Linearization in Infinite Dimensions. Some Previous and some New Results.

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Outline.

- Some previous and auxiliary results.
- An example for a resonant saddle (Hartman) and for a resonant contraction (Sternberg).
- An example of an analytic contraction that is not $\mathcal{C}^1$-linearizable in an infinite dimensional Hilbert space. This contraction is the time one map of an analytic ODE that is asymptotic stable.
- A Hartman-Grobman Theorem with parameters.
- On the relationship between norms of bounded linear operators and their spectral radius.

Some previous results.

The Hartman-Grobman Theorem.

C^0-Linearization for flows.

$$\begin{aligned} \dot{y} &= Ly + f(y) \quad (NL) \\ \dot{z} &= Lz \qquad\qquad\quad (H) \end{aligned}$$

L hyperbolic, $|f| << 1$, $Lip(f) << 1 \Rightarrow (NL)$ conjugate with (H). There exists a homeomorphism $H = I + \cdots$ such that if $\phi(t, x)$ is the solution of (NL), $\phi(0, x) = x$

$$H(\phi(t, H^{-1}x)) = e^{Lt}x$$

C^0-Linearization for maps.

$$\phi(1, x) = e^L x + \int_0^1 e^{L(1-s)} f(\phi(s, x))ds := Ax + g(x) := F(x)$$

Find local homeomorphism h such that

$$h \circ F \circ h^{-1} = A$$

On the Hartman-Grobman Theorem with Parameters.

$\mathbb{X}$=Banach space

Θ=metric space. $\Theta \ni \theta \mapsto L(\theta) \in \mathcal{L}(\mathbb{X})$ continuous

$f(\theta) : \mathbb{X} \to \mathbb{X}$.

Problem: Find $R(\theta)$ such that

$\theta \in \Theta$, $R(\theta) := I + g(\theta)$ is a homeomorphism

$$R(\theta)(L(\theta) + f(\theta)) = L(\theta)R(\theta).$$

A Hartman-Grobman Theorem with Parameters.

$$\text{Suppose} \quad \begin{cases} |L(\theta)^k P(\theta)| \leq M r^k, & 0 < r < 1 \\ |L(\theta)^{-k}(I - P(\theta))| \leq M r^k \\ L(\theta)P(\theta) = P(\theta)L(\theta) \\ |L(\theta)^{-1}| \leq N \end{cases}$$

$\mathbb{X}$=Banach space, Θ=metric space. $\Theta \ni \theta \mapsto L(\theta) \in \mathcal{L}(\mathbb{X})$ continuous $0 \notin \sigma(L(\theta))$

$Lip(f(\theta)) \leq \mu, \quad |f(\theta)| \leq \mu$

If μ is sufficiently small then there exists a unique map $\Theta \ni \theta \mapsto g(\theta) \in BUC(\mathbb{X}, \mathbb{X})$

$\theta \in \Theta$, $R(\theta) := I + g(\theta)$ is a homeomorphism

$$R(\theta)(L(\theta) + f(\theta)) = L(\theta)R(\theta).$$

Furthermore $\Theta \ni \theta \mapsto R(\theta)$ is continuous.

A Banach-Caccioppoli Theorem with parameters and variable distances.

$(\mathbb{M}, d)$=complete metric space, Θ=metric space.
$T : (x, \theta) \in \mathbb{M} \times \Theta \ \mapsto T(x, \theta) \in \mathbb{M}$,
d_θ a metric in $\mathbb{M}$

$$q \ d(x, y) \leq d_\theta(x, y) \leq Q \ d(x, y),$$

$$d_\theta(T(x, \theta), T(y, \theta)) \leq \rho \ d_\theta(x, y), \qquad \rho \in (0, 1)$$

Let $x = g(\theta)$ be the unique fixed point of $T(\cdot, \theta)$.
Suppose that for each $\theta_0 \in \Theta$ the function:
$\Theta \ni \theta \mapsto T(g(\theta_0), \theta) \in \mathbb{M}$ is continuous in $\theta = \theta_0$.

Then the function $g(\theta)$ in continuous in Θ.

To simplify things, as as before we prove instead the stronger-looking assertion: to each pair of continuous maps

$$\theta \mapsto f(\theta) \in \mathcal{F}_{\mu,0}(\mathbb{X}),\ \theta \mapsto f'(\theta) \in \mathcal{F}_{\mu,0}(\mathbb{X})$$

there corresponds a unique map $\theta \mapsto g(\theta) \in BUC(\mathbb{X}, \mathbb{X})$ such that

$$(I + g(\theta))(L(\theta) + f(\theta)) = (L(\theta) + f'(\theta))(I + g(\theta)) \qquad (1)$$

This map $g(\theta)$ is unique for each θ, and $I + g(\theta)$ is a homeomorphism. Furthermore it depends continuously on θ in the norm of $BUC(\mathbb{X}, \mathbb{X})$.
Projecting equation (1) by $P(\theta)$ and $I - P(\theta)$ we see that it is equivalent to the following system, with $g = g_s + g_u$ and $g_s = P(\theta)g$:

$$\begin{aligned} g_s(\theta) &= T_s(\theta, g) := [L_s(\theta)g_s(\theta) + f_s'(\theta)(I + g(\theta)) - f_s(\theta)](L(\theta) + f(\theta))^{-1} \\ g_u(\theta) &= T_u(\theta, g) := L_u(\theta)^{-1}[g_u(\theta)(L(\theta) + f(\theta)) + f_u(\theta) - f_u'(\theta)(I + g(\theta))] \end{aligned} \qquad (2)$$

There exists a norm $|\cdot|_\theta$ on $\mathbb{X}$ and constant $K > 0$ such that
$\frac{1}{2}|x| \leq |x|_\theta \leq K|x|, \ \forall \ x \in \mathbb{X}.$
This norm induces a norm on $\mathcal{L}(\mathbb{X})$ that we indicate also by $|\cdot|_\theta$ and there exists $\rho \in [0, 1)$ such that

$$|L_s(\theta)|_\theta \leq \rho, \quad |L_u(\theta)^{-1}|_\theta \leq \rho, \ \forall \theta \in \Theta.$$

And the operator $T(\theta) = (T_s(\theta),)T_u(\theta)$ will be an uniform contraction in the norm $|\cdot|_\theta$.
Then we use the Banach-Cacciopoli Fixed Point Theorem with parameters to obtain the conjugacy $R(\theta) = I + g(\theta)$.

The global Hartman-Grobman for flows with parameters.

Suppose also that for each $\theta \in \Theta$ there is a continuous projection $P(\theta) \in \mathcal{L}(\mathbb{X})$, that commutes with $e^{tA(\theta)}$, for every $t \in \mathbb{R}$ such that the uniform exponential dichotomy

$$\begin{aligned} |e^{tA(\theta)}P(\theta)| &\leq Me^{-\omega t}, \text{ for all } t \geq 0 \\ |e^{tA(\theta)}(I - P(\theta))| &\leq Me^{\omega t}, \text{ for all } t \leq 0 \end{aligned} \tag{3}$$

holds with M and ω independent of θ.
If moreover μ is sufficiently small and

$$\Theta \ni \theta \mapsto f(\theta) \in \mathcal{F}_{\mu,0}(\mathbb{X})$$

is continuous, then there exists a unique $g(\theta) \in BUC(\mathbb{X}, \mathbb{X})$ depending continuously on $\theta \in \Theta$ such that $I + g(\theta)$ is a homeomorphism that conjugates the flow $\Phi_{(\theta,f)}$ generated by

$$\dot{x} = A(\theta)x + f(\theta)(x)$$

with the linear flow $\Phi_\theta := \Phi_{(\theta,0)}$ generated by

$$\dot{x} = A(\theta)x,$$

or in other words, $(I + g(\theta))\Phi_{(\theta,f)}(t, x) = \Phi_\theta(t)(I + g(\theta))x$.

Proof: Using the Hartman-Grobman theorem for maps we can prove that the time one map $\Phi_{(\theta,f)}(1,\cdot)$ conjugates with $\Phi_\theta(1)$ via a conjugation $R(\theta) = I + g(\theta)$, that is $R(\theta)\Phi_{(\theta,f)}(1,\cdot) = \Phi_\theta(1)R(\theta)(\cdot)$, for every $\theta \in \Theta$. Using a result of S. Sternberg one can show that if we take $H(\theta)(\cdot) := \int_0^1 \Phi_\theta(-s)R(\theta)\Phi_{(\theta,f)}(s,\cdot)\, ds$ then

$$\Phi_\theta(t)H(\theta)(\cdot) = H(\theta)\Phi_{(\theta,f)}(t,\cdot)$$

for every $t \in \mathbb{R}$ and every $\theta \in \Theta$. By taking $t = 1$ and using the uniqueness of the conjugation given by the Hartman Theorem for maps, one concludes that $R(\theta) = H(\theta)$ for every $\theta \in \Theta$.

Linearization in the class C^1. Some historic facts

- N.H. Abel (1881).

- Hartman, $\alpha > \gamma > 0$:

$$\begin{cases} \dot{x} = \alpha x \\ \dot{y} = (\alpha - \gamma) y + xz \\ \dot{z} = -\gamma z \end{cases}$$

that does not admit a linearizing local map or class $\mathcal{C}^1$.

- (Sternberg),

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -2y + x^2 \end{cases}$$

where the nonresonance conditions are violated and there need not exist a linearizing map of class $\mathcal{C}^2$.

Mora &Sola-morales.

X. Mora, J. Solà-Morales, Existence and non-existence of finite dimentional globally attracting invariant manifolds in semilinear damped wave equations, Dynamics of Infinite-dimentional systems (Lisbon 1986), NATO Advanced Science Institute Series F: Computer and Systems Sciences, Vo. 37, Springer, Berlin, New York, (1987), pp. 187-210.

H. M. Rodrigues, J. G. Ruas Filho,
The Hartman-Grobman theorem for reversible systems on Banach Spaces, J. Nonlinear Sci. 7 (1997) pp. 271-280.

Some related results.

More References on Linearization

- G. R. Sell, Smooth linearization near a fixed point, Ame. J. Math. 107 (1985) 1035-1091
- V. Rayskin, α-Hölder linearization, J. Differential Equations, 147, pp. 271-284 (1998).
- B. Tan, σ-Hölder continuous linearization near hyperbolic fixed points in $\mathbb{R}^n$, J. Differential Equations, 162, pp. 251-259 (2000).
- B.Tan, Invariant manifolds, invariant foliations and linearization theorems in Banach space, PhD. Thesis, Georgia Institute of Technology, (1998).
- M. ElBialy, Local Contractions of Banach Spaces and Spectral Gap Conditions, J. Functional Analysis, 182, 108-150 (2001)

The first result in infinite dimensions.

Theorem *(Mora and Solà-Morales, 1987)*

X = Banach space, $T = A + \mathcal{X} : X \to X$: $A, A^{-1} \in \mathcal{L}(X)$, $\mathcal{X} \in \mathcal{C}^1(X, X)$ $\mathcal{X}(0) = 0$ and that there exists an $\eta > 0$ such that:

$$\|A^{-1}\|\|A\|^{1+\eta} < 1 \tag{4}$$

$$D\mathcal{X}(x) = o(\|x\|^{\eta}) \quad \text{as } x \to 0. \tag{5}$$

Then, there exists a local map $\phi \in \mathcal{C}^1(X, X)$ with $\phi(0) = 0$ and $D\phi(x) = o(\|x\|^{\eta})$ as $x \to 0$ such that, if $R := I + \phi$ then we have

$$RT = AR$$

(or $RTR^{-1} = A$) in a neighborhood of zero.

The Hartman Theorem for contractions.

$A = n \times n$ invertible matrix: $\|A\| < 1$ and $\mathcal{X} \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$

$$\mathcal{X}(0) = 0, \qquad D\mathcal{X}(0) = 0.$$

Then, for the map $Tx = Ax + \mathcal{X}(x)$ there exists a map $Rx = x + \phi(x)$ in a neighborhood of zero with $\phi \in \mathcal{C}^{1,\beta}$ $(0 < \beta < 1)$, $\phi(0) = 0$ and $D\phi(0) = 0$ such that $RTR^{-1} = A$

P. Hartman, On local homeomorphisms of euclidean spaces, *Boletin de la Sociedad Matematica Mexicana, Volumen* **5** *Numero* **2**, *220-241 (1960).*

A simple example.

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

with $0 < \alpha < \beta < 1$.

$$\|A\| = \beta, \ \|A^{-1}\| = \frac{1}{\alpha}$$

$$\|A^{-1}\|\|A\|^{1+\eta} < 1, \ \Leftrightarrow \ \beta^{1+\eta} < \alpha \ \textbf{(N.R.)}$$

Remarks:

- Hartman's Theorem does not require **N.R.**, Mora-S.Morales require.
- The infinite dimension case where $|\sigma(A)| = [\alpha, \beta]$ is answered by Mora-S.Morales and not by Hartman's.

Smooth Banach Space.

A Banach space $\mathbb{X}$ is $C^{1,1}$ if there exists a $C^{1,1}$-function $\gamma : \mathbb{X} \to \mathbb{R}$:

$$\begin{cases} \gamma(x) = 1, & |x| \leq 1/2, \\ \gamma(x) = 0, & |x| \geq 1. \end{cases}$$

Example: Hilbert space.

Theorem(Solà-Morales & H.M.R.)

$\mathbb{X}= C^{1,1}$ *Banach space:* $A,\ A^{-1} \in \mathcal{L}(\mathbb{X})$

$$0 < \nu_n^- < \nu_n^+ < \nu_{n-1}^- < \nu_{n-1}^+ < \cdots \nu_1^- < \nu_1^+ < 1$$

$$|\sigma(A)| \subset \bigcup_{j=1}^{n} (\nu_j^-, \nu_j^+) := \bigcup_{i=1}^{n} I_j$$

$$(I_1.I_j) \cap I_j = \emptyset, j = 1, \cdots, n \quad \textbf{(N.R.)}$$

Let $\mathcal{X} = \mathcal{X}(x)$ *be a* $C^{1,1}$*-function in a neighborhood of the origin, such that* $\mathcal{X} = 0,\ \partial_x \mathcal{X} = 0$ *at* $x = 0$*.*

Then, for the map $Tx = Ax + \mathcal{X}(x)$ *there exists a* C^1*-map* $R\ x = x + \phi(x)$ *satisfying* $\phi = 0,\ \partial_x \phi = 0$ *at* $x = 0$*, such that* $RTR^{-1} = A$ *in a sufficiently small neighborhood of the origin.*

Hildebrando M. Rodrigues and Joan Solà-Morales, *Linearization of class* C^1 *for contractions on Banach spaces*,J. Differential Equations 201, pp. 351-382 (2004)

Remarks about NR.

- In the finite dimensional case they are automaticaly satisfied and the above theorem coincides with Hartman's Theorem.
- In the infinite dimensional case it essentially extends X. Mora-S.Morales.

Linearization Theorem for a Saddle. S-M & H.M.R.

$\mathbb{Z}$=$\mathcal{C}^{1,1}$ Banach space, $L,\ L^{-1} \in L(\mathbb{Z})$.

$\exists$ real numbers $s^-,\ s^+,\ u^-,\ u^+$:

$$\left.\begin{array}{l} 0 < s^- < s^+ < 1 < u^- < u^+ \\ |\sigma(L)| \subset (s^-, s^+) \cup (u^-, u^+) := I_1 \cup I_2 \\ (I_i . I_j) \cap I_k = \emptyset, \quad i,j,k = 1,2 \end{array}\right\} \textbf{NR} \qquad (6)$$

Let $Z = Z(z)$ be a $C^{1,1}$-function in a neighborhood of the origin with values in $\mathbb{Z}$, such that $Z = 0,\ \partial_z Z = 0$, at $z = 0$.
Then for the map $T : z \mapsto z^1,\ z^1 = Lz + Z(z)$, there exists a C^1-map $R : z \mapsto u,\ u = z + \phi(z)$, satisfying $\phi = 0,\ \partial_z \phi = 0$, at $z = 0$, such that $RTR^{-1} : u \mapsto u^1$ has the form $u^1 = Lu$ in a sufficiently small neighborhood of the origin.

Hildebrando M. Rodrigues and Joan Solà-Morales, *Smooth Linearization for a Saddle on Banach spaces.* J. of Dynamics and Differential Equations, Vol16, No.3, pp.767-793 (2004).

A remark on conjugacy.

If T in nonlinear, A is the linear part of T and

$RTR^{-1} = A$ then $TR^{-1} = R^{-1}A$.

S=invariant subspace for $A \Rightarrow M =: R^{-1}S$ is an invariant manifold for T.

$$TM = TR^{-1}S = R^{-1}AS = R^{-1}S = M$$

Linearization in the class C^1. Some historic facts

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that does not admit a linearizing local map or class $\mathcal{C}^1$.

- (Sternberg),

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -2y + x^2 \end{cases}$$

where the nonresonance conditions are violated and there need not exist a linearizing map of class $\mathcal{C}^2$.

An example for a resonant saddle (Hartman).

$$\text{The map } \begin{cases} x' = ax \\ y' = ac(y + \varepsilon xz) \\ z' = cz \end{cases}$$

$$a > 1 > c > 0, \quad ac > 1$$

In this case there is no conjugacy of class C^1 between the nonlinear system and the linear system.

P. Hartman *On local homeomorphisms of euclidean spaces,* Boletin de la Sociedad Matematica Mexicana, Volumen **5** Numero **2**, 220-241 (1960)

An example for a resonant contraction. S. Sternberg.

$$\text{The map } \begin{cases} x' = \lambda x \\ y' = \lambda^2 y + \varepsilon x^2 \end{cases}$$

$0 < \lambda < 1$

In this case there is no conjugation of class C^2 between the nonlinear system and the linear system.

S. Kh. Aronson, G. R. Belitskii and E. V. Zhuzhoma, Introduction to the Qualitative Theory of Dynamical Systems on Surfaces, Translation of Mathematical Monographs, Vol. 153, American Mathematical Society.

Example of an analytic contraction that is not $\mathcal{C}^1$-linearizable in ℓ_2.

The ODE: $\dot{z} = Az + G(z)$

The time one map of the ODE:

$$z(1,z) = e^A z + \int_0^1 e^{A(1-s)}\, G(z(s,z))ds := Lz + F(z)$$

The Map: $z' = Lz + F(z)$

Construction.

Space ℓ_2

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \Rightarrow \sigma(J) = D := \{\xi \in \mathbb{C} : \ |\xi| \leq 1\} \tag{7}$$

$$b \in (0,1), \qquad g(\xi) = b + b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n + \cdots$$

$$M := \begin{pmatrix} b & 0 & 0 & 0\cdots & \cdots \\ b_1 & b & 0 & 0\cdots & \cdots \\ b_2 & b_1 & b & 0\cdots & \cdots \\ b_3 & b_2 & b_1 & b\cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = bI + \sum_{i=1}^{\infty} b_i J^i = g(J)$$

Resonance.

$$L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2$$

$$\sigma(M) = \sigma(g(J)) = g(\sigma(J))$$

$$[b^2, b] \subset |\sigma(g(J))| = |\sigma(M)| = [b^2, b + \delta], \quad 0 < \delta << 1$$

$$(|\sigma(M)|.|\sigma(M)|) \bigcap |\sigma(M)| = [b^2, (b + \delta)^2] \neq \emptyset$$

$$\Rightarrow \quad \textbf{Resonance!}$$

Construction.

$$L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2$$

$b, r \in (0,1), \ b + r < 1$

$$M := \begin{pmatrix} b & 0 & 0 & 0\cdots & \cdots \\ b_1 & b & 0 & 0\cdots & \cdots \\ b_2 & b_1 & b & 0\cdots & \cdots \\ b_3 & b_2 & b_1 & b\cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = bI + \sum_{i=1}^{\infty} b_i J^i$$

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{8}$$

Construction.

Consider the Möbius map:

$$N(\xi) = N_{1-r}(\xi) := \frac{[(1-r)-\xi]}{[1-(1-r)\xi]}$$

$$N(0) = 1-r, \quad N(D) = D$$

$$g(\xi) := \frac{b-b^2}{2-r} N_{1-r}(\xi) + \frac{b+b^2-rb^2}{2-r}.$$

$$g(0) = b, \quad g(1) = b^2 \quad g(-1) = b + r\frac{b-b^2}{2-r}$$

$$g(\xi) = b + b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n + \cdots$$

$$\begin{aligned} b_1 &= -r(b - b^2) \\ b_2 &= -r(b - b^2)(1 - r) \\ &\vdots \\ b_n &= -r(b - b^2)(1 - r)^{n-1} \\ &\vdots \end{aligned}$$

$$M := \begin{pmatrix} b & 0 & 0 & 0\cdots & \cdots \\ b_1 & b & 0 & 0\cdots & \cdots \\ b_2 & b_1 & b & 0\cdots & \cdots \\ b_3 & b_2 & b_1 & b\cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \; z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2$$

Conclusion: L is a contraction and

$$|\sigma(M)| = |\sigma(L)| = \sigma(g(J))| = [b^2, b + r\frac{b - b^2}{2 - r}] \subset [b^2, b + r]$$

The Nonlinearity.

Remark: $\mathcal{R}(M - b^2 I) \subsetneq \ell_2$

$$z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2$$

$$F(z) := \varepsilon \begin{pmatrix} 0 \\ x^2 \vec{\gamma} \end{pmatrix}$$

$$\vec{\gamma} := \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \end{pmatrix} \in \ell_2 - \mathcal{R}(M - b^2 I)$$

After some lemmas and some calculation, conclusion:
The function $Lz + F(z)$ is not $\mathcal{C}^1$-linearizable.

The Ordinary Differential Equation.

$$L = \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix} = b \left[I - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix} \right] := b[I - D],$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix}$$

$$\|D\| = \frac{1}{b} \sum_{k=1}^{\infty} |b_k| = \frac{1}{b}(b - b^2) = 1 - b < 1.$$

$$A := \log L = (\log b)I + \log(I - D) =$$

$$(\log b)I - (D + \frac{D^2}{2} + \cdots + \frac{D^n}{n} + \cdots).$$

$$\| \log L\| \leq - \log b^2$$

The ODE: $\dot{z} = Az + G(z)$

$$A = \log L, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \in \ell_2, \quad G(z) := \begin{pmatrix} 0 \\ x^2 \vec{\beta} \end{pmatrix}$$

The Map: $z' = Lz + F(z)$

$$F(z) := \varepsilon \begin{pmatrix} 0 \\ x^2 \vec{\gamma} \end{pmatrix}$$

$\vec{\beta}$ depending of $\vec{\gamma}$.

$$z(1, z) = e^A z + \int_0^1 e^{A(1-s)} \, G(z(s, z)) ds := Lz + F(z)$$

On the relationship between spectral radius and norms of bounded operators.

Theorem X=Banach space with a norm $|\cdot|$.
Suppose $K \subset \mathcal{A} \subset \mathcal{L}(X)$
$\mathcal{A}$ commutative subalgebra and K compact.

- Then given $\varepsilon > 0$ $\exists$ $\|\cdot\| \simeq |\cdot|$ and $V_\varepsilon(K)$:

$$\forall\ T \in V_\varepsilon(K) \quad \Rightarrow \quad r(T) \leq \|T\| < r(T) + \varepsilon$$

- $r(T)$ is continuous in $\mathcal{A}$.