


Lecture 4: CR Geometry and $\bar{\partial}_b$ today. Application: Quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$

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10th Workshop on the Geometric Analysis of PDEs and Several Complex Variables




Outline of the Talk

- To discuss quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$
- To discuss the $\Box_b$-equation and $\Box_b$-heat equation on quadrics

Research in CR Geometry

The possibilities for discussing research in CR Geometry are nearly endless! Directions that I could talk about include:

- CR mappings between CR manifolds
- CR mappings between SPHEREs
- Normal forms for CR mappings
- Extensions of CR functions
- Boundary behavior of holomorphic functions
- L^2 theory of the $\bar{\partial}_b$ operator
- L^p theory of the $\bar{\partial}_b$ operator
- Canonical kernels for the $\bar{\partial}_b$ operator: existence, regularity, geometry (e.g., pointwise size estimates)
- Integral kernels for the $\bar{\partial}_b$ operator: existence, regularity, geometry

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Solving the $\Box_b$-equation

Question: Can we explicitly solve the $\Box_b$-equation?

Answer: Sometimes. However, techniques for establishing the existence of G_q are typically function theoretic. We want to understand the structure and singularities of G_q.

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Quadric submanifolds

Let $\phi : \mathbb{C}^m \to \mathbb{C}^m$ be a sesquilinear form. This means

$$\phi(w, w') = \overline{\phi(w', w)}.$$

The quadric submanifold M of $\mathbb{C}^n \times \mathbb{C}^m$ associated to ϕ is

$$M = \{(z, w) : \operatorname{Im} w = \phi(z, z)\}.$$

We identify $(z, t + i\phi(z, z)) \in M$ with the point $(z, t) \in \mathbb{C}^n \times \mathbb{R}^m$.

M has the group structure

$$(z, t) \star (z', t') = (z + z', t + t' + 2 \operatorname{Im} \phi(z', z)).$$

On M there is the nonisotropic dilation structure

$$\delta \circ (z, t) = (\delta z, \delta^2 t)$$

and we expect to see the dilation structure in the geometry and integral kernels.

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More on quadrics

- $M \cong \mathbb{C}^n \times \mathbb{R}^m$ under the identification

$$\big(z, t + i\phi(z,z)\big) \longleftrightarrow (z,t)$$

- For $v \in \mathbb{R}^{2n} \cong \mathbb{C}^n$, the CR structure on M is spanned by vectors of the form:

$$Z_v(g) = (1/2)(\partial_v - i\partial_{Jv}) - i\overline{\phi(z,v)} \cdot D_t$$

and

$$\bar{Z}_v(g) = (1/2)(\partial_v + i\partial_{Jv}) + i\phi(z,v) \cdot D_t$$

where $D_t = \begin{pmatrix} \frac{\partial}{\partial t_1} & \cdots & \frac{\partial}{\partial t_m} \end{pmatrix}^t = \begin{pmatrix} T_1 & \cdots & T_m \end{pmatrix}^t$.

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The Levi forms

In an appropriate metric (chosen so $\frac{\partial}{\partial t_\ell}$ is orthogonal to the CR structure), the Levi form is the mapping

$$z \mapsto \phi(z, z) \in \mathbb{R}^m$$

For $\lambda \in \mathbb{R}^m$, the scalar-valued Levi form in the direction λ is the map

$$z \mapsto \phi_\lambda = \phi(z, z) \cdot \lambda \in \mathbb{R}.$$

Note: $X(M)$ is spanned by $\{T_\ell\}$ and we identify $\lambda \cdot (T_1, \ldots, T_m)$ with λ.

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Examples

- The Heisenberg group. $m = 1$, $\phi(z, z') = \sum_{j=1}^{n} z_j \bar{z}'_j$. $\phi(z, z) = |z|^2$ so that Range $\phi(z, z) = [0, \infty)$.
- Codimension Two Examples in $\mathbb{C}^4$. $n = m = 2$. For $(z, t) \in \mathbb{C}^2 \times \mathbb{R}^2$, with $z = (z_1, z_2)$
  1. $\phi(z, z) = \begin{pmatrix} |z_1|^2 \\ |z_2|^2 \end{pmatrix}$. Range $\phi(z, z)$ is the closed first quadrant.
  2. $\phi(z, z) = \begin{pmatrix} 2\,\mathrm{Re}(z_1 \bar{z}_2) \\ |z_1|^2 - |z_2|^2 \end{pmatrix}$. Range $\phi(z, z) = \mathbb{R}^2$.
  3. $\phi(z, z) = \begin{pmatrix} 2|z_1|^2 \\ 2\,\mathrm{Re}(z_1 \bar{z}_2) \end{pmatrix}$. Range $\phi(z, z)$ is open half-space and the origin.

For quadrics, $\Box_b$ was computed by Peloso and Ricci (2003) and is a tridiagonal operator in the ambient coordinates.

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Research Topics

- Solvability of $\Box_b u = f$; $f \in \mathcal{D}_q(M)$, $u \in \mathcal{D}'_q(M)$; (Peloso-Ricci 2003)
- Hypoellipticity of $\Box_b u = f$; (Peloso-Ricci 2003)
- If $\ker \Box_b \neq \{0\}$, then we denote the orthogonal projection onto $\ker \Box_b$ by S_q. S_0 is called the Szegö projection. What are the mapping properties of G_q and S_q?
- Asymptotics (or computations) of the Szegö kernel and the convolution kernel of G_q, particularly in terms of the intrinsic metrics on M; Side Question: What are the intrinsic metrics on M?
- Our focus – explicit calculations of G_q and S_q in the spirit of Folland-Stein (1974).

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The $\Box_b$-heat equation

Main Technique/Auxiliary Problem: The $\Box_b$-heat equation

$$\begin{cases} \dfrac{\partial u}{\partial s} + \Box_b u = 0 & \text{in } (0,\infty) \times M \\ u(0,x) = f(x) & \text{on } \{s = 0\} \times M. \end{cases}$$

By functional analysis (Spectral Theorem for Self-Adjoint Operators, Riesz Representation Theorem),

$$u(s,x) = e^{-s\Box_b} f(x) = \int_{\mathbb{C}^n \times \mathbb{R}^m} H(s,x,y) f(y)\, dy.$$

$H(s,x,y)$ is called the $\Box_b$-heat kernel.

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Heat equation facts

1. $\Box_b$ is self-adjoint (so $\sigma(\Box_b) \subset \mathbb{R}$)
2. $\Box_b$ is a positive operator.

$$(\Box_b f, f) = (\bar\partial_b f, \bar\partial_b f) + (\bar\partial_b^* f, \bar\partial_b^* f) \geq 0$$

so $\sigma(\Box_b) \subset [0, \infty)$

3. $\lim_{s\to\infty} e^{-s\Box_b} = S$ since $\lim_{s\to\infty} e^{-s\lambda} = \begin{cases} 0 & \lambda > 0 \\ 1 & \lambda = 0 \end{cases}$
4. $S(x,y) = \lim_{s\to\infty} H(s,x,y)$
5. $G = \int_0^\infty e^{-s\Box_b}(I - S)\, ds$, that is,

$$G(x,y) = \int_0^\infty H(s,x,y) - S(x,y)\, ds$$

6. If $S = 0$, then $G(x,y) = \int_0^\infty H(s,x,y)\, ds$.

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I'm not lying

In $\mathbb{R}^n$, the $\triangle$-heat kernel is the Gaussian

$$H(s,x,y) = \frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}}.$$

Note: $\lim_{s\to\infty} H(s,x,y) = 0$ (that is, there are no L^2 harmonic functions on $\mathbb{R}^n$). Also, if $n \geq 3$, a simple change of variables ($\frac{|x-y|^2}{s} \mapsto \frac{1}{s}$) shows

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In $\mathbb{R}^n$, the $\triangle$-heat kernel is the Gaussian

$$H(s,x,y) = \frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}}.$$

Note: $\lim_{s\to\infty} H(s,x,y) = 0$ (that is, there are no L^2 harmonic functions on $\mathbb{R}^n$). Also, if $n \geq 3$, a simple change of variables ($\frac{|x-y|^2}{s} \mapsto \frac{1}{s}$) shows

$$\int_0^\infty \frac{1}{(4\pi s)^{n/2}} e^{-\frac{|x-y|^2}{4s}}\, ds = \frac{c_n}{|x-y|^{n-2}},$$

the Newtonian potential and the inverse to $\triangle$!

The $\Box_b$-heat equation on "the" Heisenberg group

Let M be defined by

$$M = \left\{(z, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = \frac{1}{2}|\operatorname{Im} z|^2\right\}.$$

Under the usual identification $(x, y, t + is) \longleftrightarrow (x, y, t)$, then we have "the" Heisenberg group $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group multiplication

$$g * g' = (x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x \cdot y'),$$

and tangential vector fields

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j}, \quad 1 \leq j \leq n, \quad \text{and} \quad T = \frac{\partial}{\partial t}.$$

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$\Box_b$ on this Heisenberg group

Then

$$\Box_b f = \sum_{J \in \mathcal{I}_q} \mathcal{L}_{n-2q} f_J \, d\bar{z}_J$$

where

$$\mathcal{L}_\gamma = -\frac{1}{4} \left(\sum_{j=1}^{n} (X_j^2 + Y_j^2) + i\gamma T \right) .$$

We follow (Boggess-Raich, 2009).

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Tools

- The Fourier transform. Let

$$\hat{f}(\alpha, \beta, \tau) = \int_{\mathcal{H}^n} f(x, y, t)\, e^{-i(\alpha \cdot x + \beta \cdot y + \tau t)}\, dx\, dy\, dt$$

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$$f(x, y, \hat{\tau}) = \int_{\mathbb{R}} e^{-it\tau} f(x, y, t)\, dt.$$

This means: $\frac{\partial}{\partial t} \hat{\mapsto} i\tau$,

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$$\psi_m(r) = \frac{(-1)^m}{\quad} e^{r^2/2} \frac{d^m}{\quad} \{ e^{-r^2} \}$$

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The $\Box_b$-heat kernel, almost

Theorem (Boggess-Raich, 2009)

For any $\gamma \in \mathbb{C}$, $\tau \in \mathbb{R}$, the function

$$\rho^{\gamma}(s,x,y,\hat{\tau}) = \frac{\tau^n e^{-\gamma\frac{s\tau}{4}}}{(4\pi)^n \sinh^n(\frac{s\tau}{4})} e^{-\frac{\tau}{2}\coth(\frac{s\tau}{4})(|x|^2+|y|^2)-i\frac{\tau}{2}x\cdot y}.$$

is the fundamental solution to the $\mathcal{L}_{\gamma,\hat{\tau}}$-heat equation:

$$\begin{cases} \left(\dfrac{\partial}{\partial s} + \mathcal{L}_{\gamma,\hat{\tau}}\right)\rho^{\gamma}(s,x,y,\hat{\tau}) = 0 & \text{in } (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \\ \rho^{\gamma}(s=0,x,y,\hat{\tau}) = \delta_{(0,0)}(x,y). & \text{on } \{s=0\}\times\mathbb{R}^n\times\mathbb{R}^n \end{cases}$$

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More on Hermite functions

- $\{\psi_m\}$ is an orthonormal basis of $L^2(\mathbb{R})$
- Hermite functions satisfy the ODE

$$\psi_m''(x) = x^2\psi_m(x) - (2m+1)\psi_m(x).$$

- Mehler's formula. If $|S| < 1$, then

$$\sqrt{\pi}\sum_{m=0}^{\infty}(-iS)^m\psi_m(x)\psi_m(y) = \frac{1}{(1+S^2)^{1/2}}\,e^{-\frac{1}{2}\frac{1-S^2}{1+S^2}(x^2+y^2)}\,e^{ixy\frac{-2S}{1+S^2}}.$$

Note: Mehler's formula is the main reason that the $\Box_b$-heat equation is easier to solve than the $\Box_b$-equation on quadrics.

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Sketch of the Proof

Step 1: Apply $\widehat{\ }$ and observe

$$\widehat{X_j^2 f}(\alpha, \beta, \tau) = (-\alpha_j^2 - 2i\alpha_j \tau \frac{\partial}{\partial \beta_j} + \tau^2 \frac{\partial^2}{\partial \beta_j^2})\hat{f}$$

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Step 2: From the form of $\mathcal{L}_{\gamma,\hat{\tau}}$, we have two simplifying observations!

$$\hat{\rho}^{\gamma}(s, \alpha, \beta, \tau) = \prod_{j=1}^{n} \hat{\rho}^{\gamma/n,1}(s, \alpha_j, \beta_j, \tau)$$

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Step 3: Eliminate the red term. Set

$$u(s,\alpha,\beta,\tau) = \hat{\rho}(s,\alpha,\beta,\tau)e^{-i\frac{\alpha\beta}{\tau}}.$$

Then we want to find u so that

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Step 5: Solve for ρ (easy, at this point)

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The proof is done! □

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History

For the Heisenberg group

- Hulanicki (1976), Gaveau (1977) obtained the formulas for the sub-Laplacian, i.e., $\gamma = 0$ (so no imaginary term)
- Chang-Tie (2000) and Tie (2006) computed $\rho(s, z, w, \hat{\tau})$ for $\mathcal{H}^n$ through a similar method.
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Returning to Quadrics

Recall: The map

$$z \mapsto \phi(z, z)$$

gives the Levi form and for $\lambda \in \mathbb{R}^m$,

$$z \mapsto \phi^\lambda(z, z) = \phi(z, z) \cdot \lambda$$

is the directional Levi form.

1. $\Box_b$ is solvable if given any smooth $(0, q)$-form ψ on M with compact support, there exists a $(0, q)$-current u on M so that $\Box_b u = \psi$;
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1. $\Box_b$ is solvable if given any smooth $(0, q)$-form ψ on M with compact support, there exists a $(0, q)$-current u on M so that $\Box_b u = \psi$;
2. $\Box_b$ is hypoelliptic if given any $(0, q)$-current ψ on M, ψ is smooth on any open set on which $\Box_b \psi$ is smooth.

Solvability and the Levi form – Peloso-Ricci (2003)

Theorem (Peloso-Ricci, 2003)

Let $n^+(\lambda)$, resp., $n^-(\lambda)$, be the number of positive, resp., negative eigenvalues of ϕ^λ. Then

1. $\Box_b$ *is solvable on* $(0,q)$*-forms if and only if there does not exist* $\lambda \in \mathbb{R}^m \setminus \{0\}$ *for which* $n^+(\lambda) = q$ *and* $n^-(\lambda) = n - q$.
2. $\Box_b$ *is hypoelliptic on* $(0,q)$*-forms if and only if there does not exist* $\lambda \in \mathbb{R}^m \setminus \{0\}$ *for which* $n^+(\lambda) \leq q$ *and* $n^-(\lambda) \leq n - q$.

Codimension Two Model Examples in $\mathbb{C}^4$, Example 1

Example 1 – a Cartesian product of Heisenberg groups

$$\phi(z,z) = \begin{pmatrix} |z_1|^2 \\ |z_2|^2 \end{pmatrix}.$$

The image of ϕ is the closed first quadrant and

$$\phi^\lambda(z,z') = \begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \bar{z}'_1 \\ \bar{z}'_2 \end{pmatrix}$$

Since $\{(n^+(\lambda), n^-(\lambda))\} = \{(2,0),(1,1),(2,0),(1,0),(0,1)\}$, both solvability and hypoellipticity FAIL at *all* form levels.

The Szegö projection and projection onto the null space of $\Box_b$ are explicitly computable.

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Example 2 – The nondegenerate example

$$\phi(z,z) = \begin{pmatrix} 2\,\mathrm{Re}(z_1\bar{z}_2) \\ |z_1|^2 - |z_2|^2 \end{pmatrix}.$$

The image of $\phi = \mathbb{R}^2$ and

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Since $\{(n^+(\lambda), n^-(\lambda))\} = \{(1,1)\}$. Both solvability and hypoellipticity HOLD when $q = 0, 2$ and FAIL when $q = 1$.

Theorem (Boggess-Raich, 2013)

The fundamental solution kernel to the Kohn Laplacian $\Box_b$ for $q = 0,\ 2$ is given by

$$G(z,t) = \frac{1}{2\pi^3} \frac{1}{(|z|^4 + |t|^2)^{3/2}}.$$

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The image of ϕ is open half-space and the origin (that is, $\mathrm{Range}\,\phi = \{(x,y) \in \mathbb{R}^2 : x > 0\} \cup \{(0,0)\}$) and

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When $q = 0$ and $|z_2|^2 \geq |z_1|^2 + |t|$, then

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Heat Equation Technique

Simplicity assumption: ϕ^λ has n nonzero eigenvalues almost everywhere.

Notation: μ_j^λ are the eigenvalues of ϕ^λ and $L \in \mathcal{I}_q$ is *fixed*, set

$$\varepsilon_j^\lambda = \begin{cases} \operatorname{sgn}(\mu_j^\lambda) & \text{if } j \in L \\ -\operatorname{sgn}(\mu_j^\lambda) & \text{if } j \notin L. \end{cases}$$

Coordinates: For each $\lambda \in \mathbb{R}^m$, let $\mu_1^\lambda, \ldots, \mu_n^\lambda$ be the eigenvalues of ϕ^λ and $v_1^\lambda, \ldots, v_n^\lambda$ be unit eigenvectors. This means

$$\phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk}\mu_j^\lambda$$

and for $z \in \mathbb{C}^n$, we can express

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Theorem (Boggess-Raich, 2011)

Let $L \in \mathcal{I}_q$ and fix $\lambda \in \mathbb{R}^m$. Then

1. *The heat kernel acting on $d\bar{z}^L$ is*

$$H_L(s,z,\hat{\lambda}) = \frac{1}{(2\pi)^{\frac{m}{2}+n}} \prod_{j=1}^{n} \frac{2e^{s\varepsilon_j^\lambda |\mu_j^\lambda|}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} e^{-|\mu_j^\lambda| \coth(s|\mu_j^\lambda|)|z_j^\lambda|^2}$$

2. *If $\mu_j^\lambda > 0$ for all $j \in L$ and $\mu_j^\lambda < 0$ for all $j \notin L$. Then the kernel of the projection onto the null space is*

$$S_L(z,\hat{\lambda}) = \lim_{s\to\infty} H_L(s,z,\hat{\lambda}) = \frac{4^n}{(2\pi)^{\frac{m}{2}+n}} \prod_{j=1}^{n} |\mu_j^\lambda| e^{-|\mu_j^\lambda||z_j^\lambda|^2}$$

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General Formulas on Quadrics

For each q-tuple $L \in \mathcal{I}_q$, set

$$\Gamma_L = \{\alpha \in S^{m-1} : \mu_j^\alpha > 0 \text{ for all } j \in L \text{ and } \mu_j^\alpha < 0 \text{ for all } j \notin L\}$$

and (recall)

$$\varepsilon_j^\alpha = \begin{cases} \operatorname{sgn}(\mu_j^\alpha) & \text{if } j \in L \\ -\operatorname{sgn}(\mu_j^\alpha) & \text{if } j \notin L. \end{cases}$$

Comments:

1. If $\alpha = \lambda/|\lambda|$, then $\mu_j^\lambda = |\lambda|\mu_j^\alpha$ but $\epsilon_j^\lambda = \epsilon_j^\alpha$.
2. If $z \in \mathbb{C}^n$ is expressed in terms of the unit eigenvectors of ϕ^λ, then $z_j^\lambda = z_j^\alpha$.

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Relative inverses

Definition

If $\ker \Box_b \neq \{0\}$ and $S : L^2_{0,q}(M) \to \ker \Box_b$ is the orthogonal projection, a relative inverse is an operator N so that

$$\Box_b G f = (I - S) f.$$

General Formulas on Quadrics

Theorem (Boggess-Raich, '13, Boggess-Raich)

Suppose M is a quadric CR submanifold of $\mathbb{C}^{n+m}$. Then
1. If $|\Gamma_L| = 0$, then the fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}^L$ is given by convolution with the kernel

$$\frac{2(2\pi)^{m+n}}{4^n} G_L(z,t)$$

$$= \int_{r=0}^{1} \int_{\alpha \in S^{m-1}} \prod_{j=1}^{n} \frac{r^{\frac{1}{2}(1-\varepsilon_j^\alpha)|\mu_j^\alpha|} |\mu_j^\alpha|}{1 - r^{|\mu_j^\alpha|}} \frac{(n+m-2)!}{(A(r) - i\alpha \cdot t)^{\nu(\alpha)+m-1}} \, d\alpha \, \frac{dr}{r}$$

where

$$A_\alpha(r) = \sum_{j=1}^{n} |\mu_j^\alpha| \left(\frac{1 + r^{|\mu_j^\alpha|}}{1 - r^{|\mu_j^\alpha|}} \right) |z_j^\alpha|^2.$$

2. If $|\Gamma_L| > 0$, then the projection of $\Box_b$ on forms spanned by $d\bar{z}^L$ onto $\ker \Box_b$ is given by

$$S_L(z,t) = \frac{4^n(n+m-1)!}{(2\pi)^{m+n}} \int_{\alpha\in\Gamma_L} \frac{\prod_{j=1}^n |\mu_j^\alpha|}{(\sum_{j=1}^n |\mu_j^\alpha||z_j^\alpha|^2 - i\alpha\cdot t)^{n+m}}\, d\alpha.$$

When $L = \emptyset$, $S_\emptyset(z,t)$ is the Szegö kernel.

3. If $|\Gamma_L| > 0$, then a relative fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}^L$ is given by

$$\frac{2(2\pi)^{m+n}}{4^n(n+m-2)!} G_L(z,t)$$

$$= \int_{r=0}^1 \int_{\alpha\notin\Gamma_L} \prod_{j=1}^n \frac{r^{\frac{1}{2}(1-\varepsilon_j^\alpha)|\mu_j^\alpha|}|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}} \frac{1}{(A(r)-i\alpha\cdot t)^{n+m-1}}\, d\alpha\, \frac{dr}{r}$$

$$+ \int_{r=0}^1 \int_{\alpha\in\Gamma_L} \Big[\Big(\prod_{j=1}^n \frac{|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}} \Big) \frac{1}{(A_\alpha(r)-i\alpha\cdot t)^{n+m-1}}$$

$$- \frac{\prod_{j=1}^n |\mu_j^\alpha|}{(A_\alpha(0)-i\alpha\cdot t)^{n+m-1}} \Big]\, d\alpha\, \frac{dr}{r}.$$

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3. If $|\Gamma_L| > 0$, then a relative fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}^L$ is given by

$$\begin{aligned}
&\frac{2(2\pi)^{m+n}}{4^n(n+m-2)!} G_L(z,t) \\
&= \int_{r=0}^1 \int_{\alpha\notin\Gamma_L} \prod_{j=1}^n \frac{r^{\frac{1}{2}(1-\varepsilon_j^\alpha)|\mu_j^\alpha|}|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}} \frac{1}{(A(r)-i\alpha\cdot t)^{n+m-1}}\, d\alpha\, \frac{dr}{r} \\
&+ \int_{r=0}^1 \int_{\alpha\in\Gamma_L} \Bigg[\Big(\prod_{j=1}^n \frac{|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}}\Big) \frac{1}{(A_\alpha(r)-i\alpha\cdot t)^{n+m-1}} \\
&\qquad - \frac{\prod_{j=1}^n |\mu_j^\alpha|}{(A_\alpha(0)-i\alpha\cdot t)^{n+m-1}}\Bigg]\, d\alpha\, \frac{dr}{r}.
\end{aligned}$$

The Heisenberg group, to conclude

Theorem (Boggess-Raich)

On the Heisenberg group $\mathcal{H}^n$,

1. *The relative fundamental solution to $\Box_b = \bar{\partial}_b^* \bar{\partial}_b$ on functions is given by the integration kernel*

$$N_\emptyset(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2+it)^n} \left[\log\left(\frac{|z|^2+it}{|z|^2-it}\right) - \sum_{j=1}^{n-1} \frac{1}{j} \right].$$

2. *The relative fundamental solution to $\Box_b = \bar{\partial}_b \bar{\partial}_b^*$ on $(0,n)$-forms is given by the integration kernel*

$$N_{\{1,\ldots,n\}}(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2-it)^n} \left[\log\left(\frac{|z|^2-it}{|z|^2+it}\right) - \sum_{j=1}^{n-1} \frac{1}{j} \right].$$

Thank you for your attention!

THANK YOU!!