


Lecture 1: An Introduction to CR Geometry

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10th Workshop on the Geometric Analysis of PDEs and Several Complex Variables




Outline of the Talk

- CR Geometry – What is it?
- Boundary values of holomorphic functions in $\mathbb{C}$
- Holomorphic functions in several variables
- Differential geometry preliminaries
- An explicit calculation of the tangential Cauchy-Riemann operator on model graphs.
- A general extrinsic calculation of $\bar{\partial}_b$.

CR Geometry – what is it?

By CR geometry on a embedded manifold in $\mathbb{C}^n$, I mean the geometric and analytic structures imposed on/inherited by the manifold from ambient holomorphic functions.

On an abstract manifold, CR geometry (if there is any) is a geometry that exhibits the same structure as CR geometry on an embedded manifold.

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Holomorphic functions in $\mathbb{C}$

Notation: $z = x + iy$, $dz = dx + idy$,

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \qquad \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

A function u is holomorphic on $\Omega \subset \mathbb{C}$ if $\dfrac{\partial u}{\partial \bar{z}} = 0$ on Ω.

Theorem

If $f \in C^1(\bar{\Omega})$, then

$$f(z) = \frac{1}{2\pi i}\left\{ \int_{b\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \iint_{\Omega} \frac{\partial f/\partial \bar{z}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \right\}$$

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Notation in $\mathbb{C}^n$

Coordinates: $z = (z_1, \ldots, z_n)$. $z_j = x_j + iy_j$.

Vectors: for $1 \leq j \leq n$,

$$\frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right) \qquad \frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right)$$

The Cauchy-Riemann operator

Let $\Omega \subset \mathbb{C}^n$ be a domain and f a smooth function on Ω.

The Cauchy-Riemman operator, $\bar{\partial}$, on f is

$$\bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} \, d\bar{z}^k.$$

Let $\mathcal{I}_q = \{J = (j_1, \ldots, j_q) \in \mathbb{N}^q : 1 \leq j_1 < j_2 < \cdots < j_q \leq n\}$. A $(0, q)$-form

$$f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$$

where $d\bar{z}^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$. In general, the Cauchy-Riemann operator $\bar{\partial} : C^\infty_{0,q}(\Omega) \to C^\infty_{0,q+1}(\Omega)$ is defined by

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Holomorphic functions in several variables

Definition

A function $f : \Omega \to \mathbb{C}$ is holomorphic if $\bar{\partial} f = 0$ on Ω. Denote by $H(\Omega)$ the space of holomorphic functions on Ω.

Motivating Question: If $f \in H(\bar{\Omega})$, then what equations do $f|_{b\Omega}$ satisfy?

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Vector fields

Definition

A vector field X on Ω is a differential operator

$$X = \sum_{j=1}^{n} \left(a_j(z) \frac{\partial}{\partial z_j} + b_j(z) \frac{\partial}{\partial \bar{z}_j} \right).$$

X is called an anti-holomorphic vector field if $a_j = 0$ for all j and a holomorphic vector field if $b_j = 0$ for all j.

Note that if f, g are functions, then X satisfies the Leibniz rule

$$X\{fg\} = gXf + fXg.$$

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Differential Operators on $\mathrm{b}\Omega$

Question: How do we define differential operators on $\mathrm{b}\Omega$?

Idea: Given a vector field X on $\bar{\Omega}$ and a function $f \in \mathrm{b}\Omega$,

1. Extend f smoothly to F on Ω.
2. Apply X to F.
3. Define $Xf = XF|_{\mathrm{b}\Omega}$ via restriction.

Question: Under what conditions is this procedure well-defined?

We need to check when Xf is independent of the extension.

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Defining Functions

A defining function for Ω is a C^1 function ρ so that

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0 \text{ and } d\rho \neq 0 \text{ on } b\Omega\}.$$

That a defining function exists for Ω follows immediate from the Implicit Function Theorem!

Let F_1 and F_2 be extensions of f. Then

$$(F_1 - F_2)\big|_{b\Omega} = 0.$$

By advanced calculus, there exists a function G so that

$$F_1 - F_2 = \rho G.$$

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Tangential Vector Fields

Then if Xf is well-defined,

$$0 = X(F_1 - F_2)\big|_{b\Omega} = GX\rho\big|_{b\Omega} + \rho XG\big|_{b\Omega} = GX\rho\big|_{b\Omega}.$$

This means Xf is well-defined if and only if

$$X\rho\big|_{b\Omega} = 0.$$

Such vector fields are called tangential. Observe that in real coordinates, if $X = \sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j} + d_j \frac{\partial}{\partial y_j}$, then

$$0 = X\rho = \sum_{j=1}^{n} \left(c_j \frac{\partial \rho}{\partial x_j} + d_j \frac{\partial \rho}{\partial y_j} \right) = (\vec{c}, \vec{d}) \cdot \nabla \rho$$

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Boundary values of holomorphic functions

Theorem

Let $F \in H(\bar{\Omega})$ and $f = F\big|_{b\Omega}$. Then $\overline{Z} f = 0$ whenever

$$\overline{Z} = \sum_{j=1}^{n} b_j \frac{\partial}{\partial \bar{z}_j}$$

is an anti-holomorphic, tangential vector field (AHTVFs).

Question: And the converse?

That is much harder. . .

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An instructive example

Suppose $p : \mathbb{C}^n \to \mathbb{R}$ is a smooth function and

$$\Omega = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w > p(z)\right\}.$$

Then a defining function for Ω is

$$\rho(z, w) = p(z) - \frac{w - \bar{w}}{2i}.$$

Side Note: There are n anti-holomorphic vectors $\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$. That we need $\overline{Z}\rho = 0$ is *one* constraint, thus a basis for the AHTVFs is n-dimensional.

Side Note 2: Ω is sometimes called a model domain.

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More when $\Omega = \{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w > p(z)\}$

Observe that $\frac{\partial \rho}{\partial \bar{w}} = \frac{1}{2i}$. A basis for AHTVFs is given by

$$\bar{L}_k = \frac{\partial}{\partial \bar{z}_k} - 2i \frac{\partial \rho}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}, \qquad 1 \leq k \leq n.$$

Given a function f on $b\Omega$, we define the tangential Cauchy-Riemann operator, $\bar{\partial}_b$,

$$\bar{\partial}_b f = \sum_{k=1}^{n} \bar{L}_k f \, d\bar{z}^k.$$

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$ is a $(0, q)$-form, then we set

$$\bar{\partial}_b f = \sum_{J \in \mathcal{I}_q} \bar{\partial}_b f_J \wedge d\bar{z}^J.$$

Note: That $\bar{\partial}_b$ can be expressed using $\{d\bar{z}^k\}$ is very special to this case.

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The Heisenberg group

The example *par excellence* is the Heisenberg group

$$\mathcal{H}^n = \{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = |z|^2\}$$

so $p(z) = |z|^2$ and $\rho(z) = |z|^2 - \dfrac{w - \bar{w}}{2i}$. In this case,

$$\bar{L}_k = \frac{\partial}{\partial \bar{z}_k} - 2iz_k \frac{\partial}{\partial \bar{w}}$$

and

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We will return to the Heisenberg group repeatedly!

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and

$$\bar{\partial}_b f = \sum_{k=1}^{n} \bar{L}_k f \, d\bar{z}^k.$$

We will return to the Heisenberg group repeatedly!

Working towards a general framework – the setup

We work on a smooth submanifold $M \subset \mathbb{C}^n$ and follow the construction in Boggess (1991). We use the following notation:

- For $p \in M$, $T_p(M)$ is the real tangent space of M at p.
- The complex structure map $J : T_p(\mathbb{R}^{2n}) \to T_p(\mathbb{R}^{2n})$ is defined by

$$J_p\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j} \quad \text{and} \quad J_p\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}.$$

  Note: $T_p(M)$ may or may not be J-invariant. If $M = \mathrm{b}\Omega$, then $T_p(M)$ is never J-invariant!
- Define the maximal complex tangent space $H_p(M)$ by

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

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Working towards a general framework – the setup

The totally real part of the tangent space $X_p(M)$ is defined to be

$$X_p(M) = T_p(M)/H_p(M).$$

Definition

M is an embedded CR manifold if $\dim_{\mathbb{R}} H_p(M)$ is independent of $p \in M$.

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If $M = b\Omega$ and $\Omega \subset \mathbb{C}^n$, then

$$\dim_{\mathbb{R}} T_p(M) = \dim_{\mathbb{R}} J\{T_p(M)\} = 2n - 1$$

and $T_p(M) \neq JT_p(M)$ since $J^2 = -I$. Thus,

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Let $M = \{z \in \mathbb{C}^n : |z| = 1 \text{ and } \operatorname{Im} z_1 = 0\}$ (the equator of the unit sphere). Then

$$T_{(0,1,0,\ldots,0)} = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \text{ for } j = 3, \ldots, n \right\}$$

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Types of CR submanifolds

Let M be an embedded CR submanifold.

- If $H_p(M) = \{0\}$, then M is called totally real.
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Complexifying

Recall: When $M = \{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = p(z)\}$, we had the AHTVFs

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - 2i \frac{\partial p}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}}, \qquad j = 1, \ldots, n$$

Question: Where do these vectors fit in with $T_p(M), H_p(M)$, etc?

Problem: $\bar{L}_j$ are not *real* vectors!

Solution: Complexify!

Look at

$$T_p(M) \otimes \mathbb{C} \qquad H_p(M) \otimes \mathbb{C} \qquad X_p(M) \otimes \mathbb{C}.$$

Note that if $\dim_{\mathbb{R}} T_p(M) = k$ then $\dim_{\mathbb{R}} T_p(M) \otimes \mathbb{C} = 2k$.

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$H^{1,0}(M)$ and $H^{0,1}(M)$

Since $H_p(M)$ is J-invariant, so is $H_p(M) \otimes \mathbb{C}$.

It turns that $H_p(\mathbb{C}^n) \otimes \mathbb{C}$ (and therefore $H_p(M) \otimes \mathbb{C}$) splits into $+i$ and $-i$ eigenspaces of J, denoted $T_p^{1,0}(\mathbb{C}^n)$ and $H_p^{0,1}(\mathbb{C}^n)$.

In fact

$$H_p^{1,0}(M) = T_p^{1,0}(\mathbb{C}^n) \cap \{T_p(M) \otimes \mathbb{C}\}$$
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A word of caution

Just a comment about language:

- The (maximal) complex tangent space: $H_p(M)$
- The complexified tangent space: $T_p(M) \otimes \mathbb{C}$
- The complexified (maximal) complex tangent space: $H_p(M) \otimes \mathbb{C}$.
- The set of anti-holomorphic, tangent vectors: $H_p^{0,1}(M)$.
- To switch from spaces to bundles, union over $p \in M$, e.g., the tangent bundle $T(M) = \bigcup_{p \in M} T_p(M)$.

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An involutive structure

Given vector fields X, Y, the Lie bracket of X and Y is

$$[X, Y] = XY - YX.$$

Definition

A subbundle is involutive if it is closed under the Lie bracket.

We have the following critically important lemma!

Lemma

Let M be a CR submanifold of $\mathbb{C}^n$. Then

a. $H^{1,0}_p(M) \cap H^{0,1}_p(M) = \{0\}$ *for each* $p \in M$.

b. *The subbundles* $H^{0,1}(M)$ *and* $H^{1,0}(M)$ *are involutive.*

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Working towards $\bar{\partial}_b$ on embedded CR manifolds

- The bundle of 1-forms on $\mathbb{C}^n$ is $T^*(\mathbb{C}^n)$.
- The bundle of $(0,q)$-forms on $\mathbb{C}^n$ is denoted by $\Lambda^{0,q}T^*(\mathbb{C}^n)$.
- We can put a metric on $\Lambda^{0,q}_p T^*(\mathbb{C}^n)$ by declaring that the set $\{d\bar{z}^J : J \in \mathcal{I}_q\}$ is an orthonormal basis.

Let M be a smooth CR submanifold of $\mathbb{C}^n$ with real dimension $2n-d$.

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- This is *not* $\{j^*\omega : \omega \in \Lambda^{0,q}T^*(\mathbb{C}^n)\}$. Why?
- A smooth section of $\Lambda^{0,q}T^*(\mathbb{C}^n)|_M$ is an element

$$f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$$

where f_J are restricted to M.

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From non-tangential to tangential

Recall: A key fact in the analysis of boundary values of holomorphic functions was that only *tangential* vector fields were well-defined on M.

Step 1: Eliminate the non-tangential pieces from $\Lambda^{0,q}T^*(\mathbb{C}^n)|_M$.

Let I^q be the ideal in $\Lambda^{0,q}T^*(\mathbb{C}^n)$ generated by sums of forms

$$f_1\rho + f_2 \wedge \bar{\partial}\rho$$

where $\rho : \mathbb{C}^n \to \mathbb{R}$ vanishes on M, $f_1 \in \Lambda^{0,q}T^*(\mathbb{C}^n)$ and $f_2 \in \Lambda^{0,q-1}T^*(\mathbb{C}^n)$.

Set

$$\Lambda^{0,q}T^*(M) = \Lambda^{0,q}T^*(\mathbb{C}^n)|_M \ominus I^{p,q}|_M.$$

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From non-tangential to tangential

Step 2: Extracting the tangential component of a form.

Let $\mathcal{E}^{0,q}_M$ by the space of smooth sections of $\Lambda^{0,q}T^*(M)$.

We set the tangential projection operator $\tau_M : \Lambda^{0,q}T^*(\mathbb{C}^n)|_M \to \Lambda^{0,q}T^*(M)$ to be the orthogonal projection map.

Step 3: Defining $\bar{\partial}_b f$ for $f \in \mathcal{E}^{0,q}_M$.

1. For an open set $U \subset M$, extend f to $\tilde{f}$ on an open neighborhood in $\mathbb{C}^n$ of U.
2. Apply $\bar{\partial}$ to $\tilde{f}$.
3. Toss the non-tangential pieces.

For $p \in U$, set

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Thank you for your attention!

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