Introduction to CR Geometry and the Tangential Cauchy-Riemann Operator

Andrew Raich

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Lecture 1: An Introduction to CR Geometry

In this lecture, I will discuss boundary values of holomorphic functions and how it naturally leads to the tangential Cauchy-Riemann operator. As part of the discussion, I will review tangential vector fields and bundles, holomorphic and anti-holomorphic vector fields and bundles. For a hands-on calculation, I will give an extrinsic and concrete calculation of both $T^{1,0}(M)$ and $\bar{\partial}_b$ for (p,q)-forms where M is either the sphere S^{d-1} or the Heisenberg group.

Lecture 2: The tangential Cauchy-Riemann operator and the Levi form

In this lecture, I have several goals. First, I will show what type of structure is needed on a manifold in order for $\bar{\partial}_b$ to be defined and give many examples. I will also present an abstract (that is, intrinsic) way to formulate $\bar{\partial}_b$ on a general CR manifold. Second, I will contrast the behavior of $\bar{\partial}_b$ with the more familiar operator $\bar{\partial}$. For example, $\bar{\partial}$ commutes with the pullbacks of holomorphic mappings while $\bar{\partial}_b$ does not. I will also remind people of the definitions and properties of pullbacks and pushforwards. Finally, I will show how the CR geometric structure forms the prototype for Involutive Structures to connect the geometry discussed in these lectures with a structure with which many of the Brazilian graduate students attending the lectures will be familiar.

Lecture 3: The $\bar{\partial}_b$-problem and the Heisenberg group

In this lecture, I will introduce the $\bar{\partial}_b$-problem and review the beautiful calculations of Folland and Stein ('74) on the Heisenberg group in which they completely solve the $\bar{\partial}_b$-problem.

Lecture 4: CR Geometry and $\bar{\partial}_b$ today

In the final lecture, I will outline several open problems in CR Geometry and the study of $\bar{\partial}_b$. Problems included in this list may include:

1. The L^2 and L^2-Sobolev theory of the $\bar{\partial}_b$-problem, including pseudo-convexity, compactness, finite type, and condition $Y(q)$.
2. Techniques to solve the $\bar{\partial}_b$-problem on spaces other than L^2.
3. Problems in the existence (or non-existence) of CR mappings between CR manifolds.