Microlocal Representation of Solutions of Analytic Pseudodifferential Equations of Principal Type

François Treves

December 2022

The contents of the lecture is based on material in the book
F. Treves, Analytic Partial Differential Equations, Grund. Math. Wiss., Springer 1922.
The lecture outlines a proof of Theorem **26.3.17** in the book (Th. **3.1** below) that is slightly simpler than the proof in the book and corrects some misprints. The detailed proofs of the other claims in the lecture can all be found in the book.
For a PDF copy of this talk please write to me at *treves.jeanfrancois@Gmail.com*.
The next two pages are intended for students.

Basic Notation and Terminology

$x = (x_1 ..., x_N)$: the variable in $\mathbb{R}^N$ $(N \geq 2)$; $\xi = (\xi_1, ..., \xi_N)$: the variable in $\mathbb{R}^N \setminus \{0\}$;
$z = (z_1, ..., z_N)$: the variable in $\mathbb{C}^N$; $\zeta = (\zeta_1, ..., \zeta_N)$: the variable in $\mathbb{R}^N \setminus \{0\}$;
$D_{x_k} = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_k}$, $D_x = (D_{x_1}, ..., D_{x_N})$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_N}^{\alpha_N}$,
$\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$, $\alpha! = \alpha_1! \cdots \alpha_N!$;
likewise with ξ, $z = x + iy$ or $\zeta = \xi + i\eta$ in lieu of x;
$\frac{\partial}{\partial z_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right)$, $\frac{\partial}{\partial \zeta_k} = \frac{1}{2} \left(\frac{\partial}{\partial \xi_k} - i \frac{\partial}{\partial \eta_k} \right)$;
$\mathbb{S}^{N-1} = \left\{ \xi \in \mathbb{R}^N ; \ |\xi| = 1 \right\}$;
Ω: a domain in $\mathbb{R}^N$; Γ an open cone in $\mathbb{R}^N \setminus \{0\}$ with vertex at 0;
$\mathcal{C}_c(\Omega)$ [resp., $\mathcal{C}_c^\infty(\Omega)$]: the ring of complex-valued, compactly supported, continuous (resp., $\mathcal{C}^\infty$) functions in Ω;
$\mathcal{C}^\omega(\Omega)$: the ring of real-analytic, complex-valued functions in Ω;
$\mathcal{D}'(\Omega)$: the space of distributions in Ω;
$\mathcal{E}'(\Omega)$: the space of compactly supported distributions in Ω.

This presentation cannot avoid the terminology of **germs**.
Suppose that to each open set $\mathcal{U}$ in a topological space $\mathbb{X}$ we attach an object $\mathbf{E}(\mathcal{U})$ belonging to a certain class (a *category*) $\mathbf{E}$, say, of functions defined in $\mathcal{U}$ - or of subsets of $\mathcal{U}$, or of algebraic structures (vector spaces, rings,...) based on $\mathcal{U}$ - or of "whatever". At this stage we only need a notion of *restriction* from every open $\mathcal{U}$ to every open $\mathcal{V} \subset \mathcal{U}$, i.e., a map $\rho^{\mathcal{U}}_{\mathcal{V}} : \mathbf{E}(\mathcal{U}) \longrightarrow \mathbf{E}(\mathcal{V})$ such that $\mathcal{W} \subset \mathcal{V} \Longrightarrow \rho^{\mathcal{U}}_{\mathcal{W}} = \rho^{\mathcal{V}}_{\mathcal{W}} \circ \rho^{\mathcal{U}}_{\mathcal{V}}$.
Let then S be an arbitrary subset of $\mathbb{X}$. We posit as *equivalent at* S any two elements $\mathbf{e}_{\mathcal{U}} \in \mathbf{E}(\mathcal{U})$ and $\mathbf{e}_{\mathcal{V}} \in \mathbf{E}(\mathcal{V})$ with $\mathcal{U} \cap \mathcal{V} \supset S$ such that $\exists \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$, $\mathcal{W} \supset S$, $\rho^{\mathcal{U}}_{\mathcal{W}}(\mathbf{e}_{\mathcal{U}}) = \rho^{\mathcal{V}}_{\mathcal{W}}(\mathbf{e}_{\mathcal{V}})$. An equivalence class $(\mathbf{e}_{\mathcal{U}})_{\mathcal{U} \supset S}$ is referred to as a **germ** at S, (let us say, clumsily) in the **E**-class.
Example: *Let each* $\mathbf{E}(\mathcal{U})$ *be the set of closed subsets of* $\mathcal{U}$; *if* $S \in \mathbf{E}(\mathcal{U})$ *and* $\mathcal{V} \subset \mathcal{U}$ *then* $\rho^{\mathcal{U}}_{\mathcal{V}}(S) = \mathcal{V} \cap S$.
Most often, $S = \{x\}$, x a point in $\mathbb{X}$. But it may be important in more general cases:
Example: *Let* $S \subset \mathbb{R}^n = \mathbb{X}$ *be a closed set; let* $\mathbf{E}(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^n$ *open:* $(\mathbf{e}_{\mathcal{U}})_{\mathcal{U} \supset S}$ *is the germ of a* $\mathcal{C}^\infty$ *function at* S.

Definition 1.1. *By an* **exponentially decaying analytic symbol** *in* $\Omega \times \Gamma$ *we mean a function* $h \in \mathcal{C}^\infty(\Omega \times \Gamma)$ *that satisfies the following condition: Every compact subset* $\mathbf{K}$ of $\Omega \times \Gamma$ *has an open neighborhood* $\mathcal{N}_{\mathbf{K}}^{\mathbb{C}}$ *in* $\mathbb{C}^N \times \left(\mathbb{C}^N \setminus \{0\}\right)$ *to which* h *extends holomorphically and there are constants* $M > 0$, $\varepsilon > 0$, *such that* $|h(z, \lambda\zeta)| \leq M \exp(-\varepsilon\lambda)$ for all $(z, \zeta) \in \mathcal{N}_{\mathbf{K}}^{\mathbb{C}}$, $\lambda > 0$. *We denote by* $S_{\mathrm{a}}^{-\omega}(\Omega \times \Gamma)$ *the algebra (for ordinary addition and multiplication) of exponentially decaying analytic symbols* in $\Omega \times \Gamma$.

The **FBI transform** of $u \in \mathcal{E}'(\Omega)$ is the duality bracket

$$Fu(z, \zeta) = \frac{1}{(2\pi)^n} \int u(\tilde{x}) \exp\left(i(z - \tilde{x}) - \langle\zeta\rangle \langle z - \tilde{x}\rangle^2\right) d\tilde{x},$$

where $z = x + iy \in \mathbb{C}^N$, $|y| < \kappa |x|$, $\zeta = \xi + i\eta \in \mathbb{C}^N$, $|\eta| < \kappa |\xi|$, $\kappa > 0$ suitably small, and $\langle\zeta\rangle$ is the main branch of the square root of $\langle\zeta\rangle^2 = \zeta \cdot \zeta = \sum_{j=1}^N \zeta_j^2$.

Definition 1.2. *We say that* $u \in \mathcal{D}'(\Omega)$ *is* **microanalytic at** $(x^\circ, \xi^\circ) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$ *if there exist an open neighborhood of* x°, $U \subset\subset \Omega$, $\chi \in \mathcal{C}^\infty_c(\Omega)$, $\chi \equiv 1$ in U, *and an open cone* $\Gamma \subset \mathbb{R}^N \setminus \{0\}$, $\Gamma \ni \xi^\circ$, *such that* $\mathcal{F}(\chi u) \in S^{-\omega}_a(U \times \Gamma)$. *The set of points* $(x, \xi) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$ *at which* u *is* not *microanalytic is the* **analytic wave-front set** *of* u, *denoted by* $WF_a(u)$.

The base projection $WF_a(u) \ni (x, \xi) \longmapsto x \in \Omega$ is the **analytic singular support** of u, $\operatorname{singsupp}_a u$, the smallest closed subset E of Ω such that $u \in \mathcal{C}^\omega(\Omega \setminus E)$. We may refer to $WF_a(u) \cap \mathbb{S}^{N-1}$ as the **microsupport** of $u \in \mathcal{D}'(\Omega)$; this concept pertains to the *germ* of $\mathcal{F}u$ at points $(x, \xi) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$ if u is defined in a domain containing x.

Notation. *We denote by* $\mathbf{D}'_{x^\circ}$ *the vector space of germs of distributions (in* $\mathbb{R}^N$*) at* x°, *and by* $\mathbf{A}_{(x^\circ, \xi^\circ)}$ *the set of germs of distributions in* $\mathbb{R}^N$ *microanalytic at* (x°, ξ°).

Definition 1.3. *We refer to each element of the quotient vector space* $\mathbf{D}'_{x^\circ}/\mathbf{A}_{(x^\circ, \xi^\circ)}$ *as a* **microdistribution at** (x°, ξ°).

Definition 1.4. *A linear operator* $\mathfrak{T} : \mathcal{D}'(\Omega) \longleftarrow$ *is said to be* **analytic regularizing at** $(x^\circ, \xi^\circ) \in \Omega \times \left(\mathbb{R}^N \setminus \{0\}\right)$ *if*

$$\forall u \in \mathcal{D}'(\Omega), \ (x^\circ, \xi^\circ) \notin WF_{\mathrm{a}}(\mathfrak{T}u).$$

We denote by $\Psi^{-\omega}_{(x^\circ,\xi^\circ)}(\Omega)$ *the vector space of the linear operators* $\mathcal{D}'(\Omega) \longleftarrow$ *analytic regularizing at* (x°, ξ°).

Proposition 1.1. *Let* $h \in S^{-\omega}_{\mathrm{a}}(\Omega \times \Gamma)$, $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ *and* $\Gamma^\flat$ *be an open cone in* $\mathbb{R}^N \setminus \{0\}$ *such that* $\Gamma^\flat \cap \mathbb{S}^{N-1} \subset\subset \Gamma$*; then the operator*

$$\mathcal{D}'(\Omega) \ni u \mapsto \int_{\Gamma^\flat} e^{ix\cdot\xi} h(x,\xi)\, \widehat{\chi u}(\xi)\,(\xi)\, d\xi,$$

$(\widehat{\chi u}$*: Fourier transform of* $\chi u)$ *is analytic regularizing at every point of* $\Omega \times \left(\mathbb{R}^N \setminus \{0\}\right)$.

Definition 2.1. *By a* **classical analytic symbol of order** $m \in \mathbb{Z}$ *in* $\Omega \times \Gamma$ *we mean a formal series* $\sum_{j=0}^{\infty} P_j(x,\xi)$, $P_j \in C^\infty(\Omega \times \Gamma)$ *such that* $P_j(x,\lambda\xi) = \lambda^{m-j} P_j(x,\xi)$ *for every* $j \in \mathbb{Z}_+$ *and all* $(x,\xi) \in \Omega \times \Gamma$, $\lambda > 0$, *satisfying the following condition:*
To every compact set $\mathbf{K} \subset \Omega \times \Gamma$ *there are constants* $M > 0$, $C > 0$, *such that*

$$\forall\,(x,\xi) \in \mathbf{K},\ \forall \lambda > 0,\ \left| D_x^\alpha D_\xi^\beta P_j(x,\lambda\xi) \right| \leq M C^j j!\alpha!\beta!\lambda^{m-j-|\alpha+\beta|}. \quad (1)$$

We denote by $S^m_{\text{class}}(\Omega \times \Gamma)$ *the vector space of classical analytic symbols of order* $m \in \mathbb{Z}$ *in* $\Omega \times \Gamma$.

We have $S^m_{\text{class}}(\Omega \times \Gamma) \subset S^{m'}_{\text{class}}(\Omega \times \Gamma)$ if $m' < m$.
Definition 2.2. *By a* **finite realization** *of* $\sum_{j=0}^{\infty} P_j(x,\xi) \in S^m_{\text{class}}(\Omega \times \Gamma)$ *we mean a finite sum*

$$P_{(R,g)}(x,\xi) = g(\xi) \sum_{j<R/|\xi|} P_j(x,\xi),\ (x,\xi) \in \Omega \times \Gamma,$$

where $R > 0$ *is large and* $g \in \mathcal{C}^\infty(\mathbb{R}^N)$, $g(\xi) = 0$ *if* $|\xi| < \rho_1$, $g(\xi) = 1$ *if* $|\xi| > \rho_2$ $(0 < \rho_1 < \rho_2)$.

Theorem 2.1. *Let* $\sum_{j=0}^{\infty} P_j(x,\xi) \in S^m_{\text{class}}(\Omega\times\Gamma)$ *and* $P_{(R,g)}$ *be one of its finite realizations. Let* $U \subset\subset \Omega$ *be an open set,* $\Gamma^\flat$ *an open cone in* $\mathbb{R}^N\setminus\{0\}$ *such that* $\Gamma^\flat \cap S^{N-1} \subset\subset \Gamma$. *If* $u\in\mathcal{E}'(U)$ *the duality bracket*

$$P_{(R,g)}(x,D_x)\,u = \lim_{\varepsilon\searrow 0} \frac{1}{(2\pi)^n}\int_{\Gamma^\flat} e^{ix\cdot\xi-\varepsilon|\xi|^2}\, P_{(R,g)}(x,\xi)\,\widehat{u}(\xi)\,d\xi$$

defines a distribution in U. *We have*

$$WF_a\left(P_{(R,g)}(x,D_x)\,u\right) \subset WF_a(u)\,. \tag{2}$$

Proposition 2.1. *If* $P_{(R,g)}$ *and* $P_{\left(R^\flat,g^\flat\right)}(x,\xi)$ *are finite realizations of* $\sum_{j=0}^{\infty} P_j(x,\xi)$ *then* $P_{(R,g)} - P_{\left(R^\flat,g^\flat\right)} \in S^{-\omega}_{\mathsf{a}}(\Omega\times\Gamma)$ (cf. Definition **1.1**).

The proofs of these claims are standard.

Let $u \in \mathcal{E}'(U)$. Propositions **1.1** and **2.1** imply that

$$\frac{1}{(2\pi)^n} \int_{\Gamma^\flat} e^{ix\cdot\xi} \left(P_{(R,g)}(x,\xi) - P_{\left(R^\flat,g^\flat\right)}(x,\xi) \right) \widehat{u}(\xi)\, d\xi$$

is a $\mathcal{C}^\infty$ function in U holomorphically extendible to the "sector" $\{z \in \mathbb{C}^n;\ \operatorname{Re} z \in U,\ |\operatorname{Im} z| < \kappa |\operatorname{Re} z|\}$, $\kappa > 0$ suitably small. It follows that $P_{(R,g)}(x, D_x)\, u$ defines a microdistribution at an arbitrary point $(x^\circ, \xi^\circ) \in U \times \Gamma^\flat$, independent of g and R, as a consequence of (2) only depending on the microdistribution $\mathbf{u}$ at (x°, ξ°) defined by u, which we denote by $\mathbf{P}_{(x^\circ,\xi^\circ)}\mathbf{u}$.

Definition 2.3. *We refer to* $\mathbf{P}_{(x^\circ,\xi^\circ)} : \mathbf{D}'_{(x^\circ,\xi^\circ)} \longleftarrow$ *as a* **classical microdifferential operator of order** m **at** (x°, ξ°). *The set of these operators shall be denoted by* $\Psi^m_{(x^\circ,\xi^\circ)}$.

Let $\mathbf{S}^m_{(x^\circ,\xi^\circ)}$ denote the vector space of germs at (x°, ξ°) of classical analytic symbols of order m. As $U \times \Gamma^\flat$ contracts about (x°, ξ°) the map $S^m_{\text{class}}\left(U \times \Gamma^\flat\right) \ni P(x,\xi) \mapsto \mathbf{P}_{(x^\circ,\xi^\circ)}$ induces a *bijection* $\mathbf{S}^m_{(x^\circ,\xi^\circ)} \longrightarrow \Psi^m_{(x^\circ,\xi^\circ)}$.

Standard terminology in dealing with the set $\left\{\mathbf{P}_{(x,\xi)}\right\}_{(x,\xi)\in\Omega\times\Gamma}$:
its **total symbol**, $P(x,\xi)=\sum_{j=0}^{\infty}P_j(x,\xi)$;
its **principal symbol**, $P_0(x,\xi)$.

Definition 2.4. *The microdifferential operator* $\mathbf{P}_{(x^\circ,\xi^\circ)}\in\Psi^m_{(x^\circ,\xi^\circ)}$ *is said to be* **elliptic of order** m *if* $P_0(x^\circ,\xi^\circ)\neq 0$.

The ellipticity of $\mathbf{P}_{(x^\circ,\xi^\circ)}$ implies the existence of an open neighborhood $U\subset\Omega$ of (x°,ξ°), a cone $\Gamma^\flat\subset\Gamma$, $\Gamma^\flat\ni\xi^\circ$, a constant $c_\circ>0$, such that $|P_0(x,\lambda\xi)|\geq c_\circ\lambda^m$ for all $(x,\xi)\in U\times\Gamma^\flat$, $\lambda>0$. It is then easy to derive that there is $Q\in S^{-m}_{\text{class}}\left(U\times\Gamma^\flat\right)$ such that $P\#Q=Q\#P=1$ in $U\times\Gamma^\flat$; Q is elliptic of order $-m$; Q defines the *inverse* of $\mathbf{P}_{(x^\circ,\xi^\circ)}$, that may be denoted by $\mathbf{P}^{-1}_{(x^\circ,\xi^\circ)}$. The elliptic, classical, microdifferential operators at (x°,ξ°) form a *group* for composition.

Let $\mathbf{P}_{(x^\circ,\xi^\circ)} \in \Psi^m_{(x^\circ,\xi^\circ)}$, $\mathbf{Q}_{(x^\circ,\xi^\circ)} \in \Psi^{m'}_{(x^\circ,\xi^\circ)}$; the *compose* $\mathbf{P}_{(x^\circ,\xi^\circ)} \circ \mathbf{Q}_{(x^\circ,\xi^\circ)}$ belongs to $\Psi^{m+m'}_{(x^\circ,\xi^\circ)}$, its total symbol is

$$P(x,\xi) \# Q(x,\xi) = \sum_{\alpha \in \mathbb{Z}^N_+} \frac{1}{\alpha!} \left(D^\alpha_\xi P(x,\xi) \right) \partial^\alpha_\xi Q(x,\xi)$$

where $\partial^\alpha_\xi = \partial^{\alpha_1}_{\xi_1} \cdots \partial^{\alpha_N}_{\xi_N}$, $\partial_{\xi_k} = \partial/\partial\xi_k$.

Proposition 2.2. *The union* $\Psi_{(x^\circ,\xi^\circ)} = \bigcup_{m \in \mathbb{R}} \Psi^m_{(x^\circ,\xi^\circ)}$ *is a ring for ordinary addition and the composition law* $\#$.

We will focus on the *surjectivity* of the map $\mathbf{P}_{(x^\circ,\xi^\circ)} : \mathbf{D}'_{(x^\circ,\xi^\circ)} \longleftarrow$; in more concrete terms, it means the following:
There is an open neighborhood U *of* x° *in* $\mathbb{R}^N$, *an open cone* Γ *in* $\mathbb{R}^N \setminus \{0\}$, $\Gamma \ni \xi^\circ$, *and a finite realization* $P_{(R,g)}(x,\xi)$ *of the total symbol of* $\mathbf{P}_{(x^\circ,\xi^\circ)}$ *in* $U \times \Gamma$, *such that to every* $f \in \mathcal{D}'(U)$ *there is* $u \in \mathcal{E}'(U)$ *verifying* $x^\circ \notin WF_{\mathrm{a}}(f - P_{(R,g)}(x, D_x)\, u)$.
If $\mathbf{P}_{(x^\circ,\xi^\circ)}$ is elliptic it is bijective.

Microdifferential Operators of Principal Type

Definition 3.1. *The microdifferential operator* $\mathbf{P}_{(x^\circ,\xi^\circ)} \in \Psi^m_{(x^\circ,\xi^\circ)}$ *and its total symbol are said to be* **of principal type** at (x°,ξ°) if $P_0(x^\circ,\xi^\circ) = 0 \Longrightarrow d_\xi P_0(x^\circ,\xi^\circ) \neq 0$ (d_ξ: exterior derivative in ξ-space).

The Euler homogeneity identities imply $\xi \cdot dP_0(x,\xi) = mP_0(x,\xi)$. Thus, if $m \neq 0$, $P(x,\xi)$ is of principal type at (x°,ξ°) iff $d_\xi P_0(x^\circ,\xi^\circ) \neq 0$.
Since $d_\xi P_0 \neq 0$ is an open property, if $\mathbf{P}_{(x^\circ,\xi^\circ)}$ is of principal type the same is true of $\mathbf{P}_{(x,\xi)}$ provided (x,ξ) is sufficiently close to (x°,ξ°).
If $\mathbf{P}_{(x^\circ,\xi^\circ)}$ is of principal type it does not ensue that $\operatorname{Char} P$ is a $\mathcal{C}^\infty$ hypersurface in a neighborhood of (x°,ξ°) since P_0 is complex-valued.
If $\mathbf{E}_{(x^\circ,\xi^\circ)}$ is elliptic and $\mathbf{P}_{(x^\circ,\xi^\circ)}$ is of principal type then $\mathbf{P}_{(x^\circ,\xi^\circ)}\mathbf{E}_{(x^\circ,\xi^\circ)}$ and $\mathbf{E}_{(x^\circ,\xi^\circ)}\mathbf{P}_{(x^\circ,\xi^\circ)}$ are of principal type. An arbitrary distribution in Ω is of the form $(1-\Delta)^{-N} f$ in an open neighborhood of x°, with $f \in \mathcal{C}_c(\Omega)$ and $N \in \mathbb{Z}_+$. It follows that, in order to prove the solvability of $\mathbf{P}_{(x^\circ,\xi^\circ)}$ it suffices to prove that to every germ $\mathbf{f}$ at (x°,ξ°) of a *continuous function* there is $\mathbf{u} \in \mathbf{D}'_{(x^\circ,\xi^\circ)}$ such that $\mathbf{P}_{(x^\circ,\xi^\circ)}\mathbf{u} = \mathbf{f}$.

We apply the Weierstrass Preparation Theorem: If $P(x,\xi) \in S^m_{\text{class}}(\Omega \times \Gamma)$ is of principal type at $(x^\circ, \xi^\circ) \in \Omega \times \left(\Gamma \cap \mathbb{S}^{N-1}\right)$ there is an open neighborhood $U \subset \Omega$ of x° in $\mathbb{R}^N$, an open cone $\Gamma^\flat$ in $\mathbb{R}^N \setminus \{0\}$, $\xi^\circ \in \Gamma^\flat \subset \Gamma$, such that (possibly after relabeling the coordinates)

$$P_0(x,\xi) = E_0(x,\xi)\left(\xi_N + q\left(x,\xi'\right)\right) \text{ in } U \times \left(\Gamma^\flat \cap \mathbb{S}^{N-1}\right), \tag{3}$$

with $\xi' = (\xi_1, ..., \xi_{N-1})$, $E_0, q \in \mathcal{C}^\omega\left(U \times \left(\Gamma^\flat \cap \mathbb{S}^{N-1}\right)\right)$. We extend (3) to $U \times \Gamma^\flat$ assuming E_0 (resp., q) homogeneous of degree $m-1$ (resp, 1; "homogeneous" always meant with respect to ξ).
One can derive from (3) that there are an *elliptic* symbol $A \in S^{m-1}_{\text{class}}\left(U \times \Gamma^\flat\right)$ with principal symbol E_0 and $Q \in S^1_{\text{class}}\left(U \times \Gamma^\flat\right)$ with principal symbol q such that

$$P(x,\xi) = A(x,\xi) \# \left(\xi_N + Q\left(x,\xi'\right)\right). \tag{4}$$

To (4) correspond the operator composition

$$\mathbf{P}_{(x^\circ,\xi^\circ)} = \mathbf{A}_{(x^\circ,\xi^\circ)} \left(D_{x_N} + \mathbf{Q}_{(x^\circ,\xi^\circ)} \right).$$

Since $\mathbf{A}_{(x^\circ,\xi^\circ)}$ is elliptic, hence invertible within $\Psi_{(x^\circ,\xi^\circ)}$, the solvability of $\mathbf{P}_{(x^\circ,\xi^\circ)}$ is equivalent to that $D_{x_N} + \mathbf{Q}_{(x^\circ,\xi^\circ)}$. It is convenient to change the notation for the coordinates: x_N becomes t, a kind of time variable, x shall now stand for $(x_1, ..., x_n)$, $n = N - 1$ (≥ 1); the dual coordinates shall be ξ_j, $1 \leq j \leq n$, τ. It is also convenient to take $(x^\circ, 0)$ as the central point and replace Ω by $\Omega \times (-T_\circ, T_\circ)$, where now Ω *is a domain in* $\mathbb{R}^n_x$ (contracted about x° as needed) and $T_\circ > 0$.
We redefine $\mathbf{P}_{(x^\circ,\xi^\circ)} = D_t + \mathbf{Q}_{(x^\circ,\xi^\circ)}$. The total symbol of $\mathbf{Q}_{(x^\circ,\xi^\circ)}$ is $q(x,t,\xi) + \sum_{j=0}^{\infty} Q_j(x,t,\xi)$; $Q^{(0)}(x,t,\xi) = \sum_{j=0}^{\infty} Q_j(x,t,\xi)$ is a classical analytic symbol of order zero in Ω depending analytically on $t \in (-T_\circ, T_\circ)$.

Using symbolic calculus it can be proved that there is an *elliptic* classical symbol $E(x,t,\xi)$ *of order zero* in Ω depending analytically on $t \in (-T_\circ, T_\circ)$ solving the following Cauchy problem:

$$D_t E + E\#Q^{(0)} + q\#E - E\#q, \;\; E|_{t=0} = 1.$$

This is made possible by the fact that $\text{order}\,(q\#E - E\#q) \leq 0$. This yields, for $\mathbf{u} \in \mathbf{D}'_{(x^\circ,\xi^\circ)}$ arbitrary,

$$\begin{aligned}\mathbf{P}_{(x^\circ,\xi^\circ)}\mathbf{E}_{(x^\circ,\xi^\circ)}\mathbf{u} &= D_t\left(\mathbf{E}_{(x^\circ,\xi^\circ)}\mathbf{u}\right) + \mathbf{q}_{(\mathbf{x}^\circ,\boldsymbol{\xi}^\circ)}\mathbf{E}_{(x^\circ,\xi^\circ)}\mathbf{u} + \mathbf{Q}^{(0)}_{(x^\circ,\xi^\circ)}\mathbf{E}_{(x^\circ,\xi^\circ)}\mathbf{u} \\ &= \mathbf{E}_{(x^\circ,\xi^\circ)}\left(D_t\mathbf{u} + \mathbf{q}_{(\mathbf{x}^\circ,\boldsymbol{\xi}^\circ)}\mathbf{u}\right),\end{aligned}$$

i.e.,

$$\mathbf{E}^{-1}_{(x^\circ,\xi^\circ)}\mathbf{P}_{(x^\circ,\xi^\circ)}\mathbf{E}_{(x^\circ,\xi^\circ)} = D_t + \mathbf{q}_{(\mathbf{x}^\circ,\boldsymbol{\xi}^\circ)}.$$

The next reduction is a much deeper result. It is an application of the celebrated Egorov Theorem: *There is an invertible operator* $\mathbf{T} : \mathbf{D}'_{(x^\circ,\xi^\circ)} \longleftrightarrow$ *such that*

$$\mathbf{T}^{-1}\left(D_t+\mathbf{q}_{(\mathbf{x}^\circ,\xi^\circ)}\right)\mathbf{T} = D_t + i\mathbf{T}^{-1}\left(\operatorname{Im}\mathbf{q}_{(\mathbf{x}^\circ,\xi^\circ)}\right)\mathbf{T}. \tag{5}$$

As reflected in the equation (5) rewritten in terms of the total symbols, $\mathbf{T}$ is associated with a *local symplectic* automorphism in (x,ξ)-space, preserving (x°,ξ°); $\mathbf{T}$ is represented by an invertible *Fourier Integral Operator (***FIO***)* attached to this *symplectomorphism.* Importantly, the corresponding transformation preserves real symbols. Both the operator and the symplectomorphism depend analytically on t, allowing us to assume that

$$\mathbf{P}_{(x^\circ,\xi^\circ)} = D_t + i\mathbf{q}_{(\mathbf{x}^\circ,\xi^\circ)} \tag{6}$$

and that the *total* symbol of $\mathbf{q}_{(\mathbf{x}^\circ,\xi^\circ)}$ is the *real symbol* $q\,(x,t,\xi)$ *homogeneous of degree* 1.

From here on we assume that $q \in \mathcal{C}^\omega(\Omega \times (-T_\circ, T_\circ) \times \Gamma)$. We can now state the distribution version of the Trépreau Theorem:

Theorem 3.1. *For the microdifferential operator* $\mathbf{P}_{(x^\circ,\xi^\circ)} : \mathbf{D}'_{(x^\circ,\xi^\circ)} \hookleftarrow$ [see (6)] *to be surjective it is necessary and sufficient that the following condition be satisfied:*

(Psi)$_{(x^\circ,\xi^\circ)}$ *There exist an open neighborhood* $U \subset \Omega$ *of* (x°, ξ°)*, a number* δ*,* $0 < \delta < T_\circ$*, an open cone* $\Gamma^\flat \subset \Gamma$*, such that the function* $(-\delta, \delta) \ni t \mapsto q(x, t, \xi)$ *does not change sign from* $-$ *to* $+$ *at any point of* $(-\delta, \delta)$ *whatever* $(x, \xi) \in U \times \Gamma^\flat$.

Several remarks are in order.

1. As is well-known to specialists there is a coordinate-free formulation of **(PSI)**, therefore valid for the germ at a point of phase-space of an arbitrary pseudodifferential operator P of principal type. This would have required introducing the **null bicharateristics** of $\mathrm{Re}\, P$ or $\mathrm{Im}\, P$ and more time.
2. We shall not prove the necessity part in Theorem **3.1**, a 1969 result. The sufficiency is a relatively banal consequence of Dencker's Theorem that the validity of $(\mathbf{Psi})_{(x,\xi)}$ for every $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ entails the *local* solvability of P. Dencker's proof is very hard. That of Theorem **3.1** outlined below is relatively straightforward and describes an approach tilted towards the *representation* of the solutions of the equation $Pu = f$.
3. Theorem **3.1** is not a direct consequence of Trépreau's Theorem, in which distributions are replaced by hyperfunctions. We should point out, however, that the route followed to arrive at the *statement* of Theorem **3.1** could be straightfordwardly adapted to microfunctions. It would have sufficed to replace, wherever they occur, compactly supported distributions by analytic functionals in $\mathbb{R}^N$. What is not clear is how to adapt to analytic functionals our *proof* of Theorem **3.1**.

It is fair to say that Trépreau's proof of his theorem is hard. It relies on three important and distinct "technologies:

1. The definition of hyperfunctions as *relative cohomology classes* and that of the symplectic transformation in Egorov's Theorem as a *quantized transformation* of these classes, all done within Homological Algebra and the Oka-Serre theory of *coherent analytic sheaves*.
2. The symplectomorphic transformation of the cotangent bundle $T^*\mathbb{R}^n$ near (x°, ξ°) into the *conormal bundle* of the $\mathcal{C}^\omega$ boundary of a strictly pseudoconvex domain $\Omega^{\mathbb{C}}$ at $0 \in \partial\Omega^{\mathbb{C}}$, transforming P into $\partial/\partial z_1$ (tangent to $\partial\Omega^{\mathbb{C}}$ at 0). This relies on the concepts (due to P. Schapira) of **R**-Lagrangian and **I**-symplectic real submanifolds of the complex symplectic space $\left(\mathbb{C}^N_z \times \mathbb{C}^N_\zeta, d\zeta \wedge dz\right)$. It transforms the microlocal solvability of P at (x°, ξ°) into the solvability of $\partial/\partial z_1$ in the space of germs at 0 of holomorphic functions in $\Omega^{\mathbb{C}}$.
3. Characterizing the solvability of $\partial/\partial z_1$ in the space of germs at 0 of holomorphic functions in $\Omega^{\mathbb{C}}$ is fairly elementary but delicate (and lengthy). It culminates with a new formulation of Condition **(Psi)** (which reappears in our approach).

We outline the proof of Theorem **3.1**. We assume that $q(x,t,\xi) \in \mathcal{C}^\omega(\Omega \times (-T_\circ, T_\circ) \times \Gamma)$ has a holomorphic extension $(z,\zeta) \mapsto q(z,t,\zeta)$ to a conic neighborhood of (x°, ξ°) in $\mathbb{C}^n \times (\mathbb{C}^n \setminus \{0\})$. Keep in mind that $q(x,t,\xi) \in \mathbb{R}$, $q(x,t,\lambda\xi) = \lambda q(x,t,\xi)$ if $\lambda > 0$.
We seek a microlocal right-inverse in $U \times (-\delta,\delta) \times \Gamma^\flat$ of $P = D_t + iq(x,t,D_x)$ acting $\mathcal{C}_c(U \times (-\delta,\delta))$ $[x^\circ \in U \subset \Omega,\ 0 < \delta < T_\circ,$ $\xi^\circ \in \Gamma^\flat \subset \Gamma]$ in the form of an FIO with a complex phase function φ, specifically the solution, holomorphic with respect to (z,w,ζ) in a conic neighborhood of $(x^\circ, x^\circ, \xi^\circ)$ in $\mathbb{C}^{2n} \times (\mathbb{C}^n \setminus \{0\})$, of the *eikonal equation*

$$\frac{\partial\varphi}{\partial t}(z,t,w,s,\zeta) + iq(z,t,\partial_z\varphi(z,t,w,s,\zeta)) = 0 \tag{7}$$

$[(s,t) \in \mathbb{R}^2,\ |s| < \delta,\ |t| < \delta]$ satisfying the Cauchy condition

$$\varphi(z,t,w,s,\zeta)|_{t=s} = (z-w)\cdot\zeta + \frac{1}{2}i\langle\zeta\rangle\langle z-w\rangle^2 \tag{8}$$

($\langle\zeta\rangle = \sqrt{\zeta\cdot\zeta}$); (8) ensures that φ is an *FBI phase-function at* (x°,ξ°) in the sense of Sjöstrand, depending analytically on (s,t).

The crux of the proof of Theorem **3.1** is the interpretation of Condition **(Psi)** in terms of the following function

$$\pi_\varphi(w,t,s,\zeta) = i\varphi(w,t,w,s,\zeta).$$

The following equations are direct consequences of (7)-(8):

$$\frac{\partial \pi_\varphi}{\partial t}(w,t,s,\zeta) = q\left(w,t,\partial_z\varphi(w,t,w,s,\zeta)\right), \tag{9}$$

$$\pi_\varphi(w,s,s,\zeta) = 0.$$

The following relation is most important

$$\frac{\partial \pi_\varphi}{\partial t}(w,s,s,\zeta) = q(w,s,\zeta). \tag{10}$$

We also define

$$\varphi^\natural(z,t,w,s,\zeta) = \varphi(z,t,w,s,\zeta) + i\pi_\varphi(w,t,s,\zeta), \tag{11}$$

$$\mathcal{I}_q(y,t,s,\xi) = \int_s^t q(y,t',\xi)\,dt'. \tag{12}$$

We use the notation

$$\nabla^{\mathrm{h}}_{y,\xi}\mathcal{I}_q\left(y,t,s,\xi\right)=\left(\partial_y\mathcal{I}_q\left(y,t,s,\xi\right),\left|\xi\right|\partial_\xi\mathcal{I}_q\left(y,t,s,\xi\right)\right);$$

$\nabla^{\mathrm{h}}_{y,\xi}$ preserves the homogeneity degree. All the statements that follow presuppose $\operatorname{diam} U$, $\operatorname{diam}\left(\Gamma^\flat\cap\mathbb{S}^{n-1}\right)$ and δ to be suitably small. They are consequences of Equations (7) to (12). The first two do not rely on **(Psi)**.

Lemma 3.1. *We have, for all* $(y,t,s,\xi)\in U\times\left(-\delta,\delta\right)^2\times\Gamma^\flat$,

$$\left|\partial_y\varphi\left(y,t,y,s,\xi\right)-\xi\right|\lesssim\left|\nabla^{\mathrm{h}}_{y,\xi}\mathcal{I}_q\left(y,t,s,\xi\right)\right|+\int_s^t\left|\nabla^{\mathrm{h}}_{y,\xi}\mathcal{I}_q\left(y,t',s,\xi\right)\right|dt'$$

Lemma 3.2. *We have, for all* $(y,t,s,\xi)\in U\times\left(-\delta,\delta\right)^2\times\Gamma^\flat$,

$$\begin{aligned}\left|\operatorname{Re}\pi_\varphi\left(y,t,s,\xi\right)-\mathcal{I}_q\left(y,t,s,\xi\right)\right|\quad\lesssim\quad&\left|\nabla^{\mathrm{h}}_{y,\xi}\mathcal{I}_q\left(y,t,s,\xi\right)\right|^2\\&+\int_s^t\left|\nabla^{\mathrm{h}}_{y,\xi}\mathcal{I}_q\left(y,t',s,\xi\right)\right|^2dt'.\end{aligned}$$

The link (13) between π_φ and $\mathcal{I}_q$ can be made much stronger under the hypothesis **(Psi)**.
Relying on the analyticity of $t \longmapsto q(y,t,\xi)$ we see that the validity of **(Psi)** in $U \times \Gamma^\flat$ implies that $U \times \Gamma^\flat$ is the union of the following disjoint four sets:

- $(\mathfrak{N})$ the set $\mathfrak{N}$ of points (y,ξ) such that $q(y,t,\xi) = 0$ for all t;
- $(\mathfrak{A}^\pm)$ the set $\mathfrak{A}^+$ (resp., $\mathfrak{A}^-$) of points (y,ξ) such that $q(y,t,\xi) > 0$ (resp., < 0) in the whole interval $(-\delta,\delta)$ except, possibly, on a discrete set;
- $(\mathfrak{A})$ the set $\mathfrak{A}$ of points (y,ξ) such that $t \mapsto q(y,t,\xi)$ changes sign from $+$ to $-$ at a single point $t(y,\xi) \in (-\delta,\delta)$.

It is obvious that $\mathcal{I}_q(y,t,s,\xi) \equiv 0$ if $(y,\xi) \in \mathfrak{N}$. In the sequel we preclude the trivial case $q \equiv 0$.

The following statement is a direct consequence of the the partition $U \times \Gamma^\flat = \mathfrak{N} \cup \mathfrak{A}^+ \cup \mathfrak{A}^- \cup \mathfrak{A}$ and the equation $\frac{\partial \mathcal{I}_q}{\partial s}(y,t,s,\xi) = -q(y,s,\xi)$.

Lemma 3.3. *The validity of* **(Psi)** *in* $U \times (-\delta,\delta)^2 \times \Gamma^\flat$ *is equivalent to the following property:*
If $(y,\xi) \notin \mathfrak{N}$ *then* $\mathcal{I}_q(y,t,s,\xi) < 0$ *in every one of the following subsets of* $U \times (-\delta,\delta)^2 \times \Gamma^\flat$:

(a) $(y,\xi) \in \mathfrak{A}^+$, $-\delta < t < s < \delta$,

(b) $(y,\xi) \in \mathfrak{A}^-$, $-\delta < s < t < \delta$,

(c) $(y,\xi) \in \mathfrak{A}$, $t(y,\xi) \leq s < t < \delta$,

(d) $(y,\xi) \in \mathfrak{A}$, $-\delta < t < s \leq t(y,\xi)$.

Moreover, we have $\frac{\partial \mathcal{I}_q}{\partial s}(y,t,s,\xi) \leq 0$ *in* **(a)** *and* **(d)**, $\frac{\partial \mathcal{I}_q}{\partial s}(y,t,s,\xi) \geq 0$ *in* **(b)** *and* **(c)**.

Lemma **3.3** implies

Lemma 3.4. *If* **(Psi)** *holds in* $U \times (-\delta, \delta)^2 \times \Gamma^\flat$ *then*

$$\left|\nabla^{\mathsf{h}}_{y,\xi} \mathcal{I}_q (y, t, s, \xi)\right|^2 \lesssim - |t - s| \, \mathcal{I}_q (y, t, s, \xi)$$

in each of the sets **(a)**, **(b)**, **(c)**, **(d)**, *of Lemma* **3.3**.

Lemmas **3.2** and **3.4** imply

$$\begin{aligned} \left|\operatorname{Re} \pi_\varphi (y, t, s, \xi) - \mathcal{I}_q (y, t, s, \xi)\right| \lesssim & - |t - s| \, \mathcal{I}_q (y, t, s, \xi) \\ & - \int_s^t \left|t' - s\right| \mathcal{I}_q \left(y, t', s, \xi\right) dt' \end{aligned} \tag{13}$$

for every (y, t, s, ξ) in each of the sets **(a)**, **(b)**, **(c)**, **(d)**, of Lemma **3.3**.

Combining *Lemma* **3.4** with (13) yields

Lemma 3.5. *If Condition* **(Psi)** *holds in* $U \times (-\delta, \delta)^2 \times \Gamma^\flat$ *we have*

$$\left|\operatorname{Re} \pi_\varphi (y, t, s, \xi) - \mathcal{I}_q (y, t, s, \xi)\right| \lesssim -\delta \mathcal{I}_q (y, t, s, \xi)$$

in each of the sets **(a)**, **(b)**, **(c)**, **(d)**, *of Lemma* **3.3**.

All this allows us to prove the result we were aiming for:

Proposition 3.1. *If* $\operatorname{diam} U$, $\operatorname{diam}\left(\Gamma^\flat \cap \mathbb{S}^{n-1}\right)$, δ, *are sufficiently small the validity of Condition* **(Psi)** *in* $U \times (-\delta, \delta)^2 \times \Gamma^\flat$ *implies to the following property:*

(R) $\operatorname{Re} \pi_\varphi (y, t, s, \xi) \leq 0$ *for every* (y, t, s, ξ) *in each of the sets* **(a)**, **(b)**, **(c)**, **(d)**, *of Lemma* **3.3** *or such that* $(y, \xi) \in \mathfrak{N}$.

The parametrix, acting on an arbitrary $f \in \mathcal{C}_c\left(U \times (-\delta, \delta)\right)$, will be the limit, as $\varepsilon \searrow 0$, of the integrals

$$A_{(R,g)}{}^{(\varepsilon)} f(x,t) = \tag{14}$$

$$\frac{1}{(2\pi)^n} i \int_{\mathfrak{A}^+} \int_{-\delta}^{t} e^{i\varphi(x,t,y,s,\xi) - \varepsilon|\xi|^2} a_{(R,g)}(x,t,y,s,\xi) f(y,s)\, ds d\xi dy$$

$$-\frac{1}{(2\pi)^n} i \int_{\mathfrak{A}^-} \int_{t}^{\delta} e^{i\varphi(x,t,y,s,\xi) - \varepsilon|\xi|^2} a_{(R,g)}(x,t,y,s,\xi) f(y,s)\, ds d\xi dy$$

$$+\frac{1}{(2\pi)^n} i \int_{\mathfrak{A}} \int_{t(y,\xi)}^{t} e^{i\varphi(x,t,y,s,\xi) - \varepsilon|\xi|^2} a_{(R,g)}(x,t,y,s,\xi) f(y,s)\, ds d\xi dy$$

where φ is the solution of (7)-(8), $a_{(R,g)}$ is a finite realization of a classical analytic amplitude of order zero $a(x,t,y,s,\xi) = \sum_{j=0}^{\infty} a_j(x,t,y,s,\xi)$; $t(y,\xi)$ is the point in $(-\delta,\delta)$ where $t \mapsto q(x,t,\xi)$ changes sign from $+$ to $-$ [in **(c)** and **(d),** Lemmas **3.3**]; thus, in the third integral in (14), we can have $-\delta < t < t(y,\xi)$ as well as $t(y,\xi) < t < \delta$.

We can determine $a(z,t,w,s,\xi) = \sum_{j=0}^{\infty} a_j(z,t,w,s,\xi)$ in a conic neighborhood $\mathcal{U}^{\mathbb{C}}$ of $U \times (-\delta,\delta) \times U \times (-\delta,\delta) \times \Gamma$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times (\mathbb{C}^n \setminus \{0\})$ as the holomorphic solution of the Cauchy problem

$$(D_t + iq(z,t,D_z)) \left(e^{i\varphi(z,t,w,s,\zeta)} a(z,t,w,s,\zeta) \right) = 0, \tag{15}$$

$$a(z,s,w,s,\zeta) = 1 + \frac{1}{2} i \langle \zeta \rangle^{-1} (z-w) \cdot \zeta. \tag{16}$$

The homogeneous parts $a_j \in \mathcal{O}(\mathcal{U}^{\mathbb{C}})$ are recursively determined by the transport equations under the initial conditions

$$a_0(z,s,w,s,\zeta) = 1 + i \langle \zeta \rangle^{-1} (z-w) \cdot \zeta,$$
$$a_j(z,s,w,s,\zeta) = 0 \text{ if } j \geq 1.$$

With the above choice of the amplitude a the convergence of (14) to a distribution $A_{(\mathbf{R},\mathbf{g})}f$ in $U\times(-\delta,\delta)$ is a consequence (as proved by the standard method of dealing with oscillatory integrals) of the following facts:

i) (11), i.e., $\varphi(x,t,y,s,\xi)=\varphi^\natural(x,t,y,s,\xi)-i\pi_\varphi(y,t,s,\xi)$;

ii) **(Psi)** via Proposition **3.1**, i.e., $\operatorname{Re}\pi_\varphi(y,t,s,\xi)\leq 0$ in each domain of integration in (14);

iii) $\varphi(z,t,w,s,\zeta)$ is an FBI phase-function.

We can state:

Proposition 3.2. *Let $\chi\in\mathcal{C}^\infty_{\rm c}(U)$, $\chi\equiv 1$ in a neighborhood of x°. It follows from (15) that replacing $e^{i\varphi(x,t,y,s,\xi)}\chi(x)\,a_{(R,g)}(x,t,y,s,\xi)$ by*

$$(D_t+iq(x,t,D_x))\left(e^{i\varphi(x,t,y,s,\xi)}\chi(x)\,a_{(R,g)}(x,t,y,s,\xi)\right)$$

in (14) yields an analytic regularizing operator $\mathfrak{T}$ at (x°,ξ°) in $\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$ depending analytically on $t\in(-\delta,\delta)$.

It follows from the Cauchy conditions (8) and (16) that

$$\mathfrak{T}f\left(x,t\right)=\left(D_t+iq\left(z,t,D_w\right)\right)\left(\chi\left(x\right)A_{\mathbf{(R,g)}}f\left(x,t\right)\right)-$$

$$\frac{1}{\left(2\pi\right)^n}\chi\left(x\right)\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi-|\xi|\left|x-y^2\right|}g\left(\xi\right)f\left(y,t\right)\left(1+i\left(x-y\right)\cdot\frac{\xi}{|\xi|}\right)dyd\xi.$$

with $\mathfrak{T}$ as in Proposition **3.2**. The inversion formula for the FBI transform implies

$$f\left(x,t\right)=\frac{1}{\left(2\pi\right)^n}\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi-|\xi|\left|x-y^2\right|}f\left(y,t\right)\left(1+i\left(x-y\right)\cdot\frac{\xi}{|\xi|}\right)dyd\xi.$$

The proof of Theorem **3.1** is completed by noting that

$$f\left(x,t\right)-$$

$$\frac{1}{\left(2\pi\right)^n}\chi\left(x\right)\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi-|\xi|\left|x-y^2\right|}g\left(\xi\right)f\left(y,t\right)\left(1+i\left(x-y\right)\cdot\frac{\xi}{|\xi|}\right)dyd\xi$$

is microanalytic at $\left(\left(x^\circ,0\right),\left(\xi^\circ,\tau^\circ\right)\right)$ in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\setminus\left\{0\right\}$, $\tau^\circ=q\left(x^\circ,0,\xi^\circ\right)$.

By Corollary **3.1** and Lemma **3.5** the partition $U \times \Gamma^\flat = \mathfrak{N} \cup \mathfrak{A}^+ \cup \mathfrak{A}^- \cup \mathfrak{A}$ is equivalent to the *quasiconvexity* of $t \longmapsto -\operatorname{Re} \pi_\varphi (y, t, s, \xi) \in [0, +\infty)$. If $(y, \xi) \in \mathfrak{A}$ the graph of $t \longmapsto -\operatorname{Re} \pi_\varphi (y, t, s, \xi)$ resembles a parabola in the upper-half plane with its minimum at $t(x, \xi)$. If $(y, \xi) \in \mathfrak{A}^\pm$ then $t \longmapsto -\operatorname{Re} \pi_\varphi (y, t, s, \xi)$ is monotone, either increasing or decreasing.

Definition 3.1. *A continuous real-valued function f in an open subset $\mathcal{A}$ of $\mathbb{R}$ is said to be* **quasiconvex** *if given any compact segment $[t_1, t_2] \subset \mathcal{A}$ we have $f(t) \leq \max(f(t_1), f(t_2))$ for every $t \in [t_1, t_2]$.*

The solvability of $\partial/\partial z_1$ in the space of germs at 0 of holomorphic functions in $\Omega^{\mathbb{C}}$ requires a crucial part of the defining function of $\partial\Omega^{\mathbb{C}}$ at 0 to be quasiconvex, and that is how the hypothesis **(Psi)** enters in Trépreau's proof.

The open question, unanswered even for a *differential* operator of principal type with analytic coefficients, P, in dimension $N \geq 3$, is how to transition from microlocal to local respresentations (up to a $\mathcal{C}^\omega$ error) of the solutions of the equation $Pu = f$. Back to the original coordinates in $\mathbb{R}^N$, after contracting $\Omega \times \left(\Gamma \cap \mathbb{S}^{N-1}\right)$ about $\left(x^\circ, \frac{\xi^\circ}{|\xi^\circ|}\right)$ we have found a linear operator $G : \mathcal{E}'(\Omega) \longrightarrow \mathcal{E}'(\Omega)$ [recall the cut-off $\chi(x)$] such that $\left(U \times \Gamma^\flat\right) \cap WF_{\mathrm{a}}(PGf - f) = \varnothing$ for every $f \in \mathcal{E}'(\Omega)$, where U (resp. $\Gamma^\flat$) is an open set (resp., an open cone in $\mathbb{R}^N \setminus \{0\}$), with $x^\circ \in U \subset\subset \Omega$, $\frac{\xi^\circ}{|\xi^\circ|} \in \Gamma^\flat \cap \mathbb{S}^{N-1} \subset\subset \Gamma$. In suitable coordinates, varying with (x°, ξ°), G can be represented as a Fourier Integral Operator with complex phase-function reflecting Property **(Psi)**. There are finite coverings $\left(\Gamma_j^\flat\right)_{j=1,\ldots,r}$ of $\mathbb{R}^N \setminus \{0\}$ by cones like $\Gamma^\flat$, associated $\mathcal{C}^\infty$ partitions of unity $(h_j)_{j=1,\ldots,r}$ of $\mathbb{R}^N \setminus \{0\}$ of the Ehrenpreis-Hörmander type, and "right-inverses" G_j of P in $U_j \times \Gamma_j^\flat$ for each j. It is tempting to patch the G_j's together by forming $G = \sum_{j=1}^r h_j(D_x)\, G_j$ but, then,

$$PGf - f \cong \sum_{j=1}^r \left[P, h_j(D_x)\right] G_j f \bmod \mathcal{C}^\omega(U), f \in \mathcal{E}'(\Omega).$$

What is to be done with the (definitely not analytic regularizing) operator $\sum_{j=1}^{r} \left[P, h_j\left(D_x\right)\right] G_j$?