Dynamics of a Class of ODEs more general than almost periodic.

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UFSC, fevereiro, 2012.

Introduction

This lecture is based in the paper:
P. E. Kloeden & H. M. Rodrigues:
Dynamics of a class of ODEs more general than almost periodic, Nonlinear Analysis 74 (2011)2695-2719.

Outline

- Some classical results of nonlinear oscillations. (See J. K. Hale, Ordinary Differenial Equations.)
  - A special solution of $\dot{x} = A(t)x + f(t)$.
  - The periodic case.
  - The almost periodic case.
  - Weakly Nonlinear Equations - Noncritical Case. The Saddle Point Property.
  - $\dot{x} = A(t)x + f(t, x, \varepsilon)$. The Periodic Case.
  - $\dot{x} = A(t)x + f(t, x, \varepsilon)$. The Almost Periodic Case.
- A More General Class of ODEs.
  - A special solution of $\dot{x} = A(t)x + f(t)$.
  - $\dot{x} = A(t)x + f(t, x, \varepsilon)$. The Saddle Point Property.
  - Properties of the stable and the unstable manifold. Sequential Sets.
  - Applications.

A special solution of $\dot{x} = A(t)x + f(t)$.

Suppose $A(t), n \times n$ and $f(t), n \times 1$, continuous functions for all $t \in \mathbb{R}$.

We assume that $A(t)$ generates an evolution operator $T(t,s)$ and that for each $s \in \mathbb{R}$ there exist projection $P(s)$, such that the following exponential dychotomy holds:

$$\begin{array}{rcll} |T(t,s)P(s)| & \leq & Ke^{-\alpha(t-s)}, & \text{for } t \geq s, \\ |T(t,s)(I-P(s))| & \leq & Ke^{\alpha(t-s)} & \text{for } t \leq s. \end{array}, \tag{1}$$

where K and α are positive numbers.

If $f(t)$ is bounded in $\mathbb{R}$ then the equation $\dot{x} = A(t)x + f(t)$ has a unique solution $\bar{x}(t) := (\mathcal{K}f)(t)$ which is bounded on $\mathbb{R}$ and it is given by:

$$(\mathcal{K}f)(t) := \int_{-\infty}^{t} T(t,s)P(s)f(s)\,ds - \int_{t}^{\infty} T(t,s)(I-P(s))f(s)\,ds. \tag{2}$$

The Periodic Case

Suppose that $A(t)$ and $f(t)$ are ω-periodic and the exponential dichotomy (1). Then the special solution $(\mathcal{K}f)(t)$ of $\dot{x} = A(t)x + f(t)$ is also ω-periodic.

The Almost-periodic Case

Suppose that $A(t)$ is periodic, $f(t)$ is almost-periodic and the exponential dichotomy (1) holds. Then the special solution $(\mathcal{K}f)(t)$ of $\dot{x} = A(t)x + f(t)$ is also almost-periodic.

Weakly Nonlinear Equations - Noncritical Case. The Saddle Point Property.

The Periodic Case

Example of Motivation.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + \varepsilon \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{3}$$

Theorem. Suppose that $A(\cdot)$ and $f(\cdot, x, \varepsilon)$ are ω-periodic, the exponential dichotomy (1) holds and ε is small. Suppose also that $f(t, x, \varepsilon)$ is small in the sense expecified below.
Then the equation $\dot{x} = A(t)x + f(t, x, \varepsilon)$
has a small bounded solution $x^\star(t, \varepsilon)$, $x^\star(t, 0) \equiv 0$. This solution $x^\star(\cdot, \varepsilon)$ is ω-periodic. Moreover $x^\star(t, \varepsilon)$ has local stable S and unstable U manifolds with sections given for each $t_0 \in \mathbb{R}$, respectively $S(t_0, \varepsilon)$ $U(t_0, \varepsilon)$. Both manifolds are ω- periodic in the t variable. The last must be understood in the following sense: for each $t_0 \in \mathbb{R}$ there exists a function $g(x_-, t_0, \varepsilon)$ that defines the stable manifold, for $x_- \in P(t_0)\mathbb{X}$, $g(x_-, t_0, \varepsilon) \in S(t_0, \varepsilon)$. Then $g(x_-, \cdot, \varepsilon)$ is ω- periodic. The same holds for the unstable manifold.

Smallness Condition.

$$\mathcal{L}ip(\eta, M) := \Big\{ f : \mathbb{R} \times \Omega(\rho_0, \sigma_0) \to \mathbb{X}, |f(t,0,\varepsilon)| \le M(\varepsilon), \\ |f(t,x,\varepsilon) - f(t,y,\varepsilon)| \le \eta(\rho,\sigma)|x-y| \Big\}$$

for all (t,x,ε), $(t,y,\varepsilon) \in \mathbb{R} \times \Omega(\rho,\sigma)$, $\rho \le \rho_0$ and $0 < \sigma \le \sigma_0$, where

$$\Omega(\rho,\sigma) := \{(x,\varepsilon) \in \mathbb{X} \times \mathbb{R}^m \ : \ |x| \le \rho, \ |\varepsilon| \le \sigma\},$$

and let $\eta(\rho,\sigma)$, $M(\sigma)$ for $\rho \ge 0$, $\sigma \ge 0$, be continuous functions which are nondecreasing in both variables with $\eta(0,0) = 0$ and $M(0) = 0$.

Idea of the proof:

We prove that the operator below has a unique fixed point in a neighborhood of the origin. We call it $x^\star(t,\varepsilon)$.

$$(\mathcal{T}_\varepsilon x(\cdot))(t) = \int_{-\infty}^{t} T(t,s)P(s)f(s,x(s),\varepsilon)\,ds - \int_{t}^{\infty} T(t,s)(I-P(s))f(s,x(s),\varepsilon)\,ds,$$

Then we prove that $x^\star(t,\varepsilon)$ is ω-periodic. Now we return to our original nonlinear equation and we can assume that $x^\star(t,\varepsilon)\equiv 0$ (by a change of variables $y = x - x^\star(t,\varepsilon)$).
Therefore any solution of the new nonlinear equation, which is bounded in $[t_0,\infty)$ will have the form:

$$\begin{aligned} x(t) &= T(t,t_0)x_- + \int_{t_0}^{t} T(t,s)P(s)f(s,x(s),\varepsilon)\,ds \qquad (4)\\ &+ \int_{\infty}^{0} T(t,t+s)(I-P(t+s))f(t+s,x(t+s),\varepsilon)\,ds \end{aligned}$$

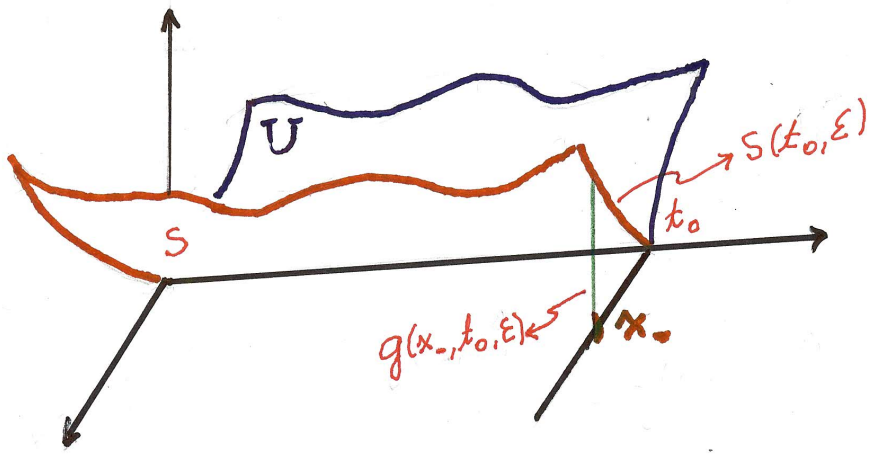
for $t \geq t_0$, where $x_- \in P(t_0)\mathbb{X}$.

We have again a problem of a fixed point of an operator. These solutions will generate the local stable manifold. The section of this manifold at $t = t_0$ will be given by the function $g(x_-, t_0, \varepsilon)$ given by:

$$g(x_-, t_0, \varepsilon) = x_- + \int_{\infty}^{0} T(t_0, t_0 + s)(I - P(t_0+s))f(t_0+s, x(t_0+s), \varepsilon)\, ds$$

One can prove that the function $t_0 \in \mathbb{R} \to g(x_-, t_0, \varepsilon)$ is ω-periodic.

The Periodic Case



The Almost-periodic Case

Example of Motivation.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + \varepsilon \begin{pmatrix} \cos t + \cos(\sqrt{2}t) \\ \sin t + \sin(\pi t) \end{pmatrix} \qquad (5)$$

Almost Periodic Funtions.

Definition 1: $f : \mathbb{R} \to \mathbb{C}$ is *almost periodic* it for every sequence $\{\alpha_n\}$ there is a subsequence $\{\alpha'_n\}$ such that $f(t + \alpha'_n)$ converges uniformly on $\mathbb{R}$.
Definition 2: A subset $S \subset \mathbb{R}$ is relatively dense if there is an $L > 0$ such that $[a, a + L] \cap S \neq \emptyset$ for any $a \in \mathbb{R}$.
Definition 3: If f is any continuous complex valued function on $\mathbb{R}$, the ε-translation set of f is defined as

$$T(f, \varepsilon) = \{\tau : \ |f(t + \tau) - f(t)| < \varepsilon, \forall \ t \in \mathbb{R}\}.$$

Definition 4: A continuous complex valued function f is *Bohr almost periodic* if for every $\varepsilon > 0$, $T(f, \varepsilon)$ is relatively dense.

Remark: **Definition 1** is equivalent to **Definition 4**.

$f : \mathbb{R} \to \mathbb{C}$ almost- periodic

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n it}$$

The module of f, $m[f]$, is the set of finite linear combinations with constants belonging to $\mathbb{Z}$ of the elementes of the sequence (λ_n).

Example: If $f(t) = \cos t + \cos(\sqrt{2}t)$,

$m[f] = \{n + m\sqrt{2} : n,\ m \in \mathbb{Z}\}$.

Theorem. Suppose that $A(\cdot)$ is periodic, $f(\cdot, x, \varepsilon)$ is almost-periodic, the exponential dichotomy (1) holds and ε is small. Suppose also that $f(t, x, \varepsilon)$ is small in the sense expecified previously .
Then the equation

$$\dot{x} = A(t)x + f(t, x, \varepsilon)$$

has a small bounded solution $x^\star(t, \varepsilon)$, $x^\star(t, 0) \equiv 0$. This solution $x^\star(\cdot, \varepsilon)$ is almost-periodic. Moreover $x^\star(t, \varepsilon)$ has local stable and unstable manifolds given for each $t_0 \in \mathbb{R}$, respectively $S = S(t_0, \varepsilon)$ $U = U(t_0, \varepsilon)$. The function $g(x_-, t_0, \varepsilon)$ that defines the stable manifold is almost periodic in the t_0 variable. The same holds for the unstable manifold.

A More General Class of ODEs.

Let $(\mathbb{X}, |\cdot|)$ be a Banach space. let $BUC(\mathbb{R}, \mathbb{X})$ denote the space of bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{X}$, which is a Banach space with the supremum norm $\|f\| := \sup_{x \in \mathbb{X}} |f(x)|$.

$$\mathcal{F} = \{f \in BUC(\mathbb{R}, \mathbb{X}) \text{ with precompact range } \mathcal{R}(f)\}.$$

is a closed subspace of $BUC(\mathbb{R}, \mathbb{X})$ and it is also a Banach space.

Roughness of Exponential Dichotomy.

We assume that A generates a $\mathcal{C}_0$-semigroup $U(t)$ and there exists a projection E and positive constants $M,\ \beta$, such that

$$\begin{aligned} |U(\sigma)E| &\leq Me^{-\beta\sigma},\ \sigma \geq 0 \\ |U(\sigma)(I-E)| &\leq Me^{\beta\sigma},\ \sigma \leq 0 \end{aligned} \tag{6}$$

We will consider the equation

$$\dot{u} = Au + B(t)u \tag{7}$$

Suppose the function $t \in \mathbb{R} \to B(t) \in \mathcal{L}(\mathbb{X})$ is continuous and that there exists $\varepsilon > 0$ such that $\sup_{t\in\mathbb{R}} |B(t)| < \varepsilon$.

Theorem

Under the above assumptions $A(t) := A + B(t)$ *defines evolution operators* $T(t,s)$ *and if* ε *is sufficiently small here exist* $\alpha \leq \beta$ *and* $K \geq M$ *and projections* $P(s) \in \mathcal{L}(\mathbb{X})$*, such that*

$$\begin{aligned} |T(t,s)P(s)| &\leq Ke^{-\alpha(t-s)}, \ t \geq s \\ |T(t,s)(I-P(s))| &\leq Ke^{\alpha(t-s)}, \ t \leq s \end{aligned} \tag{8}$$

Remark: If we add some hypotheses like:
For each compact set $D \subset \mathbb{X}$ the set $\cup_{t\in\mathbb{R}} B(t)D$ is a precompact subset of $\mathbb{X}$. For each compact $K \subset \mathbb{X}$ the function $(t,x) \in \mathbb{R} \times K \to B(t)x$ is uniformly continuous we can obtain more informations on compactness and uniform continuity of the action of $T(t,s)P(s), \ T(t,s)(I-P(s))$

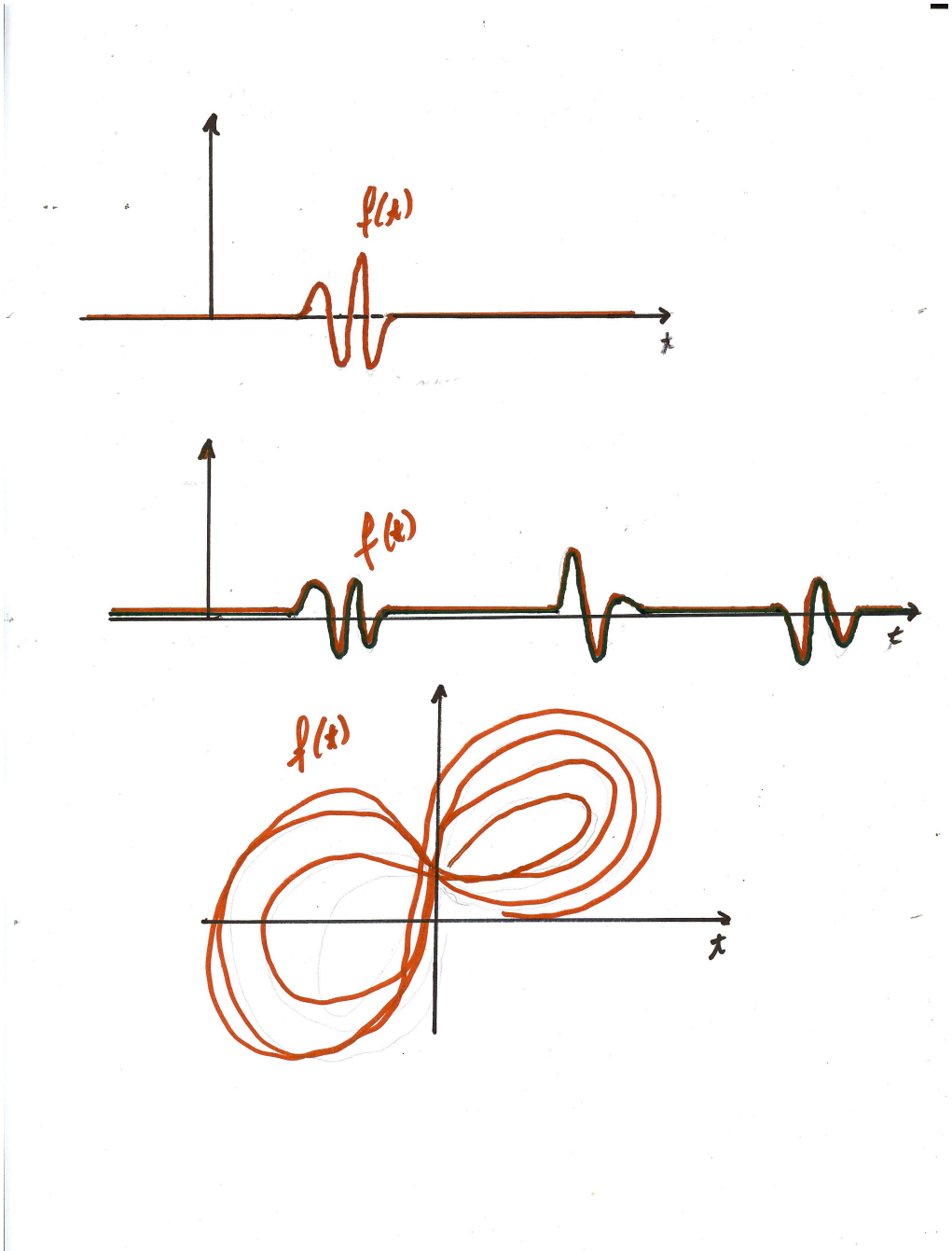
A special solution of $\dot{x} = A(t)x + f(t)$.

If the above assumptions are satisfied and $f \in \mathcal{F}$ then the special bounded solution on $\mathbb{R}$, $x^\star = \mathcal{K}f$ given by:

$$(\mathcal{K}f)(t) := \int_{-\infty}^{t} T(t,s)P(s)f(s)\,ds - \int_{t}^{\infty} T(t,s)(I-P(s))f(s)\,ds. \tag{9}$$

belongs to $\mathcal{F}$.

More General Functions.



More General Functions.

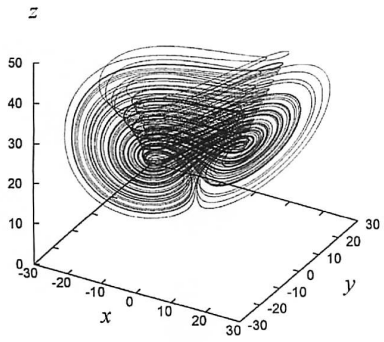


Fig. 1. The Chen attractor: $a = 35, b = 3, c = 28$.

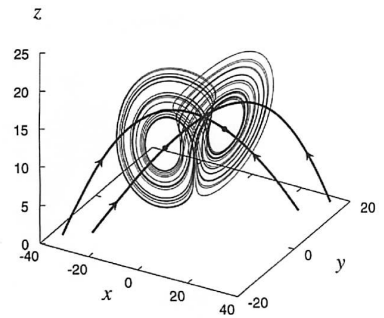


Fig. 2. Stable manifold of the Lorenz system.

2. The Chen System and Its New Form

The Chen system is described in [Chen & Ueta, 1999]

$$\begin{aligned} \frac{dx}{dt} &= a(y - x) \\ \frac{dy}{dt} &= (c - a)x - xz + cy \\ \frac{dz}{dt} &= xy - bz \end{aligned} \tag{1}$$

where a, b, c are real parameters. Typically, when $a = 35, b = 3$ and $c = 28$, the system is chaotic with a unique attractor as shown in Fig. 1.

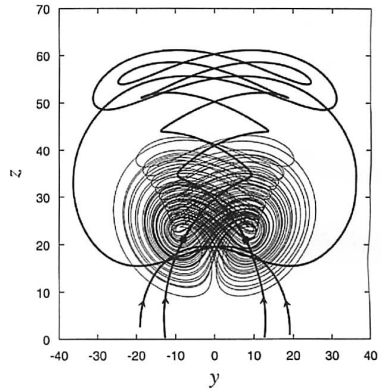


Fig. 3. Stable manifold of the Chen system.

Before proceeding to technical discussions, it is emphasized that all figures given in Appendix E are used for visualization and illustration, which are actually not needed for all the mathematical proofs given below.

Now, consider the following transformation of variables:

$$\begin{aligned} bX &= x \\ bY &= y - \frac{c+b}{a}x \\ bZ &= 2c + b - \frac{c+b}{a}b - z \\ \tau &= bt \end{aligned} \tag{2}$$

It follows from (1) and (2) that

$$\begin{aligned} b^2\frac{dX}{d\tau} &= a\left[\left(bY + \frac{c+b}{a}x\right) - x\right] \\ &= b[aY - (a - c - b)X] \end{aligned} \tag{3a}$$

$$\begin{aligned} b^2\frac{dY}{d\tau} &= -\frac{c+b}{a}\frac{dx}{dt} + (c - a)x \\ &\quad - x\left(2c + b - \frac{c+b}{a}b - bZ\right) \\ &\quad + c\left(bY + \frac{c+b}{a}x\right) \end{aligned}$$

The General Case

Theorem

Under the above assumptions, suppose that $f(t, x, \varepsilon)$ *is small in certain sense and for each compact* $D \subset \mathbb{X}$ *the set* $\{f(t, x, \varepsilon) : \forall t \in \mathbb{R}, \forall x \in D\}$ *is precompact.*
Then the equation

$$\dot{x} = A(t)x + f(t, x, \varepsilon)$$

has a small bounded solution $x^\star(t, \varepsilon)$, $x^\star(t, 0) \equiv 0$. *This solution* $x^\star(\cdot, \varepsilon)$ *belongs to* $\mathcal{F}$. *Moreover it has local stable and unstable manifolds given for each* $t_0 \in \mathbb{R}$, *respectively* $S = S(t_0, \varepsilon)$ $U = U(t_0, \varepsilon)$. *The function* $g(x_-, t_0, \varepsilon)$ *that defines the stable manifold is such that* $g(x_-, \cdot, \varepsilon) \in \mathcal{F}$. *The same holds for the unstable manifold.*

The Almost-periodic Case again.

Suppose that $A(\cdot)$ is periodic, $f(\cdot, x, \varepsilon)$ is almost-periodic, the exponential dichotomy (1) holds and ε is small. Suppose also that $f(t, x, \varepsilon)$ is small in certain sense .
Then the equation

$$\dot{x} = A(t)x + f(t, x, \varepsilon)$$

has a small bounded solution $x^\star(t, \varepsilon)$, $x^\star(t, 0) \equiv 0$. This solution $x^\star(\cdot, \varepsilon)$ is almost-periodic. Moreover it has local stable and unstable manifolds given for each $t_0 \in \mathbb{R}$, respectively $S = S(t_0, \varepsilon)$ $U = U(t_0, \varepsilon)$. The function $g(x_-, t_0, \varepsilon)$ that defines the stable manifold is almost periodic in the t_0 variable. The same holds for the unstable manifold. Moreover the module of $g(x_-, \cdot, \varepsilon)$ is contained in the module generated by the union of the module of $A(t)$ and the module of f.

Sequential Sets.

If $f \in \mathcal{F}$ we define the sequential set associated to f as

$$\mathcal{S}(f) := \{(s_n)_{n\in\mathbb{N}} : \ |f(t+s_n) - f(t)| \to 0, \quad \text{uniformly with respect to } t \in \mathbb{R} \text{ as } n \to \infty\}.$$

Example (Bochner):

$$\mathcal{S}(f) = \{(s_n)_{n\in\mathbb{N}} : \ e^{\imath\lambda_j s_n} \to 1, \text{ as } n \to \infty, \text{ for each } j \in \mathbb{N}\}$$

for a complex valued almost periodic function
$f(t) \sim \sum_{j=-\infty}^{\infty} a_j e^{\imath\lambda_j t}$

A special solution of $\dot{x} = A(t)x + f(t)$, $A(t) = A + B(t)$.

We define
$\mathcal{S}(B) := \{(s_n)_{n\in\mathbb{N}} : sup_{t\in\mathbb{R}}|B(t+s_n) - B(t)| \to 0, \text{ as } n \to \infty\}$.

Theorem

If the above assumptions are satisfied and $f \in \mathcal{F}$ then the special bounded solution on $\mathbb{R}$, $x^\star = \mathcal{K}f$ given by:

$$(\mathcal{K}f)(t) := \int_{-\infty}^{t} T(t,s)P(s)f(s)\,ds - \int_{t}^{\infty} T(t,s)(I-P(s))f(s)\,ds. \tag{10}$$

belongs to $\mathcal{F}$ and $\mathcal{S}(f) \cap \mathcal{S}(B) \subset \mathcal{S}(\mathcal{K}f)$.

The General Case again.

$$\mathcal{S}(f) := \Big\{ (s_n)_{n\in\mathbb{N}} : \ |f(t+s_n,x,\varepsilon) - f(t,x,\varepsilon)| \to 0, \ \text{as}, \ n\to\infty$$
$$\text{uniformly on} \ \ t\in\mathbb{R}, \ x\in K \ \text{for any compact set} \ K\subset\mathbb{X} \Big\}.$$

Theorem

Under the above assumptions, suppose that $f(t,x,\varepsilon)$ is small in certain sense and for each compact $D\subset\mathbb{X}$ the set $\{f(t,x,\varepsilon) : \forall t\in\mathbb{R}, \forall x\in D\}$ is precompact.
Then the equation

$$\dot{x} = A(t)x + f(t,x,\varepsilon)$$

has a small bounded solution $x^\star(t,\varepsilon)$, $x^\star(t,0)\equiv 0$. This solution $x^\star(\cdot,\varepsilon)$ belongs to $\mathcal{F}$ and $\mathcal{S}(f)\cap\mathcal{S}(B)\subset\mathcal{S}(x^\star(\cdot,\varepsilon))$
The function $g(x_-,t_0,\varepsilon)$ that defines the stable manifold is such that $g(x_-,\cdot,\varepsilon)\in\mathcal{F}$. $\mathcal{S}(f)\cap\mathcal{S}(B)\subset\mathcal{S}(g)$. Similar result holds for the unstable manifold.

Applications.

Example: Coupled Heat Equations Following Henry, page 119, we consider Consider first the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u, \qquad 0 < x < \pi, \quad u = 0 \text{ at } x = 0, \pi \tag{11}$$

Take the space $\mathbb{X} = L^2(0, \pi)$, $-A\phi(x) = \phi''(x)$ for smooth ϕ which vanish at $x = 0, \pi$ and let -A be extended to a positive defined operator in $\mathbb{X}$. Then $\mathcal{D}(A) = H_0^1(0, \pi) \cap H^2(0, \pi)$.
The spectrum of A is $\sigma(A) = \{-n^2 + \lambda;\ n = 1, 2, 3, \cdots\}$.

$$\frac{\partial v}{\partial t} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \\ & \frac{\partial^2}{\partial x^2} + \lambda \end{pmatrix} v + \varepsilon B(t)v + F(v) + f(t), \tag{12}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function such that $F(0) = 0,\ F'(0) = 0$ and $f : \mathbb{R} \to \mathbb{R}^2$ belongs to our class. Taking $\varepsilon > 0$, sufficiently small and appropriate conditions on $B(t)$ then this example will satisfy on the conditions of the previous chapter.

The Beam Equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^4 u}{\partial x^4} + \Gamma \dfrac{\partial^2 u}{\partial x^2} + \alpha \dfrac{\partial u}{\partial t} = \kappa \left(\displaystyle\int_0^1 \left| \dfrac{\partial u}{\partial x} \right|^2 \, dx \right) \left(\dfrac{\partial^2 u}{\partial x^2} \right) + \varepsilon f(t), \\ 0 < x < 1 \\ \\ u(0) = \dfrac{\partial^2 u}{\partial x^2}(0) = u(1) = \dfrac{\partial^2 u}{\partial x^2}(1) = 0 \end{cases} \tag{13}$$

Smallness Condition.

$$\mathcal{L}ip(\eta, M) := \Big\{ f : \mathbb{R} \times \Omega(\rho_0, \sigma_0) \to \mathbb{X}, |f(t,0,\varepsilon)| \le M(\varepsilon), \\ |f(t,x,\varepsilon) - f(t,y,\varepsilon)| \le \eta(\rho,\sigma)|x-y| \Big\}$$

for all (t,x,ε), $(t,y,\varepsilon) \in \mathbb{R} \times \Omega(\rho,\sigma)$, $\rho \le \rho_0$ and $0 < \sigma \le \sigma_0$, where

$$\Omega(\rho,\sigma) := \{(x,\varepsilon) \in \mathbb{X} \times \mathbb{R}^m \ : \ |x| \le \rho, \ |\varepsilon| \le \sigma\},$$

and let $\eta(\rho,\sigma)$, $M(\sigma)$ for $\rho \ge 0$, $\sigma \ge 0$, be continuous functions which are nondecreasing in both variables with $\eta(0,0) = 0$ and $M(0) = 0$.

Almost Periodic Funtions.

Definition 1: $f : \mathbb{R} \to \mathbb{C}$ is *almost periodic* it for every sequence $\{\alpha_n\}$ there is a subsequence $\{\alpha'_n\}$ such that $f(t + \alpha'_n)$ converges uniformly on $\mathbb{R}$.
Definition 2: A subset $S \subset \mathbb{R}$ is relatively dense if there is an $L > 0$ such that $[a, a + L] \cap S \neq \emptyset$ for any $a \in \mathbb{R}$.
Definition 3: If f is any continuous complex valued function on $\mathbb{R}$, the ε-translation set of f is defined as

$$T(f, \varepsilon) = \{\tau : \ |f(t + \tau) - f(t)| < \varepsilon, \forall \ t \in \mathbb{R}\}.$$

Definition 4: A continuous complex valued function f is *Bohr almost periodic* if for every $\varepsilon > 0$, $T(f, \varepsilon)$ is relatively dense.

Remark: **Definition 1** is equivalent to **Definition 4**.

The Almost-automorphic functions.

Let $f : \mathbb{R} \to \mathbb{C}$ continuous.

Definition: f is *almost-automorphic* if given a sequence of real numbers (α'_n) we can extract a subsequence (α_n) such that

$$\lim_{m\to\infty} \lim_{n\to\infty} f(t + \alpha_n - \alpha_m) = f(t)$$

for each $t \in \mathbb{R}$.

Pullback Attractor.

We now consider more general nonautonomous ODEs of the form

$$\frac{dx}{dt} = f(t,x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{14}$$

where the vector field f belongs to the class $\mathcal{F}_{ODE}$ defined by

$$\mathcal{F}_{ODE} := \Big\{ f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \text{ ; is uniformly continuous in } t \in \mathbb{R}$$
$$\text{uniformly in } x \text{ in compact subsets } C, \text{ with precompact}$$
$$\text{range } \mathcal{R}_C(f) := \cup\{f(t,x),\ t \in \mathbb{R},\ \forall x \in C\}, \text{ compact } C \Big\}.$$

Functions in the class $\mathcal{F}$ belong trivially to the class $\mathcal{F}_{ODE}$.

Elementary (autonomous) examples show that existence and uniqueness of solutions is not guaranteed for the class $\mathcal{F}_{ODE}$ without additional assumptions. For simplicity, we will assume that the existence and uniqueness of solutions does hold and that the solution mapping $x(t, t_0, x_0)$ generates a *process* [**?**, **?**], i.e. a continuous mapping $(t, t_0, x_0) \mapsto x(t, t_0, x_0) \in \mathbb{R}^d$ for $t \geq t_0$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, with the initial value and evolution properties

- (i) $x(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$,
- (ii) $x(t_2, t_0, x_0) = x(t_2, t_1, x(t_1, t_0, x_0))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}$ and $x_0 \in \mathbb{R}^d$.

A *pullback attractor* of a process consists of a family of nonempty compact sets $\mathcal{A} := \{A(t), t \in \mathbb{R}\}$ of $\mathbb{R}^d$ which is *invariant* in the sense that

$$x(t, t_0, A(t_0)) = A(t) \quad \text{for all} \quad t \geq t_0, t_0 \in \mathbb{R} \tag{15}$$

and *pullback attracting* in the sense that

$$\lim_{t_0 \to -\infty} H^*_{\mathbb{R}^d}(x(t, t_0, D), A(t)) = 0 \tag{16}$$

for all bounded subsets D of $\mathbb{R}^d$, where $x(t, t_0, D) := \cup\{x(t, t_0, x_0) : x_0 \in D\}$. Here $H^*_{\mathbb{R}^d}$ is the Hausdorff semi distance, i.e.

$$H^*_{\mathbb{R}^d}(A, B) := \max_{a \in A} \operatorname{dist}(a, B) = \max_{a \in A} \min_{b \in B} |a - b|$$

for nonempty compact subsets A and B of $\mathbb{R}^d$. The corresponding Hausdorff distance is defined by

$$H_{\mathbb{R}^d}(A, B) = \max\left\{H^*_{\mathbb{R}^d}(A, B), H^*_{\mathbb{R}^d}(B, A)\right\}$$

and is a metric on the space $\mathcal{H}(\mathbb{R}^d)$ of nonempty compact sets of $\mathbb{R}^d$.

The weakly nonlinear ODE again

We return to the weakly nonlinear ODE , i.e.,

$$\frac{dx}{dt} = A(t)x + f(t, x,), \tag{17}$$

We claim that this equation has a (local) pullback attractor given by the unique bounded solution in $\mathbb{R}$, which satisfies the integral equation:

$$\bar{x}(t) = \int_{-\infty}^{t} T_t^s f(s, \bar{x}(s), \varepsilon)\ ds,$$

i.e., the components sets are $A(t) = \{\bar{x}(t)\}$.
The solution of ODE with sufficiently small initial condition x_0 at t_0 is given by

$$x(t, t_0, x_0) = T_t^{t_0} x_0 + \int_{t_0}^{t} T_t^s f(s, x(s, t_0, x_0))\ ds,$$

Subtracting, we have

$$x(t, t_0, x_0) - \bar{x}(t)$$

$$= \quad T_t^{t_0} x_0 + \int_{t_0}^{t} T_t^s f(s, x(s, t_0, x_0))\ ds - \int_{-\infty}^{t} T_t^s f(s, \bar{x}(s))\ ds$$

$$= \quad T_t^{t_0} x_0 + \int_{t_0}^{t} T_t^s \left[f(s, x(s, t_0, x_0)) - f(s, \bar{x}(s)) \right],\ ds - \int_{-\infty}^{t_0} T_t^s f(s, \bar{x}(s))\, ds.$$

Hence

$$\begin{aligned} |x(t, t_0, x_0) - \bar{x}(t)| \quad \leq \quad & K e^{-\alpha(t-t_0)} |x_0| + \int_{t_0}^{t} K e^{-\alpha(t-s)} \eta |x(s, t_0, x_0) - \bar{x}(s)|\, ds \\ & + \int_{-\infty}^{t_0} K e^{-\alpha(t-s)} |f(s, \bar{x}(s))|\, ds \end{aligned}$$

and therefore

$$e^{\alpha t}|x(t,t_0,x_0)-\bar{x}(t)| \leq Ke^{\alpha t_0}|x_0| + \int_{t_0}^{t} Ke^{\alpha s}\eta|x(s,t_0,x_0)-\bar{x}(s)|\,ds + \int_{-\infty}^{t_0} Ke^{\alpha s}|f(s,\bar{x}(s))|\,ds.$$

Then, using Gronwall's inequality, we obtain:

$$e^{\alpha t}|x(t,t_0,x_0)-\bar{x}(t)| \leq \left[Ke^{\alpha t_0}|x_0| + \int_{-\infty}^{t_0} Ke^{\alpha s}|f(s,\bar{x}(s))|\,ds\right] e^{K\eta,(t-t_0)}$$

from which it follows that

$$|x(t,t_0,x_0)-\bar{x}(t)| \leq e^{-(\alpha-K\eta)t}\left[Ke^{(\alpha-K\eta)t_0}|x_0| + e^{-K\eta t_0}\int_{-\infty}^{t_0} Ke^{\alpha s}|f(s,\bar{x}(s))|\,d\right. \tag{18}$$

If we assume that $\eta > 0$ is sufficiently small so that $\alpha > K\eta$, then it is not difficult to prove that the second member of the last inequality tends to zero as $t_0 \to -\infty$, for the term involving the integral we use L'Hospital's Rule. This is pullback attraction, so the pullback attractor consists of just the special solution $\bar{x}(t)$. It inherits properties of the equation such as periodicity, almost periodicity or the more general as indicated in Theorem **??**.
In fact, from inequality (18) we also obtain convergence in the forwards sense, i.e.,

$$\lim_{t\to\infty} |x(t, t_0, x_0) - \bar{x}(t)| = 0,$$

so the pullback attractor is also a forward attractor in this example. The advantage of pullback attraction is that it allows us to obtain an explicit expression for the solution $\bar{x}(t)$.