New Tools – New Ways of Thinking!? – Mathematics (and) Tools: Yesterday, Today and Tomorrow.

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Abstract. Tools played an important role in the history of mathematics. Whether you think of compass, ruler, protractor, abacus, slide rules, calculating machine, calculator or computer - new tools have always changed ways of working and, consequently, thinking. New tools also often inspired the hope to teach and learn mathematics more efficiently and to gain a better understanding of mathematical contents. In this talk, some examples from the history of mathematics will be analyzed retrospectively and will be evaluated concerning the influence of tools on mathematical thinking. Then we will answer the question what digital tools will mean for mathematics teaching in the future, for the understanding of mathematical contents and for the further development of mathematics education.

> *"Humanity's progress is documented in its tools. Tools are on the one hand results of realizations, but on the other hand, new realizations are not possible without them. We often say that the time had not been ready for certain insights – but often it should rather read that time did not have the appropriate tools to gain relevant insights." (Claus 1990, p. 43)*

In this quote by computer scientist Volker Claus, we can clearly identify a duality, namely the correlation between *tool* on the one hand and *realization*, *insight* and *progress* on the other. In the following, this correlation will be examined with special focus on the importance of mathematical tools for the teaching and learning of mathematics. In other words, we are dealing with the *didactic meaning of mathematical tools*.

1. Mathematical Tools

A tool is designed for a specific purpose. It enhances human skills (like hammer, pliers or spade) or presents people with new abilities (like binoculars, microscopes or airplanes). We want to distinguish between *tools* and *aids* (even if the boundaries are blurred): while tools are versatile and open to different results, aids have been designed for a stricter purpose (like a corkscrew, for example).

What are *mathematical tools*? Firstly, they include *real tools* such as compass and ruler, set square, parabola template, calculator and computer. But the tool concept can also be interpreted more generally. An *algorithm* can be a tool, for example Newton's method for the calculation of zeros, but also *mathematical theorems*, like the intermediate value theorem or the mean value theorem, can be used as tools to solve a problem. Even *Computer programs* may be referred to as tools: Stephan Kaufmann for example titled his book about Mathematica: 'Mathematica as a tool'. Finally, we can also consider *writing conventions* and *notations* as tools for mathematical thinking.

Mathematical tools can be found on the *real* or *enactive level* (compass, ruler, parabolic compass), the *object level* (theorems, algorithms, computer programs), and on the *symbolic level* (writing conventions, notations).

Real mathematical tools are usually produce mathematical objects (like curves or numbers) on the symbolic level (for examples, see next section). They are designed for a specific *purpose*, namely, to produce or create mathematical objects, such as geometric curves or the calculation of the greatest common divisor of any given numbers. Tools on the mathematical object level transform mathematical objects into other mathematical objects. This includes computer programs in particular, for example dynamic geometry software (DGS) or computer algebra systems (CAS).

Tools are built on a *mathematical idea*, and have a *constructive* or *technical idea* as the basis of their design. Additionally, tools can be attributed with a *didactic meaning*, as they are used to gain knowledge about the created objects and their properties.

This last aspect addresses the importance of a tool as a medium, as a mediator: By using a tool, a learner can grasp and understand a mathematical idea, a theorem or a relationship. Regarding this concept, the term "Semiotic Mediation" has established itself in international discourse, but cannot be discussed here in detail (cf. Bartolini Bussi and Mariotti 1999).

3. Historical Examples

Instruments for drawing curves were already developed in ancient times. It is said that *Menaechmu*s (about 350 BC), for example, already knew how to draw a parabola using a piece of string – the same method as the one used by Johannes Kepler (1571-1630).

3.1 String Construction of a Parabola

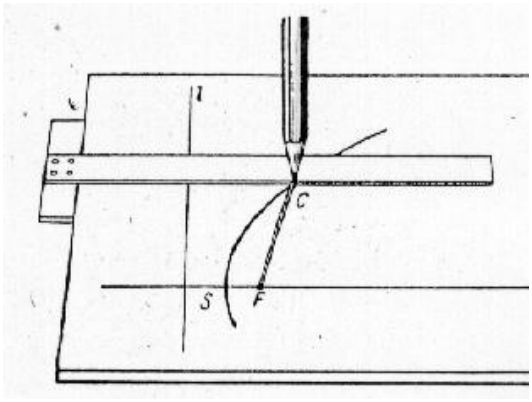


Fig. 1: String construction of a parabola

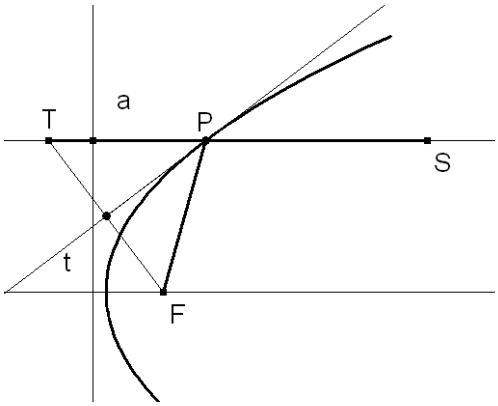


Fig. 2: Computer simulation of a parabola construction

For the string construction of a parabola, we attach a piece of string with length L to a right angle which can be moved along the rail. One end of the string is then fixed to the rail at the point S (see Fig. 2), the other end at the point F. Now we position a pen next to the rail at the point P so that the sting is taut. In this structure, we obtain $|SP| + |PF| = L = \text{const}$. Let T be a point on SP with $|TP| = |PF|$, then: $|ST| = L = \text{const}$. Hence T lies on the directrix of the parabola.

The advantages of computer representation over mechanical construction become immediately apparent when one considers the construction dynamically. The length of the string does not limit the construction and the locus can still be drawn if $P \notin [ST]$. In

addition, in a mechanical construction it is only possible to draw (one) part of the parabola at a time without lifting the pen off the paper. This limitation does not apply to a computer drawing.

3.2 Frans van Schooten's Instrument for Drawing Parabolas (1615-1660)

In 1646, Frans *van Schooten* (1615 - 1660) translated the "*Geometry*" of *Descartes* into Latin and subsequently submitted numerous proposals regarding mechanical instruments which can be used to generate conic sections. His aim was to replace the inaccuracy of string constructions of conic sections by introducing joint constructions which were more stable.

Van Schooten's parabola construction is seen in Fig. 3 (cf. van Maanen 1995). Given a rhombus supplied with joints at the four vertices, point *B* remains fixed and point *G* is moved along a rail *S*. The pen which draws the parabola is attached to the point of intersection of the rail lying alongside the diagonal *FH* and of the rail lying perpendicular to *S*. The drawn curve is in fact a parabola, as $|DG| = |DB|$. This can be seen either from the fact that *FH* is the midperpendicular of the line segment [*BG*], or from the fact that the two triangles ΔBHD and ΔDHG are congruent. *FH* is therefore also a tangent to the parabola.

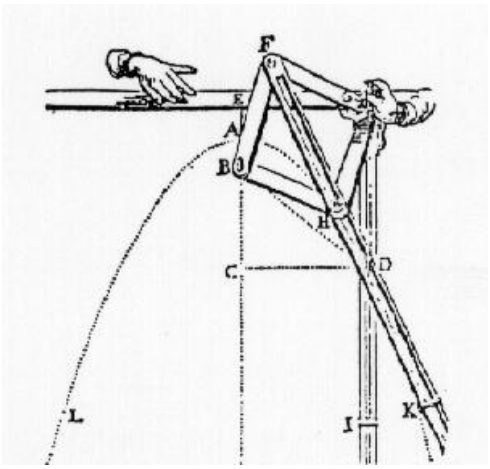


Fig. 3: Instrument for drawing parabolas by van Schooten [1]

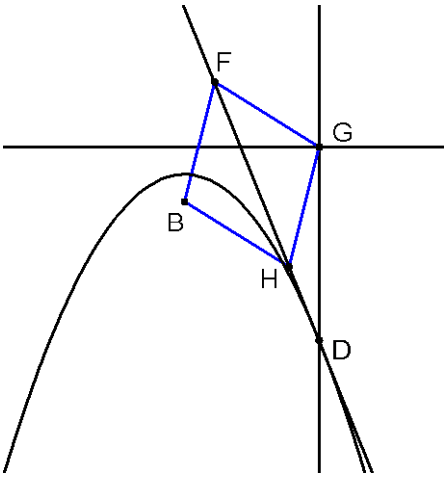


Fig. 4: Computer simulation of the parabola drawer

[1] The drawings of the instruments by van Schooten are taken from Maanen (1995).

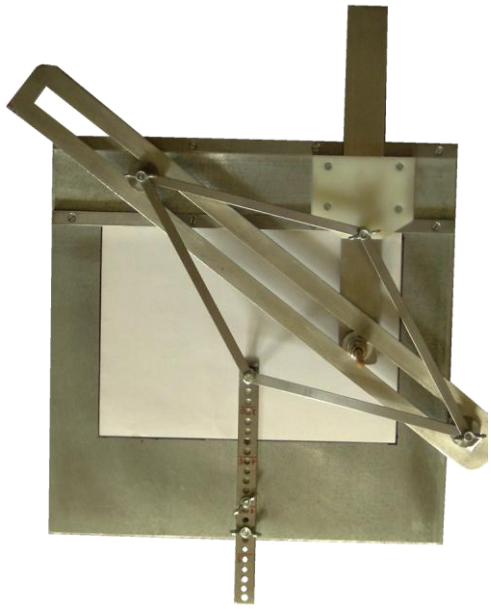


Fig. 5: Real reconstruction of van Schooten's parabola drawer (both by Christian van Randenborgh).

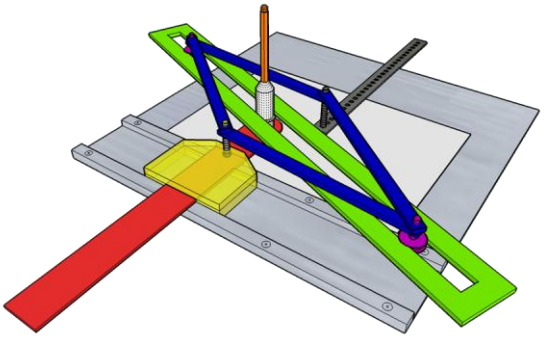


Fig. 6: Virtual reconstruction with Geogebra[2] of van Schooten's parabola drawer

Again, the limited scope of the real instrument is apparent: the drawing of the parabola must be interrupted when *D* approaches *H* or *F*. In the computer simulation, these restrictions do not occur if we use the line *FH* instead of the ray [*FH*.

When we move objects dynamically in a computer simulation, many questions arise. We can confirm their answers experimentally with either the real or the virtual instrument. For example, we can show that the construction is independent of the length of *BF* (see Fig. 4) or that the shape of the parabola depends on the position of the point *B* (focus!). We can also ask how a particular parabola can be drawn with this instrument, for instance a parabola that satisfies the equation $y = 2\,x^2$ (with an appropriate coordinate system).

To incorporate the parabola drawer in mathematics lessons, one can take the original drawing by van Schooten as a starting point. The students can then create real reconstructions of the instrument - for example in a simplified form with strips of paper, or they can work with computer simulations. By creating and using the instrument, students shall grasp the underlying mathematical concept and gain a new and deeper understanding of the properties of the given mathematical object (in this case the parabola). However, beyond this, we should not underestimate the historic-cultural and aesthetic aspect of working with historical instruments (see van Randenborgh 2012).

3.3 The Mechanical Calculating Machine

At the beginning of the 17th Century, the first plans to mechanize the number calculation process emerged. The first calculating machine originated from Professor William

[2] www.geogebra.org/cms/en/

Schickard in Tübingen (1592-1635), and was probably designed for his friend Johannes Kepler. In 1642, the nineteen-year-old Blaise Pascal (1623-1662) built several adding machines with an automatic tens-carry mechanism. In 1672, G.W. Leibniz (1646-1716) designed a calculating machine, because "it is unworthy of a human being to lose hours, like a slave, with monotonous stupid calculations ". Leibniz came up with the idea for his calculating machine when he saw a step counter with an automatic tens-carry:

> *"A few years ago, when I first saw an instrument by means of which one can count one's own steps without thinking, a thought came to me immediately: It could be helpful for the whole of arithmetic to have similar kind of tool." (Mackensen 1969 , p. 41)*

Towards the end of the 19th Century, calculating machines finally went into mass production.



Fig. 7: A calculating machine from the beginning of the 20th century

With the use of calculators, thinking and working methods regarding the calculation with numbers changed. People were no longer concerned with calculating in the sense of using a multiplication table, but with understanding a calculation algorithm expressed in the turn of a handle or the shift of a slide. Thinking in operational sequences became more important than arithmetic thinking. These calculating machines, however, were too expensive and too unwieldy to be actually used in a class room.

The mechanical calculating machine is an example of a tool that enhances human skills by enabling us to perform arithmetic calculations faster. But calculating machines also have a didactic function: they visualize the algorithmic calculation during multiplication, division or square root extraction.

This is why Felix Klein (1908) wrote:

> *"Above all, every teacher of mathematics should be familiar with it." (p. 22)*

The calculating machine is an example of the dual nature of tools in mathematics education. On the one hand, it is a "number cruncher" that performs mechanical calculations quickly, but on the other hand it is also a visualizer for arithmetic calculation methods.

4. Digital Tools

> "*The progress of science requires both new concepts and new tools. … Most of the recent scientific revolutions have been tool-driven." (Dyson 2000)*

This view of tools in science holds particularly true in mathematics when looking at the computer tool. Not only does it enable us to perform proofs (Four-color Theorem!) or work through algorithms much faster, but it also gives us entirely new possibilities in the context of computer simulations, technical processes (computed tomography) and visualizations.

4.1 Instrumental Genesis

Digital tools can be found on the representational level (hardware) and on the mathematical and symbolic level (software). Regarding their usage, the *Theory of instrumental genesis* has emerged in recent years (e.g. Trouche 2005). It examines the question how new technologies – especially handheld computers – can become *instruments* or *tools* that can help us find solutions to certain problems. For this purpose, it is - according to this theory - necessary to establish an interrelation between the device (or artifact) and the user.

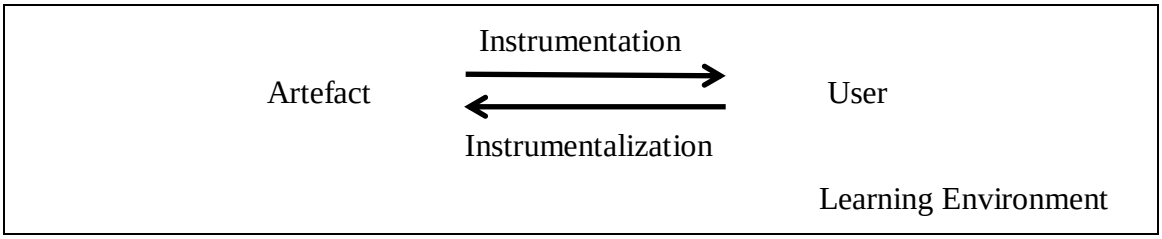


Fig. 8: Basic scheme of the theory of the instrumental genesis

During the *instrumentation process*, the user develops mental schemata or models regarding the possibilities and limits of the device within the boundaries of a given problem (e.g. regarding the menu structure, command syntax, limits of the internal accuracy of the computer or the screen resolution). During the *instrumentalization process*, the device is adjusted to the user's activity (e.g. by changing menus, setting up macros, programming or defining commands). During the *development of this instrumental* genesis, an artifact becomes an *instrument* or a *tool*, which comes both with its own technical or operational knowledge and with knowledge about how it should be used appropriately. The central task now is to develop the *learning environment* or the *instrumental orchestration* in which this tool development happens.

4.2 Expectations

There have always been expectations and hopes when it came to the introduction of new media and tools. This is what Thomas Alva Edison (1847-1931) thought after the invention of sound films:

> *„I believe that the motion picture is destined to revolutionize our educational system, and that in a few years it will supplant largely, if not entirely, the use of textbooks"(1922).*

Governor Arnold Schwarzenegger's statement on 9 June 2009 went along the same lines:

> *"The textbooks are outdated ... and there's no reason why our schools should have our students lug around these antiquated and heavy and expensive textbooks."*

The same expectations were also associated with mathematical tools. At the beginning of the 20th Century, for example, the introduction of the slide rule in mathematics teaching was controversial.

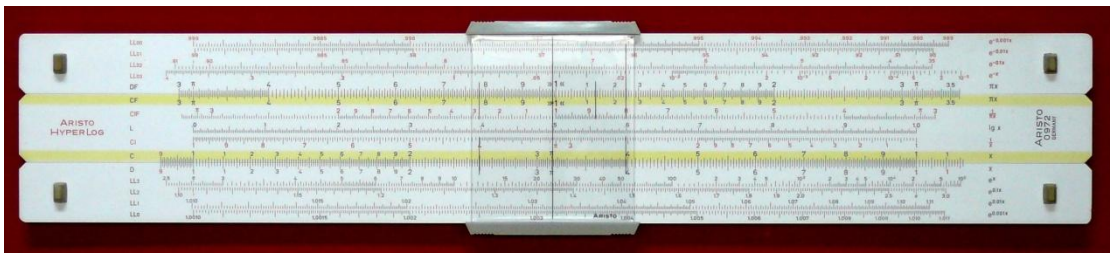


Fig. 9: The slide rule

In his book "The Slide Rule in School Education", published in 1929, Albert Rohrberg presented the special educational value of working with this device. He opposed the claim that the slide rule is only a "number cruncher" which mechanizes computational work. Instead, he spoke of the possibility that the slide rule challenges its user to do the exact opposite, leading to a spiritualization of the calculation process. Everyone who has ever used a slide rule during their years of school and training can understand what he meant by this statement. Rohrberg speaks of the unavoidable anticipation (estimation) of the calculation process which is necessary to set the decimal point correctly in the end.

A similar controversy reappeared with the introduction of the calculator.

4.3 Visions

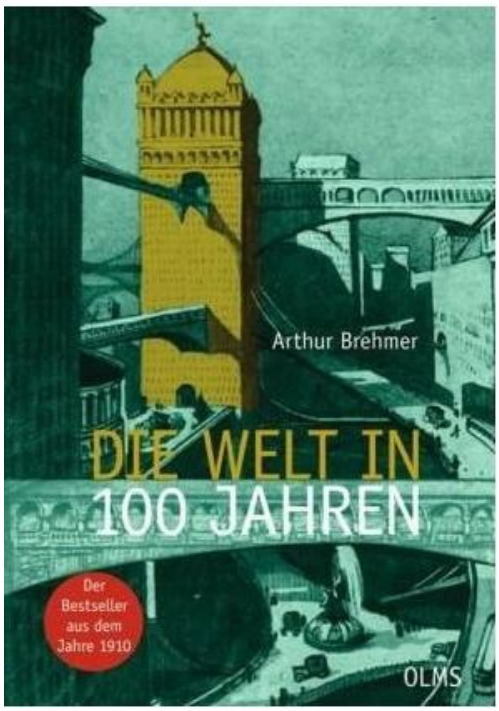


Fig. 10: Arthur Brehmer, The World in 100 Years (1910)

For a book published in 1910, Arthur Brehmer asked influential scientists of his time to describe "The World in 100 Years." In one article of this book, entitled "The Wireless Century", Robert Sloss describes "The telephone in the vest pocket" (p. 35ff):

> *"The citizen of the wireless century will walk everywhere with his 'receiver' ... On his way to work, in the underground, everywhere he will listen to the „spoken newspaper" and he will get all the news he wants ... And if he wants, he will be able to connect with every theater, every church, every concert hall and he will be able to take part in the lecture, the sermon, the music session. .... The events of the whole world will be open to him ..."*

This vision became reality about 100 years later – with the introduction of the "iPhone".

Similarly visionary were (are) the NCTM standards of 1989 (or in the revised version of 2000): being "ideal standards", they represent a vision for the future of mathematics education. This applies in particular to the use of new technologies. For this purpose, the *Technology Principle* was created:

> ***"****Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." (p. 24)*

And also

> *"... Calculators and computers are reshaping the mathematical landscape ... Students can learn more mathematics more deeply with the appropriate and responsible use of technology. ..." (p. 25)*

The first ICMI study, entitled "The Influence of Computers and Informatics on Mathematics and its Teaching" (Churchhouse 1986), also conveyed great enthusiasm regarding the development perspectives of mathematics teaching due to the availability of new technologies. Many, like Jim Kaput, predicted that new technologies would quickly change all areas of teaching mathematics:

> *"Technology in mathematics education might work as a newly active volcano – the mathematical mountain is changing before our eyes" (1992, p. 515).*

4.4 Calculator Tool



Fig. 11: The first calculator (1972): The HP 35

In 1972, the first pocket calculator was introduced into the market. Between 1976 and 1978 its usage was permitted in many schools or was even made mandatory.

In a statement given on 28 February 1978, the German Society for Mathematics Education demanded the "controlled use of calculators from the 7th grade on for all school types "(p. 117). The aim was to

- enable students to perform experimental activities following the learning through discovery and problem solving approach,
- form a concrete numerical basis for concept formation,
- treat application tasks in a realistic way by using numbers taken from reality,
- reduce activities that are not of central importance for the solution of a given problem,
- gain access to algorithmic thinking,
- make exercise phases correspond to the problems at hand.

But the focus of the discussion was on the question of how "global learning goals for mathematics teaching ... can be achieved in a better way ", and even then some demanded a "profound change in the objectives of teaching mathematics ". It is obvious from today's point of view that these hopes, demands and objectives were too far-reaching and have probably only been achieved to a small extent. Different reasons can be given as the causes for this development (see Weigand 2003).

4.5 Handheld Computer Tool



Fig. 12: Texas Instruments TI-Nspire (2012)



Fig. 13: Casio fx CP400 (2013)

When the first handheld computers "TI-92" and "Casio FX 2.0" were introduced, expectations were just as high as they had been regarding arithmetic calculators. The fact that students would now have computers available at any time at their desk in the classroom and that the "hike" to the computer room would hence become obsolete, made people expect profound changes in teaching contents and methods, but also brought fears and anxieties with it. A discussion started: What should happen to content and tests in mathematics classes, considering that students now had a device at hand that could perform exactly those calculus-based calculations which – in the reality of classrooms - were the main elements of mathematics teaching and examinations? There was a lively debate as to when the right time was to use a computer, whether it should be permitted in examinations and how the importance of traditional mathematical skills would change in the future (cf. Herget et. al. 2000).

4.6 Disappointments

Today, however, expectations regarding the impact of using computers on goals, contents and methods of teaching have been brought down to a realistic level: *Revolution* in

mathematics does not take place due the existence of a new tool, but *evolution* or a successive meaningful integration of new technologies will continue on.

In the current ICME study 17, "Mathematics Education and Technology - Rethinking the Terrain" (Hoyles and Lagrange 2010), disappointment is evident in many places: despite of countless ideas, practical teaching experience and research reports about educational usage, new technologies did not prevail as expected or hoped for at the beginning of the 1990s. Some quotes from the ICME Study:

> *"Technology still plays a marginal role in mathematics classrooms" (p. 312)*
>
> *"The impact of this technology (CAS) on most curricula is weak today" (p. 426)*
>
> *"The situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study (20 years ago) have been fulfilled." (p. 464)*

This study gives a good overview of the many activities which took place in recent years regarding the usage of new technologies in mathematics teaching and learning (cf. Weigand 2010). The book does not provide a vision (anymore), but instead lists questions which are quite similar to those from 20 years ago. One may interpret this - partly - as a form of resignation, but one can also see it as a sign that these questions are very difficult to answer. Finally, one can understand it as an invitation to develop new ideas - visions - to advance the real integration of new technologies into the teaching of mathematics.

The future use of handheld computers in mathematics education is up to speculation at the moment. How they will be used in class will certainly stand or fall by the answer to the question whether they will be approved for examinations – especially centralized examinations – or not.

Below, we will take a look at some theses which want present a more pragmatic view and which will give instructions to increase the impact of new technologies on the reality of teaching mathematics.

5. Theses for the Future

5.1 Convincing Others of the Benefits of Digital Technologies (DT)

Despite of the countless ideas, classroom suggestions, lesson plans and research reports, nowadays the use of DT has not succeeded, as many had expected in the last decades. In the ICMI Study 17 "Mathematics Education and Technology – Rethinking the terrain" (Hoyles & Lagrange 2010), we can often find disappointment about the fact that "technology still plays a marginal role in mathematics classrooms" (p. 312). In the near future, we will have to think of convincing examples for the adequate usage of digital technologies in the mathematics classroom and we will have to evaluate their benefits.

> ***Thesis 1*: We underestimated the difficulties of DT-use – in a technical sense and in relation to the contents – and we have not been able to convince teachers, lectures at university and parents of the benefit of DT in the classrooms.**

There are many empirical investigations concerning the use of CAS and SC in mathematical teaching and there is a general agreement regarding some of the main results: the use of CAS gives working with diagrams a higher significance and reinforces experimental work, in which assumptions are made through systematic testing. In addition, technology brings an increase in individual work as well as cooperative forms of work (e. g. Guin, Ruthven and Trouche 2005, Kieran and Drijvers 2006, Zbiek 2007, Hattie 2009). Above all, empirical research proves a slight benefit of the use of computers and new technologies in the classroom (see Hattie 2009).

Many investigations are well designed and evaluated, taking scientific aspects into account. However, most of them are restricted to applications of the use of DT for just a few weeks and do not show the long-term effects on students' knowledge and abilities. From a scientific point of view, the primarily long-term effects on knowledge and student ability and performance are admittedly difficult to prove. It is therefore necessary to develop a specialized educational concept to evaluate teaching, to assess exam questions and their findings and ultimately to evaluate changes in performance over a longer period of time.

5.2 Limits of the Tool

Computers are quite limited tools compared to the properties of mathematical concepts. Numeric calculations can only be done with a fixed accuracy; graphic representations on a screen show only a small part of the whole object representation in a discrete pixel world. Students have to be aware of these limitations of DT.

> ***Thesis 2*: Working with DT requires the knowledge of the limits and restrictions of DT concerning the mathematical contents.**

Example: Given is the function f_c with $f_c(x) = \frac{1}{2}x^3 - 6x + c$, $c \in IR$. For which values of c does f_c have exactly one zero? Give reasons!

Solving the problem on the symbolic level (Fig. 14) presents the students with a result they cannot interpret. A solution on the graphic level requires a dynamic point of view and the knowledge about a connection between equation and graph. The values for which there is only one zero can be found by changing the parameter c using a "slider".

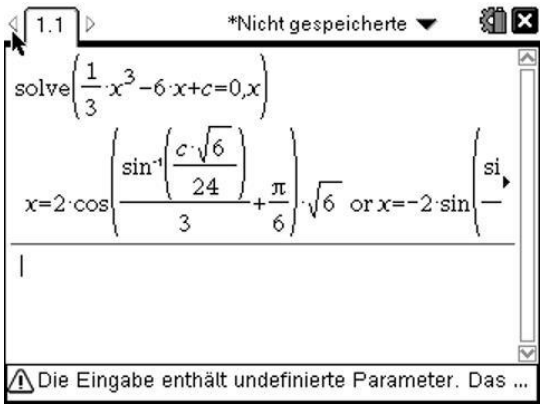


Fig. 14: Symbolic Solution of f_c (x) = 0

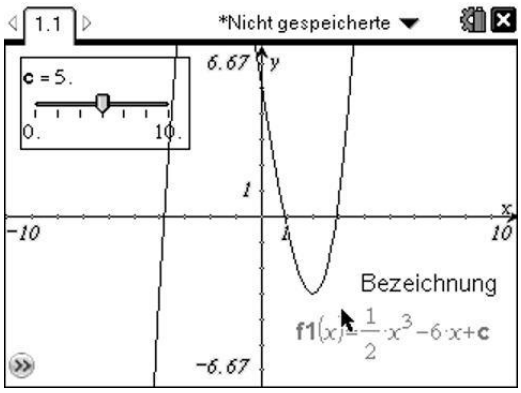


Fig. 15: Graph of f_c with c = 5

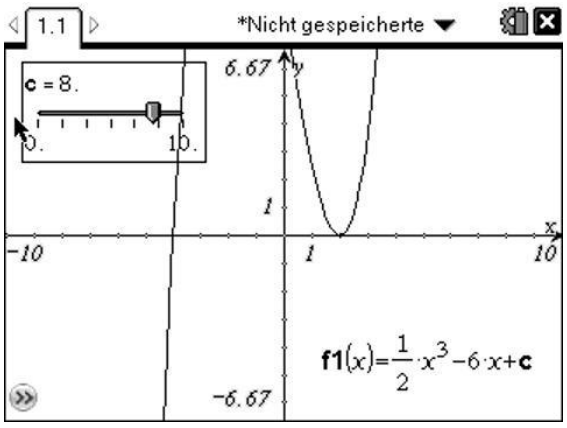


Fig. 16: Graph of f_c with c = 8

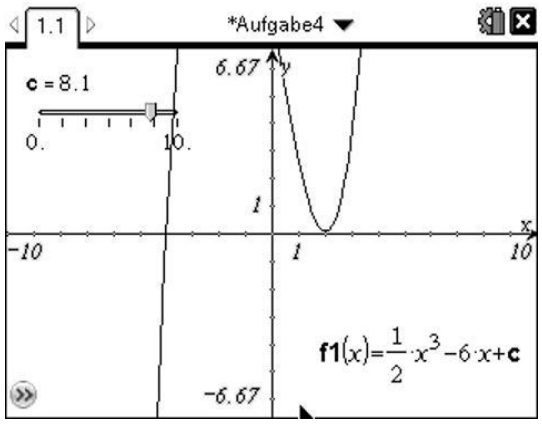


Fig. 17: Graph of f_c with c = 8.1

The solution of this problem: The equation f_c (x) = 0 has exactly one solution for c > 8 or c < –8. Due to the preset step size of the slider – for the TI-Nspire – the c-value changes in steps of 0.1. This discrete change pattern can then be found in many (wrong) student solutions. Examples of student solutions:

• For c > 8.1, the function has exactly one zero

• Slider c = 8.1. From c = 8.1 (and higher), the function $f_c(x)$ has only one zero.

• c ≥ 8.1

These solutions show the students' lack of ability to interpret the displayed (discrete) values properly and to translate them into the continuum of real numbers.

5.3 The Relation between Real and Mental Representations

The main reason for working with DT in the classroom is the goal of a better understanding of mathematical concepts. Understanding means developing *mental structures*

or *representations*. They are based on the knowledge, the abilities and the skills of using and interpreting real – symbolic, graphic and numeric – representations.

***Thesis 3*: In spite of the existence of interactive, dynamic and multiple digital representations, the main challenge is the development of mental representations.**

Example: Given are f and g with $f(x) = sin(x) + 1$ and $g(x) = 2^x$. How many intersection points do the graphs have a) in the interval $x \in [10; -10]$ and b) if $x \in IR$? Give reasons!

Solving this equation on the symbolic level with the SC gives the result (Fig. 18):

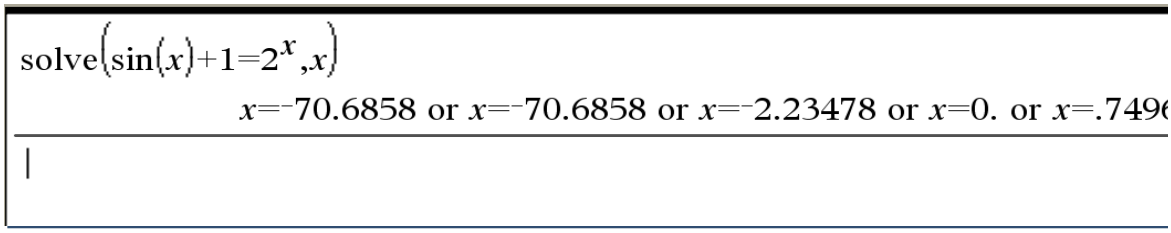


Fig. 18: The display of the TI-Nspire. A warning sign appears: "Some more solutions may exist".

It is quite difficult – and for students (nearly) impossible – to interpret this display's numeric solution. Users (students) of DT need to have strategies to control, verify and revise solutions obtained with the help of DT. Changing to the graphics screen and zooming into interesting sections is a good strategy. This gives the following graphs:

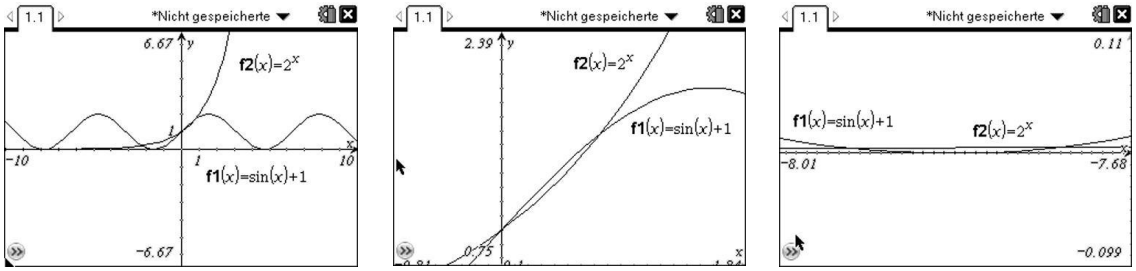


Fig. 19: Screen shots of the functions with $f(x) = sin(x) + 1$ and $g(x) = 2^x$.

To indicate the "interesting sections" and to interpret the graphs–for the area $x < 0$ especially– requires an advanced basic knowledge of the properties of the sine function and the exponential function. The solution in question 1 b) – an infinite number of intersection points – cannot be obtained from the calculator screens, it has to arise from the basic knowledge of "key structural elements" (Greer 2009, 698) in an implied *mental representation* of the functions.

This example shows on the one hand that the question concerning the adequate representation depends – of course – on the problem and the expected level of accuracy or

ability. On the other hand, this example also emphasizes the meaning of the perpetual construction of the *relationship* between *mathematical objects* and *concepts* and their *real* and *mental representations.* Despite the importance of different interactive multiple (digital) representations, the most important representation, which has to be developed in mathematics, is the *mental representation*.

5.4 Mathematical Connectivity

There are repeatedly calls for a stronger networking of mathematical areas, especially between algebra and geometry, in the teaching process and in mathematics lessons. The program "Geogebra" (**Geo**metry and Al**gebra**) supports this algebraic-geometric relationship of mathematical objects.

The demand for a "strengthening of the spatial perceptivity" through greater consideration of space "as the carrier of planimetric relations" can already be found in the Merano reform (1905). Since then, the development of this relationship has been called for repeatedly, but it has led to only little or partial use in geometry lessons. With the introduction of interactive 3-D programs such as Cabri3D or Archimedes Geo3D, it is now possible to create 3-D objects (on the screen) and to modify and experiment with them interactively. However, handling 3-D objects is significantly more complex than handling 2-D objects.

Overall, the hope becomes clear that new technological developments will allow us a step towards a better representation of geometric relations and that they will provide us with an adequate visualization of our environment. New technologies can be tools for a better integration of geometry in other areas of mathematics – and concerning mathematics teaching, especially in the area of algebra.

> ***Thesis 4*: New technologies can lead to further improvement and development of the interaction between algebra and geometry, but also between 2-D and 3-D geometry.**

5.5 External Connectivity

An integrated global concept of the use of new technologies has to follow different aspects; it concerns the interaction of different digital components such as laptops, netbooks, the Internet and pocket computers under technical aspects; it concerns the use of classroom materials, and it should support the cooperation between the teachers of a school, between school and school administration, and also between school and university.

Fig. 20 expresses the necessity to realize the connectivity of handheld technology with a lot of other media and tools in mathematics classroom learning: laptops, smartphones, interactive whiteboards, the Internet and the connectivity via a CMS to the working place at home, traditional books and e-books. Moreover, the head of the school has to support the use of DT in classrooms, all (mathematics) colleagues and the parents of the students have to be convinced of the benefit of the new tools, and it is necessary to develop regular teacher trainings.



Fig. 20: External Connectivity

***Thesis 5*: Connectivity and interconnectedness will be key words in the future. The acceptance of new technologies and their beneficial use require a global concept of teaching and learning.**

6. Outlook

Key questions regarding the use of new technologies and a change in teaching and learning cannot be answered conclusively, of course. With new and ever changing tools, "old" questions can be asked in ever new ways. The list below shows some questions which on the one hand are “old” questions requiring new answers, but which on the other hand can be asked constantly in a new way, due to our rapid technological progress.

- What defines basic knowledge and what is its significance in the context of the increased use of new technologies?
- How important are central traditional ways of working, e.g. proving, modeling and presenting, when we are faced with digital tools? How can these working methods (or some of them) be maintained and developed further? Which methods will become obsolete, which will have to be added?
- How will the relationship between "traditional" teaching (board, book, ...) and the "digital world of teaching" look like?
- What impact will the progress of mobile technology have on the methodology of teaching mathematics in and beyond the classroom?
- What is meant by students having "tool competencies" regarding the effective use of new technologies and how they can be developed (cf. Weigand & Bichler

2010)?

Let us hope that in the next few years we will get at least one step closer to answering these questions.

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