


Lecture 2: The tangential Cauchy-Riemann operator and the Levi form

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10th Workshop on the Geometric Analysis of PDEs and Several Complex Variables




Outline of the Talk

- To give several constructions of the $\bar{\partial}_b$-operator
- To relate the $\bar{\partial}_b$ operator to more general constructions, e.g., Involutive Structures.
- To introduce the Levi form $\mathcal{L}$.
- To discuss the $\bar{\partial}_b$ and $\Box_b$ problems

Geometry: The tangential Cauchy-Riemann operator

Recap

Definition

M is an embedded CR manifold if $\dim_{\mathbb{R}} H_p(M)$ is independent of $p \in M$.

Definition

For $p \in U \cap M$, set

$$\bar{\partial}_b f = \tau_M\{\bar{\partial}\tilde{f}\}.$$

where $\tau_M : \Lambda^{0,q} T^*(\mathbb{C}^n)|_M \to \Lambda^{0,q} T^*(M)$ is the orthogonal projection map and $\tilde{f}$ is any smooth extension of f.

The $\bar{\partial}_b$-problem is the the problem of finding a form u so that

$$\bar{\partial}_b u = f$$

for a given form f (which necessarily has $\bar{\partial}_b f = 0$)

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Say what?

Questions:

- What did we just do?
- Is $\bar{\partial}_b$ well-defined?
- How do I compute τ_M?
- Does this agree with the calculation on model domains?
- Why should we care about the $\bar{\partial}_b$-problem?

Easy question first: Yes, $\bar{\partial}_b$ is well-defined because τ_M forces everything to be tangential and hence independent of the extension (like earlier).

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More on the $\bar{\partial}_b$-problem

Question: Why should we care about the $\bar{\partial}_b$-problem?

Answer:

1. The $\bar{\partial}_b$-problem is fundamentally tied into the boundary values of holomorphic functions.
2. The $\bar{\partial}_b$-operator encodes all of the complex structures and complex geometry on the CR manifold M.
3. Hans Lewy was studying the $\bar{\partial}_b$-equation on the Heisenberg when he formulated the first nonsolvable PDE.
4. We have seen repeatedly that closed range of $\bar{\partial}_b$ is a critical ingredient to prove embeddability of abstract CR manifolds!!

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Computing τ_M if $M = b\Omega$

Recall: If X is a vector and ϕ is an r-form, then the contraction $X \lrcorner \phi$ is a $(r-1)$-form defined by

$$\langle X \lrcorner \phi, L_1 \wedge \cdots \wedge L_{r-1} \rangle = \langle \phi, X \wedge L_1 \cdots \wedge L_{r-1} \rangle$$

Philosophically, $X \lrcorner$ removes the dual of X from ϕ.

Suppose ρ is a defining function for $M = b\Omega$ with $|d\rho| = 1$ on M. Let $\mathbb{N}$ be the unit outward complex normal (this means $\mathbb{N} = 4 \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$).

Lemma

$$\tau_M\{\phi\} = \mathbb{N} \lrcorner (\bar{\partial}\rho \wedge \phi) \qquad \text{for } \phi \in \Lambda^{0,q} T^*(\mathbb{C}^n)|_M$$

and

$$\bar{\partial}_b f = \mathbb{N} \lrcorner (\bar{\partial}\rho \wedge \bar{\partial}\tilde{f}) \qquad \text{for } f \in \mathcal{E}^{p,q}_M$$

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Let $M = \{\rho = 0\} \subset \mathbb{C}^n$, $p \in M$, and $U \subset M$ a neighborhood of p. And suppose $|d\rho| = 1$ on M.

Let $\{L_1, \ldots, L_{n-1}\}$ be an orthonormal basis for $T^{1,0}(M \cap U)$. Then $\{\bar{L}_1, \ldots, \bar{L}_{n-1}\}$ is an orthonormal basis for $T^{0,1}(M \cap U)$.

Let $T = i \operatorname{Im} \mathbb{N}$. Then $\{L_j, \bar{L}_k, T : 1 \leq j, k \leq n-1\}$ is a basis of $T(M) \otimes \mathbb{C}$. Let $\{\omega_1, \ldots, \omega_{n-1}, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1}, \gamma\}$ be a dual basis (in $T^*(M) \otimes \mathbb{C}$).

This means

$$\omega_j\{L_j\} = \bar{\omega}_k\{\bar{L}_k\} = \gamma\{T\} = 1$$

and all other pairings are 0.

Let $\omega_n = \partial\rho$. Then $\bar{\partial}\rho(\mathbb{N}) = 1$.

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$\bar\partial_b f = \mathbb{N} \lrcorner (\bar\partial \rho \wedge \bar\partial \tilde f)$

A section of $\mathcal{E}^{0,q}_U$ is a $(0,q)$-form

$$f = \sum_{\substack{J \in \mathcal{I}_q \\ n \notin J}} f_J \, \bar\omega^J.$$

Since $\bar\omega_n = \bar\partial \rho$, we can write an extension of f by

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$$= \sum_{J \in \mathcal{I}_q} \sum_{k=1}^{n} \bar{L}_k \tilde{f}_J \, \bar{\omega}^k \wedge \bar{\omega}^J + O(\tilde{f}) + \bar{\partial}\rho \wedge (\cdots)$$

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Finally, since $\bar{\partial}\rho(\mathbb{N}) = 1$, it follows that

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The terms that comprise $O(\tilde{f})$ *are* important for many applications.

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The Key Fact

Recall: If M is a CR submanifold of $\mathbb{C}^n$, then

1. $H^{1,0}(M) \cap \overline{H^{1,0}(M)} = \{0\}$.
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Abstract CR Manifolds

Definition

Let M be a C^∞ manifold and suppose $\mathbb{L}$ is a subbundle of $T(M) \otimes \mathbb{C}$. The pair $(M, \mathbb{L})$ is called an abstract CR manifold or CR structure if

1. $\mathbb{L}_p \cap \overline{\mathbb{L}_p} = \{0\}$ for each $p \in M$
2. $\mathbb{L}$ is involutive, that is, $[L_1, L_2] \in \mathbb{L}$ whenever $L_1, L_2 \in \mathbb{L}$.

Remarks

- If $M \subset \mathbb{C}^n$, then $\mathbb{L} = H^{1,0}(M)$ provides the CR structure.
- The number $d = \dim_{\mathbb{C}}(T(M) \otimes \mathbb{C})/\mathbb{L} \oplus \overline{\mathbb{L}}$ is the CR codimension of $(M, \mathbb{L})$.

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Towards an intrinsic $\bar{\partial}_b$

Key Notation

- For $p \in M$, define X_p to be the orthogonal complement of $\mathbb{L}_p \oplus \overline{\mathbb{L}_p}$ in $T_p(M) \otimes \mathbb{C}$. This is not $X_p(M)$.
- The totally real part of the tangent bundle is $X(M) = \bigcup_{p \in M} X_p$.
- Define the subbundles

$$T^{0,1}(M) = \overline{\mathbb{L}} \qquad \text{and} \qquad T^{1,0}(M) = \mathbb{L} \oplus X(M).$$

Do not confuse $T^{1,0}(M)$ with $H^{1,0}(M)$!!

Towards an intrinsic $\bar{\partial}_b$

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Forms on M

Dual to $T^{0,1}(M)$ and $T^{1,0}(M)$ are $T^{0,1}(M)^*$ and $T^{1,0}(M)^*$, respectively and the bundle of (p,q)-forms is

$$\Lambda^{p,q}T^*(M) = \Lambda^p(T^{1,0}(M)^*)\hat{\otimes}\Lambda^q(T^{0,1}(M)^*)$$

where $\hat{\otimes}$ is the anti-symmetric tensor product (e.g., $\omega^I \wedge \bar{\omega}^J$).

The bundle of r forms is

$$\Lambda^r(T^*(M)\otimes\mathbb{C}) = \Lambda^{r,0}T^*(M)\oplus\cdots\oplus\Lambda^{0,r}T^*(M).$$

Denote the natural projection

$$\pi^{p,q}_M : \Lambda^r(T^*(M)\otimes\mathbb{C}) \to \Lambda^{p,q}(T^*(M)\otimes\mathbb{C}).$$

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Sections on M and an intrinsically defined $\bar{\partial}_b$

Given an open set $U \subset M$, let $\mathcal{E}^{p,q}(M)$ be the space of sections of (p,q)-forms over U. Let d_M be the exterior derivative operator on M.

Definition

The tangential Cauchy-Riemann operator $\bar{\partial}_b : \mathcal{E}^{p,q}_M \to \mathcal{E}^{p,q}_M$ is defined by

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Extrinsic vs. intrinsic $\bar{\partial}_b$ operators

The intrinsic $\bar{\partial}_b$ operator shares many of the properties that the extrinsically defined $\bar{\partial}_b$ operator does.

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Comments

- $X(M)$ was defined to be the orthogonal complement to $\mathbb{L} \otimes \overline{\mathbb{L}}$. This requires a metric. A different metric will give a different, but isomorphic, $\bar{\partial}_b$-complex.
- We could avoid metrics altogether using quotients. For example, if $M \subset \mathbb{C}^n$, we could have defined

$$\Lambda^{p,q} T^*(M) = \Lambda^{p,q} T^*(\mathbb{C}) / I^{p,q}.$$

  There's a similar construction in the abstract CR manifold case.
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$\bar{\partial}_b$ vs. $\bar{\partial}$ and pullbacks

Warning! CR geometry is subtle! Suppose $\Omega_1, \Omega_2 \in \mathbb{C}^n$ and $F : \Omega_1 \to \Omega_2$ is a biholomorphism. A basic result in SCV says

$$\bar{\partial} F^* = F^* \bar{\partial}.$$

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Example

Let

$$M = \{(z, u + iv) \in \mathbb{C}^2 : v = 0\}.$$

Then $\overline{\mathbb{L}} = \text{span}\{\frac{\partial}{\partial \bar{z}}\}$ and if g is a function

$$\bar{\partial}_b g = \frac{\partial g}{\partial \bar{z}}\, d\bar{z}$$

Let $f : M \to M$ be defined by

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Suppose M and N are CR manifolds and $f : M \to N$ is a CR map. then

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Aside – Involutive Structures

CR structures are only one of a class of structures that are studied on smooth manifolds M.

Definition

Let M be a C^∞ manifold and suppose $\mathbb{L}$ is a subbundle of $T(M) \otimes \mathbb{C}$. The pair $(M, \mathbb{L})$ is called an abstract CR manifold or CR structure if

- $\mathbb{L}$ is involutive, that is, $[L_1, L_2] \in \mathbb{L}$ whenever $L_1, L_2 \in \mathbb{L}$.
- $\mathbb{L}_p \cap \overline{\mathbb{L}_p} = \{0\}$ for each $p \in M$

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Examples beyond CR structures:

- A complex structure if $\mathbb{L}_p \otimes \overline{\mathbb{L}_p} = T_pM \otimes \mathbb{C}$ for all $p \in M$
- An essentially real structure if $\mathbb{L}_p = \overline{\mathbb{L}_p}$ for all $p \in M$.

Geometry: The Levi Form

The Levi form

Let $(M, \mathbb{L})$ be a CR structure.

Fact: The subbundle $\mathbb{L} \oplus \overline{\mathbb{L}} \subset T(M) \otimes \mathbb{C}$ may not be involutive!

Recall:

$$\pi_p : T_p(M) \otimes \mathbb{C} \to T_p(M) \otimes \mathbb{C}/(\mathbb{L} \oplus \overline{\mathbb{L}})$$

is the natural projection map.

Definition

The Levi form at $p \in M$ is the map
$\mathcal{L}_p : \mathbb{L}_p \to T_p(M) \otimes \mathbb{C}/(\mathbb{L} \oplus \overline{\mathbb{L}})$ defined by

$$\mathcal{L}_p(L_p) = \frac{1}{2i}\pi_p\{[\bar{L}, L]_p\} \quad \text{for } L_p \in \mathbb{L}_p.$$

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Facts about the Levi form

- The Levi form is well-defined, that is, if $L, Z \in \mathbb{L}$ and $L_p = Z_p$, then

$$\pi_p\{[\bar{L}, L]_p\} = \pi_p\{[\overline{Z}, Z]_p\}.$$

- The Levi form is a CR invariant, that is, if $(M, \mathbb{L}_M), (N, \mathbb{L}_N)$ are CR structures with Levi forms $\mathcal{L}^M$ and $\mathcal{L}^N$, and $F : M \to N$ is a CR diffeomorphism, then

$$F_*(p) \circ \mathcal{L}^M_p = \mathcal{L}^N_{F(p)} \circ F_*(p)$$

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Computing the Levi form

Question: How do we compute the Levi form?

Answer:

Step 1 Choose a metric for $T(M) \otimes \mathbb{C}$.

Step 2 Identify $T_p(M) \otimes \mathbb{C}/(\mathbb{L} \oplus \overline{\mathbb{L}})$ with $X_p(M) \otimes \mathbb{C} = T_p(M)/H_p(M) \otimes \mathbb{C}$.

Step 3 If $M \subset \mathbb{C}^n$ and $p \in M$,

$$\mathcal{L}_p : H_p^{1,0}(M) \to X_p(M) \otimes \mathbb{C}$$

given by

$$\mathcal{L}_p(L_p) = \frac{1}{2i}\pi_p\{[\bar{L}, L]_p\}, \quad L_p \in H_p^{1,0}(M).$$

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The Heisenberg group

Recall: $\mathcal{H}^n = \{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = |z|^2\}$ and a basis for $H^{1,0}(\mathcal{H}^n)$ is

$$L_j = \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}.$$

We choose a metric so that $X(\mathcal{H}^n)$ is spanned by $T = \frac{\partial}{\partial t}$ where $w = t + is$. We compute

$$[\bar{L}_k, L_j] = 2i\delta_{jk} T$$

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The Levi matrix

Let $M \subset \mathbb{C}^{n+1}$ be the boundary of a domain. Let $\{L_1, \ldots, L_n\} \in H^{1,0}(M)$ be an orthonormal basis near a point $p_0 \in M$. We identify the Levi form with the Levi matrix (c_{jk}) given by

$$c_{jk} = \frac{1}{2i}\pi_p\{[\bar{L}_k, L_j]\}.$$

The Levi form/Levi matrix is the most important geometric structure intrinsic to M.

If $M = \{(z, t + is) \in \mathbb{C}^n \times \mathbb{C} : s = p(z)\}$ then π_p picks off the $\frac{\partial}{\partial t}$ component.

$$c_{jk} = \frac{1}{2i}\left[\frac{\partial}{\partial \bar{z}_k} - 2i\frac{\partial p}{\partial \bar{z}_k}\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z_j} + 2i\frac{\partial p}{\partial z_j}\frac{\partial}{\partial w}\right] = \frac{\partial^2 p}{\partial z_j \partial \bar{z}_k}.$$

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Note: Here, $M \subset \mathbb{C}^{n+1}$ so (c_{jk}) is an $n \times n$ matrix!

If the Levi matrix has

- only nonnegative (resp., positive) eigenvalue, then M is called pseudoconvex, (resp., strictly pseudoconvex)
- at least $n-(q+1)$ positive or at least $q+1$ negative eigenvalues, then M satisfies $Z(q)$.
- $Z(q)$ and $Z(n-q)$, then M satisfies $Y(q)$.

Geometry Remark: The Levi form is to the complex Hessian what the Second Fundamental Form is to the real Hessian.

One more remark: The Hodge-* operator induces symmetry between levels $(0,q)$ and $(0,n-q)$ so symmetric requirements for $\bar{\partial}_b$ are necessary.

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Setting up the problem

The $\bar{\partial}_b$-problem is

$$\bar{\partial}_b u = f.$$

Standard approach: Via the Kohn Laplacian

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$$

and the $\Box_b$-equation

$$\Box_b v = f.$$

Note: $\Box_b$ is neither elliptic nor constant coefficient!

Notation: $H_{p,q}(M) = \text{null}\,\Box_b$ on (p,q)-forms is the space of harmonic forms.

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The $\Box_b$-problem

Theorem

Suppose $(M, \mathbb{L})$ is a hypersurface type CR manifold of dimension $2n+1$, $0 \leq p \leq n$, and $1 \leq q \leq n$. If $\Box_b : L^2_{p,q}(M) \to L^2_{p,q}(M)$ has closed range, then

- *The complex Green operator. $\Box_b$ is invertible on $H_{p,q}(M)^\perp$ with inverse $G_{p,q}$.*
- *Hodge decomposition.*

$$L^2_{p,q}(M) = \text{Range}\,\bar{\partial}_b \oplus \text{Range}\,\bar{\partial}_b^* \oplus H_{p,q}(M)$$

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The Levi form and solvability

- If $\Omega \subset \mathbb{C}^n$ and $M = \mathrm{b}\Omega$ is bounded and pseudoconvex, then $\bar{\partial}_b$ is solvable in $L^2_{p,q}(\Omega)$, all p, $1 \leq q \leq n-1$ and $H_{p,q}(M) = \{0\}$ if $1 \leq q \leq n-2$. (Shaw '85, Boas-Shaw '86).
- If $M = \mathrm{b}\Omega$ is bounded and pseudoconvex in a Stein manifold, then $\bar{\partial}_b$ is solvable (same range of p, q) (Kohn '86)
- M satisfies $Y(q)$ if and only if the solution to $\bar{\partial}_b$ gains $1/2$ derivatives. (Kohn)
- If $M = \mathrm{b}\Omega$ in a Stein manifold satisfies weak $Y(q)$, then $\bar{\partial}_b$ is solvable at levels q and $q-1$ and $H_{p,q}(M) = \{0\}$ (Harrington-Raich, 2015)
- If $M \subset \mathbb{C}^n$ and is a CR manifold of hypersurface type and satisfies weak $Y(q)$, then $\bar{\partial}_b$ is solvable at levels q and $q-1$ and $\dim H_{p,q}(M) < \infty$ (Coacalle-Raich, *submitted*)

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Thank you for your attention!

THANK YOU!!