THE FBI TRANSFORMS AND THEIR USE IN MICROLOCAL ANALYSIS

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Abstract. Using a more general class of FBI transforms introduced by S. Berhanu and J. Hounie in [13] we completely characterize regularity and microregularity in Denjoy-Carleman (non quasi analytic) classes, which includes the Gevrey classes and M. Christ version of the FBI transform defined in [24] as examples. We also exhibit a result on microlocal analyticity for solutions of first order nonlinear partial differential equations in these classes, that do not seem possible to prove using the classical FBI transform.

Contents

1. Introduction

Motivated by a recent work of S. Berhanu and J. Hounie, [13], we show how to characterize Denjoy-Carleman (local and microlocal) regularity in terms of an appropriated exponential decay of a class of FBI transforms introduced by them. From the celebrated Paley-Wiener-Schwartz theorem it is possible to characterize smooth regularity of compact supported distributions by the (rapid) decay of its Fourier transform. This is one of the main reasons to study the set of smooth microlocal singularities (the collection of directions which the Fourier transform does not decay rapidly), which is known as (smooth) wave front set. The analytic (and Denjoy-Carleman) wave front set for distributions is defined analogously (see [35, Definition 8.4.3]). The study of these singularities is called microlocal analysis.

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Microlocal analysis was studied independently and from different points of view by several mathematicians around 1970. Sato, in [42, 43] introduced and studied the analytic singular spectrum for hyperfunctions. Hörmander studied wave front sets (smooth), $WF(u)$ for distributions u, using pseudodifferential operators in [31], and later by multiplying by cutoff functions and taking the Fourier transform in [32, 34]. Hörmander also studied the wave front set $WF_M(u)$ with respect to the Denjoy-Carleman class associated with a sequence $M = (M_j)_{j\in\mathbb{N}}$ (see Definition 2.1 and [35, Definition 8.4.3]) including the analytic and Gevrey wave front sets as special cases, [33, 34]. However, the latter is based on a family of estimates using a special sequence of cut-off functions (satisfying certain decay on the derivatives up to a given order) making it more difficult in applications. Subsequently, Brós and Iagolnitzer introduced a particularly well suited transform to study the (real) analyticity [20, 21], the so called `Fourier-Brós-Iagolnitzer` transform, FBI in short. It was also shown to be the right tool to study the microlocal analyticity. In 1976 Bony (see [16]) unified the different notions: i) essential spectrum of Brós and Iagolnitzer; ii) the analytic singular spectrum of Sato; and iii) the analytic wave front set of Hörmander.

In a celebrated paper of S. Baouendi, D. Chang and F. Trèves, [6] (see also [45]), the authors introduced the notion of FBI transform adapted to an overdetermined system of (smooth) locally integrable complex vector fields $L_1, \dots, L_n$, linearly independent in $\mathbb{R}^{m+n}$

$$Fu(y; z, \zeta) = \int_{\mathbb{R}^m} e^{i\zeta\cdot(z-Z(x',y))-\langle\zeta\rangle(z-Z(x',y))^2} g(x')u(x', y)\, dZ(x', y)$$

where $z, \zeta \in \mathbb{C}^m$, $Z(x, y) = (Z_1(x, y), \dots, Z_m(x, y))$ and for each $k \in \{1, \dots, m\}$ the functions Z_k are first integrals of L_j, i.e., $L_jZ_k = 0$, for all $j \in \{1, \dots, n\}$. Using this FBI transform they introduced the concept of hypo-analyticity which has been used extensively to study problems in Several Complex Variables, (microlocal) regularity of solutions of nonlinear and linear PDE's among others applications. See, for instance [1, 6, 8, 9, 10, 15, 27] and the references therein.

In [44] Sjöstrand considered a more general version of the FBI transform

$$Fu(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\Phi(x,x')} a(x, x', \xi)\chi(x')u(x')\, dx', \quad x, \xi \in \mathbb{R}^m;$$

where Φ is holomorphic with the assumption that the real part of the Hessian of Φ is required to be `negative definite`, a is a analytic symbol and $\chi \in C_0^\infty(\mathbb{R}^m)$. Recently, in [13], S. Berhanu and J. Hounie generalized the FBI transform (we will refer to it as FBI-BH, in short) including a class of transforms where the `real part of the Hessian of the phase function may degenerate at the point of interest` and characterized smooth and analytic wave-front set by the FBI-BH decay of a given distribution. Some examples of the transforms introduced in [13] include, but are not limited to the following: for each $k = 2, 3, \dots,$

$$\mathcal{F}_k u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^{2k}} u(x')\, dx, \quad x, \xi \in \mathbb{R}^m, \quad u \in \mathcal{E}'(\mathbb{R}^m). \tag{1.1}$$

Our goal is to use the FBI-BH transforms to characterize local and microlocal Denjoy-Carleman regularity for `ultradistributions`. Let $(M_j)_{j\in\mathbb{N}}$ be a sequence of real numbers, satisfying (2.3), (2.4), (2.5) and (2.6), and $M(t)$ its associated function, given in Definition 4.1. The local characterization proved here attest that a `ultradistribution` u is in a Denjoy-Carleman class (associated to the sequence $(M_j)_{j\in\mathbb{N}}$) on a neighborhood of a given point x_0 if and only if the FBI-BH transforms evaluated at (x, ξ) decay as $e^{-M(|\xi|)}$ for x in a neighborhood of x_0 and for all $\xi \in \mathbb{R}^m$ (see Theorem 4.1).

The local characterization together with the known suitable definitions of wave-front sets (e.g. [11, Definition V.2.5, Definition V.2.11 and Theorem V.3.7] for smooth and analytic) allow us to define Denjoy-Carleman wave front sets of a given `ultradistribution`. In Theorem 5.3 it is proven that the definition of Denjoy-Carleman wave front set presented here is equivalent to the definition presented by L. Hörmander in [35, Definition 8.4.3], for `distributions`. Our definition relies in writing the given ultradistribution as boundary values of almost analytic functions defined in certain wedges. That such extensions always exist was proved by Z. Adwan and G. Hoepfner in [3], inspired by the earlier original ideas of J. Bruna, [22].

As a first application, we will show that M. Christ's version of the Sjostrand, [44], FBI can be seen as examples of the FBI-BH transforms, see Theorem 5.2. Second, inspired by results first given in [8, theorem 1.1] and [13, Theorem 5.1] we prove a result on microlocal analyticity for solutions of first order nonlinear partial differential equations that can be seen as extension of quite recent results, see for instance [1, 9, 10], for these classes of functions and that has interest in its own right.

The organization of this paper is as follows: in Section 2 we recall the definition of Denjoy-Carleman classes. The definition of the FBI-BH transform for ultradistributions and the inversion formula is given in Section 3. The characterization of local Denjoy-Carleman regularity via decay of the FBI-BH transform for `ultradistributions` will be given in Section 4. The Denjoy-Carleman wave front set characterization via FBI-BH transform is given in Section 5. In Section 6 we present the application on microlocal analyticity for solutions of first order nonlinear partial differential equations. We conclude with two appendixes: the first contains a useful result concerning the associated function and in the second we present estimates on derivatives of the phase function and an approximation result fundamental to prove the inversion formula for the FBI-BH transforms.

In this paper, we use the convention that a constant C may increase from line to line, a finite number of times.

2. Definitions and Preliminary results

We start this section with the definition of Denjoy-Carleman classes of functions.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^m$ an open subset and $M = (M_j)_{j\in\mathbb{N}}$ be an increasing sequence of positive real numbers. The Denjoy-Carleman spaces (DC spaces), $\mathcal{E}^M(\Omega)$, is defined as the set of all functions f in $C^\infty(\Omega)$ that satisfies the following property: for each $K \subset\subset \Omega$ there exist a positive constant C, depending on K and f, such that*

$$\sup_{x\in K} |f^{(\alpha)}(x)| < C^{|\alpha|+1} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}^m. \tag{2.1}$$

The subclass composed of all functions $f \in \mathcal{E}^M(\Omega)$ supported in a compact set is denoted by $\mathcal{D}^M(\Omega)$.

E. Borel ([17] and [18]) gave examples of sets $E \subset C^\infty(\mathbb{R})$, containing nowhere analytic functions, satisfying the following property

$$f^{(j)}(0) = 0,\ j = 0, 1, \ldots \Rightarrow f \equiv 0, \qquad \forall f \in E. \tag{2.2}$$

The spaces satisfying the property (2.2) are called quasianalytic (QA), otherwise non-quasianalytic (NQA). Motivated by Borel's examples, in 1912 J. Hadamard asked if the notion of quasianalyticity could be characterized in terms of the derivative's growth. Independently A. Denjoy ([26]) and T. Carleman ([23]) answered Hadamard's question. They proved that $\mathcal{E}^M(\Omega)$ is QA if and only if

$$\sum_{j\geq 1} \frac{M_{j-1}}{M_j} = +\infty.$$

Example 2.1. *If $s > 1$ and $M_j = j!^s$, then $\mathcal{E}^M(\Omega) = G^s(\Omega)$ denote the $s-$Gevey space (see [41]), which is NQA. If $M_j = j!$ then $\mathcal{E}^M(\Omega)$ ($= C^\omega(\Omega)$, the space of real-analytic functions) is QA. Other examples of QA spaces are given when*

$$M_j = (j!(\log j)^j),\ (j!(\log j)^j(\log j)^j),\ \ldots.$$

We recall that (see [5]),

$$C^\omega = G^1 = \bigcap_{M \text{ is } NQA} \mathcal{E}^M \qquad \text{and} \qquad C^\omega \subsetneq \mathcal{E}^N \subsetneq \bigcap_{s>1} G^s.$$

for $N_j = j!(\log j)^j$.

We will consider the spaces $\mathcal{E}^M(\Omega)$ and $\mathcal{D}^M(\Omega)$ equipped with a special topology, as described in H. Komatsu in [36]. The topological dual of $\mathcal{E}^M(\Omega)$ and $\mathcal{D}^M(\Omega)$ will be denoted by $\mathcal{E}^{M\prime}(\Omega)$ and $\mathcal{D}^{M\prime}(\Omega)$, respectively.

Following [36] (see also [37, 38]), given a sequence $M = (M_j)_{j\in\mathbb{N}}$, we will assume the following properties:

- Initial condition,
$$M_0 = M_1 = 1. \tag{2.3}$$
- Strong logarithmic convexity: for some fixed $A > 0$ and for any r, with $0 \leq r < 1/A$, if we set $P_j = M_j/(j!)^r$, then the sequence
$$\left(\frac{P_j}{jP_{j-1}}\right) \quad \text{is increasig.} \tag{2.4}$$
- There exist A, $H > 0$ such that
$$M_{j+k} \leq AH^{j+k} M_j M_k, \qquad j, k \in \mathbb{N}. \tag{2.5}$$
- The strong non-quasi analyticity condition: there exists a constant $A > 0$ such that
$$\sum_{j=k+1}^{\infty} \frac{M_{j-1}}{M_j} \leq A \frac{k \cdot M_k}{M_{k+1}}. \tag{2.6}$$

Note that, if the sequence M satisfies (2.4), then

$$\binom{p}{q} M_{p-q} M_q \leq M_p, \quad 0 \leq q \leq p. \tag{2.7}$$

Moreover, if the sequence $M = (M_j)_{j\in\mathbb{N}_0}$ satisfies (2.3) and (2.7) then $C^\omega(\Omega) \subset \mathcal{E}^M(\Omega)$, i.e.,

$$p! \leq M_p, \quad \forall p = 0, 1, 2, \ldots. \tag{2.8}$$

Furthermore, using Theorem B.2, it follows that $\mathcal{E}^M$ is closed under compositions (see also [36, 38]). Condition (2.4) insures that the class $\mathcal{E}^M(\Omega)$ is inverse-closed; i.e., if $f \in \mathcal{E}^M(\Omega)$ and $f(x) \neq 0$, for each $x \in \Omega$, then $1/f \in \mathcal{E}^M(\Omega)$ (see [36]).

Remark 1: Note that the Gevrey sequences $M_p = p!^s$ satisfies (2.3), (2.4) and (2.5) for all $s \geq 1$ and (2.6) for all $s > 1$.

Through this paper we will always assume that a sequence $M = (M_j)_{j\in\mathbb{N}}$ satisfies conditions (2.3)-(2.6).

3. A class of generalized FBI transforms

3.1. **FBI-BH.** Consider $\psi \in \mathcal{E}^M(\mathbb{R}^m)$ such that $0 \neq \int |\psi(x)|dx < +\infty$. Following [13] we define a FBI transform, denoted by FBI-BH transform, with generating function ψ and parameter λ of a compactly supported ultradistribution $u \in \mathcal{E}^{M\prime}(U)$, by

$$\mathcal{F}_{\psi,\lambda}u(x,\xi) = \left\langle u(x'), e^{i\xi(x-x')}\psi(|\xi|^\lambda(x-x'))\right\rangle. \tag{3.1}$$

The right-hand side of the equation above is understood in the duality sense.

Example 3.1 (Classical FBI transform)**.** *Observe that, if $\psi(x) = e^{-|x|^2}$ then $\mathcal{F}_{\psi,\frac{1}{2}}$ is the Fourier-Bros-Iagolnitzer (FBI) transform.*

Next we present a sub-class of FBI-BH transforms which we will use to characterized the Denjoy-Carleman regularity (and micro regularity).

Definition 3.1 ($\mathcal{F}_p^\lambda$ transforms)**.** *Let $k \in \mathbb{N}$ and p be a real, homogeneous, positive elliptic polynomial of degree $2k$, i.e.,*

$$p(y) = \sum_{|\alpha|=2k} a_\alpha y^\alpha, \quad y \in \mathbb{R}^m, \tag{3.2}$$

$a_\alpha > 0$ ($\alpha \in \mathbb{N}^m$ and $|\alpha| \leq 2k$), satisfies

$$c|y|^{2k} \leq p(y) \leq C|y|^{2k}, \quad \forall\, y \in \mathbb{R}^m, \tag{3.3}$$

for some $0 < c \leq C$. For $c_p = \left[\int e^{-p(y)}dy\right]^{-1}$, $\psi(y) = c_p e^{-p(y)} \in \mathcal{E}^M(\mathbb{R}^m)$ so that $\int \psi = 1$, we can define the FBI-BH transform with generating function ψ and parameter $\frac{\lambda}{2k}$ ($\lambda > 0$),

$$\mathcal{F}_p^\lambda u(x,\xi) \doteq \mathcal{F}_{\psi,\frac{\lambda}{2k}}u(x,\xi) = c_p\left\langle u(x'), e^{i\xi\cdot(x-x')}e^{-|\xi|^\lambda p(x-x')}\right\rangle,$$

$x, \xi \in \mathbb{R}^m$. When $p(y) = |y|^{2k}$ and $\lambda = 1$ we have the FBI-BH transform, $\mathcal{F}_k$, given by (1.1)*.*

Before we proof the FBI-BH inversion formula for *ultradistributions* in $\mathcal{E}^{M\prime}(\mathbb{R}^m)$ we will need the following Fourier transform inversion formula for ultradistributions.

Lemma 3.1. *Let $\chi \in \mathcal{D}^M(\mathbb{R}^m)$, such that $\chi \geq 0$ and $\int \chi(x)dx = 1$, and $\sigma(\xi) \doteq \frac{\widehat{\chi}(\xi)}{(2\pi)^m}$. If $u \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ then,*

$$\int e^{i\xi x}\sigma(\epsilon\xi)\hat{u}(\xi)\, d\xi \longrightarrow u, \quad in \quad \mathcal{E}^{M\prime}(\mathbb{R}^m) \quad as \quad \epsilon \to 0^+. \tag{3.4}$$

Proof: Let $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$. since u is compactly supported there exists a convex, bounded and open subset $V \subset \mathbb{R}^m$ such that $u \in \mathcal{E}^{M'}(V)$ and it follows from Theorem A.1 that there exist $g \in C^0(\overline{V})$ and constants $a_\alpha \in \mathbb{C}$ (for each $\alpha \in \mathbb{N}^m$) satisfying (A.1) and (A.2). Hence, if $\phi \in \mathcal{D}^M(V)$ is such that $\phi \equiv 1$ in a neighborhood of $\operatorname{supp} u$, we have

$$\int e^{i\xi\cdot x}\sigma(\epsilon\xi)\hat{u}(\xi)d\xi = \int e^{i\xi x}\sigma(\epsilon\xi)\langle u(x'); e^{-i\xi\cdot x'}\rangle\, d\xi = \int e^{i\xi x}\sigma(\epsilon\xi)\left\langle u(x'); e^{-i\xi\cdot x'}\,\phi(x')\right\rangle d\xi$$
$$= \int e^{i\xi\cdot x}\sigma(\epsilon\xi)\sum_{\alpha}(-1)^{|\alpha|}a_\alpha\int_V \partial^\alpha_{x'}\left(e^{-i\xi\cdot x'}\,\phi(x')\right) g(x')\,dx'\,d\xi. \tag{3.5}$$

In order to apply Fubini's theorem (for each $\epsilon > 0$ fixed) we will show that for each $\epsilon > 0$ and $\alpha \in \mathbb{N}^m$ there exists a positive function $g_\epsilon(\xi, \alpha) \in L^1(\mathbb{R}^m, d\xi)$ such that

$$\left|e^{i\xi\cdot x}\sigma(\epsilon\xi)\,a_\alpha\;\partial^\alpha_{x'}\left(e^{-i\xi\cdot x'}\phi(x')\right)g(x')\right| \le g_\epsilon(\xi,\alpha) \tag{3.6}$$

and

$$\int_{\xi\in\mathbb{R}^m}\sum_\alpha\int_V g_\epsilon(\xi,\alpha)|a_\alpha\, g(x')|dx'\,d\xi < +\infty. \tag{3.7}$$

In fact, if we denote the Fourier transform of a function f by $F(f)$ then, $F(\partial^\alpha\chi)(\xi) = (i\xi)^\alpha F(\chi)(\xi)$. Hence, since $\sigma(\xi) = \frac{F(\chi)(\xi)}{(2\pi)^m}$,

$$\sigma(\epsilon\xi)\,\partial^\alpha_{x'}\{e^{-i\xi x'}\phi(x')\} = \frac{F(\chi)(\epsilon\xi)}{(2\pi)^m}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(-i\xi)^\gamma(\partial^{\alpha-\gamma}\phi)(x')$$
$$= \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\frac{(-1)^{|\gamma|}\;F(\partial^\gamma\chi)(\epsilon\xi)}{(2\pi)^m\,\epsilon^{|\gamma|}}(\partial^{\alpha-\gamma}\phi)(x'), \tag{3.8}$$

Writing $(1+|\xi|^2)^m = \sum_{|\beta|\le m} c_\beta\xi^{2\beta}$, it follows that one can estimate the above expression by

$$\left|\sigma(\epsilon\xi)\partial^\alpha_{x'}\{e^{-i\xi\cdot x'}\phi(x')\}\right| = \left|\sum_{|\beta|\le m}c_\beta\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\frac{F\left\{\partial^{\gamma+2\beta}\chi\right\}(\epsilon\xi)}{(2\pi)^m\,\epsilon^{|\gamma|+2|\beta|}(1+|\xi|^2)^m}(\partial^{\alpha-\gamma}\phi)(x')\right|. \tag{3.9}$$

In addition, since $\chi \in \mathcal{D}^M(\mathbb{R}^m)$, one can use (2.5) to obtain a positive constant $C = O(M_{2m})$ such that

$$|F\{\partial^{\gamma+2\beta}\chi\}(\xi)| \le \int_{\operatorname{supp}\chi}|\partial^{\gamma+2\beta}\chi(x)|dx \le C^{|\gamma|+1}\cdot M_{|\gamma|}, \quad \forall|\beta|\le m \text{ and } \gamma\in\mathbb{N}^m. \tag{3.10}$$

Choosing a bigger positive constant $C > 0$ so that $\sum_{|\beta|\le m}c_\beta \le C$, $\sup_\xi|\sigma(\xi)| \le C$ and $\sup_x|\partial^\alpha\phi(x)| \le C^{|\alpha|+1}M_{|\alpha|}$ (for each $\alpha\in\mathbb{N}^m$) we define $g_\epsilon(\xi,\alpha)$, $0<\epsilon<1$, by

$$g_\epsilon(\xi,\alpha) = \frac{C^{|\alpha|+2}M_{|\alpha|}}{(2\pi)^m\,\epsilon^{|\alpha|+2m}}(1+|\xi|^2)^{-m}, \quad \xi\in\mathbb{R}^m,\ \alpha\in\mathbb{N}^m. \tag{3.11}$$

With this definition it is a consequence of (3.9) and (3.10) that (3.6) holds. Moreover, for $0<\epsilon<1$ fixed and $L<\epsilon/2C$, we have

$$\int_{\xi\in\mathbb{R}^m}\sum_\alpha\int_V g_\epsilon(\xi,\alpha)|a_\alpha\, g(x')|dx'd\xi \le \int_{\xi\in\mathbb{R}^m}\sum_\alpha \frac{C^{|\alpha|+2}M_{|\alpha|}}{(2\pi)^m\cdot\epsilon^{|\alpha|+2m}}\cdot\frac{1}{(1+|\xi|^2)^m}\cdot\frac{C_L L^{|\alpha|}}{M_{|\alpha|}}d\xi$$

$$\le \frac{C_L\, C^2}{(2\pi)^m\epsilon^{2m}}\sum_\alpha\left(\frac{CL}{\epsilon}\right)^{|\alpha|}\int_{\xi\in\mathbb{R}^m}\frac{1}{(1+|\xi|^2)^m}d\xi<+\infty$$

showing (3.7), where in the first inequality in the right hand side we consider $C>0$ such that $\sup_{x'\in\overline{V}} g(x')\le C$. Thus, we can apply Fubini's theorem to rewrite (3.5) as

$$\begin{aligned}\int e^{i\xi\cdot x}\sigma(\epsilon\xi)\hat{u}(\xi)d\xi &= \sum_\alpha\int_V\int e^{i\xi\cdot x}\sigma(\epsilon\xi)(-1)^{|\alpha|}a_\alpha\,\partial^\alpha_{x'}\big(e^{-i\xi\cdot x'}\phi(x')\big)g(x')d\xi dd x'\\ &=\sum_\alpha\int_V(-1)^{|\alpha|}\partial^\alpha_{x'}\left(\int e^{i\xi\cdot x}\sigma(\epsilon\xi)e^{-i\xi\cdot x'}\phi(x')\,d\xi\right)a_\alpha\, g(x')dx'\\ &=\left\langle u(x');\phi(x')\int\frac{\hat{\chi}(\epsilon\xi)}{(2\pi)^m}e^{i\xi\cdot(x-x')}\,d\xi\right\rangle=\left\langle u(x');\int\frac{\hat{\chi}(\epsilon\xi)}{(2\pi)^m}e^{i\xi\cdot(x-x')}\,d\xi\right\rangle,\end{aligned}$$

recalling that $\phi\equiv 1$ in a neighborhood of $\operatorname{supp} u$. Since $\chi\in\mathcal{D}^M(\mathbb{R}^m)$, we can perform a change of variables and use the Fourier transform inversion formula (see [35]), to obtain

$$\begin{aligned}\int e^{i\xi\cdot x}\sigma(\epsilon\xi)\hat{u}(\xi)d\xi &= \left\langle u(x');\frac{1}{(2\pi)^m\epsilon^m}\int\hat{\chi}(\xi)e^{i(\xi/\epsilon)\cdot(x-x')}d\xi\right\rangle\\ &=\left\langle u(x');\epsilon^{-m}\chi\left(\frac{x-x'}{\epsilon}\right)\right\rangle.\end{aligned}$$

Now the proof follows from Theorem B.1. ■

The main result of this section gives an inversion formula for the FBI-BH transform applied to `ultradistributions`.

Theorem 3.1. *Let $u\in\mathcal{E}^{M'}(\mathbb{R}^m)$, $\psi\in\mathcal{E}^M(\mathbb{R}^m)$ and*

$$u_\epsilon(x)\doteq\int\int e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathcal{F}_{\psi,\lambda}u(t,\xi)|\xi|^{\lambda m}\,dt\,d\xi. \tag{3.12}$$

If $\int\psi(x)\,dx=1$ and there exists a positive constant C such that

$$\|\partial^\alpha\psi\|_{L^1}\le C^{|\alpha|+1}M_{|\alpha|},\quad \textit{for all}\quad \alpha,\in\mathbb{N}_0^m \tag{3.13}$$

then

$$\lim_{\epsilon\to0^+}u_\epsilon=u\quad in\quad \mathcal{E}^{M'}(\mathbb{R}^m). \tag{3.14}$$

PROOF: Let V be a bounded convex open subset of $\mathbb{R}^n$ such that $\operatorname{supp} u\subset V$ and $\phi\in\mathcal{D}^M(V)$ be such that $\phi\equiv1$ in a neighborhood of $\operatorname{supp} u$. From the definition of $\mathcal{F}_{\psi,\lambda}u(t,\xi)$ given in (3.1) we can write u_ϵ given in (3.12), as

$$u_\epsilon(x)=\int\int e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\langle u_{x'};\phi(x')e^{i\xi\cdot(t-x')}\psi(|\xi|^\lambda(t-x'))\rangle|\xi|^{\lambda m}\,dt\,d\xi, \tag{3.15}$$

Using Theorem A.1, there exist $g \in C^0(\overline{V})$ and complex numbers a_α, $\alpha \in \mathbb{N}_0^m$, satisfying (A.1), such that,

$$u_\epsilon(x) = \int\int \sum_\alpha \int_V F_\epsilon^\alpha(x, x', t, \xi)\, dx'\, dt\, |\xi|^{\lambda m}\, d\xi, \tag{3.16}$$

where V is an open set that contains $\operatorname{supp} u$ and

$$F_\epsilon^\alpha(x, x', \xi) \doteq (-1)^{|\alpha|}\, a_\alpha\, g(x')\, \sigma(\epsilon\xi)\, \partial_{x'}^\alpha \{\phi(x')\, e^{i\xi\cdot(x-x')}\psi(|\xi|^\lambda(t-x'))\}.$$

Hence we can apply Tonelli's theorem, Leibniz rule and hypothesis (3.13) in the last inequality to estimate

$$\begin{aligned}
&\int \sum_\alpha \int_V |F_\epsilon^\alpha(x, x', t, \xi)|\, dx'\, dt \leq \\
&\leq \int \sum_\alpha \int_V \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \left|\sigma(\epsilon\xi)\xi^{\alpha-\beta}|\xi|^{\lambda|\gamma|}(\partial^\gamma\psi)(|\xi|^\lambda(t-x'))\, \partial^{\beta-\gamma}\phi(x')\, a_\alpha\, g(x')\right|\, dx'\, dt \\
&\leq \sum_\alpha \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} |\sigma(\epsilon\xi)||\xi|^{|\alpha-\beta|}\, |\xi|^{\lambda|\gamma|-\lambda m} \int_V \int |(\partial^\gamma\psi)(t)\, \partial^{\beta-\gamma}\phi(x')\, a_\alpha\, g(x')|dt\, dx' \\
&\overset{(A.1)}{\leq} C_L \sum_\alpha |\xi|^{-\lambda m}(|\xi| + 1 + |\xi|^\lambda)^{|\alpha|}\, C^{|\alpha|+1} M_{|\alpha|}\, \frac{L^{|\alpha|}}{M_{|\alpha|}}.
\end{aligned}$$

For fixed $\xi \in \mathbb{R}^m$, choosing $L = L(|\xi|) > 0$ sufficiently small it follows that

$$\int \sum_\alpha \int_V |F_\epsilon^\alpha(x, x', t, \xi)| dx' dt < +\infty. \tag{3.17}$$

In addition, (3.13) implies that

$$|\partial_{x'}^\alpha \{\phi(x')\, e^{i\xi\cdot(x-x')}\psi(|\xi|^\lambda(t-x'))\}|$$

is dominated by an integrable function in t. Thus, we can apply Fubini's theorem in (3.16) and differentiate under the integral sign to rewrite $u_\epsilon(x)$ given in (3.15) as

$$\begin{aligned}
u_\epsilon(x) &= \int \sum_\alpha \int_V \int F_\epsilon^\alpha(x, x', t, \xi)\, dt\, dx'\, |\xi|^{\lambda m}\, d\xi \\
&= \int \left\langle u(x'); \phi(x')\sigma(\epsilon\xi)e^{i\xi\cdot(x-x')} \int \psi(|\xi|^\lambda(t-x'))|\xi|^{\lambda m}\, dt \right\rangle d\xi \\
&= \int \left\langle u(x'); \sigma(\epsilon\xi)e^{i\xi\cdot(x-x')} \right\rangle d\xi = \int e^{i\xi x}\sigma(\epsilon\xi)\hat{u}(\xi)\, d\xi,
\end{aligned}$$

since $\int \psi(|\xi|^\lambda(t - x'))|\xi|^{\lambda m} dt = 1$ for all x', $\xi \in \mathbb{R}^m$. The conclusion now follows from Lemma 3.1. ■

The next result implies the validity of the inversion formula for the class of $\mathcal{F}_p^\lambda$ transforms given by Definition 3.1.

Lemma 3.2. *Let $\psi(y) = c_p e^{-p(y)} \in \mathcal{E}^M(\mathbb{R}^m)$ be the function given in Definition 3.1. Then ψ satisfies the hypothesis* (3.13) *of Theorem 3.1, that is, there exists a constant $C > 0$ such that*

$$\|\partial^\alpha\psi\|_{L^1} \leq C^{|\alpha|+1} M_{|\alpha|}.$$

Proof: Since $\psi \in \mathcal{E}^M(\mathbb{R}^m)$, there exists $C > 0$ such that,

$$\|\partial^\alpha \psi\|_{L^1} = \int |\partial^\alpha \psi(x)| dx = \int_{|x|\leq 1} |\partial^\alpha \psi(x)| dx + \int_{|x|>1} |\partial^\alpha \psi(x)| dx$$
$$\leq C^{|\alpha|+1} M_{|\alpha|} + \int_{|x|>1} |\partial^\alpha \psi(x)| dx.$$

Therefore, it is enough to verify that there exists $C > 0$ (independent of x and α) such that

$$|x|^{m+1} \cdot |\partial^\alpha \psi(x)| \leq C^{|\alpha|+1} M_{|\alpha|},$$

for all $|x| > 1$ and $\alpha \in \mathbb{N}^m$. Since $\psi(x) = c_p e^{-p(x)}$, using Theorem B.2 it follows that

$$\partial^\alpha \psi(x) = c_p \sum_{r=1}^{|\alpha|} e^{-p(x)} \sum_{\mathfrak{p}(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{[\partial^{\alpha_j}\{-p(x)\}]^{k_j}}{k_j! \alpha_j!^{k_j}}.$$

For $|x| > 1$, we have

$$\begin{aligned}
|x|^{m+1} |\partial^\alpha \psi(x)| &\overset{(3.3)}{\leq} |x|^{m+1} \sum_{r=1}^{|\alpha|} e^{-c|x|^{2k}} \sum_{\mathfrak{p}(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{[C|x|^{2k}]^{k_j}}{k_j! \alpha_j!^{k_j}} \\
&\overset{(B.6)}{\leq} |x|^{2k(m+1)} \sum_{r=1}^{|\alpha|} e^{-c|x|^{2k}} C^{|\alpha|} |x|^{2k|\alpha|} \frac{4^{|\alpha|}}{|\alpha|!} |\alpha|! \sum_{\mathfrak{p}(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\overset{(B.7)}{\leq} C^{|\alpha|} \left(\frac{2}{c}\right)^{m+1+|\alpha|} (m+1+|\alpha|)! e^{-\frac{c}{2}|x|^{2k}} \\
&\leq C^{|\alpha|+1} |\alpha|!,
\end{aligned}$$

as we wished to prove. ■

Remark 2: The class of smooth function for which (3.13) is satisfied has been the object of recent research, see [4, 29, 30]. It is called `global` L^q-`Gevrey/Denjoy-Carleman spaces`.

4. Characterization of DC (local) regularity via FBI-BH

In this section we will characterize Denjoy-Carleman regularity via the $\mathcal{F}_p^\lambda$ transforms (see Definition 3.1). To do so, we will first analyze the decay of the $\mathcal{F}_p^\lambda$ transform of a ultradistribution vanishing near the point of interest. To this end and future reference we will need the following definition.

Definition 4.1. *For each sequence $M = (M_j)_{j\in\mathbb{N}_0}$ of positive numbers we define its associate function $M(t)$ on $[0, +\infty)$ by*

$$M(t) = \sup_j \log \frac{t^j}{M_j}, \quad t \in [0, \infty). \tag{4.1}$$

For more information on the associate function, see [14, 36, 40] and the references therein.

Proposition 4.1. *If $u \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ vanishes in a neighborhood of x_0, then there exist constants $C, a > 0$ and a neighborhood V of x_0 such that for all $\theta > 0$ (independent of C, p, a and λ) we have*

$$|\mathcal{F}_p^\lambda u(x,\xi)| \leq Ce^{H/\theta}\ e^{\frac{1}{2}M(\theta|\xi|)-a|\xi|^\lambda}, \tag{4.2}$$

for all $x \in V$ and $|\xi| > 1$, where H is given in (2.5) *and $M(t)$ is given by* (4.1)*.*

Proof: Let $\delta > 0$ be such that $u = 0$ in $B(x_0, 3\delta) = \{x \in \mathbb{R}^m : |x - x_0| < 3\delta\}$. Thus, $K \doteq supp\, u \subset \mathbb{R}^m \setminus B(x_0, 3\delta)$. Consider $\varphi \in \mathcal{D}^M(B(x_0, 3\delta))$ so that, $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ when $|x - x_0| < 2\delta$ and define $\psi \doteq 1 - \varphi$. Choose $r > 0$ such that $K \subset B(x_0, r)$ and $2\delta < r$. Let $\phi_u \in \mathcal{D}^M(B(x_0, r))$ be such that $\phi_u \equiv 1$ in a neighborhood of K. Since $u \cdot \varphi \equiv 0$, it follows from Theorem A.1 that for all $L > 0$ there exists $C = C_L > 0$ such that,

$$|u(\phi)| = |u(\psi\,\phi)| = |u(\psi\,\phi\,\phi_u)| \leq C\sum_\alpha \frac{L^\alpha}{M_{|\alpha|}} \sup_{2\delta<|x'-x_0|<r} \partial^\alpha(\psi\,\phi\,\phi_u)(x'),\ \forall \phi \in \mathcal{E}^M(\mathbb{R}^m).$$

For $\phi(x') \doteq \phi_{\xi,x}(x') = c_p e^{i\xi\cdot(x-x')}e^{-|\xi|^\lambda p(x-x')}$ it follows from Lemma B.1 that

$$\begin{aligned} |\mathcal{F}_p^\lambda u(x,\xi)| &= |u(\phi_{\xi,x})| \\ &\leq C\sum_\alpha \frac{L^{|\alpha|}}{M_{|\alpha|}} \sup_{2\delta<|x'-x_0|<r} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} |\partial_{x'}^{\alpha-\gamma}(\psi\cdot\phi_u)(x')| \left|\partial_{x'}^\gamma \left\{e^{i\xi\cdot(x-x')-|\xi|^\lambda p(x-x')}\right\}\right| \\ &\leq C\sum_\alpha \frac{L^{|\alpha|}}{M_{|\alpha|}} \sup_{2\delta<|x'-x_0|<r} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} C^{|\alpha|+1}M_{|\alpha-\gamma|}e^{-|\xi|^\lambda p(x-x')}e^{\frac{1}{2}M(\theta|\xi|)+H/\theta}M_{|\gamma|}, \end{aligned} \tag{4.3}$$

for all $\theta > 0$ and $|\xi| > 1$. Moreover, using (3.3), we have

$$-p(x-x') \leq -c|x-x'|^{2k} \leq -c(|x'-x_0| - |x-x_0|)^{2k} \leq -c\delta^{2k}, \tag{4.4}$$

for $|x'-x_0| > 2\delta$ and $|x-x_0| \leq \delta$. Thus, it follows from (2.7), (4.3) and (4.4), for L sufficiently small, $a = c\delta^{2k}$ and $V = B(x_0, \delta)$, that

$$|\mathcal{F}_p^\lambda u(x,\xi)| \leq Ce^{H/\theta}e^{-a|\xi|^\lambda+\frac{1}{2}M(\theta|\xi|)},\quad \forall x \in V,\ |\xi| > 1,$$

proving the lemma. ■

Corollary 4.1. *Let $u, v \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ with $u = v$ in a neighborhood of x_0. If u satisfies* (4.2) *in a neighborhood of x_0 then v also satisfies* (4.2) *in a neighborhood of x_0.*

At this point we would like to analize deeper the inequality (4.2) for a special choice of λ's. It happens that for some distinctive λ's, the FBI-BH transform of an ultradistribution vanishing in a neighborhood of a given point will possesses the right decay in a (possible smaller) neighborhood of this point.

Definition 4.2 (Admissible λ's)**.** *We will say that $\lambda \in (0,1]$ is admissible for the sequence $M = (M_j)_{j\in\mathbb{N}_0}$ if there exist $c', c'' > 0$ such that*

$$t^\lambda \geq M(c't),\quad t \geq c''; \tag{4.5}$$

where $M(t)$ is the function associated with the sequence $M = (M_j)_{j\in\mathbb{N}_0}$ given by (4.1)*. It follows from* (A.4) *that $\lambda = 1$ is always admissible for any sequence $M = (M_j)_{j\in\mathbb{N}_0}$.*

Corollary 4.2. *If $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ vanishes in a neighborhood of x_0 and λ is admissible for $M = (M_j)_{j\in\mathbb{N}_0}$ then there exist constants $C, c, a > 0$ and a neighborhood V of x_0 such that for all $\theta > 0$ (independent of C, p, a and λ) we have*

$$|\mathcal{F}_p^\lambda u(x,\xi)| \le Ce^{-aM(c|\xi|)},\ \forall \xi \in \mathbb{R}^m,\ x \in V, \tag{4.6}$$

where H is given in (2.5) *and $M(t)$ is given by* (4.1).

Proof: If λ is admissible for $M = (M_j)_{j\in\mathbb{N}_0}$ then there exist constants $c', c'' > 0$ such that (4.5) is valid, that is,

$$-t^\lambda \le -M(c't), \quad \forall\, t \ge c''. \tag{4.7}$$

Also, if $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ vanishes in a neighborhood of x_0, Proposition 4.1, implies that there exist constants $C_1, a_1 > 0$ and a neighborhood V_0 of x_0 such that for all $\theta > 0$ (4.2) holds true, i.e.,

$$|\mathcal{F}_p^\lambda u(x,\xi)| \le C_1 e^{H/\theta}\ e^{\frac{1}{2}M(\theta|\xi|)-a_1|\xi|^\lambda} \quad x \in V_0,\ \xi \in \mathbb{R}^m. \tag{4.8}$$

Fix $0 < \theta_0 < \sqrt[\lambda]{a_1}$ then when $|\xi| \ge \max\{c''/\theta_0, c''\}$ it is a consequence of (4.7) and (4.8) that

$$\begin{aligned} |\mathcal{F}_p^\lambda u(x,\xi)| &\le C_1 e^{H/\theta_0}\ e^{\frac{1}{2}(\theta_0|\xi|)^\lambda - a_1|\xi|^\lambda} \le C_1 e^{H/\theta_0}\ e^{\frac{a_1}{2}|\xi|^\lambda - a_1|\xi|^\lambda} \\ &= C_1 e^{H/\theta_0}\ e^{-\frac{a_1}{2}|\xi|^\lambda} \le C_1 e^{H/\theta_0}\ e^{-\frac{a_1}{2}M(c''|\xi|)}. \end{aligned} \tag{4.9}$$

Since the function $\mathcal{F}_p^\lambda u(x,\xi) e^{\frac{a_1}{2}M(c''|\xi|)}$ is bounded for $|\xi| \le \max\{c''/\theta_0, c''\}$ and $x \in V \subset V_0$ with $\overline{V} \subset V_0$, there exists a positive constant C_2 such that

$$|\mathcal{F}_p^\lambda u(x,\xi)| \le C_2 e^{-\frac{a_1}{2}M(c''|\xi|)}, \qquad x \in V,\ |\xi| \le \max\{c''/\theta_0, c''\}, \tag{4.10}$$

The proof now follows from (4.9) and (4.10) by taking $a = \frac{a_1}{2}$, $C = \max\{C_2, C_1 e^{H/\theta_0}\}$, $c = c''$ and $V \subset V_0$ satisfying $\overline{V} \subset V_0$. ■

Proposition 4.2 (Admissible λ's versus inclusions of Gevrey spaces)**.** *Fix a sequence $M = (M_j)_{j\in\mathbb{N}}$ and $\lambda \in (0,1]$. Then λ is admissible for $M = (M_j)_{j\in\mathbb{N}}$ if and only if $G^{1/\lambda}(\Omega) \subset \mathcal{E}^M(\Omega)$. Here $G^{1/\lambda}(\Omega)$ denotes the Gevrey space of order $1/\lambda$ over Ω.*

Proof: It follows from [36, Lemma 3.8] that the validity of the inclusion $G^{1/\lambda}(\Omega) \subset \mathcal{E}^M(\Omega)$ is equivalent to the existence of positive constants $C_1 > 0, C_2 > 1$ such that

$$M(t) \le C_1 t^\lambda + \log C_2, \quad \forall\, t > 0. \tag{4.11}$$

It is easy to see that the later is equivalent to the fact that λ is admissible for the sequence $M = (M_j)_{j\in\mathbb{N}}$. ■

Example 4.1. *If $M_j = j!^s$ then $\lambda \in (0,1]$ is admissible for $M = (M_j)_{j\in\mathbb{N}}$ if and only if*

$$\frac{1}{s} \le \lambda \le 1. \tag{4.12}$$

This agrees with the M. Christ's version of the FBI transform, [24]. *In particular, in the analytic case, $s = 1$, $\lambda \in (0,1]$ is admissible for $M = (M_j)_{j\in\mathbb{N}}$ if and only if $\lambda = 1$ and this explain why in* [13] *the authors only used $\mathcal{F}_p^1$ to characterize analyticity.*

Next we will prove the main theorem of this section, concerning the equivalence of DC regularity and an exponential decay (in terms of the associated function $M(t)$) of the FBI-BH transform $\mathcal{F}_p^\lambda$, when λ is admissible for M.

Theorem 4.1. *Let $x_0 \in \mathbb{R}^m$, $u \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ and $\lambda \leq 1$ be admissible for M. Then, $u \in \mathcal{D}^M(\Omega)$ in a neighborhood $\Omega \subset \mathbb{R}^m$ of x_0 if and only if there exist positive constants C, c_1, c_2 and an open neighborhood U of x_0 such that*

$$|\mathcal{F}_p^\lambda u(x,\xi)| \leq Ce^{-c_1 M(c_2|\xi|)}, \ \ \forall x \in U, \ \xi \in \mathbb{R}^m. \tag{4.13}$$

Proof: Let $u \in \mathcal{D}^M(\Omega)$. In this case we have,

$$\begin{aligned}\xi^\alpha \mathcal{F}_p^\lambda u(x,\xi) &= c_p \int_{B(x_0,r)} u(x')e^{-|\xi|^\lambda p(x-x')} i^{|\alpha|} \partial_{x'}^\alpha [e^{i\xi(x-x')}]dx' \\ &= c_p(-i)^{|\alpha|} \int_{B(x_0,r)} \partial_{x'}^\alpha \left[u(x')e^{-|\xi|^\lambda p(x-x')}\right] e^{i\xi(x-x')}dx',\end{aligned} \tag{4.14}$$

for all $\alpha \in \mathbb{Z}_+^m$. Since $\mathcal{F}_p^\lambda u$ is bounded in compacts sets it will be enough to prove (4.13) for $|\xi| \geq 1$. Thus, using Leibniz rule, Lemma B.1, (2.7) and (3.3), there exist constants $C, c > 0$ such that for all $\theta > 0$ (independent of C and c) we have (similarly as in the proof of Lemma 4.1)

$$\left|\partial_{x'}^\alpha \{u(x')e^{-|\xi|^\lambda p(x-x')}\}\right| \ \leq C^{|\alpha|+1} M_{|\alpha|} e^{H/\theta} e^{\frac{1}{2}M(\theta|\xi|)}, \tag{4.15}$$

for $|x'-x_0| < r$, $|x-x_0| < 1$ and $|\xi| \geq 1$. Therefore, it follows from (4.15) and (4.14) that

$$\left|\xi^\alpha \mathcal{F}_p^\lambda u(t,\xi)\right| \leq C^{|\alpha|+1}\, M_{|\alpha|}\, e^{H/\theta}\, e^{\frac{1}{2}\, M(\theta|\xi|)}; \ \ |x-x_0| < 1 \ \text{ and } \ |\xi| \geq 1. \tag{4.16}$$

Then, using (4.16), for every $j \in \mathbb{Z}_+$, one has

$$\begin{aligned}|\xi|^j \left|\mathcal{F}_p^\lambda u(x,\xi)\right| &\leq \sqrt{m} \max_{1\leq k\leq m} \{|\xi_k|^j\} |\mathcal{F}_p^\lambda \phi u(x,\xi)| \\ &\leq C^{j+1} e^{H/\theta} M_j e^{\frac{1}{2}M(\theta|\xi|)}, \ |t-x_0| < 1 \ \text{ and } \ |\xi| \geq 1.\end{aligned} \tag{4.17}$$

Since (4.17) is true for all $j \in \mathbb{Z}_+$, choosing $\theta \doteq C^{-1}$ and making use of associated function's definition, (4.1), we obtain

$$\begin{aligned}|\mathcal{F}_p^\lambda u(x,\xi)| &\leq Ce^{HC} \inf_j \Big(\frac{M_j}{(C^{-1}|\xi|)^j}\Big) e^{\frac{1}{2}M(C^{-1}|\xi|)} \\ &\leq Ce^{HC} e^{-M(C^{-1}|\xi|)} e^{\frac{1}{2}M(C^{-1}|\xi|)} = Ce^{HC} e^{-\frac{1}{2}M(C^{-1}|\xi|)},\end{aligned} \tag{4.18}$$

for $|x-x_0| < 1$ and $|\xi| \geq 1$, proving that u satisfies the desired estimate (4.13).

Conversely, suppose that $u \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ satisfies (4.13) for some positive constants C, c_1, c_2 and an open neighborhood U of x_0. We will follow the ideas presented in [11, Theorem V.2.4] to prove that u is in $\mathcal{E}^M$ in a neighborhood of x_0. To this end, we first note that Corollary 4.2 (see also (4.6)), implies that it is enough to prove the latter for $v \in \mathcal{E}^{M\prime}(B(x_0,r))$ such that $v \equiv u$ in $B(x_0, r/2)$ with r small to be defined. Applying the inversion formula, Theorem 3.1, we can write $v = \lim_{\epsilon\to 0^+} u_\epsilon$ in $\mathcal{E}^{M\prime}(\mathbb{R}^m)$ where u_ϵ is defined in (3.12). Let $I_1^\epsilon(x)$, $I_2^\epsilon(x)$, $I_3^\epsilon(x)$ and $I_4^\epsilon(x)$ be the integrals of

$$e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathcal{F}_p^\lambda v(t,\xi)|\xi|^{\frac{\lambda m}{2k}}$$

on the respective sets,

$$\begin{aligned}
U_1 &\doteq \{(t,\xi):\, |t-x_0|<A_1,\, \xi\in\mathbb{R}^m\},\\
U_2 &\doteq \{(t,\xi):\, |t-x_0|>A_2,\, \xi\in\mathbb{R}^m\},\\
U_3 &\doteq \{(t,\xi):\, A_1\le |t-x_0|\le A_2,\, |\xi|\ge 1\},\\
U_4 &\doteq \{(t,\xi):\, A_1\le |t-x_0|\le A_2,\, |\xi|< 1\},
\end{aligned}$$

with $A_1,\, A_2>0$ to be defined. We shall prove that there exists a neighborhood Ω of x_0 such that I_j^ϵ is convergent in $\mathcal{E}^M(\Omega)$ for each $j\in\{1,\dots,4\}$.

The term $I_1^\epsilon(x)$: Since u satisfies (4.13) there exist $C, c_1, c_2, A_1>0$ such that,

$$|\mathcal{F}_p^\lambda u(t,\xi)|\le Ce^{-c_1M(c_2|\xi|)}\le Ce^{-M(c'|\xi|)},\quad (t,\xi)\in B(x_0,A_1)\times\mathbb{R}^n, \tag{4.19}$$

for some $c'>0$ and where the last inequality is a consequence of (A.10). Now, for all $\alpha\in\mathbb{N}^m$, $x\in\mathbb{R}^m$, $(t,\xi)\in U_1$ with $|\xi|\ge 1$ and $\ell\in\mathbb{N}$ such that $\frac{\lambda m}{2k}\le \ell<\frac{\lambda m}{2k}+1$, we will employ property (A.8) of the associated function described in Definition 4.1 (with $k=r=|\alpha|+\ell$, $\theta=c'$ and $\ell=1$) together with (2.5) and (4.19), to write,

$$\begin{aligned}
\big|\partial_x^\alpha\big\{e^{i\xi(x-t)}\,\mathcal{F}_p^\lambda u(t,\xi)\,\big\}\big||\xi|^{\frac{\lambda m}{2k}} &\le Ce^{-M(c'|\xi|)}|\xi|^{|\alpha|+\frac{\lambda m}{2k}}\\
&\le C\sqrt{A}\frac{H^{|\alpha|+\ell}}{c'^{|\alpha|+\ell}}M_{|\alpha|+\ell}\,e^{-\frac{1}{2}M(c'|\xi|)}\\
&\le C\sqrt{A}\frac{H^{|\alpha|}}{c'^{|\alpha|}}\frac{H^{\ell}}{c'^{\ell}}AH^{|\alpha|+\ell}M_{|\alpha|}\,e^{-\frac{1}{2}M(c'|\xi|)}.
\end{aligned}\tag{4.20}$$

Analogously, when $(t,\xi)\in U_1$ and $|\xi|<1$, we obtain

$$\big|\partial_x^\alpha\big\{e^{i\xi(x-t)}\mathcal{F}_p^\lambda u(t,\xi)\big\}\big||\xi|^{\frac{\lambda m}{2k}}\le C\sqrt{A}\frac{H^{|\alpha|}}{c'^{|\alpha|}}M_{|\alpha|}\,e^{-\frac{1}{2}M(c'|\xi|)},\quad \forall x\in\mathbb{R}^m. \tag{4.21}$$

Thus, it follows from (4.20) and (4.21) that there exists a constant $C>0$, such that for all $(t,\xi)\in U_1$ and $x\in\mathbb{R}^m$ it holds

$$\big|\partial_x^\alpha\big\{e^{i\xi(x-t)}\mathcal{F}_p^\lambda u(t,\xi)\big\}\big|\,|\xi|^{\frac{\lambda m}{2k}}\le C^{|\alpha|+1}M_{|\alpha|}\,e^{-\frac{1}{2}M(c'|\xi|)}. \tag{4.22}$$

Since $e^{-\frac{1}{2}M(c'|\xi|)}\in L^1(\mathbb{R}^m)$ (see Lemma A.1, (A.9)), $\hat{\chi}\in L^\infty(\mathbb{R}^m)$ and $\hat{\chi}(\epsilon\xi)\to\hat{\chi}(0)=1$ as $\epsilon\to 0^+$, we can apply the dominated convergence theorem to obtain that

$$I_1(x)\doteq\lim_{\epsilon\to 0^+}I_1^\epsilon(x)=\frac{1}{(2\pi)^m}\int_{U_1}e^{i\xi\,(x-t)}\mathcal{F}_p^\lambda u(t,\xi)|\xi|^{\frac{\lambda m}{2k}}\,dt\,d\xi\quad\text{is in}\quad\mathcal{E}^M(\Omega). \tag{4.23}$$

The term $I_2^\epsilon(x)$: For $I_2^\epsilon(x)$ we consider $A_2>\max\{2r,A_1,2\}$ and $\phi\in\mathcal{D}^M(B(x_0,r))$ such that $\phi\equiv 1$ in an open neighborhood of $\operatorname{supp} u$. Applying Theorem A.1, Leibniz's rule and

Theorem B.2, we see that for every $L > 0$ there exists a constant $C_L > 0$ such that

$$
\begin{aligned}
|\mathcal{F}_p^\lambda u(t,\xi)| &= c_p \left|\langle u(x'); \phi(x') e^{i\xi(t-x') - |\xi|^\lambda p(t-x')}\rangle\right| \\
&\leq c_p C_L \sum_\alpha \frac{L^{|\alpha|}}{M_{|\alpha|}} \sup_{|x'-x_0|<r} |\partial_{x'}^\alpha(\phi(x')\, e^{i\xi\,(t-x')} e^{-c|\xi|^\lambda p(t-x')})| \\
&\leq c_p C_L \sum_\alpha \frac{L^{|\alpha|}}{M_{|\alpha|}} \sup_{|x'-x_0|<r} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\gamma\leq\alpha-\beta} \binom{\alpha-\beta}{\gamma} |\xi|^{|\alpha-\beta|-|\gamma|} C^\gamma M_{|\gamma|} \times \\
&\qquad \times \sum_{r=1}^{|\beta|} e^{-c|\xi|^\lambda |t-x'|^{2k}} \sum_{\mathfrak{p}(\beta,r)} \beta! \prod_{j=1}^{|\beta|} \frac{[|\xi|^\lambda \cdot |\partial_{x'}^{\alpha_j} p(t-x')|]^{k_j}}{k_j!(\alpha_j!)^{k_j}},
\end{aligned}
\tag{4.24}
$$

for $|t-x_0| > A_2$ and $\xi \in \mathbb{R}^m$. Since $r < \frac{A_2}{2}$ and $A_2 > 2$, if $|t-x_0| > A_2$ and $|x'-x_0| < r$ then

$$
|t-x'| \geq |t-x_0| - |x'-x_0| > A_2 - r > \frac{A_2}{2} \geq 1. \tag{4.25}
$$

Hence, each derivative $\partial_{x'}^{\alpha_j} p(t-x')$ (with p given by (3.1)) can be bounded by

$$
|\partial_{x'}^{\alpha_j} p(t-x')| \leq C|t-x'|^{2k}, \quad \text{for} \quad |t-x_0| > A_2 \text{ and } |x'-x_0| < r. \tag{4.26}
$$

Thus using (2.8), (4.26), (A.8) and (B.6), for each $\theta > 0$, we can further estimate $|\mathcal{F}_p^\lambda u(t,\xi)|$ given in (4.24) as

$$
|\mathcal{F}_p^\lambda u(t,\xi)| \leq c_p C_L \sup_{|x-x_0|<r} e^{-\frac{c}{2}|\xi|^\lambda |t-x'|^{2k}} e^{\frac{1}{2}M(\theta|\xi|)} \sum_\alpha (CL)^{|\alpha|}(1+H/\theta)^{|\alpha|} \tag{4.27}
$$

for $|t-x_0| > A_2$ and $\xi \in \mathbb{R}^m$. Now, let $c', c'' > 0$ be as in (4.5), $a = \left[\frac{c}{2}\left(\frac{A_2}{2}\right)^{2k}\right]^{1/\lambda}$, $\theta = ac'$ and L sufficiently small so that the series in α in (4.27) converges. Then it follows from (4.5), (4.25) and (4.27) that for $|\xi| > \frac{c''}{a}$ and $|t-x_0| > A_2$,

$$
|\mathcal{F}_p^\lambda u(t,\xi)| \;\; \leq \;\; Ce^{-\frac{1}{2}M(\theta|\xi|)}. \tag{4.28}
$$

Moreover, since $\mathcal{F}_p^\lambda u$ is bounded in compact sets, we see that there exists a positive constant C for which it holds

$$
|\mathcal{F}_p^\lambda u(t,\xi)| \leq Ce^{-\frac{1}{2}M(\theta|\xi|)} \quad \text{for all} \quad |t-x_0| > A_2,\ \xi \in \mathbb{R}^m. \tag{4.29}
$$

Hence, similarly as the term $I_1^\epsilon(x)$ (see (4.20) and (4.23)), it follows that

$$
\lim_{\epsilon\to 0^+} I_2^\epsilon(x) = I_2(x) \doteq \int_{U_2} e^{i\xi\,(x-t)}(2\pi)^m \mathcal{F}_p u(t,\xi)|\xi|^{\frac{\lambda m}{2k}}\, dt\, d\xi \quad \text{is in} \quad \mathcal{E}^M(\Omega). \tag{4.30}
$$

The term $I_3^\epsilon(x)$, changing the contour of integration: Recall that $I_3^\epsilon(x)$ is defined as

$$
I_3^\epsilon(x) = \int_{U_3} e^{i\xi\,(x-t)}(2\pi)^m \sigma(\epsilon\xi)\mathcal{F}_p u(t,\xi)|\xi|^{\frac{\lambda m}{2k}}\, dt\, d\xi, \tag{4.31}
$$

where $U_3 = \{(t,\xi):\ A_1 \leq |t-x_0| \leq A_2,\ |\xi| \geq 1\}$. Let ϕ as before ($\phi \in \mathcal{D}^M(B(x_0,r))$, with $\phi \equiv 1$ in a neighborhood of $\operatorname{supp} u$) and define

$$
F_\beta^\epsilon(x,x',t,\xi) \doteq \sigma(\epsilon\xi) \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial_{x'}^\gamma \{e^{i\xi(x-x')} e^{-|\xi|^\lambda p(t-x')}\}\, (\partial^{\beta-\gamma}\phi)(x')\, |\xi|^{\frac{\lambda m}{2k}}. \tag{4.32}
$$

An application of Theorem A.1 allow us to rewrite the expression $I_3^\epsilon(x)$ given in (4.31) as

$$I_3^\epsilon(x) = \int_{|\xi|\geq 1} \int_{A(x_0,A_1,A_2)} \sum_\beta \int_{B(x_0;r)} (-1)^{|\beta|}\, a_\beta\; F_\beta^\epsilon(x,x',t,\xi)\; g(x')\, dx'\, dt\, d\xi, \tag{4.33}$$

where $A(x_0, A_1, A_2) = \{t \in \mathbb{R}^m : A_1 \leq |t - x_0| \leq A_2\}$ is the annuli in $t \in \mathbb{R}^m$, centered at x_0, small radius A_1 and big radius A_2. Since we cannot estimate $p(t - x')$ as before, we will use Stokes' theorem to change the contour of integration. To do so, note that it follows from Lemma B.1, Lemma B.2 and (2.7) that, for fixed $\epsilon > 0$, there exist constants $C, b > 0$ such that the expression $F_\beta^\epsilon(x, x', t, \xi)$ given by (4.32) can be estimate as

$$|F_\beta^\epsilon(x,x',t,\xi)| \leq C e^{-M(b\cdot\epsilon|\xi|)} e^{-|\xi|^\lambda p(t-x')} e^{\frac{1}{2}M(b\cdot\epsilon|\xi|)} e^{(H/(b\cdot\epsilon))} C^{|\beta|+1} M_{|\beta|}. \tag{4.34}$$

Thus, using (3.3) and (4.34), it is a consequence of Theorem A.1 that for $L > 0$ sufficiently small, there exists a constant $C_L > 0$ that allow us to estimate $I_3^\epsilon(x)$ defined in (4.33) by

$$\begin{aligned} |I_3^\epsilon(x)| &\leq \int_{|\xi|\geq 1} \int_{A(x_0,A_1,A_2)} \sum_\beta \int_{B(x_0,r)} \left|F_\beta^\epsilon(x,x',t,\xi)\right| |a_\beta\, g(x')| dx'\, dt\, d\xi \\ &\leq C_L\, e^{(H/(b\cdot\epsilon))} \int_{|\xi|\geq 1} \int_{A(x_0,A_1,A_2)} \sum_\beta (CL)^{|\beta|} e^{-\frac{1}{2}M(b\epsilon|\xi|)} e^{-c|\xi|^\lambda |t-x'|^{2k}}\, dt\, d\xi \\ &\leq C\, e^{(H/(b\cdot\epsilon))} \int_{|\xi|>1} e^{-\frac{1}{2}M(b\epsilon|\xi|)}\, d\xi \overset{(A.9)}{<} +\infty. \end{aligned} \tag{4.35}$$

Therefore, for each $\epsilon > 0$ fixed, if

$$I_3^{\epsilon,S}(x) \doteq \int_{A(x_0,A_1,A_2)} \sum_\beta \int_{B(x_0,r)} \int_{1\leq|\xi|\leq S} F_\beta^\epsilon(x,x',t,\xi) d\xi du_\beta(x')\, dt \tag{4.36}$$

then

$$I_3^\epsilon(x) = \lim_{S\to+\infty} I_3^{\epsilon,S}(x). \tag{4.37}$$

We are now ready to change the contour of integration in the integral in ξ in (4.36). Define

$$\zeta(\xi, s) \doteq \xi + is|\xi|^\lambda(x - x'), \qquad 0 < s \leq s' < \frac{1}{2r}$$

with $s' > 0$ small to be chosen. Since

$$|\Im\zeta(\xi,s)| \leq s|\xi|^\lambda(|x - x_0| + |x_0 - x'|) \leq |\xi| = |\mathcal{R}\zeta(\xi,s)|, \tag{4.38}$$

for $|x' - x_0| < r$, $|x - x_0| < r$, $|\xi| \geq 1$, $\lambda \leq 1$ and $s \leq s'$, the principal branch of the absolute value of $\zeta(\xi, s)$ (denoted $\langle\zeta(\xi, s)\rangle$) it is well defined and holomorphic. If we define

$$D_{x'}^S = \{\zeta(\xi,s) = \xi + is|\xi|^\lambda(x - x') : \; 0 \leq s \leq s',\; 1 \leq |\xi| \leq S\},$$

it is a consequence of Stokes' theorem and Theorem B.2 that

$$I_3^{\epsilon,S}(x) = \int_{A(x_0,A_1,A_2)} \sum_{\beta} \int_{B(x_0,r)} \left\{ \int_{A(0,1,S)} \mathcal{I}_3^{\epsilon}(x,x',t,\zeta(\xi,s'),\beta)|\zeta_\xi(\xi,s')|d\xi \right.$$
$$+ \int_{\{|\xi|=S,\ 0\le s\le s'\}} \mathcal{I}_3^{\epsilon}(x,x',t,\zeta(\xi,s),\beta)|\zeta_{(\xi,s)}(\xi,s)|d(\xi,s)$$
$$+ \int_{\{|\xi|=1,\ 0\le s\le s'\}} \mathcal{I}_3^{\epsilon}(x,x',t,\zeta(\xi,s),\beta)\,|\zeta_{(\xi,s)}(\xi,s)|\,d(\xi,s)$$
$$\left. + \sum_{j=1}^{m} \int_{D_{x'}^S} \frac{\partial}{\partial \overline{\zeta_j}} \left[\mathcal{I}_3^{\epsilon}(x,x',t,\zeta,\beta)\right] d\zeta \right\} du_\beta(x')dt \tag{4.39}$$

where, $\zeta = \zeta(\xi,s)$ and

$$\mathcal{I}_3^{\epsilon}(x,x',t,\zeta,\beta) \doteq e^{i\zeta\cdot(x-x')}\sigma(\epsilon\zeta) \sum_{\gamma\le\beta} \binom{\beta}{\gamma} (-i\zeta)^{\beta-\gamma} \sum_{r=1}^{|\gamma|} e^{-\langle\zeta\rangle^{\lambda} p(t-x')} \times$$
$$\times \sum_{\mathfrak{p}(\gamma,r)} (\gamma)! \prod_{j=1}^{|\gamma|} \frac{-\langle\zeta\rangle^{k_j\lambda}\{\partial_{x'}^{\alpha_j} p(t-x')\}^{k_j}}{k_j!(\alpha_j)!^{k_j}} \langle\zeta\rangle^{\lambda m/(2k)}, \tag{4.40}$$

where $\mathfrak{p}(\gamma,r)$ is as in Theorem B.2. Here $|\zeta_{(\xi,s)}(\xi,s)|$ denotes the Jacobian determinant of $(\zeta(\xi,s),s)$ and for s' fixed the Jacobian determinant of $\zeta(\xi,s')$ is 1denoted by $|\zeta_\xi(\xi,s')|$. Note that $|\zeta_{(\xi,s)}(\xi,s)| \le CS^m$ and $|\zeta_\xi(\xi,s')|$ is bounded for x and x' in compact sets and $1 \le |\xi| \le S$. Our goal now is to prove that each triple integral in the right hand-side of (4.39) converges to a function in $\mathcal{E}^M$ (when $S \to +\infty$ and $\epsilon \to 0^+$) in a neighborhood of x_0.

Since the third integral in the right-hand side of (4.39) occurs on a bounded domain (independent of S) and the integrand is a holomorphic function, it converges to a holomorphic function in $B(x_0,r)$ when $\epsilon \to 0^+$.

Now we shall prove that the fourth triple integral in the right-hand side of (4.39) is zero. From Paley-Wiener theorem we know that σ is a holomorphic function. Thus, it will be sufficient to prove that $\langle\zeta\rangle^{\frac{\lambda}{2k}}$ is a holomorphic function in $D_{x'}^S$. In fact, for $s \le s' < 1$, $|x'-x_0| < r$ and $|x-x_0| < r$ (with $r < 1/2$ small so that $1-4s^2r^2 \ge 1-4r^2 > 0$), we have

$$\mathcal{R}\{[\zeta(\xi,s)]^2\} = \mathcal{R}\{[\xi + is|\xi|^{\lambda}(x-x')]^2\} \ge |\xi|^2 - s^2|\xi|^{2\lambda}(r+r)^2 \ge |\xi|^2(1-4s^2r^2). \tag{4.41}$$

Moreover, since

$$\Im\{[\zeta(\xi,s)]^2\} \le 2s|\xi|^{\lambda+1}|x-x'| \le 4s|\xi|^2 r \quad \text{and} \quad \mathcal{R}\{[\zeta(\xi,s)]^2\} \le |\xi|^2,$$

for $|x-x_0| < r$, $|x'-x_0| < r$ and $|\xi| \ge 1$, we have

$$\begin{aligned} \cos\{\arg([\zeta(\xi,s)]^2)\} &= \frac{\mathcal{R}\{[\zeta(\xi,s)]^2\}}{[(\mathcal{R}\{[\zeta(\xi,s)]^2\})^2 + (\Im\{[\zeta(\xi,s)]^2\})^2]^{1/2}} \\ &\ge \frac{|\xi|^2(1-s^2 4r^2)}{[|\xi|^4 + 16s^2r^2|\xi|^4]^{1/2}} = \frac{1-4s^2r^2}{\sqrt{1+16s^2r^2}}. \end{aligned}$$

Thus,

$$\begin{aligned}\cos\frac{\arg[\zeta(\xi,s)]^2}{2} &= \left(\frac{\cos(\arg[\zeta(\xi,s)]^2)+1}{2}\right)^{1/2}\\ &\geq \left(\frac{\frac{1-4s^2r^2}{\sqrt{1+16s^2r^2}}+1}{2}\right)^{1/2} = \left(\frac{1-4s^2r^2+\sqrt{1+16s^2r^2}}{2\sqrt{1+16s^2r^2}}\right)^{1/2}.\end{aligned}$$

Moreover,

$$\lim_{s\to0^+}\left(\frac{1-4s^2r^2+\sqrt{1+16s^2r^2}}{2\sqrt{1+16s^2r^2}}\right)^{1/2}=1 \qquad \text{and} \qquad \cos\frac{\arg[\zeta(\xi,s)]^2}{2}\leq 1,$$

imply

$$\lim_{s\to0^+}\cos\frac{\arg[\zeta(\xi,s)]^2}{2}=1.$$

Therefore,

$$\lim_{s\to0^+}\frac{\arg[\zeta(\xi,s)]^2}{2}=0, \tag{4.42}$$

i.e., for each $0<\rho\leq 1$ there exists s' such that if $s<s'$ then

$$-\frac{\rho\pi}{4}<\frac{\arg[\zeta(\xi,s)]^2}{2}<\frac{\rho\pi}{4}.$$

Thus, it follows that $\langle\zeta\rangle^{\frac{\lambda}{2k}}$ is holomorphic in $D_{x'}$ and, as a consequence, the fourth integral on the right-hand side of (4.39) is zero, independent of ϵ, $S>0$.

It remains to analyze the first and second triple integral on the right-hand side of (4.39). Note that (4.42) implies, diminishing s' if necessary, that

$$\cos\{(\lambda/2)\arg[\zeta(\xi,s)]^2\}\geq\frac{1}{4}, \qquad s<s' \tag{4.43}$$

and (4.41) gives $|\mathcal{R}[\zeta(\xi,s)]^2|\geq\frac{|\xi|^2}{2}$ if $s'r<\frac{1}{\sqrt{8}}$. Thus,

$$\begin{aligned}\mathcal{R}\{\langle\zeta(\xi,s)\rangle^{\lambda}\} &= \mathcal{R}\{e^{\frac{\lambda}{2}(\log|\zeta(\xi,s)^2|+i\arg\zeta^2)}\} = e^{\frac{\lambda}{2}\log|\zeta(\xi,s)|^2}\cos\{(\lambda/2)\arg[\zeta(\xi,s)]^2\}\\ &\geq \tfrac{1}{4}|[\zeta(\xi,s)]^2|^{(\lambda/2)}\geq\tfrac{1}{4\cdot2^{\lambda/2}}|\xi|^{\lambda}\geq\frac{|\xi|^{\lambda}}{8}.\end{aligned} \tag{4.44}$$

Since $\zeta(\xi,s)=\xi+is|\xi|^{\lambda}(x-x')$ we have

$$|\langle\zeta(\xi,s)\rangle^{\rho}|=|e^{\frac{\rho}{2}(\log|\zeta(\xi,s)^2|+i\arg[\zeta(\xi,s)]^2)}|\leq|\zeta(\xi,s)|^{\rho}\leq(2|\xi|)^{\rho}, \tag{4.45}$$

for all $\rho>0$, $0\leq\lambda\leq1$, $|\xi|\geq1$, $|x-x_0|<r$, $|x'-x_0|<r$ e $s\leq s'\leq\frac{1}{2r}$. Summing up, we obtain from (4.43), (4.44) and (4.45) that the term $\mathcal{I}_3^{\epsilon}(x,x',t,\zeta,\beta)$ given in (4.40) is

estimated by

$$
\begin{aligned}
|\mathcal{I}_3^\epsilon(x,x',t,\zeta,\beta)| \le e^{-s|\xi|^\lambda|x-x'|^2} C e^{-M(c'|\epsilon\zeta(\xi,s)|)} e^{Rs|\xi|^\lambda|x-x'|} \sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\zeta(\xi,s)|^{|\beta|-|\gamma|}\times \\
\times \sum_{r=1}^{|\gamma|} e^{-c(|\xi|^\lambda/8)|t-x'|^{2k}} \sum_{\mathfrak{p}}(\gamma)! \prod_{j=1}^{|\gamma|} \frac{(2|\xi|)^{k_j\lambda}C^{k_j}}{k_j!(\alpha_j)!^{k_j}}(2|\xi|)^{\lambda m/(2k)} \\
\le e^{-s|\xi|^\lambda|x-x'|^2} e^{-M(c'\epsilon|\xi|)} e^{Rs|\xi|^\lambda|x-x'|}(2|\xi|)^{\lambda m/(2k)} \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\times \\
\times (2|\xi|)^{|\beta|-|\gamma|} \sum_{r=1}^{|\gamma|} e^{-(c/8)|\xi|^\lambda|t-x'|^{2k}} \sum_{\mathfrak{p}}(\gamma)! \prod_{j=1}^{|\gamma|} \frac{(2|\xi|)^{|\gamma|\lambda}C^{|\gamma|}}{k_j!} 4^{|\gamma|}\frac{1}{|\gamma|!} \\
\le e^{-s|\xi|^\lambda|x-x'|^2} e^{-M(c'\epsilon|\xi|)} e^{Rs|\xi|^\lambda|x-x'|}(2|\xi|)^{\frac{\lambda m}{2k}+|\beta|} C^{|\beta|} e^{-(c/8)|\xi|^\lambda|t-x'|^{2k}}, \quad (4.46)
\end{aligned}
$$

for $|\xi| \ge 1$, $A_1 \le |t-x_0| \le A_2$, $|x'-x_0| \le r$, $|x-x_0| < r$, with $r > 0$ sufficient small and $s < s' < 1$, where in the first inequality we used that p is a polynomial, (3.3), (4.44) and Lemma B.2 (since $\chi \in \mathcal{D}^M(B(0,R))$). In the second inequality we used that $M(t)$ is increasing and $|\zeta(\xi,s)| \ge |\xi| \ge 1$; and in the third inequality we used $0 \le \lambda \le 1$ and (B.7). Next we consider two cases:

- If $|x'-x_0| < \frac{A_1}{2}$ then

$$
|t-x'| \ge |t-x_0| - |x'-x_0| \ge \frac{A_1}{2},
$$

when $A_1 \le |t-x_0| \le A_2$. Since $s<1$, choosing $R < \frac{c}{16}\left(\frac{A_1}{2}\right)^{2k}$, we have

$$
\begin{aligned}
e^{-s|\xi|^\lambda|x-x'|^2+Rs|\xi|^\lambda-\frac{c}{8}|\xi|^\lambda|t-x'|^{2k}} &\le e^{R|\xi|^\lambda-\frac{c}{8}\left(\frac{A_1}{2}\right)^{2k}|\xi|^\lambda} = e^{-\gamma_1|\xi|^\lambda} \\
&\le e^{-\gamma_1 s|\xi|^\lambda}, \qquad \gamma_1 \doteq \tfrac{c}{16}\left(\tfrac{A_1}{2}\right)^{2k}. \quad (4.47)
\end{aligned}
$$

- If $A_1/2 < |x_0-x'| \le r$ then

$$
|x-x'| \ge |x'-x_0| - |x-x_0| > \tfrac{A_1}{2} - \tfrac{A_1}{4} = \tfrac{A_1}{4},
$$

when $|x-x_0| < \frac{A_1}{4}$. Thus,

$$
e^{-s|\xi|^\lambda|x-x'|^2+Rs|\xi|^\lambda-\frac{c}{8}|\xi|^\lambda|t-x'|^{2k}} \le e^{-s|\xi|^\lambda(\frac{A_1}{4})^2+Rs|\xi|^\lambda} \le e^{-\gamma_2 s|\xi|^\lambda}, \quad (4.48)
$$

for R sufficiently small satisfying $\gamma_2 = -R + (A_1/4)^2 > 0$.

Now, using (4.46), (4.47) and (4.48), it follows that there exist an open neighborhood U_{x_0} of x_0 and constants C, $\gamma > 0$ such that (diminishing R, if necessary) one can further bound the term $\mathcal{I}_3^\epsilon(x,x',t,\zeta(\xi,s),\beta)$ estimated in (4.46) by

$$
|\mathcal{I}_3^\epsilon(x,x',t,\zeta(\xi,s),\beta)| \le C^{|\beta|+1}|\xi|^{|\beta|+m} e^{-M(c'\epsilon|\xi|)} e^{-\gamma s|\xi|^\lambda} \quad (4.49)
$$

for $x \in U_{x_0}$ and $(t, \xi) \in U_3$. Now the second triple integral of the right hand-side of (4.39) is estimated as follows,

$$\int_{A(x_0,A_1,A_2)} \sum_\beta \int_{B(x_0,R)} \int_{R_{S,s'}} |\mathcal{I}_3^\epsilon(x, x', t, \zeta(\xi, s), \beta)| \cdot |\zeta_{(\xi,s)}(\xi, s)| \, d\xi \, ds \, d|u_\beta|(x') \, dt$$

$$\leq \sum_\beta \frac{C_L L^{|\beta|}}{M_{|\beta|}} \int_0^{s'} C^{|\beta|+1} e^{-M(c'\epsilon S)} e^{-\gamma s S^\lambda} \cdot S^{|\beta|+3m} \, ds$$

$$\leq \sum_\beta C_L (CL)^{|\beta|} s' M_{3m} \sqrt{A} \frac{H^{|\beta|+3m}}{(2c'\epsilon)^{|\beta|}} e^{-\frac{1}{2}M(c'\epsilon S)}$$

$$\leq C_\epsilon e^{-\frac{1}{2}M(c'\epsilon S)} \to 0, \;\; S \to +\infty, \tag{4.50}$$

where $R_{S,s'} \doteq \{|\xi| = S, \, 0 \leq s \leq s'\}$ and $\epsilon > 0$ is fixed. Here, the first inequality in (4.50) follows from (4.49) and (A.1). The second inequality in (4.50) follows from (2.5) and (A.8), while the third inequality is a consequence of taking $L = L(\epsilon) > 0$ sufficient small (for each fixed $\epsilon > 0$).

Therefore, we conclude that the second integral on the right hand-side of (4.39) converges to zero, when $S \to +\infty$ for $x \in U_{x_0}$.

It remains to analize the first triple integral on the right hand-side of (4.39). Since $J\zeta(\xi, s')$ is bounded, one can use (A.8), (4.5), (4.49) which allow us to estimate

$$\begin{aligned} |\mathcal{I}_3^\epsilon(x, x', t, \zeta(\xi, s), \beta) \cdot \zeta_\xi(\xi, s')| &\overset{(4.49)}{\leq} C^{|\beta|+1} |\xi|^{|\beta|+m} e^{-\gamma s' |\xi|^\lambda} \\ &\overset{(4.5)}{\leq} C^{|\beta|+1} |\xi|^{|\beta|+m} e^{-M(c_1'|\xi|)} \\ &\overset{(A.8)}{\leq} C^{|\beta|+1} M_{|\beta|+m} e^{-\frac{1}{2}M(c_1'|\xi|)} \end{aligned} \tag{4.51}$$

for $c_1' \doteq c'(\gamma s')^{1/\lambda}$ and $\gamma^{1/\lambda} s'^{1/\gamma} |\xi| \geq c''$. Also, since $|\xi|^{|\beta|+m} e^{\frac{1}{2}M(c''|\xi|)}$ is bounded on compact sets we see that (4.51) holds for all $\xi \in \mathbb{R}^m$ with a possible larger constant C. Hence the first integral in the right-hand side of (4.39) is bounded by

$$\int_{A(x_0,A_1,A_2)} \sum_\beta \int_{B(x_0,r)} \int_{|\xi|\geq 1} |\mathcal{I}_3^\epsilon(x, x', t, \zeta(\xi, s'), \beta)| \cdot |\zeta_\xi(\xi, s')| \, d\xi \, du_\beta(x') \, dt$$

$$\overset{(2.5),(4.51)}{\leq} \int_{A(x_0,A_1,A_2)} \sum_\beta \int_B \int_{|\xi|\geq 1} C^{|\beta|+1} M_{|\beta|} e^{-\frac{1}{2}M(c_1'|\xi|)} \, d\xi \, du_\beta(x') \, dt$$

$$\leq C_L \sum_\beta \frac{(CL)^\beta}{M_{|\beta|}} M_{|\beta|} \int_{|\xi|\geq 1} e^{-\frac{1}{2}M(c_1'|\xi|)} d\xi < +\infty, \tag{4.52}$$

where the last inequality follows from (A.1) for fixed $\epsilon > 0$, $L = L(\epsilon) > 0$ sufficiently small. Thus, we can use the dominated convergence theorem to show that the first integral on the right hand-side of (4.39) converges to a function uniformly in compact subsets of U_{x_0}. The derivatives $\partial^\alpha, \alpha \in \mathbb{N}^m$ of the first triple integral on the right hand-side of (4.39) will produce and extra term $|\xi|^{|\alpha|}$ in (4.49) and this can be bounded by $C^{|\alpha|} M_{|\alpha|}$ exactly as we did in (4.51).

Thus, for $x \in U_{x_0}$, we can differentiate, ∂^α, under the integral sign the expression defined by the first integral on the right hand-side of (4.39) and keeping in mind the estimates that we are going to obtain in (4.52) will be dominated by $C^{|\alpha|}M_{|\alpha|}$, we conclude that this term converges to a function in $\mathcal{E}^M(U_{x_0})$, when $\epsilon \to 0^+$.

Hence, writing $I_3^\epsilon(x) = \lim_{S\to+\infty} I_3^{\epsilon,S}(x)$ as in (4.37), where $I_3^{\epsilon,S}(x)$ is given by (4.39), we have that the term $I_3^\epsilon(x)$ defined by (4.31) converges to $I_3^0(x)$ in $\mathcal{E}^M(U_{x_0})$, when $\epsilon \to 0^+$.

The term $I_4^\epsilon(x)$: Recall that

$$I_4^\epsilon(x) = \int_{U_4} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathcal{F}_p^\lambda u(t,\xi)|\xi|^{\frac{\lambda m}{2k}}\,dt\,d\xi \tag{4.53}$$

where $U_4 = \{(t,\xi) : A_1 \leq |t-x_0| \leq A_2,\ |\xi| < 1\}$. Since U_4 is bounded and the integrand is real analytic in $x \in U_{x_0}$, it follows that the term $I_4^\epsilon(x)$ converges to the real analytic function $I_4^0(x)$ in U_{x_0} when $\epsilon \to 0^+$.

Therefore, there exists an open neighborhood U_{x_0} of x_0 such that,

$$u = \lim_{\epsilon\to 0^+} u_\epsilon(x) = \lim_{\epsilon\to 0^+}\sum_{j=1}^4 I_j^\epsilon = \sum_{j=1}^4 I_j \in \mathcal{E}^M(U_{x_0}) \text{ in } \mathcal{E}^M(U_{x_0}),$$

concluding the proof of the theorem. ■

5. Denjoy-Carleman wave front set

This section is dedicated to present a definition and a characterization (via certain decay in cones of the FBI-BH transform) of DC wave front sets for ultradistributions.

It is well known that tempered growth of holomorphic functions implies the existence of boundary values distributions [35, Theorems 3.1.11 and 3.1.14]. In addition, allowing a special exponential growth of holomorphic functions H. Komatsu (see [36]) gave necessary and sufficient conditions for an ultradistribution to be the boundary value of a "sum" of such holomorphic functions defined in wedges. Later, H.-J. Petzsche and D. Vogt [40], characterized ultradifferentiable functions by their almost analytic extensions. They used these extensions in order to show how growth properties of a holomorphic function $f(x+iy)$ determine the classes of ultradistributions which contains the boundary value of f.

Recently, G. Hoepfner and Z. Adwan [3], exhibited sufficient conditions for the existence of boundary values in the sense of ultradistributions for a class of functions much larger than holomorphic ones. This result will be crucial in this section, where we will introduce the definition of Denjoy-Carleman wave front set, $WF_M u$ of ultradistributions u and characterize it via certain decay of the FBI-BH transforms $\mathcal{F}_p^\lambda u$.

Let $(M_j)_{j\in\mathbb{N}_0}$ be a positive increasing sequence and let $M(t)$ be its associated function as in Definition 4.1. The Young conjugate of the associated function $w^* : [0,+\infty) \to [0,+\infty]$ is defined by $w^*(r) = \sup_{t\geq 0}\{M(t) - rt\}$. Another important function is

$$M^*(s) = -\log \inf_{p\in\mathbb{N}} \left\{\frac{s^p M_p}{p!}\right\},$$

which is comparable with w^* (see [40]). Let us recall that by this we mean that, for every $H > 1$ there exists $C > 0$ so that

$$M^*(Hs) - C \leq w^*(s) \leq M^*(s),\ \ \forall s > 0. \tag{5.1}$$

Note that, from the definitions of M^* and M one has

$$M(t) \leq \inf_s \{M^*(s) + st\}, \quad \forall t \geq 0. \tag{5.2}$$

We will now recall the definition of locally integrable structure and define a finer notion of those structures. For simplicity we will suppose that the structure is already defined on a special coordinate system. More precisely, a complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T(U \times V)$ (where U and V are respectively open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$) of rank n is called $M-$ locally integrable structure if for each $p \in U \times V$, there are an open neighborhood $\Omega_p \subset U \times V$ of p, functions $a_{k,j}, Z_k \in \mathcal{E}^M(\Omega_p)$, $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ such that $\mathcal{V}_q = \text{span}\,\{(L_1)_q, \dots (L_n)_q\}$ for each $q \in \Omega_p$, where

$$L_j = \frac{\partial}{\partial t} + \sum_{k=1}^{m} a_{k,j}(x,t) \frac{\partial}{\partial x_k}, \quad (x,t) \in \Omega_p \; j \in \{1, \dots n\}$$

and

$$\text{span}\,\{(dZ_1)_q, \dots, (dZ_m)_q\} = \mathcal{V}_q^{\perp}, \quad \forall q \in \Omega_p.$$

The following result exhibit sufficient conditions for existence of boundary values in the sense of ultradistributions of certain solutions of $M-$ locally integrable structures.

Theorem 5.1 (Theorem 3.1 in [3]). *Let V be an open neighborhood of $0 \in \mathbb{R}^m$, $\Gamma \subset \mathbb{R}^n$ be an open acute cone with vertex at the origin, $\delta > 0$, $\Gamma_\delta = \Gamma \cap \{v : |v| < \delta\}$, $f \in C^0(V \times \Gamma)$ and $\mathcal{V} = span\{L_1, \dots, L_n\}$ be a $M-$locally integrable structure of class $\mathcal{E}^M(\Omega)$. If*

1. *$L_j f \in L^\infty(V \times \Gamma)$, $1 \leq j \leq n$;*
2. *f increases M^*-exponentially, that is, for all $\varrho > 0$ we have*

$$|f(x,t)| e^{-\varrho M^*(|t|/\varrho)} < \infty. \tag{5.3}$$

then, there exists the boundary value of f in $\mathcal{D}^{M'}(V)$, i.e., there exists $bf \in \mathcal{D}^{M'}(V)$ such that,

$$\langle bf, \phi \rangle = \lim_{\Gamma \ni t \to 0} \int f(x,t)\phi(x)dx, \qquad \forall \phi \in \mathcal{D}^M(V).$$

For future reference, we will be using Theorem 5.1, under the hypothesis in which $m = n$ and $L_j = \partial_{\bar{z}_j}$.

Definition 5.1 (See [3]). *Let $\Omega \subset \mathbb{R}^m$ be an open set and $f \in \mathcal{E}^M(\Omega)$. A function $\tilde{f} \in \mathcal{E}^M(\Omega \times (-1,1)^m)$ is called an $M-$almost analytic extension of f if*

1. *$\tilde{f}(x,0) = f(x)$, for all $x \in \Omega$; and*
2. *For each compact set $K \subset \Omega$ there exists a constant $C(K) = C > 0$, independent of N and z, such that, for all $j \in \{1, \dots, m\}$ it holds*

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}_j}(z) \right| \leq C^{N+1} \frac{M_N}{N!} |y|^N, \;\; \forall N \in \mathbb{N} \text{ and } z = x + iy \in K \times (-1,1)^m. \tag{5.4}$$

Motivated by the original ideas of J. Bruna, [22], on the Whitney type theorem for Denjoy-Carleman non quasi-analytic classes, Z. Adwan and G. Hoepfner [3], proved that there always exist almost analytic extensions.

Following the characterization of analytic (and smooth) microregularity given in [11, Definition V.2.5, Definition V.2.11 and Theorem V.3.7], we define.

Definition 5.2. *Let $u \in \mathcal{D}^{M'}(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$. We say that u is $M-$micro-regular at (x_0, ξ^0) if there exist an open neighborhood V of x_0 and open acute cones $\Gamma_1, \dots, \Gamma_k \subset \mathbb{R}^m \backslash \{0\}$ such that,*

1. $\xi^0 \cdot \Gamma_j < 0$,
2. *there exist $\delta > 0$ and $M-$almost analytic functions $f_j \in \mathcal{E}^M(V + (\Gamma_j)_\delta)$ (i.e., f_j satisfies (5.4)) which increase M^*- exponentially (from Theorem 5.1 it follows that bf_j exists in $\mathcal{D}^{M'}(V)$ for each $j = 1, 2, \dots, K$) such that $u = \sum_{j=1}^k bf_j$ in a neighborhood of x_0.*

The $M-$ wave front set of u, is defined by

$$WF_M(u) \doteq \{(x, \xi) : \text{ u is not M-micro-regular at } (x, \xi)\}.$$

The following theorem shows that our definition of the M-wave front set of ultradistributions $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$, is equivalent to some exponential decay of the FBI-BH transform $\mathcal{F}_p^\lambda$.

Theorem 5.2. *Let $\xi^0 \in \mathbb{R}^m \setminus \{0\}$, $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ and $0 < \lambda \leq 1$ admissible for M. Then $(0, \xi^0) \notin WF_M(u)$ if and only if there exist a neighborhood $U \subset \mathbb{R}^m$ of 0, an open and acute cone $\xi^0 \in \Gamma \subset \mathbb{R}^m \setminus 0$ and constants $c, C > 0$ such that*

$$|\mathcal{F}_p^\lambda(u)(x, \xi)| \leq Ce^{-M(c|\xi|)}, \quad (t, \xi) \in U \times \Gamma. \tag{5.5}$$

PROOF: Let $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ and suppose $(0, \xi^0) \notin WF_M(u)$. Using the linearity of the FBI-BH transform, it is sufficient to consider the following situation:

> There exist an open acute cone $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ such that $\xi^0 \cdot \Gamma < 0$, an open neighborhood $V \subset \mathbb{R}^m$ of 0 and an $M-$almost analytic function f in $\mathcal{E}^M(V + i\Gamma_\delta)$ with $|f(x + it)|e^{-\varrho M^*(|t|/\varrho)} < +\infty$, for all $\varrho > 0$, and $u = bf$ in $\mathcal{D}^{M'}(V)$.

Fix an arbitrary $v_0 \in \Gamma$. Since $\xi^0 \cdot \Gamma < 0$ there exists $c' > 0$ such that,

$$\frac{\xi^0}{|\xi^0|} \cdot \frac{v_0}{|v_0|} \leq -c'.$$

Thus, it is possible to choose an open neighborhood $S_0 \subset \mathbb{S}^{m-1}$ of $\frac{\xi^0}{|\xi^0|}$ such that

$$\xi \cdot \frac{v_0}{|v_0|} < -\frac{c'}{2}, \quad \forall \xi \in S_0. \tag{5.6}$$

Next we shall denote $\Gamma \doteq \{\xi \in \mathbb{R}^m : \frac{\xi}{|\xi|} \in S_0\}$.

For $p_1 \doteq \sum_{|\alpha|=2k} a_\alpha$ (see (3.2)) and $r \doteq \frac{c'}{8 \cdot 2^{2k} p_1}$ consider $g \in \mathcal{D}^M(B_\mathbb{C}(0, r))$[1] such that $0 \leq g \leq 1$ and $g \equiv 1$ in $B_\mathbb{C}(0, \frac{3r}{4})$. Analogously to Corollary 4.1 it is sufficient to prove the decay for $\mathcal{F}_p^\lambda(g_{|\mathbb{R}^m} \cdot u)$. Since $u = bf$ in $\mathcal{D}^{M'}(V)$, diminishing c' if necessary such that $B(0, r) \subset V$, we have

$$\mathcal{F}_p^\lambda(gu)(x, \xi) = c_p \left\langle u_y, g(y)e^{Q(x,y,\xi)} \right\rangle = c_p \lim_{\mathbb{R} \ni \tau \to 0^+} \int f\big(y + i\tau \tfrac{v_0}{|v_0|}\big)g(y)e^{Q(x,y,\xi)}\, dy,$$

[1] $B_\mathbb{C}(0, r) = \{z = (x_1 + iy_1, \dots, x_m + iy_m) \in \mathbb{C}^m : \sum_{j=1}^m (x_j^2 + y_j^2) \leq r^2\}$.

where

$$Q(x,y,\xi) \doteq i\xi\cdot(x-y) - |\xi|^\lambda p(x-y), \tag{5.7}$$

with $(y,\xi)\in B(0,r)\times\mathbb{R}^m$ and x in a neighborhood of 0 to be defined. Let $\chi\in\mathcal{D}^M(B(0,r/2))$ be such that $0\le\chi\le 1$ and $\chi\equiv 1$ in $B(0,r/4)$. For $\sigma>0$ to be chosen, let

$$\tilde{z}(y,s) = y + is\chi(y)\frac{v_0}{|v_0|}$$

and

$$D \doteq \{\tilde{z}(y,s) : |y|\le r/2,\ 0\le s\le\sigma\}.$$

Considering $\sigma,\tau\in]0,1[$, with $\tau+\sigma<\delta$, it follows that,

$$D + i\tau\frac{v_0}{|v_0|} \subset B(0,r/2) + i\Gamma_\delta.$$

Applying Stokes' theorem, we can rewrite $\mathcal{F}_p^\lambda(gu)(t,\xi)$ as

$$\begin{aligned}
\mathcal{F}_p^\lambda(gu)(x,\xi) = \lim_{\tau\to 0^+} \Big[\sum_{j=1}^m \int_D \frac{\partial}{\partial\overline{z}_j}\big\{ f\big(z+i\tau\tfrac{v_0}{|v_0|}\big)\, g(z) e^{Q(x,z,\xi)}\big\} d\overline{z}_j\wedge dz \\
+ \int f\big(\tilde{z}(y,\sigma)+i\tau\tfrac{v_0}{|v_0|}\big) g(\tilde{z}(y,\sigma)) e^{Q(x,\tilde{z}(y,\sigma),\xi)} |\tilde{z}_y(y,\sigma)|\, dy\Big] \\
= \lim_{\tau\to 0^+} \Big[\sum_{j=1}^m \int_D \frac{\partial}{\partial\overline{z}_j}\big\{ f\big(z+i\tau\tfrac{v_0}{|v_0|}\big)\big\} e^{Q(x,z,\xi)} d\overline{z}_j\wedge dz \\
+ \int f\big(\tilde{z}(y,\sigma)+i\tau\chi(y)\tfrac{v_0}{|v_0|}\big) g(\tilde{z}(y,\sigma)) e^{Q(x,\tilde{z}(y,\sigma),\xi)} |\tilde{z}_y(y,\sigma)|\, dy\Big]
\end{aligned} \tag{5.8}$$

where $\tilde{z}_y(y,\sigma)$ denotes the Jacobian determinant of $y\mapsto\tilde{z}(y,\sigma)$, $dz = dz_1\wedge\cdots\wedge dz_m$, in the second equality we use that $e^{i\xi\cdot(x-z)-|\xi|^\lambda p(x-z)}\in\mathcal{O}(\mathbb{C}^m)$ (in z) and for all $z\in D$ it follows that $|z|\le\frac{r}{2}+\sigma<\frac{3r}{4}$ (if we choose $\sigma<\frac{r}{4}$), thus $g\equiv 1$ in D.

Now we will estimate the real part of

$$Q(x,y,s,\xi) \doteq Q(x,\tilde{z}(y,s),\xi) = i\xi\cdot(x-\tilde{z}(y,s)) - |\xi|^\lambda p(x-\tilde{z}(y,s))$$

where Q is given by (5.7).

$$\begin{aligned}
\mathcal{R}\{Q(x,\tilde{z}(y,s),\xi)\} \le & -s\chi(y)\frac{c'}{2}|\xi| - |\xi|^\lambda p(x-y) \\
& - |\xi|^\lambda \sum_{|\alpha|=2k} a_\alpha \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta}(x-y)^{\alpha-\beta}\mathcal{R}\left\{\left(-is\chi(y)\tfrac{v_0}{|v_0|}\right)^\beta\right\} \\
\le & -s\chi(y)\frac{c'}{2}|\xi| - |\xi|^\lambda c|x-y|^{2k} + |\xi|^\lambda \sum_{\substack{|\alpha|=2k\\|\beta|=1}} a_\alpha\binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|}\, s\chi(y) \\
& + |\xi|^\lambda \sum_{|\alpha|=2k} a_\alpha \sum_{\substack{0<\beta<\alpha\\|\beta|>1}} \binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|}\,(s\chi(y))^2 + s^2\chi(y)^2 p_1|\xi|^\lambda,
\end{aligned} \tag{5.9}$$

where we have used (3.2) and (5.6) in the first inequality and (3.3) in the second inequality. Since $r=\frac{c'}{8\cdot(2k)^m\cdot 2^{2km}p_1}$, choosing $c'<16\cdot(2k)^m\cdot 2^{2km}p_1$ it follows that $\frac{r}{2}<1$. Hence, if $y,x\in$

$B(0,r/4)$ then $|x-y|\le \frac{r}{2}<1$, $\chi(y)=1$ and we can further estimate $\mathcal{R}\{Q(x,\tilde z(y,s),\xi)\}$ in (5.9) as

$$
\begin{aligned}
\mathcal{R}\{Q(x,\tilde z(y,s),\xi)\} &\le -s\frac{c'}{2}|\xi| + |\xi|\sum_{|\alpha|=2k} a_\alpha \sum_{|\beta|=1}\binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|}\, s+ \\
&\quad + |\xi|\sum_{|\alpha|=2k} a_\alpha \sum_{\substack{0<\beta<\alpha\\ |\beta|>1}}\binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|}s^2 + |\xi|p_1 s^2 \\
&\le -s|\xi|\left[\frac{c'}{2} - p_1(2k)^m\cdot 2^{2km}\frac{r}{2}(1+s) - p_1 s\right] \\
&\le -s|\xi|\left[\frac{c'}{2} - \frac{c'}{16}(1+\sigma) - p_1\sigma\right] \le -s|\xi|\frac{c'}{8},
\end{aligned} \tag{5.10}
$$

for $|y|,|x|<\frac{r}{4}$, $s\le\sigma\le\min\left\{1,\frac{c'}{4p_1}\right\}$, $0<\lambda\le 1$ and $\xi\in\Gamma$ with $|\xi|>1$. In the second inequality in (5.10) we have used the standard combinatorial estimate $\sum_\beta\binom{\alpha}{\beta}\le(2k)^m\cdot 2^{2km}$ and the fact that $\sum_{|\alpha|=2k}a_\alpha = p_1>0$, while in the third inequality we used that $r=\frac{c'}{8\cdot(2k)^m\cdot 2^{2km}p_1}$. Finally, in the last inequality in (5.10) we took the advantage of the choice for σ to obtain $p_1\sigma\le c'/4$.

Now, for $\frac{r}{4}\le|y|\le r$, $|x|<\frac{r}{8}$ and $c'<8\cdot(2k)^m\cdot 2^{2km}p_1/9$, we have

$$|x-y|\le\tfrac{9r}{8}<1 \quad\text{ and }\quad |x-y|\ge\tfrac{r}{4}-\tfrac{r}{8}=\tfrac{r}{8}.$$

Thus, using (5.9), we obtain

$$
\begin{aligned}
\mathcal{R}\{Q(x,\tilde z(y,s),\xi)\} \le& -s\chi(y)\frac{c'}{2}|\xi| - c|\xi|^\lambda|x-y|^{2k}+ \\
&+ s|\xi|^\lambda\Big\{\sum_{\substack{|\alpha|=2k\\|\beta|=1}} a_\alpha\binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|} + \sum_{\substack{|\alpha|=2k\\ 0<\beta<\alpha,|\beta|>1}} a_\alpha\binom{\alpha}{\beta}|x-y|^{|\alpha-\beta|}s + sp_1\Big\} \\
\le& -s|\xi|\tfrac{c'}{2}\chi(y) - c(\tfrac{r}{8})^{2k}|\xi|^\lambda + s|\xi|^\lambda\Big\{\frac{9c'}{64}+\frac{9c'}{64}s+sp_1\Big\} \\
\le& -s|\xi|\frac{c'}{2}\chi(y) - \tfrac{c}{2}(\tfrac{r}{8})^{2k}|\xi|^\lambda,
\end{aligned} \tag{5.11}
$$

for $\frac{r}{4}\le|y|\le r$, $|x|<\frac{r}{8}$, $\xi\in\Gamma$ and $|\xi|>1$, $c'<\frac{8\cdot(2k)^m\cdot 2^{2km}p_1}{9}$,

$$\sigma<\min\left\{1,\tfrac{c}{2}\left(\tfrac{r}{8}\right)^{2k}\left(\tfrac{18c'}{64}+p_1\right)^{-1}\right\},$$

and $\lambda\le 1$. Choosing $\sigma\doteq\min\{1,c'/(4p_1),\frac{c}{2}\left(\frac{r}{8}\right)^{2k}\left(\frac{18c'}{64}+p_1\right)^{-1}\}$ and fixing $c'<16\cdot(2k)^m\cdot 2^{2km}p_1/9$, it follows from (5.9), (5.10) and (5.11) that there exists a positive constant c_1 such that, for

$$\mathcal{R}Q(x,\tilde z(y,s),\xi)<-c_1 s|\xi|, \tag{5.12}$$

for every $|y|<\frac{r}{4}$, $|x|<\frac{r}{8}$, $\xi\in\Gamma$ with $|\xi|>1$, $s\le\sigma$, and $\lambda\le 1$.

While,

$$\begin{gathered}\mathcal{R}Q(x,\tilde{z}(y,s),\xi) < -s\chi(y)c_1|\xi| - c_1 s|\xi|^\lambda, \\ \text{for every } \tfrac{r}{4} \le |y| \le r,\ |x| < \tfrac{r}{8},\ \xi \in \Gamma \text{ with } |\xi| > 1,\ s \le \sigma, \text{ and } \lambda \le 1.\end{gathered} \tag{5.13}$$

Moreover, when $s = \sigma$, setting $c_3 \doteq \min\left\{c_1\sigma;\ c'(\sigma c_1)^{1/\lambda}\right\}$, and using the fact that $\lambda \in (0,1]$ is admissible for the sequence $M = (M_j)_{j\in\mathbb{N}}$ (Definition 4.2, equation (4.5)) together with (5.12), (5.13) and (A.4), it follows that

$$\begin{gathered}\mathcal{R}Q(x,\tilde{z}(y,\sigma),\xi) < -M(c_3|\xi|), \\ \text{for every } |y| \le r,\ |x| < \tfrac{r}{8},\ \xi \in \Gamma_R \doteq \Gamma \setminus B(0,R) \text{ with} \\ R > \max\{1, c''(\sigma c_1)^{-1/\lambda}\}, \text{ and } \lambda \in (0,1] \text{ admissible.}\end{gathered} \tag{5.14}$$

To proceed, we are ready to look at the FBI-BH transform $\mathcal{F}_p^\lambda(gu)(x,\xi)$ given by (5.8).

For the first integral on the right hand-side of (5.8), Since f is an $M-$ almost analytic function there exist $C, c_2 > 0$ such that,

$$\left|\left(\partial_{\bar{z}_j} f\right)\left(y + is\chi(y)\frac{v_0}{|v_0|} + i\tau\frac{v_0}{|v_0|}\right) e^{Q(x,\tilde{z}(y,s),\xi)}\right| \le Ce^{-M^*(c_2(s\chi(y)+\tau))} e^{\mathcal{R}Q(x,\tilde{z}(y,s),\xi)}. \tag{5.15}$$

When $|y| < \frac{r}{4}$, we have $\chi(y) = 1$ and it follows from (5.2) and (5.12) that

$$\begin{aligned} e^{-M^*(c_2(s\chi(y)+\tau))} e^{\mathcal{R}Q(x,\tilde{z}(y,s),\xi)} &\le e^{-M^*(c_2(s+\tau))} e^{-c_1(s+\tau)|\xi|} e^{c_1\tau|\xi|} \\ &= e^{-\left\{M^*(c_2(s+\tau)) + c_2(s+\tau)(\frac{c_1}{c_2}|\xi|)\right\}} e^{c_1\tau|\xi|} \\ &\le e^{-\inf_\theta \left\{M^*(\theta) + \theta(\frac{c_1}{c_2}|\xi|)\right\}} e^{c_1\tau|\xi|} \\ &\le e^{-M(c_4|\xi|)} e^{c_1\tau|\xi|} \xrightarrow{\tau\to 0} e^{-M(c_4|\xi|)} \end{aligned}$$

$$\text{for every } |y| < \tfrac{r}{4},\ |x| < \tfrac{r}{8},\ s \le \sigma \le 1,\ \xi \in \Gamma \text{ with } |\xi| > 1, \text{ and } \lambda \le 1, \tag{5.16}$$

where $c_4 \doteq c_1/c_2$. Analogously, when $\frac{r}{4} \le |y| \le r$ it follows from (5.2) and (5.13) that

$$\begin{aligned} e^{-M^*(c_2(s\chi(y)+\tau))} e^{\mathcal{R}Q(x,\tilde{z}(y,s),\xi)} &\le e^{-M^*(c_2(s\chi(y)+\tau))} e^{-s\chi(y)c_1|\xi| - sc_1|\xi|^\lambda} \\ &\le e^{-M^*(c_2(s\chi(y)+\tau))} e^{-c_2(s\chi(y)+\tau)(c_1/c_2)|\xi|} e^{\tau c_1|\xi|} \\ &\le e^{-M(c_4|\xi|)} e^{c_1\tau|\xi|} \xrightarrow{\tau\to 0} e^{-M(c_4|\xi|)} \end{aligned}$$

$$\text{for every } \tfrac{r}{4} \le |y| \le r,\ |x| < \tfrac{r}{8},\ s \le \sigma \le 1,\ \xi \in \Gamma \text{ with } |\xi| > 1, \text{ and } \lambda \le 1, \tag{5.17}$$

Summing up, it is a consequence of (5.15), (5.16), and (5.17) and the dominated convergence theorem that there exists a positive constant C such that for every $|x| < \frac{r}{8}$, $\xi \in \Gamma$ with $|\xi| > 1$ and $\lambda \le 1$, it holds

$$\lim_{\tau\to 0^+}\left|\sum_{j=1}^m \int_D \frac{\partial}{\partial \bar{z}_j}\left\{f\left(z + i\tau\tfrac{v_0}{|v_0|}\right)g(z)\right\} e^{i\xi\cdot(x-z) - |\xi|^\lambda p(x-z)}\, d\bar{z}_j \wedge dz\right| \le Ce^{-M(c_4|\xi|)}. \tag{5.18}$$

For the second integral on the right-hand side of (5.8), we use the fact that $u = bf$ in $\mathcal{D}^{M\prime}(\Omega)$ so that, by definition, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and $L > 0$ there exists

$C_L > 0$ satisfying

$$\left|\left\langle f\big(\tilde z(\cdot) + i\tau\chi(\cdot)\tfrac{v_0}{|v_0|}\big), \phi(x,\cdot,\xi)\right\rangle\right| \leq C_\epsilon \sum_\alpha \frac{L^{|\alpha|}}{M_{|\alpha|}} \parallel \partial_y^\alpha \phi(x,y,\xi) \parallel_{L_y^\infty}, \tag{5.19}$$

for all $\phi(x,\cdot,\xi) \in \mathcal{D}^M(\Omega)$. In particular, for

$$\phi(x,y,\zeta) = g(\tilde z(y,\sigma)) e^{Q(x,\tilde z(y,\sigma)\xi)} \tilde z_y(y,\sigma),$$

an application of Lemma B.1 with $\theta = c_3$ together inequality (5.14), show that

$$\begin{aligned} |\partial_y^\alpha \phi(x,y,\xi)| &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} C^{|\alpha-\beta|} M_{|\alpha-\beta|} \left|\partial_y^\beta e^{Q(x,\tilde z(y,\sigma),\xi)}\right| \\ &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} C^{|\alpha|+1} M_{|\alpha-\beta|} e^{\mathcal{R}Q(x,\tilde z(y,\sigma),\xi)} e^{\frac{1}{2}M(c_3|\xi|)} e^{H/c_3} M_{|\beta|} \\ &\leq C^{|\alpha|+1} M_{|\alpha|} e^{-\frac{1}{2}M(c_3|\xi|)}, \quad \forall\, (x,y,\xi) \in U \times B(0,r)) \times \Gamma_R. \end{aligned} \tag{5.20}$$

Thus, for $c_5 \doteq \min\{c_3, c_4\}$ and $L > 0$ sufficiently small it follows from (5.8), (5.18), (5.19) and (5.20) that

$$|\mathcal{F}_p^\lambda(gu)(x,\xi)| \leq C e^{-\frac{3}{2}M(c_5|\xi|)}, \quad \forall (x,\xi) \in U \times \Gamma_R. \tag{5.21}$$

Since $\mathcal{F}_p^\lambda(gu)(x,\xi) e^{-M(c_5|\xi|)}$ is bounded in compact sets, (5.21) is true for every $(x,\xi) \in U \times \Gamma$ as we wished to prove.

Conversely, let $u \in \mathcal{E}^{M\prime}(\mathbb{R}^m)$ and suppose $(0,\xi^0)$ is such that (5.5) is satisfied, for some open neighborhood U of 0 and an open conic neighborhood of ξ_0, Γ. We want to prove that $(0,\xi^0) \notin WF_M(u)$. From Corollary 4.1 we can assume $u \in \mathcal{E}^{M\prime}(B(0,r))$, for some small $r > 0$ to be defined. Using the FBI-BH inversion formula, Theorem 3.1, we can write

$$u(x) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi) \mathcal{F}_p^\lambda u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi, \quad \text{in} \quad \mathcal{E}^{M\prime}(B(0,r)).$$

Consider open acute cones $C_1, \ldots, C_K$ in $\mathbb{R}^m$ such that,

$$\mathbb{R}^m \setminus \Gamma = \bigcup_{j=1}^K \overline{C_j}, \qquad C_j \cap C_\ell = \emptyset, \text{ for } \ j \neq \ell; \tag{5.22}$$

and

$$\Gamma_j = \{v \in \mathbb{R}^m : \xi \cdot v > 0 \ \ \forall\, \xi \in C_j \ \text{ and } \ \xi^0 \cdot v < 0\}, \qquad j = 1, \ldots, K$$

are open (not empty) acute cones. Then, there exists $0 < c' < 1$ such that

$$\xi \cdot v \geq c'|\xi||v|, \ \ (v,\xi) \in \Gamma_j \times C_j. \tag{5.23}$$

With $r > 0$ sufficiently small such that $B(0,r) \subset U$ we define,

$$u_1(x) \doteq \lim_{\epsilon\to 0^+} \int_\Gamma \int_{B(0,r)} e^{i\xi(x-t)} \sigma(\epsilon\xi) \mathcal{F}_p^\lambda(u)(t,\xi) |\xi|^{\frac{m}{2k}}\, dt\, d\xi,$$

$$u_2(x) \doteq \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^m\setminus\Gamma} \int_{B(0,r)} e^{i\xi(x-t)} \sigma(\epsilon\xi) \mathcal{F}_p^\lambda(u)(t,\xi) |\xi|^{\frac{m}{2k}}\, dt\, d\xi \tag{5.24}$$

and

$$u_3(x) \doteq \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m\setminus B(0,r)} e^{i\xi(x-t)}\sigma(\epsilon\xi)\mathcal{F}_p^\lambda(u)(t,\xi)|\xi|^{\frac{m}{2k}}\,dt\,d\xi.$$

The proof will be concluded if we show that each u_i is $M-$microregular in $(0,\xi^0)$, i.e., satisfies condition (1) and (2) from Definition 5.2.

It follows from the same arguments as in the proof of Theorem 4.1 that $u_1 \in \mathcal{E}^M(\Omega)$ in a neighborhood Ω of 0 (see term I_1, page 13) and $u_3 = I_2 + I_3 + I_4 \in \mathcal{E}^M(\Omega)$, for I_1, I_2 and I_3 presented in Theorem 4.1 (see (4.30), (4.31) and (4.53)). Since every $\mathcal{E}^M$ function has an $M-$almost analytic extension (see [2, Lemma 17]) it follows that $u_1 + u_3$ is the boundary value (restriction) of an $M-$almost analytic function. Next we will consider

$$\begin{aligned} f_j(x+iy) &\doteq \sigma(0)\int_{C_j}\int_{B(0,r)} e^{i\xi(x+iy-t)}\mathcal{F}_p^\lambda u(t,\xi)|\xi|^{\frac{m}{2k}}\,dt\,d\xi \\ f_j^\epsilon(x) &\doteq \int_{C_j}\int_{B(0,r)} e^{i\xi(x-t)}\sigma(\epsilon\xi)\mathcal{F}_p^\lambda u(t,\xi)|\xi|^{\frac{m}{2k}}\,dt\,d\xi. \end{aligned}$$

and we shall prove that the functions $f_j \in \mathcal{E}^M(V+(\Gamma_j)_\delta)$ (for some $\delta > 0$ and V neighborhood of 0) are $M-$almost analytic functions which increase M^*- exponentially (in the sense of Theorem 5.1, (5.3)) and $bf_j \doteq \lim_{\Gamma_j\ni v\to 0} f_j = \lim_{\epsilon\to 0^+} f_j^\epsilon$. In fact, by definition of $\mathcal{F}_p^\lambda$ (Definition 3.1) one can write

$$f_j(x+iy) = c_{p,\sigma}\int_{C_j}\int_{B(0,r)}\langle u(x'); e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\rangle dt\,d\xi, \tag{5.25}$$

where $c_{p,\sigma} = c_p\cdot\sigma(0)$ and

$$Q_1(x+iy,t,x',\xi) = i\xi(x+iy-x') - |\xi|^\lambda p(t-x').$$

Consider $\phi \in \mathcal{D}^M(B(0,r))$ such that $\phi \equiv 1$ in a neighborhood of $\operatorname{supp} u$. Thus, using Theorem A.1, there exist constants $u_\beta \in \mathbb{C}$, $\beta \in \mathbb{N}^m$ and $g \in C^0(B(0,r))$ satisfying (A.1), such that

$$f_j(x+iy) = c_{p,\sigma}\int_{C_j}\int_{B(0,r)}\sum_\beta\int_{|x'|<r}(-1)^{|\beta|}\partial_{x'}^\beta\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\phi(x')\right)u_\beta\, g(x')dx'\,dt\,d\xi. \tag{5.26}$$

Claim 1. *For each $j = 1,\dots,K$ we have that bf_j exists in $\mathcal{E}^{M'}$.*

Indeed, the proof of this claim will follow from an application of Theorem 5.1 with $\mathcal{V} = \left\{\frac{\partial}{\partial\bar z_1},\dots\frac{\partial}{\partial\bar z_m}\right\}$. By Lemma B.1, for $\theta > 0$ arbitrary, we obtain that for (x,y,x',t,ξ) in the set $V\times(\Gamma_j)_\delta\times B(0,3r)\times B(0,r)\times C_j$,

$$\begin{aligned} \left|\partial_{x'}^\beta\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\right)\right| &\le e^{-\xi\cdot y}e^{\frac12 M(\theta|\xi|)}e^{H/\theta}C^{|\beta|+1}M_{|\beta|}|\xi|^{\frac{m}{2k}} \\ &\le e^{-c|\xi||y|}e^{\frac12 M(\theta|\xi|)}e^{H/\theta}C^{|\beta|+1}M_{|\beta|}|\xi|^{\frac{m}{2k}} \end{aligned} \tag{5.27}$$

where V is a fixed bounded open neighborhood of 0 and $0 < \delta < 1$. For $|\xi| \ge 1$, we choose $\ell \in \mathbb{N}$ such that $\frac{m}{2k} \le \ell < \frac{m}{2k} + 1$ to obtain

$$\left|\partial_{x'}^\beta\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\right)\right| \overset{(A.8)}{\le} e^{-c|\xi||y|}e^{M(\theta|\xi|)}e^{H/\theta}C^{|\beta|+1}M_{|\beta|}\sqrt{A}\frac{H^\ell}{\theta^\ell}.$$

For $|\xi| < 1$ we have a trivial inequality (without the term $\sqrt{A}\frac{H^\ell}{\theta^\ell}$). Thus, increasing and keeping in mind that now $C = O(\theta^{-ell})$, we have that for every (x, y, x', t, ξ) in the set $V \times \Gamma_j \times B(0, 3r) \times B(0, r) \times C_j$, it holds

$$\left|\partial^\beta_{x'}\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\right)\right| \le e^{-c|\xi||y|}e^{M(\theta|\xi|)}e^{H/\theta}C^{|\beta|+1}M_{|\beta|}. \tag{5.28}$$

Since, for each $\varrho > 0$ there exists c_ϱ such that (see (A.10))

$$M(\theta|\xi|) = 2M(\theta|\xi|) - M(\theta|\xi|) \le \varrho M\left(c_\varrho\theta|\xi|\right) - M(\theta|\xi|). \tag{5.29}$$

Choosing $\theta = \theta_\varrho \doteq \frac{c}{c_\varrho}$, for each $\varrho > 0$, we have from (5.28) and (5.29) that for (x, y, x', t, ξ) in $V \times (\Gamma_j)_\delta \times B(0, 3r) \times B(0, r) \times C_j$, we have

$$\begin{aligned}\left|\partial^\beta_{x'}\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\right)\right| &\le e^{-c|\xi||y|}e^{\varrho M(c_\varrho\theta_\varrho|\xi|)-M(\theta_\varrho|\xi|)}e^{H/\theta_\rho}C_\rho^{|\beta|+1}M_{|\beta|}\\ &\le e^{\varrho\sup_{r>0}\{M(r)-r|y|/\varrho\}}e^{-M(\theta_\varrho|\xi|)}e^{H/\theta_\varrho}C_\varrho^{|\beta|+1}M_{|\beta|}\\ &= e^{\varrho\omega^*(|y|/\varrho)}e^{-M(\theta_\varrho|\xi|)}e^{H/\theta_\varrho}C_\varrho^{|\beta|+1}M_{|\beta|}\\ &\overset{(5.1)}{\le} e^{\varrho M^*(|y|/\varrho)}e^{-M(\theta_\varrho|\xi|)}e^{H/\theta_\varrho}C_\varrho^{|\beta|+1}M_{|\beta|},\end{aligned} \tag{5.30}$$

where $\omega^*(r)$ is the Young conjugate of $M(t)$ defined in page 20. Therefore, making use of (5.30) for $L = L(\varrho)$ sufficiently small, we can estimate $f(x + iy)$ given in (5.26), by

$$\begin{aligned}|f_j(x+iy)| &\le C_\varrho\int_{C_j}\int_{\{|t|<r\}}\sum_\beta\frac{(C_{L,\varrho}L)^{|\beta|}}{M_{|\beta|}}e^{\varrho M^*(|y|/\varrho)}e^{-M(\theta_\varrho|\xi|)}M_{|\beta|}dtd\xi\\ &\overset{(A.9)}{\le} C_\varrho e^{\varrho M^*(|y|/\varrho)}, \quad \forall(x,y)\in V\times(\Gamma_j)_\delta, \quad \varrho>0.\end{aligned} \tag{5.31}$$

Now we will show that for each $j = 1, \ldots, K$ we have that $f_j \in \mathcal{O}(B(0, r) + i(\Gamma_j)_\delta)$. To do so, fix $j \in \{1, \ldots, K\}$. Since analyticity is a local property it is sufficient to prove that for each $x_0 + iy_0 \in B(0, r) + i(\Gamma_j)_\delta$ there exists $0 < r_0 < \min\{|y_0|, r - |x_0|\}$, such that f_j is holomorphic in $B(x_0, r_0) + i(\Gamma_j \cap B(y_0, r_0))$. Note that $|y| \ge |y_0| - |y - y_0| > |y_0| - r_0 \doteq c_0$ for all $y \in \Gamma_j \cap B(y_0, r_0)$. Since M^* is a decreasing function, using (5.30), for (x, y, x', t, ξ) in $V \times (\Gamma_j)_\delta \times B(0, 3r) \times B(0, r) \times C_j$, we obtain

$$\left|\partial^\beta_{x'}\left(e^{Q_1(x+iy,t,x',\xi)}|\xi|^{\frac{m}{2k}}\right)\right| \le e^{\varrho M^*(c_0/\varrho)}e^{-M(\theta_\varrho|\xi|)}e^{H/\theta_\varrho}C_\varrho^{|\beta|+1}M_{|\beta|}.$$

Hence, applying Lemma A.1, item G, and Theorem A.1 (A.1), it follows that

$$\int_{C_j}\int_{B(0,r)}\sum_\beta\int_{\{|x'|<r\}}e^{\varrho M^*(c_0/\varrho)}e^{-M(\theta_\varrho|\xi|)}C^{|\beta|+1}M_{|\beta|}u_\beta|g(x')|dx'\,dt\,d\xi < +\infty.$$

Thus, we can differentiate under the integral sign in (5.26) and conclude that the funcion f_j is holomorphic in $B(x_0, r_0)+i(\Gamma_j\cap B(y_0, r_0))$. Since $j \in \{1, \ldots, K\}$ and $(x_0, y_0) \in (B(0, r)+i\Gamma_j)$ are arbitrary, we have

$$f_j \in \mathcal{O}(B(0, r) + i\Gamma_j), \quad \forall\, j \in \{1, \ldots, K\}. \tag{5.32}$$

Using (5.31), (5.32) and Theorem 5.1 we see that bf_j exists in $\mathcal{D}^{M'}(B(0, r))$. This finishes the proof of Claim 1.

Claim 2. *For each $j = 1, \ldots, K$ we have $bf_j = \lim_{\epsilon\to0} f^\epsilon_j$, in $\mathcal{D}^{M'}(B(0, r))$.*

Fix $j \in \{1, \ldots, K\}$, using Theorem A.1 we can write

$$f_j^\epsilon = c_p \int_{C_j} \int_{B(0,r)} \sigma(\epsilon\xi) \sum_\beta \int_\Omega (-1)^{|\beta|} \partial_{x'}^\beta \left(e^{i\xi(x-x')-|\xi|^\lambda p(t-x')} |\xi|^{\frac{m}{2k}} \phi(x') \right) u_\beta \, g(x') dx' \, dt \, d\xi.$$

for some constants $u_\beta \in \mathbb{C}$ for each $\beta \in \mathbb{N}^m$, such that, for every $L > 0$ there exists $C_L > 0$ satisfying (A.1). Now fix an arbitrary $\varphi \in \mathcal{D}^M(\mathbb{R}^m)$, (5.30) implies that we can use Fubini's theorem to write

$$\begin{aligned}
\langle f_j^\epsilon; \varphi \rangle &= c_p \int_{C_j} \int_{B(0,r)} \left\langle u(x'), \int_{\mathbb{R}^m} e^{i\xi(x-x')-|\xi|^\lambda p(t-x')} \sigma(\epsilon\xi) |\xi|^{\frac{m}{2k}} \varphi(x) dx \right\rangle dt \, d\xi \\
&= c_p \int_{C_j} \int_{B(0,r)} \left\langle u(x') \cdot \phi(x'); \hat{\varphi}(-\xi) e^{-i\xi x' - |\xi|^\lambda p(t-x')} \sigma(\epsilon\xi) |\xi|^{\frac{m}{2k}} \right\rangle dt \, d\xi \\
&= c_p \int_{C_j} \int_{B(0,r)} \sum_\beta \int_\Omega (-1)^{|\beta|} \partial_{x'}^\beta \left(\hat{\varphi}(-\xi) e^{-i\xi x' - |\xi|^\lambda p(t-x')} \sigma(\epsilon\xi) |\xi|^{\frac{m}{2k}} \phi(x') \right) u_\beta \, g(x') dx' \, dt \, d\xi.
\end{aligned}$$

Using Lemma B.2 together Lemma B.1 (as in page 13, (4.20), replacing $\mathcal{F}_p^\lambda$ by $\hat{\varphi}$) that there exist positive constants C, c_1 such that, for every $|\xi| > 1$ and $\frac{m}{2k} \le l \in \mathbb{N}$ it holds that

$$\left| \partial_{x'}^\beta \{ \hat{\varphi}(-\xi) e^{-i\xi x' - |\xi|^\lambda p(t-x')} \sigma(\epsilon\xi) |\xi|^{\frac{m}{2k}} \} \right| \le C^{|\beta|+1} M_{|\beta|} e^{-\frac{1}{4} M(c_1|\xi|)}. \tag{5.33}$$

Moreover, since u_β satisfies (A.1) and $e^{-M(c_1|\xi|)} \in L^1(\mathbb{R}^m)$, we have

$$\int_{C_j} \int_{B(0,r)} \sum_\beta \int_\Omega C^{|\beta|+1} M_{|\beta|} e^{-\frac{1}{4} M(c_1|\xi|)} u_\beta \, g(x') dx' \, dt \, d\xi < +\infty. \tag{5.34}$$

Hence, we can apply the dominated convergence theorem and obtain

$$\lim_{\epsilon \to 0} \langle f_j^\epsilon; \varphi \rangle = c_p \sigma(0) \int_{C_j} \int_{B(0,r)} \left\langle u(x'); \hat{\varphi}(-\xi) e^{-i\xi x' - |\xi|^\lambda p(t-x')} \right\rangle |\xi|^{\frac{m}{2k}} \, dt \, d\xi. \tag{5.35}$$

Furthermore, since $y \cdot \xi \ge c'|y||\xi|$, for every $(y, \xi) \in \Gamma_j \times C_j$ (see (5.23)), we can use (5.30) and the same ideas as in (5.33) and (5.34) which allow us to apply Fubini's and dominated convergence theorems in (5.25), to write

$$\begin{aligned}
\langle bf_j, \varphi \rangle &= c_{p,\sigma} \lim_{\Gamma_j \ni y \to 0} \int_{\mathbb{R}^m \times C_j \times B(0,r)} e^{i\xi \cdot (x+iy-t)} \left\langle u; e^{i\xi \cdot (t-\cdot) - |\xi|^\lambda p(t-\cdot)} \right\rangle |\xi|^{\frac{m}{2k}} \varphi(x) \, dt \, d\xi \, dx \\
&= c_{p,\sigma} \lim_{\Gamma_j \ni y \to 0} \int_{C_j \times B(0,r)} \left\langle u(x'); \int e^{i\xi \cdot (x+iy-x') - |\xi|^\lambda p(t-x')} \varphi(x) dx \right\rangle |\xi|^{\frac{m}{2k}} \, dt \, d\xi \\
&= c_{p,\sigma} \lim_{\Gamma_j \ni y \to 0} \int_{C_j} e^{-y \cdot \xi} \int_{B(0,r)} \left\langle u(x'); \hat{\varphi}(-\xi) e^{-i\xi \cdot x' - |\xi|^\lambda p(t-x')} \right\rangle |\xi|^{\frac{m}{2k}} \, dt \, d\xi \\
&= c_p \sigma(0) \int_{C_j} \int_{B(0,r)} \left\langle u(x'); \hat{\varphi}(-\xi) e^{-i\xi \cdot x' - |\xi|^\lambda p(t-x')} \right\rangle |\xi|^{\frac{m}{2k}} \, dt \, d\xi \\
&= \lim_{\epsilon \to 0} \langle f_j^\epsilon; \varphi \rangle
\end{aligned} \tag{5.36}$$

in view of (5.35). This proves Claim 2.

Therefore, from the expression of u_2 given by (5.24), (5.22) and (5.36), we conclude that $u_2 = \sum_{j=1}^K bf_j$. This finishes the proof of the theorem. ■

Remark 3: Theorem 4.1 follows easily from Theorem 5.2 and the Edge of the Wedge Theorem in Denjoy-Carleman classes (see [3]). However, the techniques given in the proof of the local version, Theorem 4.1, were used in full to prove the microlocal version, Theorem 5.2.

Using the same techniques, we can show that the definition of WF_M presented here is equivalent to the definition presented in [35]

Theorem 5.3. *Let $u \in \mathcal{D}^{M'}(\mathbb{R}^m)$ and $(0,\xi^0) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$, then $(0,\xi^0) \notin WF_M(u)$ if and only if there exist an open neighborhood U of 0 and conic and open neighborhood Γ of ξ^0, such that*

$$|\widehat{u\phi}(\xi)| \leq Ce^{-M(c|\xi|)}, \quad \xi \in \Gamma, \tag{5.37}$$

for some constants $C, c > 0$.

We finished this section presenting a characterization of WF_M via the same transfom used by M. Christ in [24]. Let $0 < \gamma \leq 1$ and

$$\langle y \rangle_1 \doteq \left(1 + \sum_{j=1}^{m} y_j^2\right)^{1/2}, \quad \forall y = (y_1, \ldots, y_m) \in \mathbb{C}^m,$$

which is well defined and holomorphic in a conic neighborhood $\Gamma \subset \mathbb{C}^m$ of $\mathbb{R}^m$. Define a differential form $\omega = dx_1 \wedge \cdots \wedge dx_m \wedge d(\xi_1 + ix_1\langle\xi\rangle_1^\gamma) \wedge \cdots \wedge d(\xi_m + ix_m\langle\xi\rangle_1^\gamma)$ and define a function a_γ by $\omega = a_\gamma(x,\xi)dx_1 \wedge \ldots dx_m \wedge d\xi_1 \wedge \cdots \wedge d\xi_m$. In [24], M. Christ introduced the following version of the FBI transform

$$\mathcal{F}_\gamma u(x,\xi) = \left\langle u(x'); e^{i\xi\cdot(x-x') - \langle\xi\rangle_1^\gamma (x-x')^2} a_\gamma(x-x',\xi) \right\rangle, \tag{5.38}$$

and characterized Gevrey regularity (G^s) via certain exponential decay of $\mathcal{F}_\gamma u(x,\xi)$ when $\frac{1}{s} \leq \gamma \leq 1$, i.e., when γ is admissible for G^s, see Example 4.1. Since $\gamma \leq 1$ and $1 \leq \langle\xi\rangle_1$ ($\xi \in \mathbb{R}^m$), the function $a_\gamma(x,\xi)$ can be written as a sum of holomorphic functions in x times smooth functions in ξ that are bounded with derivatives uniformly bounded, the proof of Theorem 5.2 can be used to extend M. Christ's characterization of Gevrey regularity.

Theorem 5.4. *Let $\xi^0 \in \mathbb{R}^m \setminus \{0\}$, $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ and $0 < \gamma \leq 1$ admissible for M. Then $(0,\xi^0) \notin WF_M(u)$ if and only if there exist a neighborhood $U \subset \mathbb{R}^m$ of 0, an open and acute cone $\xi^0 \in \Gamma \subset \mathbb{R}^m \setminus 0$ and constants $c, C > 0$ such that*

$$|\mathcal{F}_\gamma u(x,\xi)| \leq Ce^{-M(c|\xi|)}, \quad (t,\xi) \in U \times \Gamma. \tag{5.39}$$

6. Applications

In [13] the authors showed how to use their FBI transform to improve a result on propagation of microlocal analyticity for distribution solutions of a tube structure originally proved in [8] (see [13, Theorem 5.2]). The goal of this section is to present a twofold generalization of [13, Theorem 5.2]: a version for DC classes (WF_M) and for `ultradistribution` solutions.

Let, $U \subset \mathbb{R}^m$ be a bounded neighborhood of $0 \in \mathbb{R}^m$, $0 \in I \subset \mathbb{R}$ be an open interval and $t^* \in I^+ \doteq I \cap (0,+\infty)$. In this section we consider $\phi := (\phi_1, \ldots, \phi_m) \in (\mathcal{E}^M(I))^m$ and the vector field L defined by

$$L \doteq \frac{\partial}{\partial t} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t}(t) \frac{\partial}{\partial y_k}, \quad k = 1, \ldots m.$$

Note that if $Z_k(y,t) = y_k + i\phi_k(t)$ then $LZ_k = 0$ and $Z_k(y,0) = y_k$ $(k = 1, \ldots, m)$. Denote $Z \doteq (Z_1, \ldots, Z_m)$.

Definition 6.1. *We will say that $u \in \mathcal{D}'_M(U)$ is the boundary value of an approximate solution of L if $u = bh$ in $\mathcal{D}'_M(U)$, for an $h \in C^1(U \times I^+)$ satisfying hypothesis (1) and (2) from Theorem 5.1 (therefore bh exists in $\mathcal{D}'_M(U)$), there exists a positive constant C such that*

$$|Lh(x,t)| \leq C^{N+1} \frac{M_N}{N!} |t|^N, \ \ \forall N \in \mathbb{N} \text{ and } (x,t) \in U \times I^+. \tag{6.1}$$

Note that condition (6.1) is equivalent to

$$|Lh(x,t)| \leq C e^{-M^*(C|t|)}, \qquad \forall\, (x,t) \in U \times I^+, \tag{6.2}$$

and (6.2) is increasing in t and C and we may assume $C > 1$.

Theorem 6.1. *Let U, I and ϕ be as above. For each $\xi^0 \in \mathbb{R}^m \setminus \{0\}$ with $|\xi^0| = 1$, $0 < \rho < \frac{1}{4}$ and $\epsilon > 0$ let $(\star)_{\xi^0,\rho,\epsilon}$ denoting the following property*

$$(\star)_{\xi^0,\rho,\epsilon} : \begin{cases} \text{there exists } t^* \in I^+ \text{ such that:} \\ -\phi(t)\cdot\xi^0 \geq 2\epsilon^7 t^{4\rho}, \quad \forall t \in [0,t^*]; \quad \text{and} \\ |\phi(t)| \leq \epsilon^2 t^\rho, \quad \forall t \in [0,t^*]. \end{cases} \tag{6.3}$$

There exists $\epsilon_0 > 0$ such that for all $(\xi,\epsilon) \in \mathbb{S}^{m-1} \times (0,\epsilon_0]$ which satisfy $(\star)_{\xi^0,\rho,\epsilon}$ we have that $(0,\xi) \notin WF_M(u)$ for every $u \in \mathcal{D}'_M(U)$ that is the boundary value of an approximate solution in the sense of Definition 6.1.

Proof: Define $Q(x,y,t,\xi)$ for $(x,y,t,\xi) \in U \times U \times I \times \mathbb{R}^m$ as

$$Q(x,y,t,\xi) = i\xi \cdot (x - Z(y,t)) - K|\xi|(x - Z(y,t))^4. \tag{6.4}$$

The proof of Theorem 5.1 in [13] implies that, for $r^2 = \epsilon \leq \epsilon_0 < 1/576$, $K = 64$ and if $(\xi^0,\epsilon) \in \mathbb{S}^{m-1} \times (0,\epsilon_0]$ satisfies $(\star)_{\xi^0,\rho,\epsilon}$, then there exist a neighborhood $0 \in V \subset \mathbb{R}^m$, an open conic neighborhood $\Gamma \subset \mathbb{R}^m$ of ξ^0 and a constant $a = a(\epsilon) > 0$ such that

$$\mathcal{R}Q(x,y,t^*,\xi) \leq -a|\xi|, \quad \forall (x,\xi) \in V \times \Gamma, \quad y \in B(0,\epsilon^{1/2}) \tag{6.5}$$

and

$$\mathcal{R}Q(x,y,t,\xi) \leq -a|\xi|, \quad \forall (x,\xi) \in V \times \Gamma, \quad 0 \leq t \leq t^* \quad \text{and} \quad \frac{\epsilon^{1/2}}{2} \leq |y| \leq \epsilon^{1/2}. \tag{6.6}$$

We will now obtain a rougher estimate to $\mathcal{R}Q(x,y,t,\xi)$ for $x,y \in V$ and $\xi \in \Gamma$, possibly diminishing V and Γ and for $(\xi^0,\epsilon) \in \mathbb{S}^{m-1} \times (0,\epsilon_0]$ satisfying $(\star)_{\xi^0,\rho,\epsilon}$. To this end we note that we can use the homogeneity in ξ at the first inequality in (6.3) to extend it to a conic neighborhood of ξ^0 (which we will still denote by Γ) as

$$-\phi(t)\cdot\xi \geq \epsilon^7 t^{4\rho}|\xi|, \quad \forall t \in [0,t^*], \, \xi \in \Gamma. \tag{6.7}$$

Using the second inequality in (6.3) and proceeding as in [13, equation (5.5)] we have

$$\begin{aligned} \mathcal{R}\left\{[x - Z(y,t)]^4\right\} &\geq |x-y|^4 - 6|x-y|^2|\phi(t)|^2 + |\phi(t)|^4 \\ &\geq -8|\phi(t)|^4 \\ &\geq -8\epsilon^8 t^{4\rho}, \quad \forall\, x,y \in \mathbb{R}^n, \, t \in [0,t^*], \end{aligned} \tag{6.8}$$

where in the second inequality of (6.8) we used that $(|x-y|^2-3|\phi(t)|^2)^2 \geq 0$. Hence, it follows from (6.4), (6.7) and (6.8) that

$$\begin{aligned}\mathcal{R}\{Q(x,y,t,\xi)\} &\leq -t^{4\rho}\epsilon^7|\xi| + K|\xi|8\epsilon^8 t^{4\rho} \qquad (6.9)\\ &= -\epsilon^7 t^{4\rho}|\xi|(1-K8\epsilon), \quad \forall\,(x,y,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,t^*]\times\Gamma.\end{aligned}$$

Therefore, replacing a by $\min\{a,\epsilon^7(1-K8\epsilon)\}>0$, we can rewrite (6.9) as

$$\mathcal{R}\{Q(x,y,t,\xi)\} \leq -at^{4\rho}|\xi|, \quad \forall\,(x,y,t,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,t^*]\times\Gamma. \qquad (6.10)$$

Moving on, we continue to work with $(\xi^0,\epsilon)\in\mathbb{S}^{m-1}\times(0,\epsilon_0]$ satisfying $(\star)_{\xi^0,\rho,\epsilon}$. Let $\chi\in\mathcal{D}^M(B_r(0))$ be such that $\chi\equiv 1$ in $B_{r/2}(0)$. Using [2, Theorem 20], see also [3], there exists $\Psi\in\mathcal{D}^M(B_r(0)\times I)$ which is an approximate solution of L and such that $\Psi(y,0)=\chi(y)$ for all $y\in B_r(0)$ and it satisfies (6.2) with h replaced by Ψ, for some constant $C>0$. Specializing in the proof of [2, Theorem 20], it follows that $\Psi(y,t)\equiv 1$ for all $(y,t)\in B_{r/2}(0)\times I$.

Let $h\in C^1(U\times I^+)$ be as in Definition 6.1 and $p(y)=Ky^4$. Since $u=bh$ in the sense of ultradistributions, we can write

$$\mathcal{F}^1_p(u\chi)(x,\xi) = \lim_{\delta\to 0}\int h(y,\delta)\chi(y)e^{i\xi\cdot(x-y)-K|\xi|(x-y)^4}dy. \qquad (6.11)$$

Applying Stokes' theorem in the integral on the right hand-side of (6.11), we can write

$$\begin{aligned}\mathcal{F}^1_p(u\chi)(x,\xi) = \lim_{\delta\to+\infty}\Bigg\{&\int_{B(0,r)} h(y,t^*+\delta)\,\Psi(y,t^*)e^{Q(x,y,t^*,\xi)}dy\\ &+\int_{B(0,r)\times[0,t^*]} L\left[h(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}dZ(y,t)\right]\Bigg\}, \qquad (6.12)\end{aligned}$$

We note that (6.5) can be used to estimate the first integral on the right hand-side of (6.12) as

$$\left|\int_{B(0,r)} h(y,t^*+\delta)\,\Psi(y,t^*)e^{Q(x,y,t^*,\xi)}dy\right| \leq Ce^{-a|\xi|}, \quad \forall(x,\xi)\in V\times\Gamma. \qquad (6.13)$$

To estimate the second integral on the right hand side of (6.12) we first note that, for every $\varphi=\varphi(y,t)\in C^1(U\times I^+)$ it follows $d(\phi dZ)=L\phi dt\wedge dZ$ hence,

$$\begin{aligned}&\int\limits_{B_r(0)\times[0,t^*]} d\left[h(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}dZ(y,t)\right] \qquad (6.14)\\ &= \int\limits_{B_r(0)\times[0,t^*]} L\left[h(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}\right]dt\wedge dZ(y,t)\\ &= \int\limits_{B_r(0)\times[0,t^*]} (Lh)(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}dt\wedge dZ(y,t)\\ &\quad+\int\limits_{(B_r(0)\setminus B_{r/2}(0))\times[0,t^*]} h(y,t+\delta)(L\Psi)(y,t)e^{Q(x,y,t,\xi)}dt\wedge dZ(y,t),\end{aligned}$$

where in the last equality we used the property that $\Psi(y,t)\equiv 1$ for all $(y,t)\in B_{r/2}(0)\times I$.

We will now work to estimate the two integrals on the right most hand-side of (6.14). To do so, we note that Lh satisfies (6.2) while one can use the bound for $e^{Q(x,y,.t,\xi)}$ given by (6.10) to estimate the first integral on the right most hand-side of (6.14) as

$$\begin{aligned}
&\left|\int_{B_r(0)\times[0,t^*]}(Lh)(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}dt\wedge dZ(y,t)\right|\\
&\qquad\le C\exp\{-M^*(C(t+\delta))-at^{4\rho}|\xi|\}\\
&\qquad\le C\exp\{-M^*(C(t+\delta))-at|\xi|\},\quad -t^{\rho}<-t\,(0<t<1)\\
&\qquad\le C\exp\{-M^*(C(t+\delta))-C(t+\delta)\tfrac{a|\xi|}{C}\}\,e^{a\delta|\xi|}\\
&\qquad\le Ce^{-M(\frac{a}{C}|\xi|)}\,e^{a\delta|\xi|},\quad \forall\,(x,\xi)\in\mathbb{R}^n\times\Gamma, \qquad (6.15)
\end{aligned}$$

where the last inequality is a consequence of (5.2). It follows that $\lim_{\delta\to 0}$ of the first integral on the right most hand-side of (6.14) exists and

$$\left|\lim_{\delta\to 0}\int_{B_r(0)\times[0,t^*]}(Lh)(y,t+\delta)\Psi(y,t)e^{Q(x,y,t,\xi)}dt\wedge dZ(y,t)\right|\le Ce^{-M(\frac{a}{C}|\xi|)}. \qquad (6.16)$$

Moreover, since h satisfies (5.3) for all $\varrho>0$ and Ψ satisfies (6.1) for some positive constant $C>1$, together (6.6) to estimate the second integral in the right most hand-side of (6.14) as

$$\begin{aligned}
&\left|\int_{(B_r(0)\setminus B_{r/2}(0))\times[0,t^*]}h(y,t+\delta)(L\Psi)(y,t)e^{Q(x,y,t,\xi)}dt\wedge dZ(y,t)\right|\\
&\qquad\le C\exp\{\varrho M^*(\tfrac{t+\delta}{\varrho})-M^*(Ct)-a|\xi|\}\\
&\qquad\le C\exp\{C^{-1}M^*(Ct)-M^*(Ct)-a|\xi|\},\quad \text{for}\quad \varrho=C^{-1}\\
&\qquad\le C\exp\{(C^{-1}-1)M^*(Ct)-a|\xi|\}\\
&\qquad\le Ce^{-a|\xi|},\quad \forall(x,\xi)\in V\times\Gamma, \qquad (6.17)
\end{aligned}$$

since $C^{-1}-1<0$.

Therefore, it follows from (6.12), (6.13), (6.14) (6.16) and (6.17) that there exist positive constants $C,c>0$ such that

$$|\mathcal{F}_p^1(u\chi)(x,\xi)|\le Ce^{-M(c|\xi|)},\quad \forall\,(x,\xi)\in V\times\Gamma. \qquad (6.18)$$

The proof now is an immediate consequence of (6.18) jointly with Theorem 5.2. ■

Next we indicate why it does not seem to be easy to prove preceding theorem with the classic FBI. Denote

$$Q(x,y,t,\xi)=i\xi\cdot(x-Z(y,t))-K|\xi|(x-Z(y,t))^2, \qquad (6.19)$$

for some $K>0$ fixed. Observe that

$$Q(x,y,0,\xi)=i\xi\cdot(x-y)-K|\xi|(x-y)^2. \qquad (6.20)$$

is the phase of the classic FBI. Moreover,

$$\mathcal{R}Q(x,y,t,\xi)=\xi\cdot\phi(t)-K|\xi|[(x-y)^2-(\phi(t))^2]. \qquad (6.21)$$

Using the inequalities in (6.3) we obtain

$$\mathcal{R}Q(y,y,t,\xi) \leq -\epsilon^7 t^{4\rho}|\xi| + K|\xi|\epsilon^4 t^{2\rho} = -\epsilon^4 t^{2\rho}|\xi|(\epsilon^3 t^{2\rho} - K). \tag{6.22}$$

However, if K and ϵ are fixed then $(\epsilon^3 t^{2\rho} - K) < 0$ for small values of t, that is, for $t < (K/\epsilon^3)^{1/(2\rho)}$.

Appendix A. Denjoy-Carleman Properties

We will make use of the following structural theorem.

Theorem A.1 ([36], Theorem 10.3)**.** *Let $\Omega \subset \mathbb{R}^m$ be an open subset and $M = (M_j)$ an increasing sequence of positive real numbers satisfying* (2.4)*,* (2.5) *and* (2.6)*. $u \in \mathcal{D}^{M'}(\Omega)$ if and only if, for each relatively compact convex open set $V \subset \Omega$ there exist a continuous function g on $\overline{V}$ and $a_\alpha \in \mathbb{C}$ (for each $\alpha \in \mathbb{N}^m$) such that, for every $L > 0$ there exists $C_L > 0$ satisfying*

$$|a_\alpha| \leq \frac{C_L L^{|\alpha|}}{M_{|\alpha|}}, \;\; \forall \alpha \in \mathbb{N}^m \tag{A.1}$$

and

$$\langle u, \phi \rangle = \sum_\alpha (-1)^{|\alpha|} a_\alpha \int \partial^\alpha \phi(x) \cdot g(x)\, dx, \quad \forall \phi \in \mathcal{D}^M(V). \tag{A.2}$$

Recall, from Definition 4.1, that for each sequence $(M_j)_{j\in\mathbb{N}}$ of positive numbers we define its associate function $M(t)$ on $(0,+\infty)$ by

$$M(t) = \sup_j \log \frac{t^j}{M_j}, \quad t \in [0,\infty). \tag{A.3}$$

Lemma A.1. *Let $M = (M_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers satisfying* (2.3) *and let $M(t)$ be its associated function. It follows that,*

A. *If M satisfies* (2.8) *then, all $t \geq 0$,*

$$\log t \leq M(t) \leq t. \tag{A.4}$$

B. *If M satisfies* (2.8) *then, $M(t)$ is an increasing convex function in $\log t$ which vanishes for sufficiently small $t > 0$ and increases more rapidly than $\log t$ as $t \to +\infty$.*

C. *If M satisfies* (2.5) *then, for each $k > 0$ and $t > 0$ we have,*

$$M(kt) - M(t) \geq \frac{\log(t/A)\log k}{\log H}, \tag{A.5}$$

where A and H were defined by (2.5)*.*

D. *The property* (2.5) *is equivalent to*

$$M\left(\frac{t}{H}\right) \leq \frac{1}{2}M(t) + \log\sqrt{A}. \tag{A.6}$$

E. *If M satisfies* (2.5) *and* (2.8)*, then for $k > 0$ fixed there exists $c = c(k)$ such that,*

$$\frac{3}{2}M(kt) - M(t) \geq 0, \quad t > c. \tag{A.7}$$

F. *If M satisfies (2.5) and (2.7), then for each $\theta > 0$ and $k, r \in \mathbb{N}$ such that $k \geq r \geq 0$ we have*

$$t^r M_{k-r} \leq \sqrt{A}\frac{H^r}{\theta^r} M_k e^{\frac{1}{2}M(\theta t)}, \;\; t > 0; \tag{A.8}$$

where A and H were defined by (2.5). When $k = r$ the sequence M does not need to satisfy (2.7).

G. *Let M be a sequence satisfying (2.8). If $c, \gamma > 0$ then*

$$\int_{\mathbb{R}^m} e^{-cM(\gamma|\xi|)} d\xi < +\infty. \tag{A.9}$$

H. *Let M be a sequence satisfying (2.5). For each $c > 0$ there exists $c' > 0$ (depending on the sequence M and c) such that*

$$cM(t) \geq M(c't), \;\; \forall t > 0. \tag{A.10}$$

I. *Let M be a sequence satisfying properties (2.5) and (2.8). If $c, \gamma > 0$ and $r \in \mathbb{N}$ then,*

$$\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi < +\infty. \tag{A.11}$$

Proof: (A) From Definition 4.1 we have,

$$\log t \overset{(2.3)}{=} \log \frac{t^1}{M_1} \leq \sup_p \log \frac{t^p}{M_p} = M(t) \overset{(2.8)}{\leq} \sup_p \log \frac{t^p}{p!} \leq t,$$

for all $t \geq 0$.
(B) From (2.8) and (2.3) we obtain

$$\left(\frac{M_p}{M_0}\right)^{\frac{1}{p}} \geq \left(\frac{p!}{1}\right)^{\frac{1}{p}} \geq 1$$

and the proof follows from [36, page 49].
(C) [36, Proposition 3.4].
(D) [36, Proposition 3.6].
(E) Since M satisfies (2.8) the from item B. we have that the associated function M is increasing. Thus, it is sufficient to assume $0 < k < 1$. Applying (A.5) and Definition 4.1, we obtain,

$$\frac{3}{2}M(kt) - M(t) \geq \frac{1}{2}M(kt) + \frac{\log(t/A)\log k}{\log H} \geq \frac{1}{2}\log\frac{(kt)^j}{M_j} + \frac{\log k}{\log H}\cdot\log(t/A), \quad \forall j \in \mathbb{N}.$$

Assuming $H > 1$ in (2.5) it follows that $p \doteq -\frac{\log k}{\log H} > 0$. Since (2.8) implies $M_j \geq 1$, we have

$$\frac{3}{2}M(kt) - M(t) \geq \log\left\{t^{(j/2)-p}\, k^{j/2}\, M_j^{-1/2} A^p\right\} \overset{M_j \geq 1}{\geq} \log\left\{(M_j^{-1}kt)^{\frac{j-2p}{2}}\, A^p\, M_j^{-p}\, k^p\right\},$$

for each $j \in \mathbb{N}$ and $t \geq 0$. In addition, if we fix $j > 2p$ and consider $t > c \doteq (k^{-j} M_j^j A^{-2p})^{\frac{1}{j-2p}}$ then,

$$M_j^{-1}kt \geq (M_j^{-j+2p}k^{j-2p}k^{-j}M_j^j A^{-2p})^{\frac{1}{j-2p}} = \left(M_j^{2p}k^{-2p}A^{-2p}\right)^{\frac{1}{j-2p}} = \left(\frac{M_j}{kA}\right)^{\frac{2p}{j-2p}}.$$

Therefore, for $t > c$ we obtain

$$\frac{3}{2}M(kt) - M(t) \geq \log 1 = 0.$$

(F) Let $A > 0$ and $H > 0$ be as (2.5). For arbitrary $\theta > 0$ and $k, r \in \mathbb{N}$ such that $k \geq r \geq 0$, (2.7) implies that

$$t^r M_{k-r} \overset{(2.7)}{\leq} \frac{H^r}{\theta^r} M_k \frac{(\frac{\theta t}{H})^r}{M_r} \overset{(4.1)}{\leq} \frac{H^r}{\theta^r} M_k e^{M((\theta t)/H)}, \quad t \geq 0.$$

The conclusion now follows from (A.6).
(G) Note that

$$\int_{\mathbb{R}^m} e^{-cM(\gamma|\xi|)} d\xi = C\int_0^\infty e^{-cM(\gamma t)} t^{m-1} dt = C\int_0^{\gamma^{-1}} e^{-cM(\gamma t)} t^{m-1} dt + C\int_{\gamma^{-1}}^{+\infty} e^{-cM(\gamma t)} t^{m-1} dt.$$

From the definition of $M(t)$ we have,

$$e^{-cM(\gamma t)} = \exp\left\{-c\sup_k \left[\log\frac{(\gamma t)^k}{M_k}\right]\right\} = \exp\left\{\inf_k \left[\log\left(\frac{(\gamma t)^k}{M_k}\right)^{-c}\right]\right\}$$

Then,

$$\int e^{-cM(\gamma|\xi|)} \leq C\int_0^{\gamma^{-1}} \left[\frac{(\gamma t)^k}{M_k}\right]^{-c} t^{m-1} dt + C\int_{\gamma^{-1}}^\infty \left[\frac{(\gamma t)^j}{M_j}\right]^{-c} t^{m-1} dt,$$

for any $k, j \in \mathbb{N}$. Since $M(t)$ is increasing it is sufficient to consider the case $c, \gamma \leq 1$. Choosing $k \leq m-1 \leq \frac{m-1}{c}$ and $j > \frac{m+1}{c}$ (then, $jc > m - 1 + 2$), we have

$$\begin{aligned}\int e^{-cM(\gamma|\xi|)} d\xi \leq& C\int_0^{\gamma^{-1}} (\gamma t)^{-kc+m-1} M_k^c \gamma^{-m+1} dt + C\int_{\gamma^{-1}}^\infty (\gamma t)^{-jc+m-1} M_j^c \gamma^{-m+1} dt \\ \leq& C\int_0^{\gamma^{-1}} M_k^c \gamma^{-m+1} dt + C\int_{\gamma^{-1}}^\infty \frac{1}{(\gamma t)^2} M_j^c \gamma^{-m+1} dt < +\infty.\end{aligned}$$

(H) Since $M(t) \geq 0$, for all $t \geq 0$, it is sufficient to consider $0 < c < 1$. Let $k \in \mathbb{Z}_+$ such that $\frac{1}{k} < c$. Thus, with $A \geq 1$ in (2.5), it follows that

$$\begin{aligned}cM(t) >& \frac{1}{k}\sup_j \log\frac{t^j}{M_j} = \sup_j \log\left(\frac{t^j}{M_j}\right)^{1/k} \geq \sup_j \log\left(\frac{t^{kj}}{M_{kj}}\right)^{1/k} \\ \overset{(2.5)}{\geq}& \sup_j \log\frac{t^j}{(AH^{kj}M_{(k-1)j}M_j)^{1/k}} \\ \overset{(2.5)}{\geq}& \cdots \overset{(2.5)}{\geq} \sup_j \log\frac{t^j}{[AH^{kj}\cdot AH^{(k-1)j}\cdots AH^{2j}(M_j)^k]^{1/k}} \\ \geq& \sup \log\frac{(c't)^j}{M_j} = M(c't),\end{aligned} \tag{A.12}$$

where $c' = c_2\left\{A^{(k-1)/k}H^{(k+(k-1)+(k-2)+\cdots+2)/k}\right\}^{-1}$.

(I) Since the sequence M satisfies (2.5) and (2.8) we can apply (A.8), (A.9) and (A.10). Thus, there exist $c'>0$ such that

$$\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi \overset{(A.8),(A.10)}{\leq} \int_{\mathbb{R}^m} \sqrt{A}\frac{H^r}{(c'\gamma)^r} M_r e^{\frac{1}{2}M(c'\gamma|\xi|)} e^{-M(c'\gamma|\xi|)} d\xi$$
$$\leq \frac{\sqrt{A}H^r M_r}{(c'\gamma)^r}\int_{\mathbb{R}^m} e^{-\frac{1}{2}M(c'\gamma|\xi|)} d\xi \overset{(A.9)}{<} +\infty$$

as we wished to prove. ■

Appendix B. Useful Lemmas

Theorem B.1. *Let Ω be an open neighborhood of $\mathbb{R}^m$, $\phi \in \mathcal{D}^M(\mathbb{R}^m)$ such that $\phi \geq 0$ and $\int \phi(x)dx = 1$ and $u \in \mathcal{E}^{M\prime}(\Omega)$. If $\phi_\epsilon(x) = \epsilon^{-m}\phi(\frac{x}{\epsilon})$ then $u * \phi_\epsilon \to u$ in $\mathcal{E}^{M\prime}(\Omega)$, when $\epsilon \to 0$.*

Proof: Let $\psi \in \mathcal{E}^M(\Omega)$ be arbitrary. If V is a convex, bounded and open bounded subset of Ω with $\operatorname{supp} u \subset V$, it follows, from Theorem A.1, that there exist constants a_α satisfying (A.1) and $g \in C^0(\overline{V})$ such that

$$\langle u * \phi_\epsilon, \psi\rangle = \int \langle u_y, \phi_\epsilon(x-y)\rangle\, \psi(x)\, dx = \int \sum_\alpha (-1)^{|\alpha|} a_\alpha \int_V \partial_y^\alpha \left[\phi_\epsilon(x-y)\right]\, g(y) dy\, \psi(x)\, dx. \tag{B.1}$$

From the definition of the function ϕ_ϵ we obtain that

$$\int \sum_\alpha |a_\alpha| \int_V |\partial_y^\alpha \left[\phi_\epsilon(x-y)\right] \cdot \psi(x) \cdot g(y)| dy\, dx$$
$$= \int \sum_\alpha |a_\alpha| \int_V \left|(\partial^\alpha \phi)\left(\frac{x-y}{\epsilon}\right) \cdot \frac{\psi(x)}{\epsilon^{|\alpha|+m}}\right| |g(y)| dy\, dx. \tag{B.2}$$

In addition, there exists $r>0$ such that $V \cup \operatorname{supp}\phi \subset B(0,r)$. Thus, if $y \in V$ and $\frac{x-y}{\epsilon} \in \operatorname{supp}\phi$ it follows that

$$|x| \leq \epsilon\left|\frac{x-y}{\epsilon}\right| + |y| \leq r(\epsilon+1).$$

Since $\phi \in \mathcal{D}^M(\mathbb{R}^m)$, there exists a positive constant C so that

$$\int \sum_\alpha |a_\alpha| \int_V |\partial_y^\alpha[\phi_\epsilon(x-y)] \cdot \psi(x)|\, |g(y)| dy\, dx$$
$$= \int_{B(0,r(\epsilon+1))} \sum_\alpha |a_\alpha| \int_V \left|\frac{1}{\epsilon^{|\alpha|+m}}(\partial^\alpha\phi)\left(\frac{x-y}{\epsilon}\right)\psi(x)\right| |g(y)| dy\, dx$$
$$\leq \epsilon^{-m}\sum_\alpha \frac{C_L L^{|\alpha|}}{M_{|\alpha|}} \frac{C^{|\alpha|+1}M_{|\alpha|}}{\epsilon^{|\alpha|}} = \epsilon^{-m} C_L C \sum_\alpha \left(\frac{LC}{\epsilon}\right)^{|\alpha|} < +\infty$$

for $L < \frac{\epsilon}{C}$. Therefore, we can apply Fubini's theorem and differentiate under the integral sign,

$$
\begin{aligned}
\langle u * \phi_\epsilon, \psi \rangle &= \sum_\alpha a_\alpha \int_V \int_{B(0,r(\epsilon+1))} \frac{(-1)^{|\alpha|}}{\epsilon^{|\alpha|+m}} (\partial_y^\alpha \phi) \left(\frac{x-y}{\epsilon} \right) \psi(x)\, dx\, g(y) dy \\
&= \left\langle u_y; \int \phi_\epsilon(x-y)\psi(x)\, dx \right\rangle \\
&= \left\langle u_y; \int (\check{\phi})_\epsilon(y-x)\psi(x)\, dx \right\rangle
\end{aligned}
$$

where $\check{\phi}(x) = \phi(-x)$ (observe that $\phi_\epsilon(x-y) = \epsilon^{-m}\check{\phi}[(y-x)/\epsilon]$). To complete the proof it is sufficient to check that if $\psi \in \mathcal{E}^M(\Omega)$ then $\phi_\epsilon * \psi \to \psi$ in $\mathcal{E}^M(\Omega)$, when $\epsilon \to 0$ or, equivalently, that $(\phi_\epsilon * \psi - \psi) \to 0$ in $\mathcal{E}^M(\Omega)$, when $\epsilon \to 0$. Indeed, since $\int \phi(x)dx = 1$ then

$$\int_{\mathbb{R}^m} \phi_\epsilon(x)dx = \int_{\mathbb{R}^m} \phi(x)dx = 1; \tag{B.3}$$

thus, from the dominated convergence theorem;

$$\int_{|x|\leq\lambda} \phi_\epsilon(x)dx = \int_{|x|\leq\lambda/\epsilon} \phi(x)dx \to 1, \;\; \epsilon \to 0, \qquad \forall\lambda > 0.$$

From (B.3) and basic properties of convolution, for each $K \subset\subset \Omega$ we have

$$
\begin{aligned}
\sup_{x\in K} |\partial^\alpha\{\phi_\epsilon * \psi(x) - \psi(x)\}| &= \sup_{x\in K} \left| \int \phi_\epsilon(y) \partial_x^\alpha(\psi(x-y) - \psi(x))\, dy \right| \\
&\leq \sup_{x\in K} \int |\phi_\epsilon(y)||\partial_x^\alpha\psi(x-y) - \partial_x^\alpha\psi(x)|\, dy, \qquad \text{(B.4)}
\end{aligned}
$$

moreover, from the dominated convergence theorem, the right-hand side of (B.4) converges uniformly to zero, when $\epsilon \to 0$ (for each $\alpha \in \mathbb{N}^m$). Since $\psi \in \mathcal{E}^M(\Omega)$ for $K_1 = \{x - y : x \in K,\, y \in \operatorname{supp}\phi\} \subset\subset \mathbb{R}^m$ there exist $C, h > 0$ such that $\sup_{x\in K_1} |\partial^\alpha\psi(x)| \leq Ch^{|\alpha|}M_{|\alpha|}$. Consequently, from (B.4) there exists $C' > 0$, independent of $\epsilon \leq 1$, such that

$$\sup_{x\in K} |\partial^\alpha\{\phi_\epsilon * \psi(x) - \psi(x)\}| \leq C'h^{|\alpha|}M_{|\alpha|}.$$

Given $\delta > 0$ there exist $\alpha_0 > 0$ so that $C'2^{-|\alpha|} < \delta$, for each $|\alpha| > \alpha_0$. Thus,

$$\sup_{x\in K} \frac{|\partial^\alpha(\phi_\epsilon * \psi(x) - \psi(x))|}{(2h)^{|\alpha|}M_{|\alpha|}} < \delta, \quad \text{for all} \;\; |\alpha| > \alpha_0.$$

Otherwise, using (B.4) for $|\alpha| \leq \alpha_0$ we see that

$$\sup_{x\in K;} \frac{|\partial^\alpha(\phi_\epsilon * \psi(x) - \psi(x))|}{(2h)^{|\alpha|}M_{|\alpha|}} \to 0,$$

when $\epsilon \to 0$ independently of α (since $|\alpha| < \alpha_0$). Therefore,

$$\sup_{\alpha\in\mathbb{Z}_+^m} \sup_{x\in K} \frac{|\partial^\alpha(\phi_\epsilon * \psi(x) - \psi(x))|}{(2h)^{|\alpha|}M_{|\alpha|}} \to 0.$$

Showing that $\phi_\epsilon * \psi(x) \to \psi(x)$ in $\mathcal{E}^M$ as we wished to prove. ■

Next we recall the Faa di Bruno generalized formula.

Theorem B.2. [25, Corollary 2.10]. *Let $\gamma \in \mathbb{N}^m$ and $h(x_1, \dots, x_d) = f(g(x_1, \dots, x_d))$ with $g \in C^\gamma(U_{x_0})$ and $f \in C^{|\gamma|}(V_{y_0})$, where $y_0 = g(x_0)$, and $U_{x_0} \subset \mathbb{R}^d$ and $V_{y_0} \subset \mathbb{R}$ open neighborhoods of x_0 and y_0 respectively. Then,*

$$\partial^\gamma h = \sum_{r=1}^{|\gamma|} \partial^r f \sum_{\mathfrak{p}(\gamma,r)} (\gamma!) \prod_{j=1}^{|\gamma|} \frac{(\partial^{\alpha_j} g)^{k_j}}{k_j!(\alpha!)^{k_j}}, \tag{B.5}$$

where

$$\mathfrak{p}(\gamma, r) = \Big\{(k_1, \dots, k_{|\gamma|}; \alpha_1, \dots, \alpha_{|\gamma|}) \text{ for some } 1 \le s \le |\gamma|,\ k_i = 0, \text{ and } \alpha_i = 0 \text{ for } 1 \le i \le |\gamma| - s;\ k_i > 0 \text{ for } n - s + 1 \le i \le m; \text{ and } 0 \prec \alpha_{n-s+1} \prec \dots \prec \alpha_n \text{ are such that}$$

$$\sum_{i=1}^{|\gamma|} k_i = r, \qquad \sum_{i=1}^{|\gamma|} k_i \alpha_i = \gamma \Big\}. \tag{B.6}$$

Recall that from [25, p. 515] there exists $C > 0$ such that

$$r! \sum_{\mathfrak{p}(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \le C^{|\alpha|}. \tag{B.7}$$

Lemma B.1. *Let $\Omega \subset \mathbb{R}^m \times \mathbb{R}^m$ be an open set, $\xi \in \mathbb{R}^m \setminus \{0\}$ and $M = (M_j)$ a sequence satisfying (2.3), (2.4), (2.5) and (2.7). If $\psi \in \mathcal{E}^M(\mathbb{R}^m)$, $g(x, y)$, $f(x, y) \in \mathcal{E}^M(\Omega)$ and*

$$Q(t, x, y, \xi) \doteq -i\xi \cdot g(x, y) - |\xi|^\lambda \psi(t - f(x, y))$$

then, for each compact subset $K \subset \mathbb{R}^m \times \Omega$ there exists $C > 0$ (independent of t, x, y, ξ, θ and α) such that for all $(t, x, y) \in K$, $\alpha \in \mathbb{N}^m$, $\theta > 0$ and $|\xi| > 1$, we have

$$\left|\partial_x^\alpha \left\{e^{Q(t,x,y,\xi)}\right\}\right| \ \le\ e^{\mathcal{R}\{Q(t,x,y,\xi)\}} e^{\frac{1}{2}M(\theta|\xi|)} e^{(H/\theta)} C^{|\alpha|+1} M_{|\alpha|}. \tag{B.8}$$

Proof: Using Theorem B.2, we have

$$\partial_x^\alpha \left\{e^{Q(t,x,y,\xi)}\right\} \ =\ \sum_{r=1}^{|\alpha|} e^{Q(t,x,y,\xi)} \sum_{\mathfrak{p}(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{[\partial_x^{\alpha_j} Q(t, x, y, \xi)]^{k_j}}{k_j!(\alpha_j!)^{k_j}}.$$

Since ψ and f are in $\mathcal{E}^M(\mathbb{R}^m)$ and the composition of functions in $\mathcal{E}^M(\mathbb{R}^m)$ is in $\mathcal{E}^M(\mathbb{R}^m)$ (for (2.5) and (2.7)) there exist $C > 1$ such that,

$$\begin{aligned}\left|\partial_x^\alpha \{e^{Q(t,x,y,\xi)}\}\right| &\le \sum_{r=1}^{|\alpha|} e^{\mathcal{R}\{Q(t,x,y,\xi)\}} \sum_{\mathfrak{p}(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left[|\xi|\,|\partial_x^{\alpha_j} g(x, y)| + |\xi|^\lambda \left|\partial_x^{\alpha_j}\{\psi(t - f(x, y))\}\right|\right]^{k_j}}{k_j!(\alpha_j!)^{k_j}} \\ &\le \sum_{r=1}^{|\alpha|} e^{\mathcal{R}\{Q(t,x,y,\xi)\}} \sum_{\mathfrak{p}(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{|\xi|^{k_j} (C^{\alpha_j+1} M_{\alpha_j})^{k_j}}{k_j!(\alpha_j!)^{k_j}}\end{aligned}$$

where $|\xi| \ge 1$ and $\lambda < 1$, $\sum_{j=1}^{|\alpha|} k_j = r$ (see Theorem B.2). Since,

$$\frac{M_1}{1!} = 1, \quad \frac{M_{j+k}}{(j+k)!} \le AH^{j+k} \frac{M_j}{j!} \frac{M_k}{k!} \quad \text{and} \quad \frac{M_j}{j!} \frac{M_k}{k!} \le \frac{M_{j+k}}{(j+k)!}, \quad \forall j, k \in \mathbb{N};$$

from [2, p. 2275] we have

$$\prod_{j=1}^{|\alpha|} \frac{M_{|\alpha_j|}^{k_j}}{(\alpha_j!)^{k_j}} \leq \prod_{j=1}^{|\alpha|} \frac{M_{|\alpha_j|}^{k_j} n^{|\alpha_j|k_j}}{|\alpha_j!|^{k_j}} \leq n^{|\alpha|} H^{|\alpha|} \frac{M_{|\alpha|-r}}{(|\alpha|-r)!}. \tag{B.9}$$

From the definition of $\mathfrak{p}$ (see Theorem B.2), we obtain

$$\begin{aligned} \left|\partial_x^\alpha e^{Q(t,x,y,\xi)}\right| &\leq \sum_{r=1}^{|\alpha|} e^{\mathcal{R}\{Q(t,x,y,\xi)\}} \sum_{\mathfrak{p}(\alpha,r)} |\alpha|! \frac{|\xi|^r C^{2|\alpha|} H^\alpha M_{|\alpha|-r} n^{|\alpha|}}{(|\alpha|-r)!} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\ &\overset{(A.8)}{\leq} e^{\mathcal{R}\{Q(t,x,y,\xi)\}} e^{\frac{1}{2}M(\theta|\xi|)} e^{(H/\theta)} C^{|\alpha|+1} M_{|\alpha|}, \end{aligned}$$

for all $\theta > 0$ and a positive constant C. ■

Lemma B.2. *If $u \in \mathcal{D}^M(B(0,R))$ then there exist positive constants C, c' such that*

$$|\widehat{u}(\zeta)| \leq Ce^{-M(c'|\zeta|)} e^{R|\Im\zeta|}, \ \ \forall \zeta \in \mathbb{C}^m. \tag{B.10}$$

Proof: If $\zeta \in \mathbb{C}^m$, for each $j \in \mathbb{N}$ we have

$$|\zeta|^j |\widehat{u}(\zeta)| \leq (\sqrt{m})^j \max_k |\zeta_k|^j |\widehat{u}(\zeta)| \leq (\sqrt{m})^j \sum_{|\alpha|=j} |\zeta^\alpha \widehat{u}(\zeta)|.$$

Since $|\zeta^\alpha \widehat{u}(\zeta)| = |\widehat{D^\alpha u}(\zeta)|$ and $u \in \mathcal{D}^M(B(0,R))$ there exists a positive constant C such that,

$$|\widehat{u}(\zeta)| \ \ \leq \ \ \frac{(\sqrt{m})^j}{|\zeta|^j} \sum_{|\alpha|=j} |\widehat{D^\alpha u}(\zeta)| \leq \frac{(\sqrt{m})^j}{|\zeta|^j} \sum_{|\alpha|=j} \int_{B(0,R)} e^{\Im\zeta\cdot x} C^{|\alpha|+1} M_{|\alpha|} dx.$$

Therefore,

$$|\widehat{u}(\zeta)| \leq C \inf_j \frac{C^j M_j}{|\zeta|^j} e^{R|\Im\zeta|} \leq Ce^{-M(c'|\zeta|)} e^{R|\Im\{\zeta\}|} \tag{B.11}$$

proving the lemma. ■

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