TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

GLOBAL SOLVABILITY OF REAL ANALYTIC INVOLUTIVE SYSTEMS ON COMPACT MANIFOLDS. PART 2

JORGE HOUNIE AND GIULIANO ZUGLIANI

ABSTRACT. This work continues the study initiated in [15] of the global solvability of a locally integrable structure of tube type and co-rank one, considering a linear partial differential operator $\mathbb{L}$ associated with a real analytic closed 1-form defined on a real analytic closed n-manifold. We deal now with a general complex form and complete the characterization of the global solvability of $\mathbb{L}$. In particular, we state a general theorem, encompassing the main result of [15].

As in [15], we are also able to characterize the global hypoellipticity of $\mathbb{L}$ and the global solvability of $\mathbb{L}^{n-1}$ —the last non-trivial operator of the complex when M is orientable— which were previously considered in [1] and [2], respectively, under an additional regularity assumption on the set of the characteristic points of $\mathbb{L}$.

CONTENTS

2010 *Mathematics Subject Classification.* 35A01, 35N10, 58J10.

Key words and phrases. Global solvability, complex vector fields, involutive systems, Liouville numbers.

The first author was partially supported by CNPq [grant 303634/2014-6] and FAPESP [grant 2012/03168-7]; the second author was supported by FAPESP [grant 2014/23748-3].

1. Introduction

Suppose we are given a real analytic closed (i.e., compact and without boundary) connected n-dimensional manifold M $(n > 1)$, equipped with a Riemannian metric, where a real analytic closed 1-form c is defined. In what follows, the real and imaginary parts of c will be respectively denoted by a and b, and we will write $c = a + ib$.

Consider the vector fields

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial C}{\partial t_j}(t)\frac{\partial}{\partial x}, \quad j = 1, \dots, n,$$

where $(t_1, \dots, t_n)$ are local coordinates on M, x belongs to the unit circle $\mathbb{S}^1$, and C is a local primitive of c. They are local generators of the bundle $\mathcal{V} \doteq (T')^\perp \subset \mathbb{C} \otimes T(M \times \mathbb{S}^1)$ where T' is the line sub-bundle of $\mathbb{C} \otimes T^*(M \times \mathbb{S}^1)$ generated by the 1-form $dx - c$ (we refer to [9] and [20] for details).

Denote by $\Lambda^{p,0}$, $p = 0, \dots, n$, the sub-bundle of $\Lambda^p(\mathbb{C} \otimes T^*(M \times \mathbb{S}^1))$ locally generated by $dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_p}$, if $J = \{j_1, \dots j_p\}$ and $1 \leqslant j_1 < j_2 < \cdots < j_p \leqslant n$.

The focus of this work is the associated differential operator $\mathbb{L} : C^\infty(M \times \mathbb{S}^1) \to C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$ defined by

$$\mathbb{L}u = d_t u + c(t) \wedge \partial_x u, \tag{1.1}$$

where $d_t : C^\infty(M \times \mathbb{S}^1, \Lambda^{p,0}) \to C^\infty(M \times \mathbb{S}^1, \Lambda^{p+1,0})$ is the exterior derivative on M.

Any involutive structure defines in a natural way a complex of differential operators which in the case of $\mathcal{V}$ is given by (1.1) when acting on functions. Thus, we have a complex

$$\begin{aligned} &C^\infty(M \times \mathbb{S}^1) \xrightarrow{\mathbb{L}} C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0}) \xrightarrow{\mathbb{L}^1} \\ &\xrightarrow{\mathbb{L}^1} C^\infty(M \times \mathbb{S}^1, \Lambda^{2,0}) \xrightarrow{\mathbb{L}^2} \cdots \xrightarrow{\mathbb{L}^{n-1}} C^\infty(M \times \mathbb{S}^1, \Lambda^{n,0}) \xrightarrow{\mathbb{L}^n} 0 \end{aligned} \tag{1.2}$$

analogous to the de Rham complex.

Here we will study the smooth global solvability of the equation $\mathbb{L}u = f$, i.e., the possibility of finding a globally defined solution $u \in C^\infty(M \times \mathbb{S}^1)$ when f is smooth. Of course, if f is in the range of $\mathbb{L}$ it must satifsy two obvious conditions analogous to the fact that an exact form is both closed and orthogonal to the closed cocyles: f must be orthogonal to the kernel of the dual operator $\mathbb{L}^*$, and $\mathbb{L}^1 f = 0$ (a consequence of the involutivity $\mathbb{L}^1\mathbb{L} = 0$). They are usually referred to as compatibility conditions for f and may be formulated in different equivalent ways that are chosen to best suit the operator under consideration.

Denote by Σ_0 the set of the critical points of b, that is,

$$\Sigma_0 \doteq \{t \in M : b(t) = 0\}.$$

A global solvability result was obtained quite recently in [15] when $a \equiv 0$ and b is a real analytic closed non-exact 1-form. In fact, one of the conditions presented there that characterizes the global solvability is the so called property $(\star)$ (originally introduced in [1] to study the global hypoellipticity of $\mathbb{L}$), namely:

Every connected component Σ of Σ_0 contains a point p^* such that the local primitives of b are open at p^*. $(\star)$

We observe that when dealing with non-purely imaginary complex forms $c = a + ib$ a small divisors phenomenon appears so the global solvability as well as the global hypoellipticity are related to diophantine conditions, as in the works [12, 14] for single vector fields. More precisely, when b is not exact, an approximation property involving the real part a plays a decisive role for the components of Σ_0 where property $(\star)$ fails. As for results concerning the global solvability of systems defined by a smooth closed form, the case in which $M = \mathbb{T}^n$ has been extensively studied and we refer to [8, 5, 6, 7, 4, 3, 11].

We now state the main result of this work. Denote by $\mathscr{A}$ the set of the connected components of Σ_0 such that, if $\Sigma \in \mathscr{A}$, no point $p \in \Sigma$ has a local primitive of b open at p. Denote by $H_1(M, \mathbb{Z})$ the first homology group of M with coefficients in $\mathbb{Z}$. We will associate to Σ a vector $I(\Sigma) \in \mathbb{R}^m$, with $m = \operatorname{rank} i_*(H_1(\Sigma, \mathbb{Z}))$, where $i_* : H_1(\Sigma, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is the natural homomorphism induced by the inclusion $i : \Sigma \hookrightarrow M$.

Theorem 1.1. *Assume that the 1-form $c = a + ib$ is real analytic and closed. The following statements are equivalent:*

- (I) $\mathbb{L}$ *is globally solvable.*
- (II) *One of the two conditions below is satisfied:*
  - (II.1) $\mathscr{A} = \emptyset$, *or, for $\Sigma \in \mathscr{A}$, $I(\Sigma)$ is neither a rational nor a Liouville vector;*
  - (II.2) *the form b has a primitive $B^\sharp$ defined on M, and the semi-level sets $\{t \in M : B^\sharp(t) > r\}$ and $\{t \in M : B^\sharp(t) < r\}$ are connected for every $r \in \mathbb{R}$; in addition, a is rational, and, if $q \in \mathbb{Z}$ is such that $qI(\Sigma) \in (2\pi\mathbb{Z})^m$ for $\Sigma \in \mathscr{A}$, then qa is integral.*

The definition of integral, rational and Liouville vectors (and forms) is presented in Section 2. The compatibility conditions and the precise notion of global solvability will be given in Section 3. In the proof of the theorem we will take advantage of some special properties of real analytic functions proved in [13, 17, 18] and also make decisive use of Lojasiewicz's inequality, which states that if Φ is a real analytic function on a neighborhood of the origin and $\Phi(0) = 0$, then there exist $C_0 > 0$ and $\theta \in (0, 1)$ such that

$$\|\nabla\Phi(s)\| \geqslant C_0|\Phi(s)|^\theta$$

for every s sufficiently close to 0.

Theorem 1.1 gives another proof of the real analytic result obtained in [15] where $a \equiv 0$ and b is not exact. Indeed, property $(\star)$ is trivially equivalent to $\mathscr{A} = \emptyset$. Condition (II.1) may be roughly described as follows: it is mainly concerned with a non-exact b and in this case $\mathscr{A} = \emptyset$ implies that the semilevel sets of a primitive $\widetilde{B}$ of the pullback of b to the minimal covering of M are connected (see [15]). Thus, in this subcase, the connectedness of the semilevel sets of $\widetilde{B}$ is enough to grant alone the global solvability of $\mathbb{L}$, and the real form a plays no role. This situation occurs, for instance, if $\mathbb{L}$ is elliptic (so $\Sigma_0 = \emptyset$ which implies $\mathscr{A} = \emptyset$); this gives some ground to call $\mathbb{L}$ *degenerate elliptic* when $\mathscr{A} = \emptyset$. When $\mathscr{A} \neq \emptyset$, the components $\Sigma \in \mathscr{A}$ become important and the vector $I(\Sigma)$ must be examined to rule out the presence of small divisors.

In the case when c is smooth and exact, the authors in [10] studied the global solvability of the complex for a smooth closed orientable manifold M (in the context of linear self-adjoint operators in a Hilbert space). We notice that in this case the real form a plays no role either.

The global hypoellipticity of (1.1) was studied and characterized by Bergamasco, Cordaro and Malagutti in [1] under the additional assumption that Σ_0 consists of embedded analytic submanifolds of M, and their main result is that (II.1) is equivalent to the global hypoellipticity of (1.1). Concerning the top level of the complex, the global solvability was characterized in [2] when M is orientable, again under the assumption that Σ_0 consists of embedded analytic submanifolds of M.

In Section 7, we will see that a consequence of the proof of Theorem 1.1 is that the regularity assumption on Σ_0 can be removed from both works without changing the conclusion (although it should be noticed that we make use of the existence shown in [1] of a primitive $B^\dagger$ of b defined on a neighborhood of Σ_0). Summing up, this work generalizes and extends the main global regularity and global solvability results respectively proved in [1] and [2], as well as the analytic results in [15].

We also discuss in Section 7 the solvability of a particular structure in the smooth category —namely when $\mathcal{V}$ is a Mizohata structure, which is equivalent to considering a Morse 1-form b— and present an example.

Remark 1.2. Although the analyticity of $c = a + ib$ is assumed throughout this work, it should be noticed that it is only the analyticity of b that matters and is used in the proofs. Thus, as in [1] and [2], the real part a can be assumed to be just smooth rather than real analytic, and when $b \equiv 0$, we may as well assume that M is a smooth closed manifold.

2. Liouville vectors and preliminaries

Consider the universal covering $\Pi : \mathscr{U} \to M$ with a real analytic structure and fix A and B respectively the primitives of the pullbacks Π^*a and Π^*b obtained by integration from $t_0 \in \mathscr{U}$. We also define a continuous functional on the space $\Gamma(M)$ consisting of closed curves $\gamma \in C([0,1], M)$, which we denote by T_a. Then $\Gamma(M)$ is a metric space with the topology of the uniform convergence.

Given $\gamma \in \Gamma(M)$, let $\widetilde{\gamma}$ be a lifting of γ to $\mathscr{U}$. Define

$$T_a(\gamma) = A(\widetilde{\gamma}(1)) - A(\widetilde{\gamma}(0)).$$

Notice that

- T_a does depend neither on the covering space, nor on the liftings, and is a continuous map.
- $T_a(\gamma)$ depends only on the homotopy class of γ, that is, if $\gamma_0, \gamma_1 \in \Gamma(M)$ and there exists a continuous function $F(t,s) : M \times [0,1] \longrightarrow M$ such that $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$, then $T_a(\gamma_0) = T_a(\gamma_1)$.
- If $\widetilde{\gamma}$ is a piecewise C^1 path, we have

$$\int_\gamma a = \int_{\widetilde{\gamma}} d_t A = T_a(\gamma).$$

- By the Hurewicz Theorem, T_a induces a homomorphism on the first homology group $H_1(M, \mathbb{Z})$ (the class of γ in $H_1(M, \mathbb{Z})$ will be denoted by $[\gamma]$).

Suppose that Σ is a component of Σ_0 and that $\Sigma \in \mathscr{A}$. Consider the natural homomorphism $i_* : H_1(\Sigma, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ induced by the inclusion $i : \Sigma \hookrightarrow M$. As $i_*(H_1(\Sigma, \mathbb{Z}))$ is a subgroup of $H_1(M, \mathbb{Z})$, it is finitely generated, and we fix a linearly independent set $\{[\nu_1], \ldots, [\nu_m]\}$ generating its free part.

We will denote by $I(\Sigma)$ the vector $(2\pi)^{-1}(T_a([\nu_1]), \ldots, T_a([\nu_m]))$.

Definition 2.1. *We say that* $I(\Sigma)$ *is*

- integral *if* $I(\Sigma) \in \mathbb{Z}^m$*;*

- rational *if* $I(\Sigma) \in \mathbb{Q}^m$*;*
- Liouville *if it is not rational and there exist* $P_j \in \mathbb{Z}^m$, $Q_j \in \mathbb{Z}^+$, *with* $Q_j > 1$, *and* $C > 0$ *satisfying*

$$\left| I(\Sigma) - \frac{P_j}{Q_j} \right| < \frac{C}{Q_j^j},$$

for every $j \in \mathbb{Z}^+$.

It is plain that this definition does not depend on the choice of the generators. In [1], Σ is assumed to be an embedded analytic submanifold of M. The vector considered there is $(2\pi)^{-1}(\int_{\rho_1} a, \dots, \int_{\rho_r} a)$, where $\{\rho_1, \dots, \rho_r\}$ is a set of linearly independent generators of the free part of $H_1(\Sigma, \mathbb{Z})$, represented by smooth closed curves in Σ. If we choose smooth curves ν_l as representatives of the homology classes in $H_1(\Sigma, \mathbb{Z})$, we see that both definitions are equivalent when Σ is a submanifold. Also, since the definitions make sense if Σ is replaced by the whole manifold M, it is meaningful to say that the 1-form a itself is integral, rational or Liouville if it satisfies the respective conditions described above.

Lemma 2.2. *There is a neighborhood* V *of* Σ *such that every closed curve* $\gamma \in C([0,1], V)$ *satisfies* $T_a(\gamma) = T_a(\gamma^\sharp)$*, for some piecewise real analytic curve* $\gamma^\sharp \in \Gamma(\Sigma)$.

Proof. For every $x \in \Sigma$, let $\mathscr{B}(x, \varepsilon_x)$ be a neighborhood of x with radius ε_x sufficiently small and with the property that every pair of points in $\mathscr{B}(x, \varepsilon_x) \cap \Sigma$ can be connected by a piecewise real analytic path in $\mathscr{B}(x, \varepsilon_x) \cap \Sigma$ (see, for instance, Proposition 2.7 below).

Since Σ is compact, we have that $\Sigma \subset \cup_{j=1}^s \mathscr{B}(x_j, \varepsilon_j/4)$. Define $\varepsilon \doteq \min \varepsilon_j/4$. Suppose that $\operatorname{dist}(\gamma, \Sigma) < \varepsilon$. Consider a partition $\{0 = t_0 < t_1 < \cdots < t_r = 1\}$ of $[0,1]$ with $t_{k+1} - t_k < \delta$, where δ is such that if $t, t' \in [0,1]$ and $|t - t'| < \delta$, then $\|\gamma(t) - \gamma(t')\| < \varepsilon$. Set $p_k \doteq \gamma(t_k)$. Take $p'_k \in \Sigma$ such that $\operatorname{dist}(p_k, p'_k) < \varepsilon$ and denote by x_{j_k} the center of some neighborhood $\mathscr{B}(x_{j_k}, \varepsilon_{j_k}/4) \in \{\mathscr{B}(x_1, \varepsilon_1/4), \dots, \mathscr{B}(x_s, \varepsilon_s/4)\}$ containing p'_k.

We have:

$$\|p'_{k+1} - x_{j_k}\| \leqslant \|p'_{k+1} - p_{k+1}\| + \|p_{k+1} - p_k\| + \|p_k - p'_k\| + \|p'_k - x_{j_k}\| < \varepsilon_{j_k}.$$

This means that we can connect p'_k with p'_{k+1} by means of a piecewise real analytic path $\gamma^\sharp[t_k, t_{k+1}]$ in $\mathscr{B}(x_{j_k}, \varepsilon_{j_k}) \cap \Sigma$ and obtain in this way a curve $\gamma^\sharp \in \Gamma(M)$.

Notice that, for every $1 \le k \le r-1$, $\mathscr{B}(x_{j_k}, \varepsilon_{j_k})$ contains the curve $\beta_k \in \Gamma(M)$ obtained by means of $\gamma[t_k, t_{k+1}]$, $\gamma^\sharp[t_k, t_{k+1}]$, and two paths connecting, respectively p'_k to p_k, and p'_{k+1} to p_{k+1}. Thus, $T_a(\beta_k) = 0$, and then $T_a(\gamma) = T_a(\gamma^\sharp)$, as desired. □

Remark 2.3. Lemma 2.2 can also be proved by deforming γ into $\gamma^\sharp$ along the flow of $(B^\dagger)^2$, which is a gradient flow in a neighborhood of Σ where a semi-global primitive $B^\dagger$ of b is defined (that the gradient flow of the real analytic function $(B^\dagger)^2$ gives a continuous retraction of V onto Σ is due to Lojasiewicz ([16])).

We finish this preliminary section by presenting three key results involving real analytic functions that, as in [15], are essential in order to show the existance of certain convenient paths and to estimate their lengths.

Lemma 2.4. [17, Lemma 25] *Let O be an open set in $\mathbb{R}^m$ and $\Phi \in C^\infty(O)$ satisfying the Lojasiewicz's inequality:*

$$\|\nabla\Phi(s)\| \geqslant C_0|\Phi(s)|^\theta$$

for constants $C > 0$ and $\theta \in [0, 1)$, and every $s \in O$. For $s \in O$ with $\nabla\Phi(s) \neq 0$, the maximal solution $\gamma_s : [0, \delta(s)) \to O$ of

$$\begin{cases} y' = \dfrac{\nabla\Phi(y)}{\|\nabla\Phi(y)\|} \\ y(0) = s. \end{cases}$$

satisfies

$$\Phi(\gamma_s(\tau)) \geqslant \Phi(s) + C_0\tau^{\frac{1}{1-\theta}},$$

for $\tau \in [0, \delta(s))$.

Proposition 2.5. [18, Proposition 3] *Let h be a real analytic function defined on an open subset U of $\mathbb{R}^N$. Given a compact set $\mathscr{K} \subset U$, there exists $C_1 \doteq C_1(\mathscr{K}) > 0$ such that, for every $r \in h(\mathscr{K})$, any two points in a component of $h^{-1}(r) \cap \mathscr{K}$ can be joined by a piecewise real analytic path ς in $h^{-1}(r) \cap \mathscr{K}$ whose length is less than C_1.*

We denote by $\mathcal{B}$ the ball of radius $r > 0$ and centered at $0 \in \mathbb{R}^n$.

Definition 2.6. *A set $E \subset \mathcal{B}$ is said to be semi-analytic at $s \in E$ if there exist an open neighborhood O of s and a finite number of real analytic functions $\{g_{ij}, f_{ij}\}$ on O such that*

$$E \cap O = \cup_i\{s' \in O : g_{ij}(s') = 0, f_{ij}(s') > 0, \forall j\}.$$

Proposition 2.7. [13, p.462] *Let a^* be a non-isolated point belonging to the closure of a semi-analytic set $E \subset \mathcal{B}$. Then, for every $a \in E\backslash\{a^*\}$ sufficiently close to a^*, there exists a real analytic map $\lambda : (-1, 1) \to \mathcal{B}$ such that $\lambda(0) = a^*$ and $a \in \lambda(0, 1) \subset E$.*

Remark 2.8. The proposition above will allow us whenever necessary to join two points in a component Σ of Σ_0 by a piecewise real analytic path in Σ.

3. Compatibility conditions for the general case

Denote by D the group of deck transformations associated with the covering $\Pi : \mathscr{U} \to M$. This group is isomorphic to $\pi_1(M)$ (basepoints can be omitted).

If $C = A + iB$ (recall that A and B are respectively the primitives of the pulled back forms $\Pi^* a$ and $\Pi^* b$), we may write

$$C(\sigma(t)) = C(t) + c_\sigma, \quad t \in \mathscr{U}, \quad \sigma \in \mathsf{D}, \tag{3.1}$$

where c_σ is a constant.

The space of plausible right hand sides for the equation $\mathbb{L}u = f$ will be denoted by $\mathbb{E}$. Set $F \doteq \Pi^* f$ and denote by $\{\widehat{F}(t,\xi)\}_{\xi\in\mathbb{Z}}$ the Fourier coefficients of F with respect to $x \in \mathbb{S}^1$.

Definition 3.1 (Compatibility conditions)**.** *We say that $f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$ belongs to $\mathbb{E}$ if F satisfies the following conditions:*

- *for each $\xi \in \mathbb{Z}$ and each smooth curve γ connecting t to $\sigma(t)$ in $\mathscr{U}$ such that $i\xi c_\sigma \in 2\pi i\mathbb{Z}$,*
$$\int_\gamma e^{i\xi C(s)} \widehat{F}(s,\xi) = 0;$$
- *$d_t(e^{i\xi C(t)}\widehat{F}(t,\xi)) = 0$ for every $\xi \in \mathbb{Z}$.*

A candidate to a solution of $\mathbb{L}u = f$ should satisfy, for every $\xi \in \mathbb{Z}$, the differential equation

$$d_t\widehat{u}(t,\xi) + i\xi c(t)\widehat{u}(t,\xi) = \widehat{f}(t,\xi), \tag{3.2}$$

which can be rewritten in $\mathscr{U}$ as

$$d_t(e^{i\xi C(t)}\widehat{U}(t,\xi)) = e^{i\xi C(t)}\widehat{F}(t,\xi), \tag{3.3}$$

if $U = \Pi^* u$. Therefore, the conditions come from a computation concerning a necessary condition for a 1-form to be in the image of the operator (1.1).

Moreover, notice that the second condition guarantees that $\mathbb{L}^1 f = 0$, which is a natural condition from the involutivity of the system. We should remark that the first condition already implies that $e^{i\xi C(t)}\widehat{F}(t,\xi)$ is a closed 1-form when $i\xi c_\sigma \in 2\pi i\mathbb{Z}$, since indeed there will be a covering $\Pi_1 : M_1 \to M$ on which a primitive C_1 of c is defined and $e^{i\xi C_1(t)}\widehat{F}_1(t,\xi)$ is an exact 1-form, with $F_1 = \Pi_1^* f$.

Now, if we integrate (3.3) from $t_0 \in \mathscr{U}$ to $t \in \mathscr{U}$ for each $\xi \in \mathbb{Z}$, it yields

$$\widehat{U}(t,\xi) = \int_{t_0}^{t} \upsilon + K_\xi e^{-i\xi C(t)},$$

where $\upsilon(s,\xi) = e^{i\xi[C(s)-C(t)]}\widehat{F}(s,\xi)$ (in the integrals, $C(s)$ and $\widehat{F}(s,\xi)$ must be understood respectively, with some abuse of notation, $C(\gamma(s))$ and $\widehat{F}(\gamma(s),\xi)$ where $\gamma(s)$ is a path joining t_0 and t).

In order to find a solution on M we need that $\widehat{U}(\sigma(t),\xi) = \widehat{U}(t,\xi)$, $\sigma \in \mathsf{D}$, which uniquely determines K_ξ and the coefficients of the sought-after solution when $i\xi c_\sigma \notin 2\pi i\mathbb{Z}$, namely

$$\widehat{U}(t,\xi) = \frac{1}{e^{i\xi c_\sigma} - 1} \int_{t}^{\sigma(t)} \upsilon. \tag{3.4}$$

Sometimes we will rewrite (3.4) as

$$\widehat{U}(t,\xi) = \int_{t_0}^{t} \upsilon + e^{i\xi[C(t_0)-C(t)]}\widehat{u}(t_0,\xi). \tag{3.5}$$

Definition 3.2. *We say that the operator* (1.1) *is globally solvable if given any* 1*-form* $f \in \mathbb{E}$ *there exists* $u \in \mathscr{D}'(M \times \mathbb{S}^1)$ *such that* $\mathbb{L}u = f$*. If the solution* u *can be taken in* $C^\infty(M \times \mathbb{S}^1)$ *we say that* $\mathbb{L}$ *is globally solvable in* C^∞*. We say that the operator* (1.1) *is globally hypoelliptic if* $u \in C^\infty(M \times \mathbb{S}^1)$ *whenever* $\mathbb{L}u \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$.

4. Proof of (II.1) implies (I)

We start by recalling that $\mathbb{Z} \ni \xi \mapsto \widehat{F}(t,\xi)$ is rapidly decreasing, i.e., for every $N \in \mathbb{Z}^+$, there is a constant $C_N > 0$ such that

$$|\widehat{F}(t,\xi)| \leqslant \frac{C_N}{(1+|\xi|)^N}.$$

We wish to prove that the Fourier coefficients $\widehat{U}(t,\xi)$ of a presumed solution satisfy similar estimates. Once this is done, they will grant that there is a smooth function $u(x,t)$ on M that has these Fourier coefficients showing that $\mathbb{L}$ is globally solvable in C^∞.

The proof that (II.1) implies (I) is divided into two steps. In the first one, we will prove the following

Proposition 4.1. *Fix $t \in \mathscr{U}$ and $N \in \mathbb{Z}^+$. If $\widehat{U}(t,\xi)$ is defined by* (3.4)*, there exists a constant $C_N(t) > 0$ such that the estimate*

$$|\widehat{U}(t,\xi)| \leqslant \frac{C_N(t)}{(1+|\xi|)^N}, \quad \xi \in \mathbb{Z}, \tag{4.1}$$

holds.

Notice that, for each $t \in \mathscr{U}$ and $\xi \in \mathbb{Z}$, we are free to choose t_0 and the (finite) path used in (3.4) and (3.5), so the idea is to select them carefully. The choice will obey the following rules: (i) the term $(e^{i\xi c_\sigma} - 1)^{-1}$ will have polynomial growth with respect to $|\xi|$; (ii) the exponential term $e^{i\xi[C(s)-C(t)]}$ will remain bounded on each path by a constant independent of ξ.

In a second step, we will prove that given $t \in M$ we can actually choose $C_N(t')$ in (4.1) to be bounded on a neighborhood of t (this will imply, by compactness, that we may take C_N independent of t).

Also, to prove the estimates, it will be convenient to choose a specific $\sigma \in \mathsf{D}$ that might depend on $t \in \mathscr{U}$ and $\xi \in \mathbb{Z}\setminus\{0\}$. Because of this, we need to prove that the definition of the coefficients (3.4) is independent of $\sigma \in \mathsf{D}$.

Lemma 4.2. *Let $\phi, \sigma \in \mathsf{D}$. If $f \in \mathbb{E}$ and $i\xi c_\sigma, i\xi c_\phi \notin 2\pi i\mathbb{Z}$, then for each $t \in \mathscr{U}$,*

$$\frac{1}{e^{i\xi c_\sigma} - 1}\int_t^{\sigma(t)} \upsilon = \frac{1}{e^{i\xi c_\phi} - 1}\int_t^{\phi(t)} \upsilon.$$

Proof. Consider $\sigma, \phi \in \mathsf{D}$, with $i\xi c_\sigma, i\xi c_\phi \notin 2\pi i\mathbb{Z}$. We have

$$\int_t^{\sigma\phi(t)} \upsilon = \int_t^{\sigma(t)} \upsilon + \int_{\sigma(t)}^{\sigma\phi(t)} \upsilon,$$

where the integrals only depend on the endpoints of the corresponding paths since υ is exact because $f \in \mathbb{E}$. If we make the change of variables $s' = \sigma^{-1}(s)$ in the latter integral, we obtain

$$\int_{\sigma(t)}^{\sigma\phi(t)} \upsilon = e^{i\xi c_\sigma}\int_t^{\phi(t)} \upsilon.$$

Hence,

$$\int_t^{\sigma\phi(t)} \upsilon = \int_t^{\sigma(t)} \upsilon + e^{i\xi c_\sigma}\int_t^{\phi(t)} \upsilon.$$

Similarly,

$$\int_t^{\phi\sigma(t)} \upsilon = \int_t^{\phi(t)} \upsilon + e^{i\xi c_\phi} \int_t^{\sigma(t)} \upsilon.$$

Let $\rho = \sigma\phi\,\sigma^{-1}\phi^{-1}$. By iteration of (3.1) we conclude that $c_\rho = c_\sigma + c_\phi - c_\sigma - c_\phi = 0$, and since $f \in \mathbb{E}$, this implies that the integral of υ from $\sigma\phi(t)$ to $\phi\sigma(t)$ is zero, so the left hand sides of the last two equations coincide. Equating the right hand sides we get the desired result. □

In order to prove Proposition 4.1, we now will present some auxiliary results.

Lemma 4.3. *Suppose that $I(\Sigma)$ is neither rational nor Liouville. Then, for every $\xi \in \mathbb{Z} \setminus \{0\}$, there exist a curve $\alpha \in \Gamma(\Sigma)$ such that*

$$\left|e^{i\xi T_a(\alpha)} - 1\right| \geqslant \frac{C}{|\xi|^s}$$

for some $C > 0$ and $s \in \mathbb{Z}^+$.

Proof. We follow here the arguments in [1]. For each $\xi \neq 0$, we have that $I \doteq \{l : \xi T_a([\nu_l]) \notin 2\pi\mathbb{Z}\} \neq \emptyset$.

Suppose that there exists $\varepsilon_0 > 0$ such that, for each $\xi \neq 0$, there is $l \in I$ with $|\xi T_a([\nu_l]) - 2\pi P| \geqslant \varepsilon_0$, for every $P \in \mathbb{Z}$. In this case, the result will follow since $T_a([\nu_l]) = T_a(\alpha)$ for some closed curve $\alpha \in \Gamma(\Sigma)$.

Otherwise, as

$$\lim_{\theta\to\theta_0} \frac{|e^{i\theta} - e^{i\theta_0}|}{|\theta - \theta_0|} = 1,$$

for every $l \in I$, there will exist $P_l \in \mathbb{Z}$ such that

$$|e^{i\xi T_a([\nu_l])} - e^{2\pi i P_l}| \geqslant \frac{1}{2}|\xi T_a([\nu_l]) - 2\pi P_l|. \tag{4.2}$$

Since $I(\Sigma)$ is not Liouville, there is $s' \in \mathbb{Z}^+$, with $s' > 1$, such that, for every $(P_1, \ldots, P_m) \in \mathbb{Z}^m$ and $Q \in \mathbb{Z}^+$, with $Q > 1$,

$$\max_{1\leqslant l\leqslant m} |QT_a([\nu_l]) - 2\pi P_l| \geqslant \frac{1}{|Q|^{s'-1}}.$$

By (4.2) we will have

$$\max_{1\leqslant l\leqslant m} |e^{i\xi T_a([\nu_l])} - 1| \geqslant \frac{1}{2|\xi|^{s'-1}},$$

and the result follows. □

Lemma 4.4. *Assume that $\Pi(t) \in \Sigma$ for some $\Sigma \in \mathscr{A}$. Then estimates (4.1) hold true for t.*

Proof. Consider the curve α obtained in the previous lemma, and the lift $\widetilde{\alpha}$ of α starting at $t' \in \mathscr{F}$, where $\mathscr{F}$ is the component of $\Pi^{-1}(\Sigma)$ containing t. Thus $\widetilde{\alpha}$ connects t' to $\sigma(t')$, for some $\sigma \in \mathsf{D}$. By Lemma 2.2, we may assume that $\widetilde{\alpha}$ is a piecewise real analytic path.

Consider also a piecewise real analytic path η_0 connecting t to t' through critical points of B (see Remark 2.8) and then join t to $\sigma(t)$ by following η_0, $\tilde{\alpha}$, and $\sigma(\eta_0^{-1})$.

Plug the resulting path η in (3.4). As B is constant over η and $c_\sigma = T_a(\alpha)$, the desired decay follows by an application of Lemma 4.3. $\square$

The next lemma will be used to prove the desired estimates for the remaining points. For any $r \in \mathbb{R}$ set

$$\Omega^r \doteq \{s \in \mathscr{U} : B(s) > r\}, \qquad \Omega_r \doteq \{s \in \mathscr{U} : B(s) < r\}.$$

Lemma 4.5. *Call $\mathscr{O}$ a connected component of $\Omega_{B(t)}$ (respectively, $\Omega^{B(t)}$) such that $t \in \mathrm{cl}(\mathscr{O})$.*

- (i) *If $\sigma \in \mathsf{D}$, σ not the identity, is such that $\sigma(t) \in \mathscr{O}$, then* (4.1) *holds for t and $\xi < 0$ (respectively, $\xi > 0$).*
- (ii) *If* (4.1) *holds for $t_0 \in \mathrm{cl}(\mathscr{O})$, then* (4.1) *holds for t and $\xi < 0$ (respectively, $\xi > 0$).*

Proof. We find a curve $\gamma(s) \subset \mathrm{cl}(\mathscr{O})$ connecting t and $\sigma(t)$. Then the exponential term in $\upsilon(s,\xi) = e^{i\xi[C(s)-C(t)]}\widehat{F}(s,\xi)$ is bounded by 1 along $\gamma(s)$. Since the sequence $\{(e^{i\xi c_\sigma} - 1)^{-1}\}_{\xi \neq 0}$ is bounded when $b_\sigma \neq 0$, the estimates stated in (i) are easily checked by looking at the formula (3.4). On the other hand, for the proof of (ii) we take advantage of formula (3.5). $\square$

If s is a regular point of $\Pi^*(b)$, consider the solution of

$$\gamma' = \frac{\nabla B}{|\nabla B|}(\gamma), \quad \gamma(0) = s. \tag{4.3}$$

Denote by γ_s such solution, and by $[0,\delta)$ the maximal interval of γ_s. We then have:

Corollary 4.6. *If $\delta = \infty$, there exists $s_0 \in [0,\delta)$ such that* (4.1) *holds for $t_0 \doteq \gamma(s_0)$ and $\xi > 0$.*

Proof. First, we fix $\Pi(t)$ in the ω-limit set of $\Pi \circ \gamma_s$, and sufficiently small neighborhoods $\mathscr{B}_0 \subset \mathscr{B}_1$ of t in $\mathscr{U}$ with $\mathrm{dist}(\mathscr{B}_0, \partial\mathscr{B}_1) = d > 0$.

Suppose that there exists $\tau_0 \in [0,\delta)$ such that $\gamma_s(\tau)$ is in a ball inside $\mathscr{B}_1$ for every $\tau > \tau_0$. Then, a consequence of Lemma 2.4 applied to a local primitive B' of b defined on a neighborhood of the closure of $\Pi(\mathscr{B}_1)$ is that

$$C_0(\tau-\tau_0)^{\frac{1}{1-\theta}} \leqslant B'\circ\Pi(\gamma_s(\tau)) - B'\circ\Pi(\gamma_s(\tau_0)) \leqslant 2\|B'\|_\infty, \tag{4.4}$$

which means that δ can not be taken arbitrarily large, a contradiction.

Therefore, we conclude that if $\delta=\infty$, there exists a sequence of intervals $\{(\tau_{j,0},\tau_{j,1})\}_{j\in\mathbb{Z}^+}$ such that $\Pi(\gamma_s(\tau_{j,0}))\in\Pi(\partial\mathscr{B}_0)$, $\Pi(\gamma_s(\tau_{j,0},\tau_{j,1}))\subset \Pi(\mathscr{B}_1\setminus\mathscr{B}_0)$, and $\Pi(\gamma_s(\tau_{j,1}))\in\Pi(\partial\mathscr{B}_1)$. Again by Lemma 2.4,

$$B'\circ\Pi(\gamma_s(\tau_{j,1})) \geqslant B'\circ\Pi(\gamma_s(\tau_{j,0})) + Cd^{1/(1-\theta)}. \tag{4.5}$$

Let ℓ_n be the path $\gamma_s[\tau_{0,0},\tau_{n,0}]$. Notice that $t_1 \doteq \gamma_s(\tau_{n,0})$ belongs to $\sigma(\mathscr{B}_1)$ for some $\sigma\in\mathsf{D}$.

Since B is strictly increasing along the orbit, the inequality above shows us that we can select n so as to have $\inf_{\mathscr{B}_1} B\circ\sigma > \sup_{\mathscr{B}_1} B$. We then can connect $t_0 \doteq \gamma_s(\tau_{0,0})$ to $\sigma(t_0)$ by means of ℓ_n and an analytic path in $\sigma(\mathscr{B}_1)$. □

Corollary 4.7. *The estimates in* (4.1) *hold for t if B is open at t.*

Proof. If t is a regular point, set $t_1 \doteq t$ and consider γ_{t_1}. If not, we can connect t to a regular point t_1 through an analytic curve λ, with $B(\lambda(\tau)) > B(t)$, by Proposition 2.7, and consider γ_{t_1}.

Let $[0,\delta_1)$ be the maximal interval of γ_{t_1}. If $\delta_1=\infty$, we apply Corollary 4.6 and Lemma 4.5(ii), and the result is proved for $\xi>0$.

If $\delta_1<\infty$, then set $\bar t_1 \doteq \lim_{\tau\to\delta_1}\gamma_{t_1}(\tau)$. If $\Pi(\bar t_1)\in\Sigma_1$ and $\Sigma_1\in\mathscr{A}$, we apply Lemma 4.4 for $\bar t_1$ and Lemma 4.5(ii) for $\bar t_1$ and t, and again the result is proved for $\xi>0$.

If Σ_1 is not in $\mathscr{A}$, there is a point $p^*\in\Sigma_1$ at which a local primitive of b is open, and we connect $\Pi(\bar t_1)$ to p^* by a piecewise real analytic path in Σ_1 (see Remark 2.8). Therefore, we can proceed likewise. If the above possibilities do not occur, after a finite number of steps, we find a component Σ_k such that $\Sigma_k=\Sigma_j$ for some $j<k$. Let $\bar t_j,\bar t_k$ be the points obtained above such that $\Pi(\bar t_j),\Pi(\bar t_k)\in\Sigma_j$. We can connect $\bar t_k$ to $\sigma(\bar t_j)$, for some $\sigma\in\mathsf{D}$, σ not the identity, by a piecewise real analytic path that is projected on Σ_j. We then apply Lemma 4.5(i) for $\bar t_j$ and Lemma 4.5 (ii) for $\bar t_j$ and t.

The conclusion is the decay for $\xi>0$ in any situation.

In order to obtain the decay for $\xi<0$, we carry out the same proof by using the vector field $-\nabla B/|\nabla B|$. □

Proof of Proposition 4.1. In view of Lemma 4.4 and Corollary 4.7, it remains only to prove (4.1) for the critical points t such that $\Pi(t)$ is in a component of Σ_0 having a point p^* at which a local primitive of b is open. Since we can connect $\Pi(t)$ to p^* through Σ_0 by a piecewise real analytic path, we apply Corollary 4.7 and Lemma 4.5(ii). $\square$

Remark 4.8. The ideas in the proof of Corollary 4.6 may be used to show that the case $\delta = \infty$ can not occur when b is exact: indeed, assuming $\delta = \infty$ would lead to contradict an estimate like (4.4) for a global primitive on M.

Now our goal is to prove the following

Proposition 4.9. *Given $t \in \mathscr{U}$, for every $N \in \mathbb{Z}^+$ there is a constant $C'_N > 0$ such that*

$$|\widehat{U}(t', \xi)| \leqslant \frac{C'_N}{(1 + |\xi|)^N} \tag{4.6}$$

hold for t' in some neighborhood V of t.

Proof. **Case I.** First we suppose that t is a regular point of $\Pi^*(b)$. Take a local chart φ from V onto a ball $\mathcal{B}$ of radius r and centered at 0.

Let $\mathcal{Q}', \mathcal{Q}$ be open squares inside $\mathcal{B}$, both centered at 0, with $\mathrm{cl}(\mathcal{Q}') \subset \mathcal{Q}$ and $\mathcal{Q}'$ having side-length equal to $2A$.

Let $\ell : [0, 1] \to \mathcal{Q}'$ be the segment joining $\varphi(t') = (t_1, \dots, t_n) \in \mathcal{Q}'$ to $m \doteq (A, 0, \dots, 0)$. We have, for $\tau \in [0, 1]$,

$$\ell(\tau) = (1 - \tau)\varphi(t) + \tau m = ((1 - \tau)t_1 + \tau A, (1 - \tau)t_2, \dots, (1 - \tau)t_n),$$

and

$$B \circ \varphi^{-1}(\ell(\tau)) = (1 - \tau)t_1 + \tau A.$$

Since $A \geqslant t_1$, we have $B \circ \varphi^{-1}(\ell(\tau)) \geqslant t_1 = B(t')$. Therefore, if we consider (3.5) with $t_0 \doteq \varphi^{-1}(m)$, the result for $\xi > 0$ follows after observing that the length of $\varphi^{-1}(\ell)$ is bounded by $2A\sqrt{2}\sup_{\mathrm{cl}(\mathcal{Q}')} \|D\varphi^{-1}\|$.

Case II. Suppose t is a critical point of $\Pi^*(b)$. Consider $\mathcal{B} \subset \mathcal{B}'$ open balls centered at 0 such that $\varphi : V' \to \mathcal{B}'$ is a local chart and $t \in V = \varphi^{-1}(\mathcal{B})$.

Step 1. Suppose that $q \in V$ is such that $B(q) > B(t)$.

First we apply Lemma 2.4 for $B \circ \varphi^{-1}$ and $\mathcal{B}'$. We have that $\gamma_{\varphi(q)}$ necessarily encounters $\partial\mathcal{B}$ at $p \doteq \gamma_{\varphi(q)}(\tau)$ and

$$\tau \leqslant \left(\frac{2}{C_0} \sup_{\mathrm{cl}(\mathcal{B})} |B \circ \varphi^{-1}|\right)^{1-\theta}.$$

We now focus our analysis on the analytic set $\partial\mathcal{B}$. Denote by Σ'_0 the critical points of $\varphi_*(\Pi^* b)|_{\partial\mathcal{B}}$ and write Σ'_j for the components of Σ'_0, j in a finite set $J \subset \mathbb{Z}^+$. Fix a point $m_j \in \Sigma'_j$.

If $p \notin \Sigma_0'$, we can apply Lemma 2.4 in $\partial\mathcal{B}$ (with constant $C_0^\sharp$) and obtain $\gamma_p^\sharp : [0,\delta) \to \partial\mathcal{B}$ with

$$\lim_{\tau\to\delta} \gamma_p^\sharp(\tau) \doteq p' \in \Sigma_0'.$$

If $p \in \Sigma_0'$, we put $p' \doteq p$. We connect $\varphi(q)$ to p and p to p' respectively by using $\gamma_{\varphi(q)}$ and $\gamma_p^\sharp$. We also can connect p' to m_j by a path ς in $\partial\mathcal{B}$ and in a same level set of $B\circ\varphi^{-1}$, with $|\varsigma| \leqslant C_1^\sharp$ (see Proposition 2.5).

Hence, we have a path γ^+, connecting q to $\varphi^{-1}(m_j)$, along which B is greater than $B(q)$. If $\tilde{C}_0 \doteq \min\{C_0, C_0^\sharp\}$, put

$$\tilde{C} \doteq \left(\frac{2}{\tilde{C}_0} \sup_{\mathrm{cl}(\mathcal{B})} |B\circ\varphi^{-1}|\right)^{1-\theta},$$

and then γ^+ has length less than or equal to

$$(2\tilde{C} + C_1^\sharp) \sup_{\mathrm{cl}(\mathcal{B})} \|D\varphi^{-1}\|.$$

Therefore, if we consider (3.5) with q and $\varphi^{-1}(m_j)$, we will have for $\xi > 0$ that $|\widehat{U}(q,\xi)|$ is bounded by

$$(2\tilde{C} + C_1^\sharp) \sup_{\mathrm{cl}(\mathcal{B})} \|D\varphi^{-1}\| \sup_{s\in\gamma^+} |\widehat{F}(s,\xi)| + \sup_{j\in J} |\widehat{U}(\varphi^{-1}(m_j),\xi)|,$$

and then the result for $\xi > 0$.

Step 2. Now we will deal with those points $q \in V$ such that $B(q) \leqslant B(t)$. When q is not a critical point, a possibility for the solution $\gamma_{\varphi(q)}$ is that there is τ satisfying $\gamma_{\varphi(q)}(\tau) = s \in \partial\mathcal{B}$, and then we follow exactly the same proof in Step 1.

The second possibility is that

$$\lim_{\tau\to\delta} \gamma_{\varphi(q)}(\tau) \doteq \overline{p} \text{ is a critical point of } B\circ\varphi^{-1}.$$

By Proposition 2.7, there is a path ς in $\mathcal{B}$ connecting any critical point to $\varphi(t)$ through a same level set of $B\circ\varphi^{-1}$, and with length uniformly bounded by C_1 due to Proposition 2.5. Then we can connect q to $q^* \doteq \varphi^{-1}(\overline{p})$ by $\varphi^{-1}(\gamma_{\varphi(q)})$, and q^* to t by $\varphi^{-1}(\varsigma)$. If q is critical, we put $q^* \doteq q$.

Along the resulting path γ^+, B is greater than or equal to $B(q)$. Then, if we consider (3.5) with q and t, we will have for $\xi > 0$

$$|\widehat{U}(q,\xi)| \leqslant (\tilde{C} + C_1) \sup_{\mathrm{cl}(\mathcal{B})} \|D\varphi^{-1}\| \sup_{s\in\gamma^+} |\widehat{F}(s,\xi)| + |\widehat{U}(t,\xi)|.$$

Hence, after Step 1 and Step 2, the Proposition is proved for $\xi > 0$ also in this case.

The proof for $\xi < 0$ is obtained similarly. $\square$

Since the coefficients $\{\widehat{u}(t,\xi)\}$ of a candidate to the solution of the system satisfy
$$d_t\widehat{u}(t,\xi) + i\xi c(t)\widehat{u}(t,\xi) = \widehat{f}(t,\xi),$$
in any local chart of M, we have found a continuous function that satisfies the equation $\mathbb{L}u = f$ in the weak sense and it remains to be shown that $u(t,x)$ is smooth. This will follow by proving the appropriate decay for the derivatives of the coefficients, which involves an induction argument on the differentiation order. We refer the reader to [15] for the computations.

Then we have the infinite differentiability of u on $M \times \mathbb{S}^1$, and we finish the proof of (II.1) implies (I).

5. Proof of (II.2) implies (I)

We now intend to define the Fourier coefficients of a candidate to the global solution of the system when (II.2) holds and prove that they satisfy the estimates in (4.6). Denote by q the smallest positive integer such that qa is integral and set $J \doteq q\mathbb{Z}$.

First we will define the candidate $\widehat{u}(p,\xi)$ for $p \in M$ and $\xi \in J$. Define $\mathscr{D}'_J \doteq \{u \in \mathscr{D}'(M \times \mathbb{S}^1) : u(t,x) = \sum_{\xi\in J} \widehat{u}(t,\xi)e^{i\xi x}\}$.

Notice that for $\xi \in J$ the function $s \mapsto e^{-i\xi A(s)}$ is the lifting of a function defined on M because qa is integral. Consider the isomorphism T of $\mathscr{D}'_J$ given by
$$T\left(\sum_{\xi\in J} \widehat{u}(t,\xi)e^{i\xi x}\right) = \sum_{\xi\in J} \widehat{u}(t,\xi)e^{-i\xi A(t)}e^{i\xi x}.$$

We then have $T^{-1}\mathbb{L}T = \mathbb{L}^\sharp = d_t + ib(t)\partial_x$. Since the semilevel sets of a primitive $B^\sharp$ of b are connected, due to [10] it is possible to define the Fourier coefficients of a candidate to the global solution to $\mathbb{L}^\sharp w = g$ — provided g satisfies the respective compatibility conditions— and prove their uniformly rapid decay. Notice that if $f \in \mathbb{E}$, then $g = T^{-1}f$ is such that $e^{-\xi B^\sharp(\cdot)}\widehat{g}(\cdot,\xi)$ is exact on M for every $\xi \in J$. We then define $\widehat{u}(p,\xi) \doteq \widehat{Tw}(p,\xi)$ for $\xi \in J$.

Notice that $J = \mathbb{Z}$ if a is integral (or, equivalently, if $q = 1$).

In turn, the coefficients $\widehat{U}(t,\xi)$ for $\xi \in \mathbb{Z} \setminus J$ will be defined by (3.4) since when $\xi \notin J$ there exists $\sigma \in \mathsf{D}$ such that $i\xi c_\sigma = i\xi a_\sigma \notin 2\pi i\mathbb{Z}$.

Lemma 5.1. *The estimates in* (4.1) *hold for* $t \in \mathscr{U}$ *if* $\Pi(t)$ *is a point of* $\Sigma \in \mathscr{A}$.

Proof. First notice that $I(\Sigma)$ behaves as both a non-rational and a non-Liouville vector with respect to the denominators that are not in

J. Indeed, by (II.2),

$$\left| I(\Sigma) - \frac{P}{\xi} \right| \geqslant \frac{1}{|\xi|}$$

for every $P \in \mathbb{Z}^m$ and $\xi \notin J$. Consider then the curve α obtained by Lemma 4.3 for $\xi \notin J$. As in Lemma 4.4, by using the lift $\widetilde{\alpha}$ of α, one can join t to $\sigma(t)$, for some $\sigma \in \mathsf{D}$, by a piecewise real analytic path η through critical points of B and plug η in (3.4) to obtain the desired decay for $\xi \notin J$. $\square$

Corollary 5.2. *The estimates in* (4.1) *hold for t if B is open at t.*

Proof. If t is a regular point, set $t_1 \doteq t$ and consider γ_{t_1}. If not, we can connect t to a regular point t_1 through an analytic curve λ, with $B(\lambda(\tau)) > B(t)$ (see Proposition 2.7) and consider γ_{t_1}.

Let $[0, \delta_1)$ be the maximal interval of γ_{t_1}. Since b is exact, we have $\delta_1 < \infty$, and we set $\bar{t}_1 \doteq \lim_{\tau \to \delta_1} \gamma_{t_1}(\tau)$. Hence $q \doteq \Pi(\bar{t}_1)$ belongs to some component Σ_1 of Σ_0.

If Σ_1 is not in $\mathscr{A}$, there is a point $p^* \in \Sigma_1$ at which a local primitive of b is open, and we connect $\Pi(\bar{t}_1)$ to p^* by a piecewise real analytic path in Σ_1. Next we can reason as before with p^* in the place of t and obtain a point q^* which will belong to a component Σ_2 of Σ_0 that might be in $\mathscr{A}$ or not. If $\Sigma_2 \notin \mathscr{A}$ the process must continue. As b is exact, this procedure ends after a finite number of steps hitting a component $\Sigma_j \in \mathscr{A}$.

Applying Lemma 5.1 and Lemma 4.5 (*ii*), the conclusion is the decay for $\xi > 0$. In order to obtain the decay for $\xi < 0$, we carry out the same proof by using the vector field $-\nabla B/|\nabla B|$. $\square$

In order to prove (4.1) for the critical points t such that $\Pi(t)$ is in a component of Σ_0 having a point p^* at which a local primitive of b is open, we connect $\Pi(t)$ to p^* by a piecewise real analytic path in Σ_0, and we apply Corollary 5.2 and Lemma 4.5(*ii*). Finally, we apply Proposition 4.9.

In view of the obtained decay, we may reason as we did in the end of Section 4 to conclude the smoothness of the solution on $M \times \mathbb{S}^1$.

This finishes the proof of (II.2) implies (I), and the sufficiency part in Theorem 1.1 is complete.

6. Proof of (I) implies (II)

In this section, the following lemma will be crucial.

Lemma 6.1. *Suppose that $I(\Sigma)$ is a Liouville vector. Then there exist a sequence of real smooth closed 1-forms $\{p_j\}_{j\in\mathbb{Z}^+}$ such that*

$$T_{p_j}(\gamma') \in 2\pi\mathbb{Z}, \text{ for every } \gamma' \in \Gamma(\Sigma);$$

a sequence of integers q_j, with $q_j > 1$; and $C > 0$ satisfying

$$\left\| a - \frac{1}{q_j} \cdot p_j \right\|_\infty < \frac{C}{q_j^j}.$$

Proof. First, the vector spaces $H_1(M, \mathbb{R})$ and $H_1(M, \mathbb{Z}) \otimes \mathbb{R}$ are isomorphic of finite dimension —say N— and by viewing $\{[\nu_1], \dots, [\nu_m]\}$ (see Section 2) as a linearly independent set of $H_1(M, \mathbb{R})$, we can complete it to a basis $\{[\nu_1], \dots, [\nu_N]\}$.

Recall that de Rham's Theorem furnishes an isomorphism ϕ between $H^1_R(M)$ and $\mathrm{Hom}(H_1(M, \mathbb{R}), \mathbb{R})$ mapping the cohomology class of a to

$$[\nu_k] \mapsto \int_{[\nu_k]} a.$$

We will represent the class in $H^1_R(M)$ of a by

$$\lambda_1\mu_1 + \cdots + \lambda_N\mu_N,$$

where the class of the 1-form μ_k is mapped by ϕ to $[\nu_k] \mapsto \int_{[\nu_k]} \mu_l = \delta_{k,l}$, and, thus,

$$\lambda_l = \int_{[\nu_l]} a.$$

In other words, $a = \lambda_1\mu_1 + \cdots + \lambda_N\mu_N + d_t h$, with $h \in C^\infty(M)$. Hence, if we denote by g_j the 1-form

$$2\pi \left(\frac{P_j^1}{Q_j}\mu_1 + \cdots + \frac{P_j^m}{Q_j}\mu_m + T_a([\nu_{m+1}])\mu_{m+1} + \cdots + T_a([\nu_N])\mu_N + d_t h \right),$$

where $P_j = (P_j^1, \dots, P_j^m) \in \mathbb{Z}^m$ and $Q_j \in \mathbb{Z}$ are obtained by the fact that $I(\Sigma)$ is Liouville, we have:

$$\|a - g_j\|_\infty \leqslant C \left| I(\Sigma) - \frac{P_j}{Q_j} \right| < \frac{C'}{Q_j^j}.$$

Moreover, setting $p_j \doteq Q_j g_j$, and taking $[\nu] \in i_*(H_1(\Sigma, \mathbb{Z}))$, we have that $T_{p_j}([\nu]) \in 2\pi\mathbb{Z}$ since it is a linear combination with integral coefficients of the numbers $T_{p_j}([\nu_l])$, $l = 1, \dots, m$, and $T_{p_j}([\nu_l]) = \int_{[\nu_l]} Q_j g_j = 2\pi P_j^l$. In particular, $T_{p_j}(\gamma') \in 2\pi\mathbb{Z}$ for every $\gamma' \in \Gamma(\Sigma)$, and the result is proved. $\square$

Now we move on to the proof of the necessity in Theorem 1.1. Assume that neither (II.1) nor (II.2) holds. Then we will be in one of the three situations described below that will be labeled as (A), (B) and (C).

(A) *There is $\Sigma \in \mathscr{A}$ such that $I(\Sigma)$ is a Liouville vector.*

First we will suppose that $b \not\equiv 0$. We will consider a semi-global primitive $B^\dagger$ of b, defined on a neighborhood V of Σ, with $B^\dagger \equiv 0$ on Σ_0 (we refer the reader to [1, Proposition 3.1] and [15, Proposition 16] for details). Since $\Sigma \in \mathscr{A}$, we have that $B^\dagger > 0$ or $B^\dagger < 0$ on $V \setminus \Sigma$. Assume that $B^\dagger < 0$ on $V \setminus \Sigma$ and take another neighborhood W of Σ, with $\mathrm{cl}(W) \subset V$ and $\varepsilon \doteq -\max_{\partial W} B^\dagger > 0$.

Next let $\chi : M \to \{0, 1\}$ be the characteristic function of W and $\psi : \mathbb{R} \to [0, 1]$ be a smooth non-negative function on $\mathbb{R}$ satisfying

- $\psi^{-1}(\{1\}) = [-\varepsilon/4, \infty)$;
- $\psi^{-1}(\{0\}) = (-\infty, -\varepsilon/2]$.

We then define a smooth function $F : M \to [0, 1]$ by

$$F(t) = \chi(t)\psi(B^\dagger(t)). \tag{6.1}$$

Note that

$$B^\dagger(t) \leqslant -\varepsilon/4, \quad \forall t \in \mathrm{supp}(d_t F). \tag{6.2}$$

When $I(\Sigma)$ is a Liouville vector, Lemma 6.1 asserts that there exist a sequence of closed forms $\{p_j\}$, $j \in \mathbb{Z}^+$, and integers $q_j > 1$ such that $\{q_j^j(a - q_j^{-1}p_j)\}$ is bounded (the sequence $\{q_j\}$ can be assumed to go to the infinity).

Also, Lemma 2.2 says that there is a neighborhood V' of Σ such that $T_{p_j}(\gamma) \in 2\pi\mathbb{Z}$, for every closed curve $\gamma \in C([0, 1], V')$. If we denote by A_j a primitive of $q_j^{-1}p_j$ on $\mathscr{U}$, this allows us to define the functions $e^{-iq_jA_j(\cdot)} \in C^\omega(V')$. By shrinking V' if necessary, we can assume that $V' \subset V$. Finally we set

$$v(t, q_j) \doteq \beta_j e^{-iq_jA_j(t)} e^{q_jB^\dagger(t)} F(t),$$

and we have

$$d_t v(t, q_j) + iq_j c(t) v(t, q_j) = f(t, q_j), \tag{6.3}$$

where

$$f(t, q_j) = \beta_j e^{-iq_jA_j(t)} e^{q_jB^\dagger(t)} [iq_j \left(a - \frac{p_j}{q_j}\right) F(t) + d_t F(t)].$$

The sequence $\{\beta_j\}$ will be chosen in such a way that $\{v(t, q_j)\}$ does not have tempered growth although the $f(t, q_j)$ given by (6.3) will be the Fourier coefficients of a smooth 1-form f on $M \times \mathbb{S}^1$ after setting the remaining frequencies ($\xi \neq q_j$) equal to zero. In order to do this, define, for each $j \in \mathbb{Z}^+$, $\beta_j \doteq \min\{e^{q_j\varepsilon/8}, q_j^{j/2}\}$.

At a point $t^* \in \Sigma$, $B^\dagger(t^*) = 0$, and we have $|v(t^*, q_j)| = \beta_j$. Using that $I(\Sigma)$ is Liouville and (6.2), we obtain that

$$|f(t, q_j)| \leqslant C\beta_j \left(\frac{1}{q_j^{j-1}} + e^{-q_j \frac{\varepsilon}{4}} \right) \leqslant C \left(\frac{1}{q_j^{\frac{j}{2}-1}} + e^{-q_j \frac{\varepsilon}{8}} \right), \tag{6.4}$$

as desired.

Equations (6.3) and (6.4) also reveal that $f \in \mathbb{E}$. Moreover, since there is $\sigma \in \mathsf{D}$ such that $c_\sigma \notin 2\pi\mathbb{Q}$, by (3.4) the unique solution defined on M to the homogeneous version of the differential equation in (6.3) is null for every Fourier frequency. Hence, any candidate to solve $\mathbb{L}u = f$ must have the Fourier coefficients given by $v(t, q_j)$ in these respective frequencies. The conclusion is that we can not have a distribution solving the system.

If $b \equiv 0$, then Σ is the whole manifold. The above computations can be carried out by defining $F \equiv 1$ on M in this case.

(B) *There exist $\Sigma \in \mathscr{A}$ and $q \in \mathbb{Z}$ such that $qI(\Sigma) \in (2\pi\mathbb{Z})^m$. Further, b is not exact or qa is not integral.*

In this case, $b \not\equiv 0$, and we define F by (6.1) once again. Lemma 2.2 says that there is a neighborhood V' of Σ such that $qT_a(\gamma) \in 2\pi\mathbb{Z}$, for every closed curve $\gamma \in C([0,1], V')$. This allows us to define the functions $e^{-ijqA(\cdot)} \in C^\omega(V')$. We again can assume that $V' \subset V$ and set

$$v(t, jq) \doteq \beta_j e^{-ijqA(t)} e^{jqB^\dagger(t)} F(t).$$

Then

$$d_t v(t, jq) + ijqc(t)v(t, jq) = f(t, jq),$$

where

$$f(t, jq) = \beta_j e^{-ijqA(t)} e^{jqB^\dagger(t)} d_t F(t).$$

We also set $\beta_j \doteq e^{jq\varepsilon/8}$. As before, $\{v(t, jq)\}$ does not have tempered growth and

$$|f(t, jq)| \leqslant C\beta_j e^{-jq\frac{\varepsilon}{4}} \leqslant Ce^{-jq\frac{\varepsilon}{8}},$$

which indeed defines an element of $\mathbb{E}$ (setting zero for the remaining Fourier frequencies).

As either b is not exact or qa is not integral, there is $\sigma \in \mathsf{D}$ such that $qc_\sigma \notin 2\pi\mathbb{Z}$. Hence, the unique solution defined on M to the homogeneous version of the differential equation is null for each frequency multiple of q, and a candidate to solve $\mathbb{L}u = f$ must have the Fourier coefficients given by $v(t, jq)$ in these respective frequencies. Therefore, again we do not have a distribution solving the system.

(C) *The* 1*-forms* a *and* b *are respectively rational and exact, and there is a disconnected semilevel set of the primitive* $B^\sharp$ *of* b *on* M.

First we state a variation of a celebrated lemma of Hörmander's. The version presented here is quite similar to the standard one in [19] and need not be proved.

Lemma 6.2. *If* (1.1) *is globally solvable, in the sense of Definition* 3.2, *there exist constants* $C > 0$ *and* $m \in \mathbb{Z}^+$ *such that, for all* $f \in \mathbb{E}$ *and* $g \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$,

$$\left| \int_{M \times \mathbb{S}^1} \langle f, g \rangle \right| \leqslant C \|f\|_m \|\mathbb{L}^* g\|_m,$$

where $\mathbb{L}^*$ *is the adjoint operator of* $\mathbb{L}$.

Here $\|v\|_m = \sup_{M\times\mathbb{S}^1} \sum_{|\beta|\leqslant m} |\partial^\beta v(t, x)|$, where $|\beta|$ denotes the order of a multi-index β.

If there is a disconnected semilevel set of $B^\sharp$ on M, the operator $\mathbb{L}^\sharp = d_t + ib(t)\partial_x$ is not globally solvable as (see [10]) there exist 1-forms f_0, g_0 on M, with f_0 exact, such that, by setting $f_j^\sharp(t, x) \doteq e^{jB^\sharp(t)+ixj} f_0(t)$ and $g_j^\sharp(t, x)e^{-jB^\sharp(t)-ixj} g_0(t)$, we have

$$I_0 \doteq \int_{M\times\mathbb{S}^1} \langle f_j^\sharp, g_j^\sharp \rangle \neq 0, \text{ and}$$

$$\|f_j^\sharp\|_m \|(\mathbb{L}^\sharp)^*(g_j^\sharp)\|_m \to 0 \text{ when } j \to \infty.$$

Notice that if qa is integral, the function $s \mapsto e^{-ijqA(s)}$ can be projected on the whole manifold M.

We then consider the smooth 1-forms f_j, g_j on M having jq-Fourier coefficients equal to $e^{jqB^\sharp(t)-ijqA(t)} f_0(t)$ and $e^{-jqB^\sharp(t)+ijqA(t)} g_0(t)$, respectively, and equal to zero for the remaining frequencies.

It is plain that $f_j \in \mathbb{E}$ and that f_j, g_j jointly violate Lemma 6.2 for the operator $\mathbb{L} = d_t + c(t)\partial_x$.

Since (A), (B), and (C) above lead to the non-global solvability of $\mathbb{L}$, the implication (I) $\implies$ (II) is proved, and so is Theorem 1.1.

Remark 6.3. It should be noticed that the general result of [10] assumes that M is orientable. The sufficiency part of this result was invoked in the beginning of Section 5. At the first level of the complex, a formula for a solution to (1.1) is furnished in [10]; since it is obtained by means of integration of 1-forms along paths, the orientability of M is not required. As for the necessary part, at the first level of the complex the inequalities above can be violated in an orientable neighborhood of a certain path (as in [15], for instance), and then the orientability need not be assumed again.

7. Comments and examples

7.1. **Global hypoellitpticty.** We now discuss a global regularity result for the operator (1.1) as a consequence of the proof of Theorem 1.1, namely

Theorem 7.1. *Assume that $c = a + ib$ is real analytic and closed. The following statements are equivalent:*

1. $\mathbb{L}$ *is globally hypoelliptic, in the sense of Definition* 3.2*;*
2. $\mathscr{A} = \emptyset$*, or, for* $\Sigma \in \mathscr{A}$, $I(\Sigma)$ *is neither a rational nor a Liouville vector.*

Corollary 7.2. *Assume that b is real analytic, closed and not exact. The following statements are equivalent:*

- (a) $\mathbb{L}$ *is globally solvable;*
- (b) $\mathbb{L}$ *is globally hypoelliptic.*

Thus, when b is not exact, $\mathbb{L}$ is globally solvable precisely when it is globally hypoelliptic, a fact already known for the case $a \equiv 0$ ([15, Corollary 2]).

We recall that Theorem 7.1 was originally proved in [1] under the assumption that Σ_0 consists of embedded analytic submanifolds of M. More specifically, they assume this hypothesis in [1, Theorem 5.3] in order to prove that, when $\mathscr{A} \neq \emptyset$, (2) in Theorem 7.1 holds if and only if a solution $u \in \mathscr{D}'(M \times \mathbb{S}^1)$ to $\mathbb{L}u = f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$ is smooth at $(t, x) \in M \times \mathbb{S}^1$ if $t \in \Sigma$, with $\Sigma \in \mathscr{A}$.

We can drop this hypothesis since when (2) holds it follows that $\operatorname{Ker} \mathbb{L} \simeq \mathbb{C}$. Indeed, in this case either $\mathscr{A} = \emptyset$ or a is not rational. Note that the first possibility implies that b is not exact. Hence, in any case there exists $\sigma \in \mathsf{D}$ such that $c_\sigma = a_\sigma + ib_\sigma \notin 2\pi\mathbb{Q}$, and by (3.4) $\widehat{U}(t, \xi) = 0$ for $\xi \neq 0$.

Therefore, if $\mathbb{L}u = f$, then $f \in \mathbb{E}$, and by Theorem 1.1 there exists a smooth solution u' to the system. Thus, $u - u'$ is constant and it follows that (2) $\implies$ (1).

That the extra assumption can be dropped in the other implication is a consequence of the computations in Section 6. In fact, by assuming that (2) does not hold, we are in the situations (A) and (B) described therein. It is enough then to take $\beta_j = 1$ there in order to obtain

$u \in \mathscr{D}'(M \times \mathbb{S}^1)$ not smooth at $(t^*, x) \in \Sigma \times \mathbb{S}^1$ and such that $\mathbb{L}u = f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$.

7.2. **Global solvability of $\mathbb{L}^{n-1}$.** The main result stated in [2] is that, when M is orientable and Σ_0 consists of embedded analytic submanifolds of M, the operator $\mathbb{L}^{n-1}$ is globally solvable (that is, for every $f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{n,0})$ orthogonal to $\operatorname{Ker} \mathbb{L}$, there exists $u \in \mathscr{D}'(M \times \mathbb{S}^1, \Lambda^{n-1,0})$ satisfying $\mathbb{L}^{n-1}u = f$) if and only if (II.1) or (II.2) in Theorem 1.1 holds.

Assuming that M is orientable, there is a natural pairing on $C^\infty(M \times \mathbb{S}^1, \Lambda^{k,0}) \times C^\infty(M \times \mathbb{S}^1, \Lambda^{n-k,0})$, $0 \leqslant k \leqslant n$, which may be used to interpret the operators $\mathbb{L}^k$ and $\mathbb{L}^{n-1-k}$, $0 \leqslant k \leqslant n$, as dual of each other (recall that $C^\infty(M \times \mathbb{S}^1, \Lambda^{0,0})$ means $C^\infty(M \times \mathbb{S}^1)$).

The proof of the necessity in [2] is then achieved by violating *a priori* estimates (as in the previous section) and it bears on the fact that, under that extra assumption on Σ_0, it is possible to define certain functions in a neighborhood of a component of Σ_0 (see [2, Lemma 3.2]). Such functions can be replaced by the functions $e^{-iq_j A_j(\cdot)}$ and $e^{-ijqA(\cdot)}$ that were obtained in Section 6. The details are left to the reader.

The sufficiency, in turn, follows from a general result of functional analysis after proving the global hypoellipticity of $\mathbb{L}$, which holds now in view of the previous subsection. We therefore can state:

Theorem 7.3. *Assume that M is orientable and c is real analytic and closed. The following statements are equivalent:*

1. $\mathbb{L}$ *is globally solvable, in the sense of Definition* 3.2*;*
2. *For every $f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{n,0})$ orthogonal to the kernel of* $\mathbb{L}$ *there exists $u \in \mathscr{D}'(M \times \mathbb{S}^1, \Lambda^{n-1,0})$ satisfying $\mathbb{L}^{n-1}u = f$.*

7.3. **Mizohata structures.** In this subsection we abandon the analyticity assumptions, assume that c is a smooth closed non-exact 1-form defined on a smooth closed connected manifold M of dimension $n > 1$. We will impose additional restrictions on b that we describe now. Recall that the vector fields

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial C}{\partial t_j}(t)\frac{\partial}{\partial x}, \quad j = 1, \dots, n,$$

where $(t_1, \dots, t_n)$ are local coordinates on M and C is a local primitive of the complex form c, are local generators of $\mathcal{V} \subset \mathbb{C} \otimes T(M \times \mathbb{S}^1)$, which is orthogonal to the line sub-bundle $T' \subset \mathbb{C} \otimes T^*(M \times \mathbb{S}^1)$ generated by the 1-form $dx - c$. Denote by $T^0 = T' \cap T^*(M \times \mathbb{S}^1)$ the *characteristic set* of $\mathcal{V}$. A point $\eta = \sum_{j=1}^n \eta_j dt_j + \eta_0\, dx \in T^*_{(t,x)}(M \times \mathbb{S}^1) \setminus \{0\}$ belongs to $T^0_{(t,x)}$ if and only if $\nabla B(t) = 0$, where B is a local primitive of b, and $\eta = \eta_0 \, dx$, with $\eta_0 \in \mathbb{R} \setminus \{0\}$. Hence the set Σ_0 of critical points of b is the image of the characteristic set under the canonical projection $T^*(M \times \mathbb{S}^1) \to M$. Recall also that

Definition 7.4. *The Levi form of an involutive (or formally integrable) structure $\mathcal{V}$ at the characteristic point $\eta \in T^0_{(t,x)}$, $\eta \neq 0$, is the hermitian form on $\mathcal{V}_p$, $p = (t,x)$, defined by*

$$\mathcal{L}_{(p,\eta)}(\mathsf{v}, \mathsf{w}) = \frac{1}{2i} \eta \left([X, \overline{Y}]_p \right),$$

where X and Y are smooth sections of $\mathcal{V}$ defined in a neighborhood of $p = (t,x)$ and satisfying $X_p = \mathsf{v}$, $Y_p = \mathsf{w}$. A non-elliptic formally integrable structure of codimension 1 with non-degenerate Levi form is called a Mizohata structure.

We now will assume that our structure $\mathcal{V}$ is a Mizohata structure. Thus, if $X = v_1 L_1 + \cdots + v_n L_n$ and $Y = w_1 L_1 + \cdots + w_n L_n$, with $v_j, w_j \in \mathbb{C}$, $j = 1, \ldots n$, we have

$$\begin{aligned}
\mathcal{L}_{(p,\eta)}(X,Y) &= \frac{1}{2i} \eta \Big(\sum_{j,k=1}^{n} v_j \overline{w_k} [L_j, \overline{L_k}] \Big) \\
&= \frac{1}{2i} \eta \Big(\sum_{j,k=1}^{n} v_j \overline{w_k} (-2i) \frac{\partial^2 B}{\partial t_j \partial t_k}(t) \partial_x \Big) \\
&= -\eta_0 \, (v_1, \ldots, v_n) \, \mathrm{Hess}_t \, B(\overline{w_1}, \ldots, \overline{w_n})^t.
\end{aligned}$$

Hence, requiring that the Levi form is non-degenerate at any $\eta \in T^0_{(t,x)}$, $\eta \neq 0$, is equivalent to considering a system defined by a Morse 1-form b, i.e., a smooth closed 1-form whose local primitives have only non-degenerate critical points (the primitives defined on a covering space share the same property). The set Σ_0 is finite since there is a local chart in a neighborhood of $p \in \Sigma_0$ such that $B \circ \varphi^{-1}(t_1, \ldots, t_n) = \pm t_1^2 \pm \cdots \pm t_n^2$.

A global solvability result in this setup can be stated as

Theorem 7.5. *Assume that the form b is Morse. The following statements are equivalent:*

(I) *$\mathbb{L}$ is globally solvable, in the sense of Definition 3.2.*

(II) *One of the two conditions below is satisfied:*

- (II.1) *Property $(\star)$ holds;*
- (II.2) *the form a is integral, the form b has a primitive $B^\sharp$ defined on M, and the semilevel sets $\{t \in M : B^\sharp(t) > r\}$ and $\{t \in M : B^\sharp(t) < r\}$ are connected for every $r \in \mathbb{R}$.*

The *index* of $p \in \Sigma_0$ will be the number of negative eigenvalues of $\operatorname{Hess}_p B$. Notice that the non-existence of critical points of index 0 or n —which are points of a local maximum or a local minimum of a local primitive of b— is clearly equivalent to (II.1). Notice also that when the form b is exact the semilevel sets $\{t \in M : B^\sharp(t) > r\}$ and $\{t \in M : B^\sharp(t) < r\}$ are connected for every $r \in \mathbb{R}$ if and only if $B^\sharp$ has only one point of local maximum and only one point of local minimum on M.

Since it is readily verified that any local primitive of the real form b satisfies a Lojasiewicz's inequality, one can carry out the arguments used throughout this work in the real analytic setup, in particular, there is a version of Proposition 4.9 that is the main step in the proof of (II) implies (I) in Theorem 7.5; this version can be proved along the lines of [15, proof of Theorem 26].

Therefore, the above statements on the global hypoellipticity of $\mathbb{L}$ and the global solvability of $\mathbb{L}^{n-1}$ also hold in this setup.

7.4. **Example.** Consider the torus $M \doteq \mathbb{T}^{m+r}$ and identify $\mathbb{T}^1$ with $\mathbb{R}/(2\pi\mathbb{Z})$. Consider also a real Morse 1-form $b'(t') \doteq b_1(t')dt_1 + \cdots + b_m(t')dt_m$ defined on $\mathbb{T}^m$, $t' = (t_1, \ldots, t_m) \in \mathbb{T}^m$. The primitive of such form on $\mathbb{R}^m$ can be written as $\beta_1 t_1 + \cdots + \beta_m t_m + P(t')$, where $\beta_j \in \mathbb{R}$, and P is periodic on each variable.

On M define the 1-forms $a(t) \doteq \alpha_1 dt_1 + \cdots + \alpha_{m+r} dt_{m+r}$, with $\alpha_j \in \mathbb{R}$, and $c(t) \doteq a(t) + ib'(t')$. Define then the operator $\mathbb{L} \doteq d_t + c(t)\partial_x$ on $\mathscr{D}'(M \times \mathbb{S}^1)$. The vector fields are given by

$$\begin{cases} L_j = & \partial_j + (\alpha_j + i\beta_j + i\partial_j P(t'))\partial_x, \quad j = 1, \ldots, m, \\ L_j = & \partial_j + \alpha_j \partial_x, \quad j = m+1, \ldots, m+r, \end{cases}$$

and the critical set Σ_0 is $\{(s_k, y) : y \in \mathbb{T}^r\}$, where s_k are the isolated critical points of b'.

This example can be dealt with as in this work although without assuming the analiticity of the imaginary part, and the conclusion is that $\mathbb{L}$ will be globally solvable if and only if one of the following three conditions is satisfied:

- $\mathscr{A} = \emptyset$, i.e., there are no points of local maximum or local minimum of the local primitives of b';
- If there are critical points s_k that are local maximum or local minimum of the primitives of b', then $(\alpha_{m+1}, \ldots, \alpha_{m+r})$ is neither a rational nor a Liouville vector;
- The vector $(\alpha_1, \ldots, \alpha_{m+r})$ is rational; $\beta_j = 0$, for $j = 1, \ldots, m$; and P has only one point of local maximum and only one point

of local minimum on $\mathbb{T}^m$. In addition, if $q \in \mathbb{Z}$ is such that $q(\alpha_{m+1}, \dots, \alpha_{m+r}) \in \mathbb{Z}^r$, then $q(\alpha_1, \dots, \alpha_{m+r}) \in \mathbb{Z}^{m+r}$.

By using this example we can briefly compare the main result in this work and the analytic result in [15], where c is a purely imaginary non-exact 1-form. While in [15] $\mathbb{L}$ is globally solvable if and only if $\mathscr{A} = \emptyset$ —which is equivalent to the connectedness of the semilevel sets of a primitive $\widetilde{B}$ of the pullback of $b(t) = b'(t')$ to the minimal covering of M— here when b is not exact, $\mathbb{L}$ is globally solvable if $\mathscr{A} = \emptyset$ ($\mathbb{L}$ is degenerate elliptic), regardless of the real part a, and if $\mathscr{A} \neq \emptyset$, that is, in the presence of a disconnected semilevel set of $\widetilde{B}$, provided that $(\alpha_{m+1}, \dots, \alpha_{m+r})$ is neither a rational nor a Liouville vector.

References

1. Adalberto P. Bergamasco, Paulo D. Cordaro, and Pedro A. Malagutti, *Globally hypoelliptic systems of vector fields*, J. Funct. Anal. **114** (1993), no. 2, 267–285. MR 1223704 (94e:35048)
2. Adalberto P. Bergamasco, Paulo D. Cordaro, and Gerson Petronilho, *Global solvability for certain classes of underdetermined systems of vector fields*, Math. Z. **223** (1996), no. 2, 261–274. MR 1417431 (98e:58180)
3. Adalberto P. Bergamasco, Cleber de Medeira, Alexandre Kirilov, and Sérgio L. Zani, *On the global solvability of involutive systems*, J. Math. Anal. Appl. **444** (2016), no. 1, 527–549. MR 3523390
4. Adalberto P. Bergamasco, Cleber de Medeira, and Sérgio Luís Zani, *Globally solvable systems of complex vector fields*, J. Differential Equations **252** (2012), no. 8, 4598–4623. MR 2881049 (2012m:35364)
5. Adalberto P. Bergamasco and Alexandre Kirilov, *Global solvability for a class of overdetermined systems*, J. Funct. Anal. **252** (2007), no. 2, 603–629. MR 2360930 (2008k:35338)
6. Adalberto P. Bergamasco, Alexandre Kirilov, Wagner Vieira Leite Nunes, and Sérgio L. Zani, *On the global solvability for overdetermined systems*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4533–4549. MR 2922600
7. Adalberto P. Bergamasco, Alexandre Kirilov, Wagner Vieira Leite Nunes, and Sérgio L. Zani, *Global solutions to involutive systems*, Proc. Amer. Math. Soc. **143** (2015), no. 11, 4851–4862. MR 3391043
8. Adalberto P. Bergamasco and Gerson Petronilho, *Global solvability of a class of involutive systems*, J. Math. Anal. Appl. **233** (1999), no. 1, 314–327. MR 1684389
9. Shiferaw Berhanu, Paulo D. Cordaro, and Jorge Hounie, *An introduction to involutive structures*, New Mathematical Monographs, vol. 6, Cambridge University Press, Cambridge, 2008. MR 2397326 (2009b:32048)
10. Fernando Cardoso and Jorge Hounie, *Global solvability of an abstract complex*, Proc. Amer. Math. Soc. **65** (1977), no. 1, 117–124. MR 0463721
11. P. L. Dattori da Silva and A. Meziani, *Cohomology relative to a system of closed forms on the torus*, Math. Nachr. **289** (2016), no. 17-18, 2147–2158. MR 3583261

12. Stephen J. Greenfield and Nolan R. Wallach, *Global hypoellipticity and Liouville numbers*, Proc. Amer. Math. Soc. **31** (1972), 112–114. MR 0296508
13. Heisuke Hironaka, *Subanalytic sets*, Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, pp. 453–493. MR 0377101 (51 #13275)
14. J. Hounie, *Globally hypoelliptic vector fields on compact surfaces*, Comm. Partial Differential Equations **7** (1982), no. 4, 343–370. MR 652813
15. Jorge Hounie and Giuliano Zugliani, *Global solvability of real analytic involutive systems on compact manifolds*, Mathematische Annalen **369** (2017), no. 3, 1177–1209.
16. S. Łojasiewicz, *Sur les trajectoires du gradient d'une fonction analytique*, Geometry seminars, 1982–1983 (Bologna, 1982/1983), Univ. Stud. Bologna, Bologna, 1984, pp. 115–117. MR 771152
17. H.-M. Maire, *Hypoelliptic overdetermined systems of partial differential equations*, Comm. Partial Differential Equations **5** (1980), no. 4, 331–380. MR 567778 (81f:35027)
18. B. Teissier, *Appendice: sur trois questions de finitude en géométrie analytique réelle*, Acta Math. **151** (1983), no. 1-2, 39–48. MR 716370 (85g:32006)
19. François Treves, *Study of a model in the theory of complexes of pseudodifferential operators*, Ann. of Math. (2) **104** (1976), no. 2, 269–324. MR 0426068
20. François Trèves, *Hypo-analytic structures*, Princeton Mathematical Series, vol. 40, Princeton University Press, Princeton, NJ, 1992, Local theory. MR 1200459 (94e:35014)

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil. Tel.: +55-16-33551-8220
E-mail address: hounie@dm.ufscar.br

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil. Tel.: +55-16-33551-8220
E-mail address, Corresponding author: giuzu@dm.ufscar.br