exists $\rho \in [0, 1)$ such that
\[
|L_s(\theta)|_{\theta} \leq \rho, \quad |L_u(\theta)^{-1}|_{\theta} \leq \rho, \quad \forall \theta \in \Theta.
\]
And the operator $T(\theta) = (T_s(\theta), T_u(\theta))$ will be an uniform contraction in the norm $| \cdot |_{\theta}$.
Then we use the Banach-Cacciopoli Fixed Point Theorem with parameters to obtain the conjugacy $R(\theta) = I + g(\theta)$. 
The global Hartman-Grobman for flows with parameters.

Suppose also that for each \( \theta \in \Theta \) there is a continuous projection \( P(\theta) \in \mathcal{L}(X) \), that commutes with \( e^{tA(\theta)} \), for every \( t \in \mathbb{R} \) such that the uniform exponential dichotomy

\[
|e^{tA(\theta)}P(\theta)| \leq Me^{-\omega t}, \text{ for all } t \geq 0 \\
|e^{tA(\theta)}(I - P(\theta))| \leq Me^{\omega t}, \text{ for all } t \leq 0
\]

holds with \( M \) and \( \omega \) independent of \( \theta \).

If moreover \( \mu \) is sufficiently small and

\[
\Theta \ni \theta \mapsto f(\theta) \in \mathcal{F}_{\mu,0}(X)
\]

is continuous, then there exists a unique \( g(\theta) \in BUC(X,X) \) depending continuously on \( \theta \in \Theta \) such that \( I + g(\theta) \) is a homeomorphism that conjugates the flow \( \Phi_{(\theta,f)} \) generated by

\[
\dot{x} = A(\theta)x + f(\theta)(x)
\]

with the linear flow \( \Phi_\theta := \Phi_{(\theta,0)} \) generated by

\[
\dot{x} = A(\theta)x,
\]

or in other words,

\[
(I + g(\theta))\Phi_{(\theta,f)}(t,x) = \Phi_\theta(t)(I + g(\theta))x.
\]
**Proof:** Using the Hartman-Grobman theorem for maps we can prove that the time one map $\Phi_{(\theta,f)}(1, \cdot)$ conjugates with $\Phi_\theta(1)$ via a conjugation $R(\theta) = I + g(\theta)$, that is $R(\theta)\Phi_{(\theta,f)}(1, \cdot) = \Phi_\theta(1)R(\theta)(\cdot)$, for every $\theta \in \Theta$. Using a result of S. Sternberg one can show that if we take $H(\theta)(\cdot) := \int_0^1 \Phi_\theta(-s)R(\theta)\Phi_{(\theta,f)}(s, \cdot) \, ds$ then

$$\Phi_\theta(t)H(\theta)(\cdot) = H(\theta)\Phi_{(\theta,f)}(t, \cdot)$$

for every $t \in \mathbb{R}$ and every $\theta \in \Theta$. By taking $t = 1$ and using the uniqueness of the conjugation given by the Hartman Theorem for maps, one concludes that $R(\theta) = H(\theta)$ for every $\theta \in \Theta$. 
Linearization in the class $C^1$. Some historic facts

- N.H. Abel (1881).

- Hartman, $\alpha > \gamma > 0$:

\[
\begin{align*}
\dot{x} &= \alpha x \\
\dot{y} &= (\alpha - \gamma)y + xz \\
\dot{z} &= -\gamma z
\end{align*}
\]

that does not admit a linearizing local map or class $C^1$.

- (Sternberg),

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= -2y + x^2
\end{align*}
\]

where the nonresonance conditions are violated and there need not exist a linearizing map of class $C^2$. 
Mora & Sola-morales.

H. M. Rodrigues, J. G. Ruas Filho,
Some related results.

More References on Linearization

- G. R. Sell, Smooth linearization near a fixed point, Ame. J. Math. 107 (1985) 1035-1091
- B. Tan, $\sigma$-Hölder continuous linearization near hyperbolic fixed points in $\mathbb{R}^n$, J. Differential Equations, 162, pp. 251-259 (2000).
The first result in infinite dimensions.

**Theorem** (*Mora and Solà-Morales, 1987*)

\[ X = \text{Banach space}, \ T = A + \mathcal{X} : X \to X : A, A^{-1} \in \mathcal{L}(X), \]
\[ \mathcal{X} \in C^1(X, X) \ \mathcal{X}(0) = 0 \text{ and that there exists an } \eta > 0 \text{ such that:} \]
\[ \|A^{-1}\| \|A\|^{1+\eta} < 1 \quad (4) \]
\[ D\mathcal{X}(x) = o(\|x\|^{\eta}) \text{ as } x \to 0. \quad (5) \]

Then, there exists a local map \( \phi \in C^1(X, X) \) with \( \phi(0) = 0 \) and \( D\phi(x) = o(\|x\|^{\eta}) \) as \( x \to 0 \) such that, if \( R := I + \phi \) then we have

\[ RT = AR \]

(or \( RTR^{-1} = A \)) in a neighborhood of zero.
The Hartman Theorem for contractions.

$A = n \times n$ invertible matrix: $\|A\| < 1$ and $\mathcal{X} \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$

$\mathcal{X}(0) = 0, \quad D\mathcal{X}(0) = 0.$

Then, for the map $T\mathbf{x} = A\mathbf{x} + \mathcal{X}(\mathbf{x})$ there exists a map $R\mathbf{x} = \mathbf{x} + \phi(\mathbf{x})$ in a neighborhood of zero with $\phi \in C^{1,\beta}$ $(0 < \beta < 1)$, $\phi(0) = 0$ and $D\phi(0) = 0$ such that $RTR^{-1} = A$

A simple example.

\[ A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \]

with \( 0 < \alpha < \beta < 1 \).

\[ \| A \| = \beta, \quad \| A^{-1} \| = \frac{1}{\alpha} \]

\[ \| A^{-1} \| \| A \|^{1+\eta} < 1, \iff \beta^{1+\eta} < \alpha \quad (N.R.) \]

Remarks:

- Hartman’s Theorem does not require N.R., Mora-S.Morales require.
- The infinite dimension case where \( |\sigma(A)| = [\alpha, \beta] \) is answered by Mora-S.Morales and not by Hartman’s.
Smooth Banach Space.

A Banach space $\mathbb{X}$ is $C^{1,1}$ if there exists a $C^{1,1}$-function $\gamma : \mathbb{X} \to \mathbb{R}$:

$$
\begin{cases}
\gamma(x) = 1, & |x| \leq 1/2, \\
\gamma(x) = 0, & |x| \geq 1.
\end{cases}
$$

Example: Hilbert space.
Theorem (Solà-Morales & H.M.R.)

\[ X = C^{1,1} \text{ Banach space: } A, \ A^{-1} \in \mathcal{L}(X) \]

\[ 0 < \nu_n^- < \nu_n^+ < \nu_{n-1}^- < \nu_{n-1}^+ < \cdots \nu_1^- < \nu_1^+ < 1 \]

\[ |\sigma(A)| \subset \bigcup_{j=1}^n (\nu_j^-, \nu_j^+) := \bigcup_{i=1}^n I_j \]

\[ (I_1 \cdot I_j) \cap I_j = \emptyset, j = 1, \ldots, n \quad (\text{N.R.}) \]

Let \( X = X(x) \) be a \( C^{1,1} \)-function in a neighborhood of the origin, such that \( X = 0, \ \partial_x X = 0 \) at \( x = 0 \).

Then, for the map \( Tx = Ax + X(x) \) there exists a \( C^1 \)-map \( R \ x = x + \phi(x) \) satisfying \( \phi = 0, \ \partial_x \phi = 0 \) at \( x = 0 \), such that \( RTR^{-1} = A \) in a sufficiently small neighborhood of the origin.

Remarks about NR.

- In the finite dimensional case they are automatically satisfied and the above theorem coincides with Hartman’s Theorem.

- In the infinite dimensional case it essentially extends X. Mora-S. Morales.
Linearization Theorem for a Saddle. S-M & H.M.R.

$\mathbb{Z}=C^{1,1}$ Banach space, $L, L^{-1} \in L(\mathbb{Z})$.

$\exists$ real numbers $s^-, s^+, u^-, u^+$:

$$\left\{\begin{array}{l}
0 < s^- < s^+ < 1 < u^- < u^+ \\
|\sigma(L)| \subset (s^-, s^+) \cup (u^-, u^+):= I_1 \cup I_2 \\
(l_i,l_j) \cap l_k = \emptyset, \quad i,j,k = 1,2
\end{array}\right\} \quad \text{NR (6)}$$

Let $Z = Z(z)$ be a $C^{1,1}$-function in a neighborhood of the origin with values in $\mathbb{Z}$, such that $Z = 0$, $\partial_z Z = 0$, at $z = 0$.

Then for the map $T : z \mapsto z^1$, $z^1 = Lz + Z(z)$, there exists a $C^1$-map $R : z \mapsto u$, $u = z + \phi(z)$, satisfying $\phi = 0$, $\partial_z \phi = 0$, at $z = 0$, such that $RTR^{-1} : u \mapsto u^1$ has the form $u^1 = Lu$ in a sufficiently small neighborhood of the origin.

A remark on conjugacy.

If $T$ in nonlinear, $A$ is the linear part of $T$ and

$$RTR^{-1} = A \text{ then } TR^{-1} = R^{-1}A.$$ 

$S$ = invariant subspace for $A \Rightarrow M =: R^{-1}S$ is an invariant manifold for $T$.

$$TM = TR^{-1}S = R^{-1}AS = R^{-1}S = M$$
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that does not admit a linearizing local map or class $C^1$.

- (Sternberg),

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= -2y + x^2
\end{align*}
\]

where the nonresonance conditions are violated and there need not exist a linearizing map of class $C^2$. 
An example for a resonant saddle (Hartman).

The map

\[
\begin{align*}
x' &= ax \\
y' &= ac(y + \varepsilon xz) \\
z' &= cz
\end{align*}
\]

\[a > 1 > c > 0, \quad ac > 1\]

In this case there is no conjugacy of class $C^1$ between the nonlinear system and the linear system.

P. Hartman *On local homeomorphisms of euclidean spaces*, Boletin de la Sociedad Matematica Mexicana, Volumen 5 Numero 2, 220-241 (1960)
An example for a resonant contraction. S. Sternberg.

The map

\[
\begin{cases}
    x' = \lambda x \\
    y' = \lambda^2 y + \varepsilon x^2
\end{cases}
\]

$0 < \lambda < 1$

In this case there is no conjugation of class $C^2$ between the nonlinear system and the linear system.

Example of an analytic contraction that is not $C^1$-linearizable in $\ell_2$.

The ODE: $\dot{z} = Az + G(z)$

The time one map of the ODE:

$$z(1, z) = e^A z + \int_0^1 e^{A(1-s)} G(z(s, z)) ds := Lz + F(z)$$

The Map: $z' = Lz + F(z)$
Construction.

**Space** $\ell_2$

\[
J := \begin{pmatrix}
0 & 0 & 0 & \cdots & \cdots \\
1 & 0 & 0 & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix} \Rightarrow \sigma(J) = D := \{\xi \in \mathbb{C} : |\xi| \leq 1\}
\]

(7)

\[b \in (0, 1), \quad g(\xi) = b + b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n + \cdots\]

\[
M := \begin{pmatrix}
b & 0 & 0 & 0 & \cdots & \cdots \\
b_1 & b & 0 & 0 & \cdots & \cdots \\
b_2 & b_1 & b & 0 & \cdots & \cdots \\
b_3 & b_2 & b_1 & b & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix} = bl + \sum_{i=1}^{\infty} b_i J^i = g(J)
\]
Resonance.

\[ L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \tilde{y} \end{pmatrix} \in \ell_2 \]

\[ \sigma(M) = \sigma(g(J)) = g(\sigma(J)) \]

\[ [b^2, b] \subset |\sigma(g(J))| = |\sigma(M)| = [b^2, b + \delta], \quad 0 < \delta << 1 \]

\[ (|\sigma(M)|, |\sigma(M)|) \cap |\sigma(M)| = [b^2, (b + \delta)^2] \neq \emptyset \]

\[ \Rightarrow \quad \text{Resonance!} \]
Construction.

\[
L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ y \end{pmatrix} \in \ell_2
\]

\( b, r \in (0, 1), \; b + r < 1 \)

\[
M := \begin{pmatrix}
  b & 0 & 0 & 0 & \cdots & \cdots \\
  b_1 & b & 0 & 0 & \cdots & \cdots \\
  b_2 & b_1 & b & 0 & \cdots & \cdots \\
  b_3 & b_2 & b_1 & b & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots 
\end{pmatrix} = bI + \sum_{i=1}^{\infty} b_i J^i
\]

\[
J := \begin{pmatrix}
  0 & 0 & 0 & \cdots & \cdots \\
  1 & 0 & 0 & \cdots & \cdots \\
  0 & 1 & 0 & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots 
\end{pmatrix}
\]
Construction.

Consider the Möbius map:

\[ N(\xi) = N_{1-r}(\xi) := \frac{[(1 - r) - \xi]}{[1 - (1 - r)\xi]} \]

\[ N(0) = 1 - r, \quad N(D) = D \]

\[ g(\xi) := \frac{b - b^2}{2 - r} N_{1-r}(\xi) + \frac{b + b^2 - rb^2}{2 - r}. \]

\[ g(0) = b, \quad g(1) = b^2 \quad g(-1) = b + r \frac{b - b^2}{2 - r} \]
\[ g(\xi) = b + b_1 \xi + b_2 \xi^2 + \cdots + b_n \xi^n + \cdots \]

\[
\begin{align*}
b_1 &= -r(b - b^2) \\
b_2 &= -r(b - b^2)(1 - r) \\
&\vdots \\
b_n &= -r(b - b^2)(1 - r)^{n-1} \\
&\vdots
\end{align*}
\]

\[
M := \begin{pmatrix} b & 0 & 0 & 0 & \cdots & \cdots \\ b_1 & b & 0 & 0 & \cdots & \cdots \\ b_2 & b_1 & b & 0 & \cdots & \cdots \\ b_3 & b_2 & b_1 & b & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2
\]

**Conclusion:** \( L \) is a contraction and

\[
|\sigma(M)| = |\sigma(L)| = \sigma(g(J))| = [b^2, b + r \frac{b - b^2}{2 - r}] \subset [b^2, b + r]
\]
The Nonlinearity.

Remark: $\mathcal{R}(M - b^2 I) \not\subset \ell_2$

$$z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2$$

$$F(z) := \varepsilon \begin{pmatrix} 0 \\ x^2 \vec{\gamma} \end{pmatrix}$$

$$\vec{\gamma} := \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \end{pmatrix} \in \ell_2 - \mathcal{R}(M - b^2 I)$$

After some lemmas and some calculation, conclusion: The function $Lz + F(z)$ is not $C^1$-linearizable.
The Ordinary Differential Equation.

\[
L = \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix} = b \left[ I - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix} \right] := b[I - D],
\]

\[
D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix}
\]

\[
\|D\| = \frac{1}{b} \sum_{k=1}^{\infty} |b_k| = \frac{1}{b} (b - b^2) = 1 - b < 1.
\]

\[
A := \log L = (\log b) I + \log (I - D) = (\log b) I - (D + \frac{D^2}{2} + \cdots + \frac{D^n}{n} + \cdots).
\]

\[
\| \log L \| \leq - \log b^2
\]
The ODE: \[ \dot{z} = Az + G(z) \]

\[ A = \log L, \quad z := \begin{pmatrix} x \\ \overrightarrow{y} \end{pmatrix} \in \ell_2, \quad G(z) := \begin{pmatrix} 0 \\ x^2 \overrightarrow{\beta} \end{pmatrix} \]

The Map: \[ z' = Lz + F(z) \]

\[ F(z) := \varepsilon \begin{pmatrix} 0 \\ x^2 \overrightarrow{\gamma} \end{pmatrix} \]

\( \overrightarrow{\beta} \) depending of \( \overrightarrow{\gamma} \).

\[ z(1, z) = e^Az + \int_0^1 e^{A(1-s)} G(z(s, z))ds := Lz + F(z) \]
On the relationship between spectral radius and norms of bounded operators.

**Theorem** $X=$ Banach space with a norm $| \cdot |$.

Suppose $K \subset \mathcal{A} \subset \mathcal{L}(X)$

$\mathcal{A}$ commutative subalgebra and $K$ compact.

- Then given $\varepsilon > 0$ $\exists \| \cdot \| \simeq | \cdot |$ and $V_\varepsilon(K)$:

  \[\forall \ T \in V_\varepsilon(K) \ \Rightarrow \ \ r(T) \leq \| T \| < r(T) + \varepsilon\]

- $r(T)$ is continuous in $\mathcal{A}$.