


Lecture 3: The $\Box_b$-equation on the Heisenberg group

Andy Raich

University of Arkansas

10th Workshop on the Geometric Analysis of PDEs and Several Complex Variables




Outline of the Talk

- To discuss the Heisenberg group
- To solve the $\Box_b$-equation on the Heisenberg group
- To contextualize the form of the integral kernel of the inverse operator.

The Heisenberg group

Recall: The Heisenberg group

$$\mathcal{H}^n = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = |z|^2\}.$$

The Siegel upper half space

$$\mathcal{U}^n = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > |z|^2\}$$

so

$$\mathcal{H}^n = \mathrm{b}\mathcal{U}^n.$$

The Heisenberg group

Recall: The Heisenberg group

$$\mathcal{H}^n = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = |z|^2\}.$$

The Siegel upper half space

$$\mathcal{U}^n = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > |z|^2\}$$

so

$$\mathcal{H}^n = \mathrm{b}\mathcal{U}^n.$$

$\mathcal{U}^n$ and the unit ball

Denote the unit ball by

$$B(0,1) = \{(w, w_{n+1}) \in \mathbb{C}^{n+1} : |w|^2 + |w_{n+1}|^2 < 1\}.$$

The mappings

$$w_{n+1} = \frac{i - z_{n+1}}{i + z_{n+1}} \qquad w_k = \frac{2iz_k}{i + z_{n+1}}, \quad k = 1, \ldots, n$$

$$z_{n+1} = i\frac{1 - w_{n+1}}{1 + w_{n+1}} \qquad z_k = \frac{w_k}{1 + w_{n+1}}, \quad k = 1, \ldots, n$$

give a biholomorphism between $\mathcal{U}^n$ and $B(0,1)$.

The mapping extends to the boundary, except at the "south pole" $(0, \ldots, 0, -1)$ which maps to infinity.

$\mathcal{U}^n$ and the unit ball

Denote the unit ball by

$$B(0,1) = \{(w, w_{n+1}) \in \mathbb{C}^{n+1} : |w|^2 + |w_{n+1}|^2 < 1\}.$$

The mappings

$$w_{n+1} = \frac{i - z_{n+1}}{i + z_{n+1}} \qquad w_k = \frac{2iz_k}{i + z_{n+1}}, \quad k = 1, \ldots, n$$

$$z_{n+1} = i\frac{1 - w_{n+1}}{1 + w_{n+1}} \qquad z_k = \frac{w_k}{1 + w_{n+1}}, \quad k = 1, \ldots, n$$

give a biholomorphism between $\mathcal{U}^n$ and $B(0,1)$.

The mapping extends to the boundary, except at the "south pole" $(0, \ldots, 0, -1)$ which maps to infinity.

$\mathcal{U}^n$ and the unit ball

Denote the unit ball by

$$B(0,1) = \{(w, w_{n+1}) \in \mathbb{C}^{n+1} : |w|^2 + |w_{n+1}|^2 < 1\}.$$

The mappings

$$w_{n+1} = \frac{i - z_{n+1}}{i + z_{n+1}} \qquad w_k = \frac{2iz_k}{i + z_{n+1}}, \quad k = 1, \ldots, n$$

$$z_{n+1} = i\frac{1 - w_{n+1}}{1 + w_{n+1}} \qquad z_k = \frac{w_k}{1 + w_{n+1}}, \quad k = 1, \ldots, n$$

give a biholomorphism between $\mathcal{U}^n$ and $B(0,1)$.

The mapping extends to the boundary, except at the "south pole" $(0, \ldots, 0, -1)$ which maps to infinity.

The geometry of $\mathcal{H}^n$

For each $\delta > 0$, define the dilation δ by

$$\delta \circ (z, z_{n+1}) = (\delta z, \delta^2 z_{n+1}).$$

Then $\delta : \mathcal{U}^n \to \mathcal{U}^n$ and $\delta : \mathcal{H}^n \to \mathcal{H}^n$.

Writing $z_{n+1} = t + is$, we identify

$$\mathcal{H}^n \cong \mathbb{C}^n \times \mathbb{R}$$

with the identification

$$(z, t + i|z|^2) \longleftrightarrow (z, t)$$

The geometry of $\mathcal{H}^n$

For each $\delta > 0$, define the dilation δ by

$$\delta \circ (z, z_{n+1}) = (\delta z, \delta^2 z_{n+1}).$$

Then $\delta : \mathcal{U}^n \to \mathcal{U}^n$ and $\delta : \mathcal{H}^n \to \mathcal{H}^n$.

Writing $z_{n+1} = t + is$, we identify

$$\mathcal{H}^n \cong \mathbb{C}^n \times \mathbb{R}$$

with the identification

$$(z, t + i|z|^2) \longleftrightarrow (z, t)$$

The geometry of $\mathcal{H}^n$

For each $\delta > 0$, define the dilation δ by

$$\delta \circ (z, z_{n+1}) = (\delta z, \delta^2 z_{n+1}).$$

Then $\delta : \mathcal{U}^n \to \mathcal{U}^n$ and $\delta : \mathcal{H}^n \to \mathcal{H}^n$.

Writing $z_{n+1} = t + is$, we identify

$$\mathcal{H}^n \cong \mathbb{C}^n \times \mathbb{R}$$

with the identification

$$(z, t + i|z|^2) \longleftrightarrow (z, t)$$

Groups and measures

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, the group structure is given by

$$(z,t)\cdot(w,s) = \big(z+w, t+s+2\,\mathrm{Im}(z\cdot\bar{w})\big)$$

Note that $(z,t)^{-1} = (-z,-t)$ and

$$\delta\circ\big((z,t)\cdot(w,s)\big) = \big(\delta\circ(z,t)\big)\cdot\big(\delta\circ(w,s)\big).$$

Notes:

- We will see that the nonisotropic dilation δ permeates every aspect of the analysis of the Heisenberg group.
- Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ corresponds to Haar measure on $\mathcal{H}^n$ and this is the measure we will use. It is not equivalent to surface area measure.

Groups and measures

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, the group structure is given by

$$(z,t)\cdot(w,s) = \big(z+w, t+s+2\,\mathrm{Im}(z\cdot\bar{w})\big)$$

Note that $(z,t)^{-1} = (-z,-t)$ and

$$\delta\circ\big((z,t)\cdot(w,s)\big) = \big(\delta\circ(z,t)\big)\cdot\big(\delta\circ(w,s)\big).$$

Notes:

- We will see that the nonisotropic dilation δ permeates every aspect of the analysis of the Heisenberg group.
- Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ corresponds to Haar measure on $\mathcal{H}^n$ and this is the measure we will use. It is not equivalent to surface area measure.

Groups and measures

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, the group structure is given by

$$(z,t)\cdot(w,s) = \big(z+w, t+s+2\,\mathrm{Im}(z\cdot\bar{w})\big)$$

Note that $(z,t)^{-1} = (-z,-t)$ and

$$\delta\circ\big((z,t)\cdot(w,s)\big) = \big(\delta\circ(z,t)\big)\cdot\big(\delta\circ(w,s)\big).$$

Notes:

- We will see that the nonisotropic dilation δ permeates every aspect of the analysis of the Heisenberg group.
- Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ corresponds to Haar measure on $\mathcal{H}^n$ and this is the measure we will use. It is not equivalent to surface area measure.

Groups and measures

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, the group structure is given by

$$(z,t)\cdot(w,s) = \big(z+w, t+s+2\,\mathrm{Im}(z\cdot\bar{w})\big)$$

Note that $(z,t)^{-1} = (-z,-t)$ and

$$\delta\circ\big((z,t)\cdot(w,s)\big) = \big(\delta\circ(z,t)\big)\cdot\big(\delta\circ(w,s)\big).$$

Notes:

- We will see that the nonisotropic dilation δ permeates every aspect of the analysis of the Heisenberg group.
- Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ corresponds to Haar measure on $\mathcal{H}^n$ and this is the measure we will use. It is not equivalent to surface area measure.

Groups and measures

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, the group structure is given by

$$(z,t)\cdot(w,s) = \big(z+w, t+s+2\,\mathrm{Im}(z\cdot\bar{w})\big)$$

Note that $(z,t)^{-1} = (-z,-t)$ and

$$\delta\circ\big((z,t)\cdot(w,s)\big) = \big(\delta\circ(z,t)\big)\cdot\big(\delta\circ(w,s)\big).$$

Notes:

- We will see that the nonisotropic dilation δ permeates every aspect of the analysis of the Heisenberg group.
- Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ corresponds to Haar measure on $\mathcal{H}^n$ and this is the measure we will use. It is not equivalent to surface area measure.

Vector fields and $\bar{\partial}_b$

Recall: On $\mathcal{H}^n$

$$\bar{L}_k = \frac{\partial}{\partial \bar{z}_k} - 2iz_k \frac{\partial}{\partial \bar{w}}$$

and

$$\bar{\partial}_b f = \sum_{k=1}^{n} \bar{L}_k f \, d\bar{z}^k.$$

Under the identification,

$$\bar{L}_k \longleftrightarrow \overline{Z}_k = \frac{\partial}{\partial \bar{z}_k} - iz_k \frac{\partial}{\partial t}$$

and

$$\bar{\partial}_b f \longleftrightarrow \bar{\partial}_b f = \sum_{k=1}^{n} \bar{Z}_k f \, d\bar{z}^k.$$

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$, then

$$\bar{\partial}_b f = \sum_{J \in \mathcal{I}_q} \sum_{k=1}^{n} \bar{Z}_k f_J \, d\bar{z}^k \wedge d\bar{z}^J.$$

Vector fields and $\bar{\partial}_b$

Recall: On $\mathcal{H}^n$

$$\bar{L}_k = \frac{\partial}{\partial \bar{z}_k} - 2iz_k \frac{\partial}{\partial \bar{w}}$$

and

$$\bar{\partial}_b f = \sum_{k=1}^{n} \bar{L}_k f \, d\bar{z}^k.$$

Under the identification,

$$\bar{L}_k \longleftrightarrow \overline{Z}_k = \frac{\partial}{\partial \bar{z}_k} - iz_k \frac{\partial}{\partial t}$$

and

$$\bar{\partial}_b f \longleftrightarrow \bar{\partial}_b f = \sum_{k=1}^{n} \bar{Z}_k f \, d\bar{z}^k.$$

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$, then

$$\bar{\partial}_b f = \sum_{J \in \mathcal{I}_q} \sum_{k=1}^{n} \bar{Z}_k f_J \, d\bar{z}^k \wedge d\bar{z}^J.$$

Vector fields and $\bar{\partial}_b$

Recall: On $\mathcal{H}^n$

$$\bar{L}_k = \frac{\partial}{\partial \bar{z}_k} - 2iz_k \frac{\partial}{\partial \bar{w}}$$

and

$$\bar{\partial}_b f = \sum_{k=1}^{n} \bar{L}_k f \, d\bar{z}^k.$$

Under the identification,

$$\bar{L}_k \longleftrightarrow \overline{Z}_k = \frac{\partial}{\partial \bar{z}_k} - iz_k \frac{\partial}{\partial t}$$

and

$$\bar{\partial}_b f \longleftrightarrow \bar{\partial}_b f = \sum_{k=1}^{n} \bar{Z}_k f \, d\bar{z}^k.$$

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$, then

$$\bar{\partial}_b f = \sum_{J \in \mathcal{I}_q} \sum_{k=1}^{n} \bar{Z}_k f_J \, d\bar{z}^k \wedge d\bar{z}^J.$$

The adjoint operator $\bar{\partial}_b^*$

Fact: If $J = (j_1, \ldots, j_q)$, then

$$\frac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J = \begin{cases} 0 & \text{if } k \notin \{j_1, \ldots, j_q\} \\ (-1)^{\ell-1} d\bar{z}_{j_1} \wedge \cdots \wedge \widehat{d\bar{z}_\ell} \wedge \cdots \wedge d\bar{z}_{j_q} & \text{if } k = j_\ell. \end{cases}$$

From the relationship

$$(\bar{\partial}_b f, g) = (f, \bar{\partial}_b^* g)$$

integration by parts shows

$$\bar{\partial}_b^* f = -\sum_{J \in \mathcal{I}_q} \sum_{j=1}^n Z_j f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J.$$

If f is a $(0,1)$-form, then

$$\bar{\partial}_b^* f = -\sum_{j=1}^n Z_j f_j.$$

The adjoint operator $\bar{\partial}_b^*$

Fact: If $J = (j_1, \ldots, j_q)$, then

$$\frac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J = \begin{cases} 0 & \text{if } k \notin \{j_1, \ldots, j_q\} \\ (-1)^{\ell-1} d\bar{z}_{j_1} \wedge \cdots \wedge \widehat{d\bar{z}_\ell} \wedge \cdots \wedge d\bar{z}_{j_q} & \text{if } k = j_\ell. \end{cases}$$

From the relationship

$$(\bar{\partial}_b f, g) = (f, \bar{\partial}_b^* g)$$

integration by parts shows

$$\bar{\partial}_b^* f = -\sum_{J \in \mathcal{I}_q} \sum_{j=1}^{n} Z_j f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J.$$

If f is a $(0,1)$-form, then

$$\bar{\partial}_b^* f = -\sum_{j=1}^{n} Z_j f_j.$$

The adjoint operator $\bar{\partial}_b^*$

Fact: If $J = (j_1, \ldots, j_q)$, then

$$\frac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J = \begin{cases} 0 & \text{if } k \notin \{j_1, \ldots, j_q\} \\ (-1)^{\ell-1} d\bar{z}_{j_1} \wedge \cdots \wedge \widehat{d\bar{z}_\ell} \wedge \cdots \wedge d\bar{z}_{j_q} & \text{if } k = j_\ell. \end{cases}$$

From the relationship

$$(\bar{\partial}_b f, g) = (f, \bar{\partial}_b^* g)$$

integration by parts shows

$$\bar{\partial}_b^* f = -\sum_{J \in \mathcal{I}_q} \sum_{j=1}^{n} Z_j f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J.$$

If f is a $(0,1)$-form, then

$$\bar{\partial}_b^* f = -\sum_{j=1}^{n} Z_j f_j.$$

The Kohn Laplacian and $\mathcal{L}_\alpha$

Recall: The Kohn Laplacian

$$\Box_b = \bar\partial_b^* \bar\partial_b + \bar\partial_b \bar\partial_b^*.$$

A calculation shows (remember: $T = \frac{\partial}{\partial t}$ and $Z_j = \frac{\partial}{\partial z_j} + i\bar z_j \frac{\partial}{\partial t}$)

Proposition

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar z^J$, then

$$\Box_b f = \sum_{J \in \mathcal{I}_q} \mathcal{L}_{n-2q} f_J \, d\bar z^J$$

where

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar Z_j + \bar Z_j Z_j) + i\alpha T.$$

The Kohn Laplacian and $\mathcal{L}_\alpha$

Recall: The Kohn Laplacian

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*.$$

A calculation shows (remember: $T = \frac{\partial}{\partial t}$ and $Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$)

Proposition

If $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}^J$, *then*

$$\Box_b f = \sum_{J \in \mathcal{I}_q} \mathcal{L}_{n-2q} f_J \, d\bar{z}^J$$

where

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_{j=1}^{n} \left(Z_j \bar{Z}_j + \bar{Z}_j Z_j \right) + i\alpha T.$$

Proof Sketch

Suppose $f = f_J \, d\bar{z}^J$. Then

$$\bar{\partial}_b^* \bar{\partial}_b f = -\sum_{j,k=1}^{n} Z_j \overline{Z}_k f_J \, \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J)$$

and

$$\bar{\partial}_b \bar{\partial}_b^* f = -\sum_{j,k=1}^{n} \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right).$$

Proof Sketch

Suppose $f = f_J \, d\bar{z}^J$. Then

$$\bar{\partial}_b^* \bar{\partial}_b f = -\sum_{j,k=1}^{n} Z_j \overline{Z}_k f_J \, \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J)$$

and

$$\bar{\partial}_b \bar{\partial}_b^* f = -\sum_{j,k=1}^{n} \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right).$$

Proof Sketch

Suppose $f = f_J \, d\bar{z}^J$. Then

$$\bar{\partial}_b^* \bar{\partial}_b f = - \sum_{j,k=1}^{n} Z_j \overline{Z}_k f_J \, \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J)$$

and

$$\bar{\partial}_b \bar{\partial}_b^* f = - \sum_{j,k=1}^{n} \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right).$$

Proof Sketch, cont'd

Facts:

- $[\bar{Z}_k, Z_j] = 2i\delta_{jk}T$
- If $j \neq k$, then $\frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = -d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J\right)$
- If $j = k$, then $\frac{\partial}{\partial \bar{z}_k} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = \begin{cases} d\bar{z}^J & \text{if } k \notin J \\ 0 & \text{if } k \in J \end{cases}$
- If $j = k$, then $d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J\right) = \begin{cases} 0 & \text{if } k \notin J \\ d\bar{z}^J & \text{if } k \in J \end{cases}$

Proof Sketch, cont'd

Facts:

- $[\bar{Z}_k, Z_j] = 2i\delta_{jk}T$
- If $j \neq k$, then $\dfrac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = -d\bar{z}^k \wedge \left(\dfrac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J\right)$
- If $j = k$, then $\dfrac{\partial}{\partial \bar{z}_k} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = \begin{cases} d\bar{z}^J & \text{if } k \notin J \\ 0 & \text{if } k \in J \end{cases}$
- If $j = k$, then $d\bar{z}^k \wedge \left(\dfrac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J\right) = \begin{cases} 0 & \text{if } k \notin J \\ d\bar{z}^J & \text{if } k \in J \end{cases}$

Proof Sketch, cont'd

Facts:

- $[\bar{Z}_k, Z_j] = 2i\delta_{jk}T$
- If $j \neq k$, then $\frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = -d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J\right)$
- If $j = k$, then $\frac{\partial}{\partial \bar{z}_k} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = \begin{cases} d\bar{z}^J & \text{if } k \notin J \\ 0 & \text{if } k \in J \end{cases}$
- If $j = k$, then $d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J\right) = \begin{cases} 0 & \text{if } k \notin J \\ d\bar{z}^J & \text{if } k \in J \end{cases}$

Proof Sketch, cont'd

Facts:

- $[\bar{Z}_k, Z_j] = 2i\delta_{jk}T$
- If $j \neq k$, then $\dfrac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = -d\bar{z}^k \wedge \left(\dfrac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J\right)$
- If $j = k$, then $\dfrac{\partial}{\partial \bar{z}_k} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) = \begin{cases} d\bar{z}^J & \text{if } k \notin J \\ 0 & \text{if } k \in J \end{cases}$
- If $j = k$, then $d\bar{z}^k \wedge \left(\dfrac{\partial}{\partial \bar{z}_k} \lrcorner d\bar{z}^J\right) = \begin{cases} 0 & \text{if } k \notin J \\ d\bar{z}^J & \text{if } k \in J \end{cases}$

Proof Sketch, completed

Thus, if $j \neq k$, then

$$Z_j \overline{Z}_k f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) + \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right) = 0$$

and

$$\Box_b (f_J \, d\bar{z}^J) = -\Big(\sum_{k \in J} \overline{Z}_k Z_k + \sum_{k \notin J} Z_k \overline{Z}_k \Big) \{ f_J \} \, d\bar{z}^J .$$

Writing

$$\overline{Z}_k Z_k = \frac{1}{2} (\overline{Z}_k Z_k + Z_k \overline{Z}_k) + iT \quad \text{and} \quad Z_k \overline{Z}_k = \frac{1}{2} (\overline{Z}_k Z_k + Z_k \overline{Z}_k) - iT$$

it follows that

$$\Box_b (f_J \, d\bar{z}^J) = \left(-\frac{1}{2} \sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k \right) + i(-q + n - q) \, T \right) \{ f_J \} d\bar{z}^J . \quad \square$$

Proof Sketch, completed

Thus, if $j \neq k$, then

$$Z_j \overline{Z}_k f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) + \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right) = 0$$

and

$$\Box_b (f_J \, d\bar{z}^J) = -\Big(\sum_{k \in J} \overline{Z}_k Z_k + \sum_{k \notin J} Z_k \overline{Z}_k \Big) \{f_J\} \, d\bar{z}^J.$$

Writing

$$\overline{Z}_k Z_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) + iT \quad \text{and} \quad Z_k \overline{Z}_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) - iT$$

it follows that

$$\Box_b (f_J \, d\bar{z}^J) = \left(-\frac{1}{2} \sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k \right) + i(-q + n - q) \, T \right) \{f_J\} d\bar{z}^J. \quad \square$$

Proof Sketch, completed

Thus, if $j \neq k$, then

$$Z_j \overline{Z}_k f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) + \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right) = 0$$

and

$$\Box_b (f_J \, d\bar{z}^J) = -\Big(\sum_{k \in J} \overline{Z}_k Z_k + \sum_{k \notin J} Z_k \overline{Z}_k \Big) \{f_J\} \, d\bar{z}^J.$$

Writing

$$\overline{Z}_k Z_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) + iT \quad \text{and} \quad Z_k \overline{Z}_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) - iT$$

it follows that

$$\Box_b (f_J \, d\bar{z}^J) = \left(-\frac{1}{2} \sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k \right) + i(-q + n - q) \, T \right) \{f_J\} d\bar{z}^J. \quad \square$$

Proof Sketch, completed

Thus, if $j \neq k$, then

$$Z_j \overline{Z}_k f_J \frac{\partial}{\partial \bar{z}_j} \lrcorner (d\bar{z}^k \wedge d\bar{z}^J) + \overline{Z}_k Z_j f_J \, d\bar{z}^k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}^J \right) = 0$$

and

$$\Box_b (f_J \, d\bar{z}^J) = -\Big(\sum_{k \in J} \overline{Z}_k Z_k + \sum_{k \notin J} Z_k \overline{Z}_k \Big) \{f_J\} \, d\bar{z}^J.$$

Writing

$$\overline{Z}_k Z_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) + iT \quad \text{and} \quad Z_k \overline{Z}_k = \frac{1}{2}(\overline{Z}_k Z_k + Z_k \overline{Z}_k) - iT$$

it follows that

$$\Box_b (f_J \, d\bar{z}^J) = \left(-\frac{1}{2} \sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k \right) + i(-q+n-q) \, T \right) \{f_J\} d\bar{z}^J. \quad \square$$

More on $\mathcal{L}_\alpha$

Facts about $\mathcal{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j) + i\alpha T$:

- $\mathcal{L}_\alpha$ is order 2 with respect to the dilation δ.
- The forbidden values of α occur when $\alpha = \pm n, \pm(n+2), \ldots$. E.g., on functions ($q = 0$, $n - 2q = n$)

$$\mathcal{L}_n f = \bar{\partial}_b^* \bar{\partial}_b f = -\sum_{k=1}^{n} Z_k \bar{Z}_k f.$$

$\mathcal{L}_n$ annihilates boundary values of holomorphic functions and hence has a *very large* null space (and is therefore not invertible, even when restricted to L^2).

More on $\mathcal{L}_\alpha$

Facts about $\mathcal{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j) + i\alpha T$:

- $\mathcal{L}_\alpha$ is order 2 with respect to the dilation δ.
- The forbidden values of α occur when $\alpha = \pm n, \pm(n+2), \ldots$. E.g., on functions ($q = 0$, $n - 2q = n$)

$$\mathcal{L}_n f = \bar{\partial}_b^* \bar{\partial}_b f = -\sum_{k=1}^{n} Z_k \bar{Z}_k f.$$

$\mathcal{L}_n$ annihilates boundary values of holomorphic functions and hence has a *very large* null space (and is therefore not invertible, even when restricted to L^2).

More on $\mathcal{L}_\alpha$

Facts about $\mathcal{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\bar{Z}_j + \bar{Z}_jZ_j\right) + i\alpha T$:

- $\mathcal{L}_\alpha$ is order 2 with respect to the dilation δ.
- The forbidden values of α occur when $\alpha = \pm n, \pm(n+2), \ldots$. E.g., on functions ($q = 0$, $n - 2q = n$)

$$\mathcal{L}_n f = \bar{\partial}_b^* \bar{\partial}_b f = -\sum_{k=1}^{n} Z_k \bar{Z}_k f.$$

$\mathcal{L}_n$ annihilates boundary values of holomorphic functions and hence has a *very large* null space (and is therefore not invertible, even when restricted to L^2).

The inverse to $\mathcal{L}_\alpha$

Question: What does it mean to invert $\mathcal{L}_\alpha$ (when α is not forbidden)?

Answer: We want a function F_α so that

$$\mathcal{L}_\alpha F_\alpha = \delta_0.$$

Convolution then produces an inverse operator! That is,

$$u = v * F_\alpha(x)$$

solves $\mathcal{L}_\alpha u = v$ where

$$f * g(x) = \int_{y \in \mathbb{C}^n \times \mathbb{R}} f(y) g(y^{-1} \cdot x)\, dy$$

The inverse to $\mathcal{L}_\alpha$

Question: What does it mean to invert $\mathcal{L}_\alpha$ (when α is not forbidden)?

Answer: We want a function F_α so that

$$\mathcal{L}_\alpha F_\alpha = \delta_0.$$

Convolution then produces an inverse operator! That is,

$$u = v * F_\alpha(x)$$

solves $\mathcal{L}_\alpha u = v$ where

$$f * g(x) = \int_{y \in \mathbb{C}^n \times \mathbb{R}} f(y) g(y^{-1} \cdot x)\, dy$$

The inverse to $\mathcal{L}_\alpha$

Question: What does it mean to invert $\mathcal{L}_\alpha$ (when α is not forbidden)?

Answer: We want a function F_α so that

$$\mathcal{L}_\alpha F_\alpha = \delta_0.$$

Convolution then produces an inverse operator! That is,

$$u = v * F_\alpha(x)$$

solves $\mathcal{L}_\alpha u = v$ where

$$f * g(x) = \int_{y \in \mathbb{C}^n \times \mathbb{R}} f(y) g(y^{-1} \cdot x)\, dy$$

The Folland-Stein Theorem

Let

$$\phi_\alpha(z,t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}}(|z|^2 + it)^{-\frac{n-\alpha}{2}}.$$

Theorem (Folland-Stein (1974))

If α is not a forbidden value,

$$\mathcal{L}_\alpha \phi_\alpha = \gamma_\alpha \delta_0$$

in the sense of distributions where the constant

$$\gamma_\alpha = \frac{2^{2-n}\pi^{n+1}}{\Gamma(\frac{[n+\alpha]}{2})\Gamma(\frac{[n-\alpha]}{2})}.$$

Comments on the Theorem

Remarks:

- ϕ_α is (nonisotropically) homogeneous of degree $-2n$ in $\mathbb{C}^n \times \mathbb{R}$, hence it is in $L^1_{\text{loc}}(\mathbb{C}^n \times \mathbb{R})$
- $\mathcal{L}_\alpha \phi_\alpha$ is automatically a distribution since ϕ_α is a distribution.
- $\Gamma(s)$ has poles at $s = 0, -1, \ldots$, hence $\gamma_\alpha = 0$ exactly at the forbidden values.

Comments on the Theorem

Remarks:

- ϕ_α is (nonisotropically) homogeneous of degree $-2n$ in $\mathbb{C}^n \times \mathbb{R}$, hence it is in $L^1_{\text{loc}}(\mathbb{C}^n \times \mathbb{R})$
- $\mathcal{L}_\alpha \phi_\alpha$ is automatically a distribution since ϕ_α is a distribution.
- $\Gamma(s)$ has poles at $s = 0, -1, \ldots$, hence $\gamma_\alpha = 0$ exactly at the forbidden values.

Comments on the Theorem

Remarks:

- ϕ_α is (nonisotropically) homogeneous of degree $-2n$ in $\mathbb{C}^n \times \mathbb{R}$, hence it is in $L^1_{\text{loc}}(\mathbb{C}^n \times \mathbb{R})$
- $\mathcal{L}_\alpha \phi_\alpha$ is automatically a distribution since ϕ_α is a distribution.
- $\Gamma(s)$ has poles at $s = 0, -1, \ldots$, hence $\gamma_\alpha = 0$ exactly at the forbidden values.

Proof Sketch

There are two assertions to prove:

1. $\mathcal{L}_\alpha \phi_\alpha(z,t) = 0$ whenever $(z,t) \neq (0,0)$
2. That γ has the prescribed value.

Let $\epsilon > 0$ and set

$$\phi_{\alpha,\epsilon}(z,t) = (|z|^2 + \epsilon^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + \epsilon^2 + it)^{-\frac{n-\alpha}{2}}$$

$\phi_{\alpha,\epsilon} \in C^\infty(\mathbb{C}^n \times \mathbb{R})$ and $\phi_{\alpha,\epsilon} \to \phi_\alpha$ as $\epsilon \to 0$ in L^1_{loc} and hence in $\mathcal{D}'(\mathbb{C}^n \times \mathbb{R})$.

Proof Sketch

There are two assertions to prove:

1. $\mathcal{L}_\alpha \phi_\alpha(z,t) = 0$ whenever $(z,t) \neq (0,0)$
2. That γ has the prescribed value.

Let $\epsilon > 0$ and set

$$\phi_{\alpha,\epsilon}(z,t) = (|z|^2 + \epsilon^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + \epsilon^2 + it)^{-\frac{n-\alpha}{2}}$$

$\phi_{\alpha,\epsilon} \in C^\infty(\mathbb{C}^n \times \mathbb{R})$ and $\phi_{\alpha,\epsilon} \to \phi_\alpha$ as $\epsilon \to 0$ in L^1_{loc} and hence in $\mathcal{D}'(\mathbb{C}^n \times \mathbb{R})$.

Proof Sketch

There are two assertions to prove:

1. $\mathcal{L}_\alpha \phi_\alpha(z,t) = 0$ whenever $(z,t) \neq (0,0)$
2. That γ has the prescribed value.

Let $\epsilon > 0$ and set

$$\phi_{\alpha,\epsilon}(z,t) = (|z|^2 + \epsilon^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + \epsilon^2 + it)^{-\frac{n-\alpha}{2}}$$

$\phi_{\alpha,\epsilon} \in C^\infty(\mathbb{C}^n \times \mathbb{R})$ and $\phi_{\alpha,\epsilon} \to \phi_\alpha$ as $\epsilon \to 0$ in L^1_{loc} and hence in $\mathcal{D}'(\mathbb{C}^n \times \mathbb{R})$.

Proof Sketch, cont'd

Set

$$\psi_{\alpha,\epsilon} = \mathcal{L}_\alpha(\phi_{\alpha,\epsilon}).$$

Then calculations show that

$$\psi_{\alpha,\epsilon}(z,t) = \epsilon^{-2n-2}\psi_{\alpha,1}(z/\epsilon, t/\epsilon^2)$$

and

$$\psi_{\alpha,1}(z,t) = (n^2-\alpha^2)(|z|^2+1-it)^{-\frac{n+\alpha+2}{2}}(|z|^2+1+it)^{-\frac{n-\alpha+2}{2}}.$$

Proof Sketch, cont'd

Set

$$\psi_{\alpha,\epsilon} = \mathcal{L}_\alpha(\phi_{\alpha,\epsilon}).$$

Then calculations show that

$$\psi_{\alpha,\epsilon}(z,t) = \epsilon^{-2n-2}\psi_{\alpha,1}(z/\epsilon, t/\epsilon^2)$$

and

$$\psi_{\alpha,1}(z,t) = (n^2-\alpha^2)(|z|^2+1-it)^{-\frac{n+\alpha+2}{2}}(|z|^2+1+it)^{-\frac{n-\alpha+2}{2}}.$$

Proof Sketch, cont'd

Consequences

- $\psi_{\alpha,\epsilon}(z,t) \to 0$ as $\epsilon \to 0$ if $(z,t) \neq (0,0)$.
- $\psi_{\alpha,1} \in L^1(\mathbb{C}^n \times \mathbb{R})$
- Thus, $\{\psi_{\alpha,\epsilon}\}_{\epsilon>0}$ for a constant multiple of an approximation of the identity!

Thus, the proof is complete with

$$\gamma_\alpha = \int_{\mathbb{C}^n \times \mathbb{R}} \psi_{\alpha,1}(z,t)\, dz\, dt.$$

Proof Sketch, cont'd

Consequences

- $\psi_{\alpha,\epsilon}(z,t) \to 0$ as $\epsilon \to 0$ if $(z,t) \neq (0,0)$.
- $\psi_{\alpha,1} \in L^1(\mathbb{C}^n \times \mathbb{R})$
- Thus, $\{\psi_{\alpha,\epsilon}\}_{\epsilon>0}$ for a constant multiple of an approximation of the identity!

Thus, the proof is complete with

$$\gamma_\alpha = \int_{\mathbb{C}^n \times \mathbb{R}} \psi_{\alpha,1}(z,t)\, dz\, dt.$$

Proof Sketch, cont'd

Consequences

- $\psi_{\alpha,\epsilon}(z,t) \to 0$ as $\epsilon \to 0$ if $(z,t) \neq (0,0)$.
- $\psi_{\alpha,1} \in L^1(\mathbb{C}^n \times \mathbb{R})$
- Thus, $\{\psi_{\alpha,\epsilon}\}_{\epsilon>0}$ for a constant multiple of an approximation of the identity!

Thus, the proof is complete with

$$\gamma_\alpha = \int_{\mathbb{C}^n \times \mathbb{R}} \psi_{\alpha,1}(z,t)\, dz\, dt.$$

Proof Sketch, cont'd

Set $\beta = \frac{n+\alpha+2}{2}$ and $\gamma = \frac{n-\alpha+2}{2}$ (so $\beta + \gamma = n+2$) and

$$I = \int_{\mathbb{C}^n \times \mathbb{R}} (|z|^2 + 1 - it)^{-\beta} (|z|^2 + 1 + it)^{-\gamma} \, dz \, dt.$$

Scaling $t \mapsto (|z|^2 + 1)t$,

$$I = \int_{\mathbb{C}^n} (|z|^2 + 1)^{-n-1} \, dz \cdot J$$

where

$$J = \int_{\mathbb{R}} (1 - it)^{-\beta} (1 + it)^{-\gamma} \, dt.$$

Proof Sketch, cont'd

Set $\beta = \frac{n+\alpha+2}{2}$ and $\gamma = \frac{n-\alpha+2}{2}$ (so $\beta + \gamma = n + 2$) and

$$I = \int_{\mathbb{C}^n \times \mathbb{R}} (|z|^2 + 1 - it)^{-\beta} (|z|^2 + 1 + it)^{-\gamma} \, dz \, dt.$$

Scaling $t \mapsto (|z|^2 + 1)t$,

$$I = \int_{\mathbb{C}^n} (|z|^2 + 1)^{-n-1} \, dz \cdot J$$

where

$$J = \int_{\mathbb{R}} (1 - it)^{-\beta} (1 + it)^{-\gamma} \, dt.$$

Proof Sketch, cont'd

Since

$$\int_0^\infty e^{-As} s^n \, ds = n! A^{-n-1}, \qquad A > 0$$

it follows that

$$\int_{\mathbb{C}^n} (1 + |z|^2)^{-n-1} \, dz = \frac{\pi^{n+1}}{n!} \int_{\mathbb{C}^n} \int_0^\infty e^{-\pi s(1+|z|^2)} s^n \, ds \, dz$$

$$= \frac{\pi^{n+1}}{n!} \int_0^\infty e^{-\pi s} s^n s^{-n} \, ds = \frac{\pi^n}{n!}.$$

Proof Sketch, cont'd

Since

$$\int_0^\infty e^{-As} s^n \, ds = n! A^{-n-1}, \qquad A > 0$$

it follows that

$$\begin{aligned}\int_{\mathbb{C}^n} (1+|z|^2)^{-n-1} \, dz &= \frac{\pi^{n+1}}{n!} \int_{\mathbb{C}^n} \int_0^\infty e^{-\pi s(1+|z|^2)} s^n \, ds \, dz \\ &= \frac{\pi^{n+1}}{n!} \int_0^\infty e^{-\pi s} s^n s^{-n} \, ds = \frac{\pi^n}{n!}.\end{aligned}$$

Proof Sketch, cont'd

We have only to compute

$$J = \int_{\mathbb{R}} (1 - it)^{-\beta}(1 + it)^{-\gamma}\, dt.$$

By analytic continuation, it turns out to be enough to assume $-n \leq \alpha \leq n$ (so that $\beta, \gamma \geq 1$).

Recall:

$$(1 + it)^{-\gamma}\Gamma(\gamma) = \int_0^\infty e^{-ixt} e^{-x} x^{\gamma-1}\, dx$$

and

$$(1 - it)^{-\beta}\Gamma(\beta) = \int_{-\infty}^0 e^{-ixt} e^{-|x|} |x|^{\beta-1}\, dx = \int_0^\infty e^{ixt} e^{-x} x^{\beta-1}\, dx$$

Proof Sketch, cont'd

We have only to compute

$$J = \int_{\mathbb{R}} (1 - it)^{-\beta}(1 + it)^{-\gamma}\, dt.$$

By analytic continuation, it turns out to be enough to assume $-n \leq \alpha \leq n$ (so that $\beta, \gamma \geq 1$).

Recall:

$$(1 + it)^{-\gamma}\Gamma(\gamma) = \int_0^\infty e^{-ixt} e^{-x} x^{\gamma-1}\, dx$$

and

$$(1 - it)^{-\beta}\Gamma(\beta) = \int_{-\infty}^0 e^{-ixt} e^{-|x|} |x|^{\beta-1}\, dx = \int_0^\infty e^{ixt} e^{-x} x^{\beta-1}\, dx$$

Proof Sketch, cont'd

By Plancherel's Theorem, (specifically, $(f,g) = (\hat{\check{f}}, g) = (\check{f}, \hat{g})$)

$$\Gamma(\beta)\Gamma(\gamma)\int_{-\infty}^{\infty}(1-it)^{-\beta}(1+it)^{-\gamma}\,dt$$

$$= 2\pi\int_{0}^{\infty} e^{-x}x^{\gamma-1}e^{-x}x^{\beta-1}\,dx = \frac{\pi n!}{2^n}.$$

This completes the proof! □

Proof Sketch, cont'd

By Plancherel's Theorem, (specifically, $(f,g) = (\hat{\check{f}}, g) = (\check{f}, \hat{g})$)

$$\Gamma(\beta)\Gamma(\gamma)\int_{-\infty}^{\infty}(1-it)^{-\beta}(1+it)^{-\gamma}\,dt$$

$$= 2\pi\int_{0}^{\infty} e^{-x}x^{\gamma-1}e^{-x}x^{\beta-1}\,dx = \frac{\pi n!}{2^n}.$$

This completes the proof! □

Consequences

We assume that α is not a forbidden value (i.e., $\alpha \neq \pm n, \pm(n+2), \dots$). The fundamental solution to $\mathcal{L}_\alpha$ is

$$F_\alpha(z,t) = \gamma_\alpha^{-1}\phi_\alpha(z,t).$$

The solution operator S_α is then

$$S_\alpha f(y) = f * F_\alpha(y) = \int_{\mathbb{C}^n \times \mathbb{R}} f(y) F_\alpha(y^{-1} \cdot x)\, dy.$$

Proposition

If $f \in \mathcal{S}$ or $f \in \mathcal{E}'$, then

$$\mathcal{L}_\alpha S_\alpha f = S_\alpha \mathcal{L}_\alpha f = f.$$

Consequences

We assume that α is not a forbidden value (i.e., $\alpha \neq \pm n, \pm(n+2), \dots$). The fundamental solution to $\mathcal{L}_\alpha$ is

$$F_\alpha(z,t) = \gamma_\alpha^{-1}\phi_\alpha(z,t).$$

The solution operator S_α is then

$$S_\alpha f(y) = f * F_\alpha(y) = \int_{\mathbb{C}^n \times \mathbb{R}} f(y) F_\alpha(y^{-1} \cdot x)\, dy.$$

Proposition

If $f \in \mathcal{S}$ or $f \in \mathcal{E}'$, then

$$\mathcal{L}_\alpha S_\alpha f = S_\alpha \mathcal{L}_\alpha f = f.$$

Consequences

We assume that α is not a forbidden value (i.e., $\alpha \neq \pm n, \pm(n+2), \dots$). The fundamental solution to $\mathcal{L}_\alpha$ is

$$F_\alpha(z,t) = \gamma_\alpha^{-1}\phi_\alpha(z,t).$$

The solution operator S_α is then

$$S_\alpha f(y) = f * F_\alpha(y) = \int_{\mathbb{C}^n \times \mathbb{R}} f(y) F_\alpha(y^{-1} \cdot x)\, dy.$$

Proposition

If $f \in \mathcal{S}$ or $f \in \mathcal{E}'$, then

$$\mathcal{L}_\alpha S_\alpha f = S_\alpha \mathcal{L}_\alpha f = f.$$

Consequences, cont'd

Definition

The operator $\mathcal{L}_\alpha$ is called locally solvable if given any distribution f defined on a near a point p in $\mathcal{H}^n$, there exists a distribution u and a neighborhood of p so that

$$\mathcal{L}_\alpha u = f.$$

The operator $\mathcal{L}_\alpha$ is called hypoelliptic if whenever $f \in C^\infty$, so is u.

Proposition

If $\alpha \neq \pm n, \pm(n+2), \ldots,$ then $\mathcal{L}_\alpha$ is both locally solvable and hypoelliptic.

Corollary

When $0 < q < n$, the Kohn Laplacian $\Box_b$ is both locally solvable and hypoelliptic.

Consequences, cont'd

Definition

The operator $\mathcal{L}_\alpha$ is called locally solvable if given any distribution f defined on a near a point p in $\mathcal{H}^n$, there exists a distribution u and a neighborhood of p so that

$$\mathcal{L}_\alpha u = f.$$

The operator $\mathcal{L}_\alpha$ is called hypoelliptic if whenever $f \in C^\infty$, so is u.

Proposition

If $\alpha \neq \pm n, \pm(n+2), \ldots,$ then $\mathcal{L}_\alpha$ is both locally solvable and hypoelliptic.

Corollary

When $0 < q < n$, the Kohn Laplacian $\Box_b$ is both locally solvable and hypoelliptic.

Consequences, cont'd

Definition

The operator $\mathcal{L}_\alpha$ is called locally solvable if given any distribution f defined on a near a point p in $\mathcal{H}^n$, there exists a distribution u and a neighborhood of p so that

$$\mathcal{L}_\alpha u = f.$$

The operator $\mathcal{L}_\alpha$ is called hypoelliptic if whenever $f \in C^\infty$, so is u.

Proposition

If $\alpha \neq \pm n, \pm(n+2), \ldots,$ then $\mathcal{L}_\alpha$ is both locally solvable and hypoelliptic.

Corollary

When $0 < q < n$, the Kohn Laplacian $\Box_b$ is both locally solvable and hypoelliptic.

Consequences, cont'd

Definition

The operator $\mathcal{L}_\alpha$ is called locally solvable if given any distribution f defined on a near a point p in $\mathcal{H}^n$, there exists a distribution u and a neighborhood of p so that

$$\mathcal{L}_\alpha u = f.$$

The operator $\mathcal{L}_\alpha$ is called hypoelliptic if whenever $f \in C^\infty$, so is u.

Proposition

If $\alpha \neq \pm n, \pm(n+2), \ldots,$ then $\mathcal{L}_\alpha$ is both locally solvable and hypoelliptic.

Corollary

When $0 < q < n$, the Kohn Laplacian $\Box_b$ is both locally solvable and hypoelliptic.

Forbidden values still ok

Theorem (see Stein '93)

A relative fundamental solution to $\Box_b = \bar\partial_b^* \bar\partial_b$ *on functions is given by the integration kernel*

$$N(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2 - it)^n} \log\left(\frac{|z|^2 - it}{|z|^2 + it}\right).$$

If S is the projection onto the $\ker \Box_b$ where $\Box_b = \bar\partial_b^* \bar\partial_b$, then G_0 is given by convolution with N and satisfies

$$\Box_b G_0 f = (I - S) f.$$

Forbidden values still ok

Theorem (see Stein '93)

A relative fundamental solution to $\Box_b = \bar{\partial}_b^* \bar{\partial}_b$ *on functions is given by the integration kernel*

$$N(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2 - it)^n} \log\left(\frac{|z|^2 - it}{|z|^2 + it}\right).$$

If S is the projection onto the $\ker \Box_b$ where $\Box_b = \bar{\partial}_b^* \bar{\partial}_b$, then G_0 is given by convolution with N and satisfies

$$\Box_b G_0 f = (I - S) f.$$

Forbidden values still ok

Theorem (see Stein '93)

A relative fundamental solution to $\Box_b = \bar\partial_b^* \bar\partial_b$ *on functions is given by the integration kernel*

$$N(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2 - it)^n} \log\left(\frac{|z|^2 - it}{|z|^2 + it}\right).$$

If S is the projection onto the $\ker \Box_b$ where $\Box_b = \bar\partial_b^* \bar\partial_b$, then G_0 is given by convolution with N and satisfies

$$\Box_b G_0 f = (I - S) f.$$

Contextualizing $\phi_\alpha(z,t)$

In $\mathbb{R}^n$, $n \geq 3$, the Laplacian

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and Poisson's equation $\triangle u = f$ is solved by

$$u(x) = f * N(x) = \int_{\mathbb{R}^n} f(y) N(x-y)\, dy$$

where

$$N(x-y) = c_n \frac{1}{|x-y|^{n-2}}.$$

Contextualizing $\phi_\alpha(z,t)$

In $\mathbb{R}^n$, $n \geq 3$, the Laplacian

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and Poisson's equation $\triangle u = f$ is solved by

$$u(x) = f * N(x) = \int_{\mathbb{R}^n} f(y) N(x-y)\, dy$$

where

$$N(x-y) = c_n \frac{1}{|x-y|^{n-2}} = c_n \frac{|x-y|^2}{|x-y|^n}.$$

Contextualizing $\phi_\alpha(z,t)$

In $\mathbb{R}^n$, $n \geq 3$, the Laplacian (an order 2 operator)

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and Poisson's equation $\triangle u = f$ is solved by

$$u(x) = f * N(x) = \int_{\mathbb{R}^n} f(y) N(x-y)\, dy$$

where

$$N(x-y) = c_n \frac{1}{|x-y|^{n-2}} = c_n \frac{|x-y|^2}{|x-y|^n} = c_n' \frac{d_E(x,y)^2}{|B_E(x, d_E(x,y))|}.$$

Contextualizing $\phi_\alpha(z,t)$

In $\mathcal{H}^n$, define the "norm" function

$$\rho(z,t) = (|z|^4 + t^2)^{1/4} \sim |z| + |t|^{1/2}$$

Observe that

$$\rho((z,t)^{-1}) = \rho(-(z,t)) = \rho(z,t)$$

and

$$\rho(\delta \circ (z,t)) = \delta\rho(z,t).$$

Contextualizing $\phi_\alpha(z,t)$

In $\mathcal{H}^n$, define the "norm" function

$$\rho(z,t) = (|z|^4 + t^2)^{1/4} \sim |z| + |t|^{1/2}$$

Observe that

$$\rho\big((z,t)^{-1}\big) = \rho\big(-(z,t)\big) = \rho(z,t)$$

and

$$\rho\big(\delta \circ (z,t)\big) = \delta\rho(z,t).$$

The control distance

Define the pseudometric

$$d_{\mathcal{H}^n}\big((z,t),(w,s)\big) = \rho\big((w,s)^{-1}\cdot(z,t)\big) = \rho\big(z-w, t-s-2\,\mathrm{Im}(w\cdot\bar{z})\big)$$
$$\sim |z-w| + |t-s-2\,\mathrm{Im}(w\cdot\bar{z})|^{1/2}.$$

The term $T(w,z) = -2\,\mathrm{Im}(w\cdot\bar{z})$ is the called the twist term.

The balls

$$B_{\mathcal{H}^n}(x,r) = \{y\in\mathbb{C}^n\times\mathbb{R} : d_{\mathcal{H}^n}(y,x) < r\}.$$

We see that

$$|F_\alpha(y^{-1}\cdot x)| = c_\alpha \frac{d_{\mathcal{H}^n}(y,x)^2}{|B_{\mathcal{H}^n}(x, d_{\mathcal{H}^n}(y,x))|}$$

The control distance

Define the pseudometric

$$d_{\mathcal{H}^n}\big((z,t),(w,s)\big) = \rho\big((w,s)^{-1}\cdot(z,t)\big) = \rho\big(z-w, t-s-2\,\mathrm{Im}(w\cdot\bar{z})\big)$$
$$\sim |z-w| + |t-s-2\,\mathrm{Im}(w\cdot\bar{z})|^{1/2}.$$

The term $T(w,z) = -2\,\mathrm{Im}(w\cdot\bar{z})$ is the called the twist term.

The balls

$$B_{\mathcal{H}^n}(x,r) = \{y \in \mathbb{C}^n \times \mathbb{R} : d_{\mathcal{H}^n}(y,x) < r\}.$$

We see that

$$|F_\alpha(y^{-1}\cdot x)| = c_\alpha \frac{d_{\mathcal{H}^n}(y,x)^2}{|B_{\mathcal{H}^n}(x, d_{\mathcal{H}^n}(y,x))|}$$

The control distance

Define the pseudometric

$$d_{\mathcal{H}^n}\big((z,t),(w,s)\big) = \rho\big((w,s)^{-1}\cdot(z,t)\big) = \rho\big(z-w, t-s-2\,\mathrm{Im}(w\cdot\bar{z})\big)$$
$$\sim |z-w| + |t-s-2\,\mathrm{Im}(w\cdot\bar{z})|^{1/2}.$$

The term $T(w,z) = -2\,\mathrm{Im}(w\cdot\bar{z})$ is the called the twist term.

The balls

$$B_{\mathcal{H}^n}(x,r) = \{y \in \mathbb{C}^n \times \mathbb{R} : d_{\mathcal{H}^n}(y,x) < r\}.$$

We see that

$$|F_\alpha(y^{-1}\cdot x)| = c_\alpha \frac{d_{\mathcal{H}^n}(y,x)^2}{|B_{\mathcal{H}^n}(x, d_{\mathcal{H}^n}(y,x))|}$$

The control distance

Define the pseudometric

$$d_{\mathcal{H}^n}\big((z,t),(w,s)\big) = \rho\big((w,s)^{-1}\cdot(z,t)\big) = \rho\big(z-w, t-s-2\,\mathrm{Im}(w\cdot\bar{z})\big)$$
$$\sim |z-w| + |t-s-2\,\mathrm{Im}(w\cdot\bar{z})|^{1/2}.$$

The term $T(w,z) = -2\,\mathrm{Im}(w\cdot\bar{z})$ is the called the twist term.

The balls

$$B_{\mathcal{H}^n}(x,r) = \{y \in \mathbb{C}^n \times \mathbb{R} : d_{\mathcal{H}^n}(y,x) < r\}.$$

We see that

$$|F_\alpha(y^{-1}\cdot x)| = c_\alpha \frac{d_{\mathcal{H}^n}(y,x)^2}{|B_{\mathcal{H}^n}(x, d_{\mathcal{H}^n}(y,x))|}$$

The control distance, more generally

Question: Is it always the case that the complex Green operator is governed by a metric?

Answer: No, in general (Machedon 1988, Nagel-Stein 2006). Two non-equivalent metrics are needed in these examples and the answer in the general case is unknown!!! Nagel and Stein considered "decoupled" models of the form

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{k=1}^{n} p_k(z_k)\right\}$$

where $p_k : \mathbb{C} \to \mathbb{R}$ is a subharmonic, nonharmonic polynomial (i.e., $\triangle p_k \geq 0$ and $\triangle p_k \not\equiv 0$).

In the case of $\mathcal{H}^n$, the two metrics are equivalent.

The control distance, more generally

Question: Is it always the case that the complex Green operator is governed by a metric?

Answer: No, in general (Machedon 1988, Nagel-Stein 2006). Two non-equivalent metrics are needed in these examples and the answer in the general case is unknown!!! Nagel and Stein considered "decoupled" models of the form

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{k=1}^{n} p_k(z_k)\right\}$$

where $p_k : \mathbb{C} \to \mathbb{R}$ is a subharmonic, nonharmonic polynomial (i.e., $\triangle p_k \geq 0$ and $\triangle p_k \not\equiv 0$).

In the case of $\mathcal{H}^n$, the two metrics are equivalent.

The control distance, more generally

Question: Is it always the case that the complex Green operator is governed by a metric?

Answer: No, in general (Machedon 1988, Nagel-Stein 2006). Two non-equivalent metrics are needed in these examples and the answer in the general case is unknown!!! Nagel and Stein considered "decoupled" models of the form

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{k=1}^{n} p_k(z_k)\right\}$$

where $p_k : \mathbb{C} \to \mathbb{R}$ is a subharmonic, nonharmonic polynomial (i.e., $\triangle p_k \geq 0$ and $\triangle p_k \not\equiv 0$).

In the case of $\mathcal{H}^n$, the two metrics are equivalent.

The control distance, more generally

Question: Is it always the case that the complex Green operator is governed by a metric?

Answer: No, in general (Machedon 1988, Nagel-Stein 2006). Two non-equivalent metrics are needed in these examples and the answer in the general case is unknown!!! Nagel and Stein considered "decoupled" models of the form

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{k=1}^{n} p_k(z_k)\right\}$$

where $p_k : \mathbb{C} \to \mathbb{R}$ is a subharmonic, nonharmonic polynomial (i.e., $\triangle p_k \geq 0$ and $\triangle p_k \not\equiv 0$).

In the case of $\mathcal{H}^n$, the two metrics are equivalent.

The control distance, more generally

Question: Is it always the case that the complex Green operator is governed by a metric?

Answer: No, in general (Machedon 1988, Nagel-Stein 2006). Two non-equivalent metrics are needed in these examples and the answer in the general case is unknown!!! Nagel and Stein considered "decoupled" models of the form

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{k=1}^{n} p_k(z_k)\right\}$$

where $p_k : \mathbb{C} \to \mathbb{R}$ is a subharmonic, nonharmonic polynomial (i.e., $\triangle p_k \geq 0$ and $\triangle p_k \not\equiv 0$).

In the case of $\mathcal{H}^n$, the two metrics are equivalent.

The control distance, more generally

New Question: Are there any general cases for which the complex Green operator is governed by a metric?

Yes, if $\dim_{\mathbb{R}} M = 3$ (so hypersurfaces in $\mathbb{C}^2$) AND the M is of finite type. (late 1980's, early 1990's. Authors include Nagel, Stein, Rosay, Wainger, Christ, McNeal, Kohn, Fefferman, among others)

In $\mathbb{C}^2$, finite type means commutators of L and $\bar{L}$ produce T (in $\mathcal{H}^n$, $[\overline{Z}, Z] = 2iT$).

The "correct" metric (in $\mathbb{C}^2$) is generated when one is only allowed to flow in the directions $\operatorname{Re} L$ and $\operatorname{Im} L$ where L is the generator of $T^{1,0}(M)$. This metric is called the control metric or Carnot-Carathéodory metric

The control distance, more generally

New Question: Are there any general cases for which the complex Green operator is governed by a metric?

Yes, if $\dim_{\mathbb{R}} M = 3$ (so hypersurfaces in $\mathbb{C}^2$) AND the M is of finite type. (late 1980's, early 1990's. Authors include Nagel, Stein, Rosay, Wainger, Christ, McNeal, Kohn, Fefferman, among others)

In $\mathbb{C}^2$, finite type means commutators of L and $\bar{L}$ produce T (in $\mathcal{H}^n$, $[\overline{Z}, Z] = 2iT$).

The "correct" metric (in $\mathbb{C}^2$) is generated when one is only allowed to flow in the directions $\operatorname{Re} L$ and $\operatorname{Im} L$ where L is the generator of $T^{1,0}(M)$. This metric is called the control metric or Carnot-Carathéodory metric

The control distance, more generally

New Question: Are there any general cases for which the complex Green operator is governed by a metric?

Yes, if $\dim_{\mathbb{R}} M = 3$ (so hypersurfaces in $\mathbb{C}^2$) AND the M is of finite type. (late 1980's, early 1990's. Authors include Nagel, Stein, Rosay, Wainger, Christ, McNeal, Kohn, Fefferman, among others)

In $\mathbb{C}^2$, finite type means commutators of L and $\bar{L}$ produce T (in $\mathcal{H}^n$, $[\overline{Z}, Z] = 2iT$).

The "correct" metric (in $\mathbb{C}^2$) is generated when one is only allowed to flow in the directions $\operatorname{Re} L$ and $\operatorname{Im} L$ where L is the generator of $T^{1,0}(M)$. This metric is called the control metric or Carnot-Carathéodory metric

The control distance, more generally

New Question: Are there any general cases for which the complex Green operator is governed by a metric?

Yes, if $\dim_{\mathbb{R}} M = 3$ (so hypersurfaces in $\mathbb{C}^2$) AND the M is of finite type. (late 1980's, early 1990's. Authors include Nagel, Stein, Rosay, Wainger, Christ, McNeal, Kohn, Fefferman, among others)

In $\mathbb{C}^2$, finite type means commutators of L and $\bar{L}$ produce T (in $\mathcal{H}^n$, $[\overline{Z}, Z] = 2iT$).

The "correct" metric (in $\mathbb{C}^2$) is generated when one is only allowed to flow in the directions $\operatorname{Re} L$ and $\operatorname{Im} L$ where L is the generator of $T^{1,0}(M)$. This metric is called the control metric or Carnot-Carathéodory metric

The control distance, more generally

New Question: Are there any general cases for which the complex Green operator is governed by a metric?

Yes, if $\dim_{\mathbb{R}} M = 3$ (so hypersurfaces in $\mathbb{C}^2$) AND the M is of finite type. (late 1980's, early 1990's. Authors include Nagel, Stein, Rosay, Wainger, Christ, McNeal, Kohn, Fefferman, among others)

In $\mathbb{C}^2$, finite type means commutators of L and $\bar{L}$ produce T (in $\mathcal{H}^n$, $[\overline{Z}, Z] = 2iT$).

The "correct" metric (in $\mathbb{C}^2$) is generated when one is only allowed to flow in the directions $\operatorname{Re} L$ and $\operatorname{Im} L$ where L is the generator of $T^{1,0}(M)$. This metric is called the control metric or Carnot-Carathéodory metric

Thank you for your attention!

THANK YOU!!