ON THE CONTINUITY AND COMPACTNESS OF PSEUDODIFFERENTIAL OPERATORS ON LOCALIZABLE HARDY SPACES

G. HOEPFNER, R. KAPP, AND T. PICON

Abstract. In this paper we establish Sobolev type compact embedding theorems for Hörmander classes of pseudodifferential operators $OpS^{-\alpha}_{1,\delta}$ on localizable Hardy space. Our work include new optimal boundedness results. As application, we obtain compact embeddings for compactly supported distributions with respect to the space variables in the nonhomogeneous localizable Hardy-Sobolev spaces.

1. Introduction

In this work we obtain Sobolev compact embedding results in localizable Hardy spaces $h^p(\mathbb{R}^N)$ for the Hörmander classes of pseudodifferential operators $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$, $0 < \alpha < N$ and $0 \leq \delta < 1$, when the symbols are assumed to be supported on balls. This matter is motivated from studies on compact embedding version on localizable Hardy-Sobolev spaces.

Let $k \in \mathbb{Z}_+$ and $1 \leq p < \infty$. The classical compact embedding theorems for Sobolev spaces due to F. Rellich and V. Kondrachov ([4, Theorem 9.16]) attests that $W^{k,p}(B) \hookrightarrow L^q(B)$ for $p \leq q < p^*_k := pN/(N - pk)$, where $B \subset \mathbb{R}^N$ is a open ball and $1 \leq p < N/k$. We point out that the Sobolev-Gagliardo-Nirenberg inequality ensures the continuity of the inclusion up to the critical point $q = p^*_k$, however the compactness fails. Here we denote $W^{k,p}(B)$ to be the set of all functions $f \in L^p(B)$ such that $\partial^\gamma f \in L^p(B)$ for $|\gamma| \leq k$ endowed with the norm $\|f\|_{k,p} = \sum_{|\gamma|\leq k} \|\partial^\gamma f\|_{L^p}$.

These embedding results for Sobolev spaces may be seen as boundedness results of certain one parameter family of operators J_α for $\alpha > 0$,

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called Bessel potentials, defined by

$$J_\alpha f(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi} b_\alpha(\xi)\hat{f}(\xi)d\xi, \quad f \in \mathscr{S}(\mathbb{R}^N), \tag{1.1}$$

where $b_\alpha(\xi) := (1+4\pi^2|\xi|^2)^{-\alpha/2}$. The standard notation for the Bessel potential is usually given by $(1-\Delta)^{-\alpha/2}$. The representation (1.1) allows us to interpret a type of nonhomogeneous fractional Sobolev space also called potential space (see [19, p.135]) denoted by $\mathscr{L}^p_\alpha(\mathbb{R}^N)$, when α is a real positive number and $1 \leq p \leq \infty$, as the set of all functions $f \in L^p$ which can be written as $f = J_\alpha g$ for some $g \in L^p$ and naturally $\|f\|_{\alpha,p} := \|g\|_{L^p}$ defines a norm on this space. When $\alpha = k \in \mathbb{Z}^+$ and $1 < p < \infty$ these potential spaces coincides with classical Sobolev spaces $W^{k,p}(\mathbb{R}^N)$.

With the advent of the real Hardy space $H^p(\mathbb{R}^N)$ introduced by C. Fefferman and E. Stein ([7], [20]) the potential spaces were naturally extended for $0 < p < \infty$, called nonhomogeneous fractional Hardy-Sobolev spaces and denoted by $H^{\alpha,p}(\mathbb{R}^N)$, defined to be the set of all tempered distributions $f \in H^p(\mathbb{R}^N)$ which can be written as $f = J_\alpha g$ for some $g \in H^p(\mathbb{R}^N)$ endowed with the norm $\|f\|_{\alpha,p} := \|g\|_{H^p}$. Clearly this family of spaces includes all the previous spaces since $H^{\alpha,p}(\mathbb{R}^N) = \mathscr{L}^p_\alpha(\mathbb{R}^N)$ for $1 < p < \infty$. Analogously we may consider the homogeneous version of potential spaces and Hardy-Sobolev spaces, denoted by $\dot{\mathscr{L}}^p_\alpha(\mathbb{R}^N)$ and $\dot{H}^{\alpha,p}(\mathbb{R}^N)$ respectively, replacing the Bessel potential J_α by the Riesz potential operator I_α (see [15] and [21] for more details). These spaces have been object of study in several directions: maximal and pointwise characterizations ([14, 16, 5]), atomic decomposition and applications to div-curl lemma ([3, 6]), elliptic inequalities [2], among many others.

In view of the good properties of fractional integration on Hardy spaces the classical Sobolev embedding theorems (see [20, p.136]) are extended to $0 < p < 1$, to be precise the inclusions $\dot{H}^{\alpha,p}(\mathbb{R}^N) \subset H^q(\mathbb{R}^N)$ are continuous for $p \leq q \leq pN/(N-\alpha p)$ and $\alpha p < N$. The situation for the analogous compactness embedding results for Hardy Sobolev spaces changes dramatically since Hardy spaces are not localizable. This difficulty can be surpassed with the use of the localizable version of $H^p(\mathbb{R}^N)$ introduced by D. Goldberg in [8] and denoted by $h^p(\mathbb{R}^N)$. Since results on the continuity for fractional integration fails for localizable Hardy spaces (see [15, Rem. 2.7]) we will restrict ourselves to the subspaces $h^{\alpha,p}_c(B) \subset h^{\alpha,p}(\mathbb{R}^N)$ defined *bis in idem* as $H^{\alpha,p}(\mathbb{R}^N)$ by replacing $H^p(\mathbb{R}^N)$ by $h^p(\mathbb{R}^N)$ and where $h^{\alpha,p}_c(B) := h^{\alpha,p}(\mathbb{R}^N) \cap \mathscr{E}'(B)$[1]. Similar results for $h^{1,p}_c(B)$ where studied in [9].

Compact embedding results for $h^{\alpha,p}_c(B)$ may be interpreted by the action of modified Bessel potential supported on B into $h^p(\mathbb{R}^N)$ that may be written as a pseudodifferential operator associated to the symbol $\lambda_\alpha(x,\xi) = \psi(x)(1 + 4\pi^2|\xi|^2)^{-\alpha/2}$ belonging to the Hörmander class $S^{-\alpha}_{1,0}(\mathbb{R}^N)$, where ψ is a cut-off function such that $\psi(x) = 1$ on B. This allow us to consider general symbols supported on B with respect to the space variables and investigate compactness results for pseudodifferential operators acting on localizable Hardy spaces which we will now describe. Our main result is the following

Theorem 1.1. *Let $0 < p \leq 1$ and $\alpha \in]0, N[$. Assume $b(x,D)$ in $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ for some $0 \leq \delta < 1$. If there exist $R > 0$ such that $b(x,\xi) \equiv 0$ for $|x| > R$ then $b(x,D)$ maps compactly $h^p(\mathbb{R}^N)$ to $h^q(\mathbb{R}^N)$ for $p \leq q < p^*_\alpha := pN/(N - \alpha p)$.*

The exponent p^*_α given in the statement of Theorem 1.1 is a natural extension of Sobolev conjugate exponent when $\alpha \in]0, N[$. An important tool to prove our main result, that may have interest on its own, deals with (h^p, h^q) boundedness of pseudodifferential operators in the class $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ for $0 \leq \delta < 1$. More precisely

Proposition 1.1. *Let $0 < p \leq 1$ and $\alpha \in [0, N[$ then $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ maps continuously $h^p(\mathbb{R}^N)$ to $h^q(\mathbb{R}^N)$ for every $p \leq q \leq p^*_\alpha$. Moreover the result is optimal, in the sense that if $\alpha < N\left(\frac{1}{p} - \frac{1}{q}\right)$ then there exists $b(x,D) \in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ which is not continuous from $h^p(\mathbb{R}^N)$ to $h^q(\mathbb{R}^N)$.*

Particular cases of Proposition 1.1 were proved earlier, namely:

(i) When $\alpha = 0 = \delta$ and $q = p$ the result was first obtained by D. Goldberg in [8].
(ii) The continuity in $h^1(\mathbb{R}^N)$, for operators with symbols in $S^0_{1,\delta}(\mathbb{R}^N)$ when $0 < \delta$ was obtained by M. Taylor in [22].
(iii) When $\alpha = 0 < \delta$ and $q = p$ Proposition 1.1 is the main result in [12].
(iv) The case $\alpha = 1$ and $q = p^*_1$ was treated in [9].

A natural extension of Proposition 1.1 for operators in the class $OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ when $0 \leq \delta < 1$, $0 < \rho \leq 1$ and $\delta \leq \rho$ will be presented in Subsection 3.2 together with further results concerning continuity of pseudodifferential operators in Hörmander classes.

[1]We denote by $\mathscr{E}'(B)$ the set of distributions compactly supported in B

In Proposition 1.1 it was assumed that $\alpha \in [0, N[$ and then consequently $0 < p \leq 1 < N/\alpha$. Next we highlight a general version assuming $\alpha > 0$ adjusting the target space depending on p and N/α in the spirit of [15, Corollary 2.3].

Corollary 1.1. *Let $\alpha > 0$ and $b(x, D) \in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ for $0 \leq \delta < 1$. Then $b(x, D)$ maps continuously*

(i) *$h^p(\mathbb{R}^N)$ to $h^{p^*_\alpha}(\mathbb{R}^N)$ if $p < N/\alpha$.*
(ii) *$h^p(\mathbb{R}^N)$ to $bmo(\mathbb{R}^N)$ if $p = N/\alpha$.*
(iii) *$h^p(\mathbb{R}^N)$ to $\Lambda^{\alpha - N/p}(\mathbb{R}^N)$ if $p > N/\alpha$.*

A duality consequence of Corollary 1.1 item (i) is that operators with symbols in $S^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ with $\alpha > 0$ and $0 \leq \delta < 1$ map continuously $\Lambda^{r(p^*_\alpha)}(\mathbb{R}^N)$ to $\Lambda^{r(p)}(\mathbb{R}^N)$, where $r(s) = N(1/s - 1)$ and $0 < s < 1$ (see Section 2.2)

The Rellich-Kondrachov compactness theorem for localizable Hardy-Sobolev were studied for distributions in $h^{k,p}_c(B) := h^{k,p}(\mathbb{R}^N) \cap \mathscr{E}'(B)$ for $k \in \mathbb{Z}_+$ in the context of div-curl estimates for elliptic systems of complex vector fields. Namely, let B a generic ball and denote $\dot{h}^{k,p}_c(B)$ the subspace of all $u \in \mathscr{E}'(B)$ such that $\partial^\beta u \in h^p(\mathbb{R}^N)$ for $|\beta| = k$, then the [9, Theorem 3.1] asserts the embedding; $\dot{h}^{k,p}_c(B) \subset\subset h^q_c(\mathbb{R}^N)$ is compact for $p \leq q < p^*_k$. Our next result, which is a direct consequence of Theorem 1.1, extends [9, Theorem 3.1] for the homogeneous fractional Hardy-Sobolev space $\dot{h}^{\alpha,p}_c(B)$, as follows

Corollary 1.2. *Let $N \geq 2$, $0 < \alpha < N$ and $p^*_\alpha = pN/(N - \alpha p)$ for $0 < p \leq 1$. Then*

(i) *If $B \subset \mathbb{R}^N$ is an open ball, there is a continuous embedding $h^{\alpha,p}_c(B) \subset h^{p^*_\alpha}(\mathbb{R}^N)$.*
(ii) *For $p \leq q < p^*_\alpha$, the embedding $h^{\alpha,p}_c(B) \subset\subset h^q(\mathbb{R}^N)$ is compact.*

The organization of this paper is as follows. In Section 2 we recall some tools needed on the subject of localizable Hardy spaces $h^p(\mathbb{R}^N)$ and pseudodifferential operators. The Section 3 will be devoted to prove Proposition 1.1. The proof of Theorem 1.1 will be given in Section 4. In Section 5 we discuss applications such as Sobolev-Gagliardo-Nirenberg inequalities and Rellich-Kondrachov compactness theorem for nonhomogeneous localizable Hardy-Sobolev spaces including the proof of Corollary 1.2.

2. Preliminaries

In this paper we will consider pseudodifferential operators with symbols in Hörmander's classes $S^m_{\rho,\delta}(\mathbb{R}^N)$, which we will now describe. For

$m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, a symbol $a = a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^N)$ of order m and type (ρ,δ) is a smooth function defined on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying the following estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \quad \alpha, \beta \in \mathbb{Z}^N_+$$

with $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$. To each symbol $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^N)$ we associate the pseudodifferential operator $a(x,D) \in OpS^m_{\rho,\delta}(\mathbb{R}^N)$ given by

$$a(x,D)u(x) = \int e^{2\pi i x\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi, \quad u \in \mathscr{S}'(\mathbb{R}^N), \tag{2.1}$$

where $\hat{u}(\xi)$ is the Fourier transform of u, also denoted by $\mathscr{F}u(\xi)$.

The symbolic calculus of compositions of such operators is given by

Proposition 2.1 ([23],p.13). *Let $a_j(x,D) \in OpS^{m_j}_{\rho_j,\delta_j}(\mathbb{R}^N)$, $j = 1,2$, with $0 \leq \delta_1, \delta_2 < 1$ and $0 < \rho_1, \rho_2 \leq 1$. Assume $\delta_2 < \rho$, with $\rho = \min\{\rho_1, \rho_2\}$. Then*

$$a_1(x,D) \circ a_2(x,D) = q(x,D) \in OpS^{m_1+m_2}_{\rho,\delta}(\mathbb{R}^N), \tag{2.2}$$

with $\delta = \max\{\delta_1, \delta_2\}$. Moreover, the symbol $q(x,\xi)$ has asymptotic expansion

$$q(x,\xi) \sim \sum_{\alpha \geq 0} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a_1(x,\xi) \partial_\xi^\alpha a_2(x,\xi).$$

For composition of exotic classes $OpS^m_{\rho,\rho}(\mathbb{R}^N)$ for $0 \leq \rho < 1$ see [20, p.320].

From now on, we will always assume $0 \leq \delta < 1$, $0 < \rho \leq 1$, $\delta \leq \rho$ when dealing with operators in $OpS^m_{\rho,\delta}(\mathbb{R}^N)$. Pseudodifferential operators are bounded from $\mathscr{S}(\mathbb{R}^N)$ to $\mathscr{S}(\mathbb{R}^N)$, the Schwartz space, and it possesses distribution kernels $K(x,y) \in \mathscr{S}'(\mathbb{R}^N \times \mathbb{R}^N)$, the temperated distributions. A well known pointwise estimate for the kernel $K(x,y)$ is the context of the next result.

Proposition 2.2 (Theorem 1.1 in [1]). *Let $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^N)$. Then the distribution kernel $K(x,y)$ of $a(x,D)$ is smooth off the diagonal $\{(x,x)\} \subset \mathbb{R}^N \times \mathbb{R}^N$ and is given by*

$$K(x,y) = \lim_{\varepsilon \to 0} \int e^{2\pi i(x-y).\xi} a(x,\xi)\psi(\varepsilon\xi)d\xi \quad in \quad \mathscr{S}'(\mathbb{R}^N),$$

where $\psi \in C^\infty_c(\mathbb{R}^N)$ satisfies $\psi \equiv 1$ for $|\xi| \leq 1$. If $M \in \mathbb{N}$ and $M+m+N > 0$ then for $\alpha, \beta \in \mathbb{Z}^N_+$ the kernel $K(x,y)$ satisfies

$$\sup_{|\alpha|+|\beta|=M} |D_x^\alpha D_y^\beta K(x,y)| \leq \frac{C_{\alpha,\beta}}{|x-y|^{\frac{M+m+N}{\rho}}}, \quad x \neq y. \tag{2.3}$$

Moreover, given $\alpha,\beta\in\mathbb{Z}_+^N$ *there exists* $L_0\in\mathbb{Z}_+$ *such that*

$$\sup_{|x-y|\geq 1/2}|x-y|^L|D_x^\alpha D_y^\beta K(x,y)|\leq C_{\alpha\beta L}, \tag{2.4}$$

for each $L\geq L_0$.

Remark 2.1. *The constants* $C_{\alpha,\beta}$ *and* $C_{\alpha\beta L}$ *appearing in* (2.3) *and* (2.4)*, respectively, depend on* α,β,L,L_0 *and a finite (according with* α *and* β*) number of seminorms of* $a(x,\xi)$ *in* $S^m_{\rho,\delta}(\mathbb{R}^N)$.

We conclude this short preamble recalling the (L^p,L^q) boundedness of pseudodifferential operators in the class $OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$.

Proposition 2.3 (Theorem 3.5 in [1])**.** *Let* $b(x,D)\in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ *and* $0\leq\delta<1$. *Then* $b(x,D)$ *maps continuously* $L^p(\mathbb{R}^N)$ *into* $L^q(\mathbb{R}^N)$ *for*

$$\alpha=N\left[\frac{1}{p}-\frac{1}{q}+(1-\rho)\left(\frac{1}{q}-\frac{1}{2}\right)\right]$$

and $1<p\leq q\leq 2$.

2.1. **Localizable Hardy spaces.** Let us recall the localizable Hardy spaces $h^p(\mathbb{R}^N)$ introduced by D. Goldberg in [8]. Fix, once for all, a nonnegative function $\varphi\in C_c^\infty(\mathbb{R}^N)$ supported in the unit ball with integral equal to 1. For $u\in\mathscr{S}'(\mathbb{R}^N)$ we define the *small maximal function* $m_\varphi u$ by

$$m_\varphi u(x)=\sup_{0<t<1}|(u*\varphi_t)(x)|$$

where $\varphi_t(x)=t^{-N}\varphi(x/t)$.

Definition 2.1. *Let* $0<p<\infty$. *A tempered distribution* $u\in\mathscr{S}'(\mathbb{R}^N)$ *belongs to* $h^p(\mathbb{R}^N)$ *if and only if* $m_\varphi u\in L^p(\mathbb{R}^N)$*; i.e.,*

$$\|u\|_{h^p}:=\|m_\varphi u\|_{L^p}<\infty.$$

When $p=\infty$, *we set* $h^\infty(\mathbb{R}^N)=L^\infty(\mathbb{R}^N)$.

The spaces $h^p(\mathbb{R}^N)$ are independent of the choice of $\varphi\in\mathscr{S}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\varphi(x)\,dx\neq 0$. For $0<p\leq 1$, the space $h^p(\mathbb{R}^N)$ is a complete metric space with the distance

$$d(u,v)=\|u-v\|^p_{h^p},\quad u,v\in h^p(\mathbb{R}^N).$$

For $p=1$, $h^1(\mathbb{R}^N)$ is a normed space densely contained in $L^1(\mathbb{R}^N)$. For $p>1$, $h^p(\mathbb{R}^N)=L^p(\mathbb{R}^N)$ and $\|u\|_{h^p}$ is a norm equivalent to the usual L^p norm. Although $h^p(\mathbb{R}^N)$ is not locally convex for $0<p<1$ and $\|u\|_{h^p}$ is not truly a norm (it is a quasi-norm [24]), we will still refer to $\|u\|_{h^p}$ as the "norm" of u, as it is customary.

Another equivalent definition, is as follows. Let

$$S_{\mathscr{J}} = \{\psi \in \mathscr{S}(\mathbb{R}^N) : \|x^\alpha \partial^\beta \psi\|_{L^\infty} \leq 1, \quad |\alpha|, |\beta| \leq \mathscr{J}\} \subset \mathscr{S}'(\mathbb{R}^N)$$

and define the maximal function $\mathfrak{m}f$, $f \in \mathscr{S}'(\mathbb{R}^N)$ by

$$\mathfrak{m}f(x) = \sup_{\psi \in S_{\mathscr{J}}} m_\psi f(x), \quad x \in \mathbb{R}^N. \tag{2.5}$$

It is not difficult to see that, for $\mathscr{J}$ big enough, depending on p and N, a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^N)$ belongs to $h^p(\mathbb{R}^N)$ if and only if $\mathfrak{m}f \in L^p(\mathbb{R}^N)$; the functional

$$f \mapsto |\!|\!| f |\!|\!|_{h^p(\mathbb{R}^N)} := \|\mathfrak{m}f\|_{L^p(\mathbb{R}^N)}$$

defines a "norm" in $h^p(\mathbb{R}^N)$ that is equivalent to the norm $\|\cdot\|_{h^p}$ see, for instance [8, 20].

2.2. **Duality.**

Definition 2.2. *Let $0 < r < 1$. A continuous function f belongs to the homogeneous Hölder space $\dot{\Lambda}^r(\mathbb{R}^N)$ if there exists $c > 0$ such that*

$$|f(x+h) - f(x)| \leq c|h|^r,$$

for every $x, h \in \mathbb{R}^N$. For $r = 1$, $f \in \dot{\Lambda}^1(\mathbb{R}^N)$ if there exists $c > 0$ such that

$$|f(x+h) + f(x-h) - 2f(x)| \leq c|h|,$$

and if $r = k + s$, $k = 1, 2, \ldots$, $0 < s \leq 1$, $f \in \dot{\Lambda}^r(\mathbb{R}^N)$ if all derivatives $\partial^\alpha f \in \dot{\Lambda}^s(\mathbb{R}^N)$ for every $\alpha \in \mathbb{N}^N$, $|\alpha| = k$.

This is a locally convex topological vector space with the seminorm

$$|f|_{k+s} := \sum_{|\alpha|=k} \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|\partial^\alpha f(x+h) - \partial^\alpha f(x)|}{|h|^s}, \quad 0 < s < 1$$

or

$$|f|_{k+1} := \sum_{|\alpha|=k} \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|\partial^\alpha f(x+h) + \partial^\alpha f(x-h) - 2\partial^\alpha f(x)|}{|h|}, \quad s = 1.$$

modulus the subspace of those functions such that $|f|_r = 0$ which are the polynomials of degree $\leq m$ if m is an integer such that $m - 1 < r \leq m$.

When $0 < p < 1$ the dual space of $h^p(\mathbb{R}^N)$ may be identified with the nonhomogeneous Hölder space $\Lambda^r(\mathbb{R}^N) := \dot{\Lambda}^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $r = N\left(\frac{1}{p} - 1\right)$ equipped with the norm $\|f\|_r = |f|_r + \|f\|_{L^\infty}$. The dual of $h^1(\mathbb{R}^N)$ can be identified with the space $\mathrm{bmo}(\mathbb{R}^N)$, which we define as follows.

Definition 2.3. *A function* $f \in L^1_{loc}(\mathbb{R}^N)$ *belongs to* $\mathrm{bmo}(\mathbb{R}^N)$ *when*

$$\sup_{|B|<1} \inf_{c\in\mathbb{R}} \frac{1}{|B|}\int_B |f(x)-c|dx + \sup_{|B|\geq 1} \frac{1}{|B|}\int_B |f(x)|dx < \infty. \tag{2.6}$$

Here B denote a generic ball in $\mathbb{R}^N$ and

$$f_B := \frac{1}{|B|}\int_B f(x)dx := \fint_B f(x)dx,$$

where $|B|$ is the Lebesgue measure of B. Equivalently, a locally integrable function $f \in \mathrm{bmo}(\mathbb{R}^N)$ when

$$\sup_{|B|<1} \frac{1}{|B|}\int_B |f-f_B| + \sup_{|B|\geq 1} \frac{1}{|B|}\int_B |f| < \infty. \tag{2.7}$$

Throughout the paper we will use the notation B^r_x for a ball in $\mathbb{R}^N$ centered at x and radius r and B will denote a generic ball.

The quantities appearing in (2.6) and (2.7) define equivalent norms in $\mathrm{bmo}(\mathbb{R}^N)$. Also, one can use cubes Q with sides parallel to the coordinate axes, instead of balls and the resulting quantities are comparable.

2.3. **Atomic decomposition.** We now describe the atomic decomposition for $h^p(\mathbb{R}^N)$ ([8, 20]). A bounded, compactly supported function $a(x)$ is an $h^p(\mathbb{R}^N)$ atom if satisfies the following properties: there exists a cube Q with sides parallel to the coordinate axes containing the support of a such that

(1) $|a(x)| \leq |Q|^{-1/p}$, a.e., with $|Q|$ denoting the Lebesgue measure of Q;
(2) if $|Q| < 1$, we further require that $\int_{\mathbb{R}^N} x^\alpha a(x)\,dx = 0$, $\alpha \in \mathbb{N}^N$, $|\alpha| \leq N_p := \lfloor N(p^{-1}-1)\rfloor$.

Any $f \in h^p(\mathbb{R}^N)$ can be written as an infinite linear combination of h^p-atoms, more precisely, there exist scalars λ_j and h^p-atoms a_j such that $\sum_j |\lambda_j|^p < \infty$, the series $\sum_j \lambda_j a_j$ converges to f both in $h^p(\mathbb{R}^N)$ and in $\mathscr{S}'(\mathbb{R}^N)$, and $\|f\|^p_{h^p} \sim \inf \sum_j |\lambda_j|^p$, where the infimum is taken over all atomic representations.

Therefore, in order to prove that a continuous linear operator $T : \mathscr{S}'(\mathbb{R}^N) \to \mathscr{S}'(\mathbb{R}^N)$ can be extended as a bounded operator from $h^p(\mathbb{R}^N)$, $0 < p \leq 1$, to a complete metric space $(X, \|\cdot\|_X)$ it will be enough to check that there exists a constant $C > 0$ such that $\|m_\varphi Ta\|_X \leq C$ uniformly for all h^p-atoms $a(x)$. We will be using X for either $h^q(\mathbb{R}^N)$ or $\mathrm{bmo}(\mathbb{R}^N)$.

3. Proof of Proposition 1.1

We first start by noticing that the fundamental part of Proposition 1.1 is the content of the next result.

Proposition 3.1. *Let* $0 < p \leq 1$, $\alpha \in \ [0, N[$ *and* $0 \leq \delta < 1$. *Then* $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ *maps continuously* $h^p(\mathbb{R}^N)$ *to* $h^{p^*_\alpha}(\mathbb{R}^N)$.

Assume that Proposition 3.1 has been proved and let $a(x, D) \in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$, with p, α and q as in Proposition 1.1, then from one hand we have that $a(x, D)$ is $(h^p, h^{p^*_\alpha})$ bounded. From the other hand, since $OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N) \subset OpS^0_{1,\delta}(\mathbb{R}^N)$, we see that $a(x, D)$ is (h^p, h^p) bounded. Now Proposition 1.1 follows from the (real) interpolation theorem on localizable Hardy spaces (see [24, p. 66]).

Proof of Proposition 3.1: Given $\alpha \in \ ]0, N[$ there is $r \in \ ]1, \infty[$ such that

$$\frac{1}{r^*_\alpha} = \frac{1}{r} - \frac{\alpha}{N} \in \left]0, \frac{1}{2}\right].$$

Let $a(x)$ be an h^p-atom supported in a cube Q centered at x_0 with side length ℓ and denote by Q^* the cube with the same center and side length 2ℓ. Consider the notation $Ta := b(x, D)a$.

To estimate $\|m_\varphi Ta\|_{L^{p^*_\alpha}}$ we may write $\int |m_\varphi Ta(x)|^{p^*_\alpha} dx = I_1 + I_2$ with

$$I_1 = \int_{Q^*} |m_\varphi Ta(x)|^{p^*_\alpha} dx \quad \text{and} \quad I_2 = \int_{\mathbb{R}^N \setminus Q^*} |m_\varphi Ta(x)|^{p^*_\alpha} dx.$$

Since T maps continuously $L^r(\mathbb{R}^N)$ into $L^{r^*_\alpha}(\mathbb{R}^N)$ (Proposition 2.3) and $m_\varphi f(x)$ is bounded a.e. by the Hardy-Littlewood maximal function $Mf(x)$ which is bounded in $L^s(\mathbb{R}^N)$ for $1 < s \leq \infty$, it follows by Hölder's inequality with $1 < \gamma := r^*_\alpha / p^*_\alpha$ that

$$\begin{aligned}
I_1 &\leq \left(\int_{Q^*} |m_\varphi Ta(x)|^{p^*_\alpha \gamma} dx \right)^{\frac{1}{\gamma}} |Q^*|^{1-\frac{1}{\gamma}} \\
&\leq \|M(Ta)\|^{p^*_\alpha}_{L^{r^*_\alpha}} |Q^*|^{1-\frac{p^*_\alpha}{r^*_\alpha}} \\
&\leq \|Ta\|^{p^*_\alpha}_{L^{r^*_\alpha}} |Q^*|^{1-\frac{p^*_\alpha}{r^*_\alpha}} \\
&\leq C\|a\|^{p^*_\alpha}_{L^r} |Q^*|^{1-\frac{p^*_\alpha}{r^*_\alpha}} \\
&\leq C\|a\|^{p^*_\alpha}_{L^\infty} |Q^*|^{\frac{p^*_\alpha}{r}} |Q^*|^{1-\frac{p^*_\alpha}{r^*_\alpha}} \\
&\leq C|Q|^{-\frac{p^*_\alpha}{p}} |Q^*|^{1+p^*_\alpha\left(\frac{1}{r}-\frac{1}{r^*_\alpha}\right)} \\
&= c_N,
\end{aligned}$$

with $c_N > 0$ independent of the atom $a(x)$. When $\alpha = 0$ then $p_\alpha^* = p$ and the same control holds using that T is bounded from $L^2(\mathbb{R}^N)$ to itself.

Before we estimate I_2, we will recall some important facts that might be stated in a more general situation.

Remark 3.1. *Let $\alpha \geq 0$ and $b = b(x, D) \in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ be a pseudo-differential operator with symbol $b(x, \xi) \in S^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ which is bounded in $L^2(\mathbb{R}^N)$ (Proposition 2.3) and denote $b^\varepsilon = b^\varepsilon(x, D)$ the operator $\mathscr{S}(\mathbb{R}^N) \ni f \to \Phi_\varepsilon * bf$, for some $\Phi \in \mathscr{S}(\mathbb{R}^N)$. Note that $\Phi_\varepsilon * f$ may be written as (inverse Fourier transform)*

$$\Phi_\varepsilon * f(x) = \int e^{2\pi i x\cdot\xi} \hat{\Phi}(\varepsilon\xi) \hat{f}(\xi) d\xi, \quad f \in \mathscr{S}(\mathbb{R}^N)$$

and regarded as a pseudodifferential operator with symbol $\hat{\Phi}(\varepsilon\xi)$. Moreover $\xi \to \hat{\Phi}(\varepsilon\xi)$ belongs to $S^0_{1,0}(\mathbb{R}^N)$ uniformly in $0 < \varepsilon \leq 1$ and $b^\varepsilon(x, D)$ is obtained by composing on the left the pseudodifferential operator $f \to \Phi_\varepsilon * f$ with $b(x, D)$. The symbol of the composition will be denoted by $b^\varepsilon(x, \xi) \in S^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ and kernel $K^\varepsilon(x, y)$. Therefore, it follows from Remark 2.1 that $K^\epsilon(x, y)$ also satisfies estimates (2.3) *and* (2.4) *but with constants $C_{\alpha,\beta}$ and $C_{\alpha\beta N}$ independent of $0 < \varepsilon \leq 1$.*

Moving on, to estimate I_2 we will divide in two cases, depending on the type of the atoms considered. First we will consider atoms a that satisfies $supp(a) \subset Q$ along with conditions (1) and (2) in Section 2.3 for some cube Q of side length $\ell \leq 1$, in particular a has vanishing moments up to the order $N_p = [N(p^{-1} - 1)]$, the greatest integer smaller than or equal to $N(p^{-1} - 1)$.

We denote Q by $Q(x_0, \ell)$ the cube centered at x_0 and side length ℓ and $b^\varepsilon(x, D)$ the composition operator $a \mapsto \varphi_\varepsilon * b(x, D)a$, with kernel $K^\varepsilon(x, \cdot)$. We have

(3.1)

$$\begin{aligned} b^\varepsilon(x, D)a &= \int_Q \left(K^\varepsilon(x, y) - \sum_{|\gamma| \leq N_p} \partial_y^\gamma K^\varepsilon(x, x_0) \frac{(y - x_0)^\gamma}{\gamma!} \right) a(y) dy \\ &= \int_Q R^\varepsilon(x, y) a(y) dy. \end{aligned}$$

By Taylor's formula and estimate (2.3) applied for $M = |\beta| = N_p + 1$ and $m = -\alpha$ we have

$$\begin{aligned} \sup_{y\in Q} |R^{\varepsilon}(x,y)| &\leq c\ell^{N_p+1} \sum_{|\beta|=N_p+1} \sup_{y\in Q} |\partial_y^{\beta} K^{\varepsilon}(x,y)| \\ &\leq c\frac{\ell^{N_p+1}}{|x-x_0|^{N_p+1-\alpha+N}}, \quad x \notin Q^*, \end{aligned} \tag{3.2}$$

therefore

$$|b^{\varepsilon}(x,D)a| \leq c(N,p)|Q|^{-\frac{1}{p}+1} \frac{\ell^{N_p+1}}{|x-x_0|^{N_p+N-\alpha+1}} \tag{3.3}$$

with $c(N,p)$ independent of $0 < \epsilon < 1$ (see Remark 3.1). Since

$$N_p + N - \alpha + 1 > N\left(\frac{1}{p} - 1\right) + N - \alpha = \frac{N}{p} - \alpha = \frac{N}{p_{\alpha}^*},$$

the function $|x-x_0|^{-(N_p+N-\alpha+1)p_{\alpha}^*}$ is integrable on $\mathbb{R}^N \backslash Q^*$ and so

$$\begin{aligned} I_2^{\varepsilon} &= \int_{\mathbb{R}^N\backslash Q^*} |b^{\varepsilon}(x,D)a|^{p_{\alpha}^*} dx \\ &\leq c_1 |Q|^{-\frac{p_{\alpha}^*}{p}+p_{\alpha}^*} \ell^{(N_p+1)p_{\alpha}^* - (N_p+N-\alpha+1)p_{\alpha}^* + N} \\ &= C \end{aligned}$$

uniformly in ε and ℓ.

Now we shall consider atoms a that satisfies $supp(a) \subset Q$ along with condition (1) in Section 2.3 for some cube Q of side length $\ell \geq 1$, in particular $\|a\|_{\infty} \leq 1$. As before, denote Q by $Q(x_0, \ell)$ the cube centered at x_0 and side length ℓ. In this case, it will be enough to prove that

$$m_{\varphi} Ta(x) \leq c\frac{|Q|^{-\frac{1}{p}+1}}{|x-x_0|^L}, \quad x \notin Q^*, \tag{3.4}$$

where $c > 0$ and $L > \max\{N/p_{\alpha}^*, L_0\}$ are fixed and independent of the atom $a(x)$. In fact, if (3.4) is proved, then

$$I_2 \leq \int_{\mathbb{R}^N\backslash Q^*} \left(\frac{|Q|^{-\frac{1}{p}+1}}{|x-x_0|^L}\right)^{p_{\alpha}^*} dx \lesssim \ell^{\, p_{\alpha}^* N\left(1-\frac{1}{p}\right) + (N - Lp_{\alpha}^*)} \lesssim 1.$$

To prove (3.4), we must show that

$$|b^{\varepsilon}(x,D)a| \leq c\frac{|Q|^{-\frac{1}{p}+1}}{|x-x_0|^L}, \quad x \notin Q^*, \tag{3.5}$$

with $c > 0$ and $L > \max\{N/p_{\alpha}^*, L_0\}$ independent of $0 < \varepsilon \leq 1$ and the atom $a(x)$.

Let us proof (3.5). For $x \notin Q^*$ and $y \in Q$ we have $|x-y| \sim |x-x_0|$ and $|x-y| \geq \ell/2 \geq 1/2$. Then, one can apply (2.4) from Proposition 2.2 for $m=0$ and $L > \max\{N/p^*_\alpha, L_0\}$ to obtain

$$|b^\varepsilon(x,D)a| = \left|\int_Q K^\varepsilon(x,y)a(y)dy\right| \leq C_L \frac{|Q|^{-\frac{1}{p}+1}}{|x-x_0|^L}, \quad x \notin Q^*$$

showing that (3.5) holds.

Summing up, we have proved that $b(x,D)$ is bounded from $h^p(\mathbb{R}^N)$ to $h^{p^*_\alpha}(\mathbb{R}^N)$, as we wished to prove. □

3.1. **Proof of Proposition 1.1: optimality.** Let us now discuss the necessary condition for the (h^p, h^q) continuity of operators in the class $OpS^{-\alpha}_{1,0}(\mathbb{R}^N)$. First we note that

$$q \leq p^*_\alpha \Longrightarrow \alpha \geq N\left(\frac{1}{p}-\frac{1}{q}\right) \Longleftrightarrow -\alpha \leq N\left(\frac{1}{q}-\frac{1}{p}\right).$$

Suppose $m > N(1/q - 1/p)$. We claim that the pseudodifferential operator J_m defined by symbol $\langle\xi\rangle^m := (1+4\pi^2|\xi|^2)^{m/2} \in S^m_{1,0}(\mathbb{R}^N)$ is unbounded from $h^p(\mathbb{R}^N)$ to $h^q(\mathbb{R}^N)$.

Denote by (x_1, x'), $x' = (x_2, \ldots, x_N)$ a generic point in $\mathbb{R}^N \simeq \mathbb{R} \times \mathbb{R}^{N-1}$. Choose $\psi(x') \in C^\infty_c(\mathbb{R}^{N-1})$ supported in the unit cube Q' in $\mathbb{R}^{N-1}$ with $\widehat{\psi}(0) = 1$, define

$$\alpha(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ -1 & \text{for } -1 \leq t < 0, \\ 0 & \text{for } |t| > 1, \end{cases} \tag{3.6}$$

and set $f_\varepsilon(x) := 2^{-1/p}\varepsilon^{-N/p}\alpha(x_1/\varepsilon)\psi(x'/\varepsilon)$, for some fixed $N/(N+1) < p \leq 1$. Then $f_\varepsilon(x)/\|\psi\|_{L^\infty}$ is an h^p-atom and we see that $\|f_\varepsilon\|_{h^p} \simeq \|f_\varepsilon\|_{H^p} \leq C\|\psi\|_{L^\infty} \leq C'$ for $0 < \varepsilon \leq 1$. If $g_\varepsilon(x) = J_m f_\varepsilon(x)$, then its Fourier transform is given by

$$\begin{aligned} \mathscr{F} g_\varepsilon(\xi_1, \xi') &= \langle\xi\rangle^m\, 2^{-1/p}\, \varepsilon^{N(1-1/p)}\, \widehat{\psi}(\varepsilon\xi'_1)\widehat{\alpha}(\varepsilon\xi_1) \\ &= \langle\xi\rangle^m\, 2^{-1/p}\, \varepsilon^{N(1-1/p)}\, \widehat{\psi}(\varepsilon\xi')\left(\frac{1-\cos(2\pi\varepsilon\xi_1)}{i\pi\varepsilon\xi_1}\right). \end{aligned} \tag{3.7}$$

Hence, $|\mathscr{F} g_\varepsilon(1/4\varepsilon, 0')| \simeq \varepsilon^{N(1-1/p)-m}$ and

$$\frac{|\mathscr{F} g_\varepsilon(1/4\varepsilon, 0')|}{\langle(1/4\varepsilon, 0')\rangle^{N\left(\frac{1}{q}-1\right)}} \geq C\varepsilon^{N\left(1-\frac{1}{p}\right)-m+N\left(\frac{1}{q}-1\right)}, \ \ 0 < \varepsilon \leq 1$$

thus, we conclude that

$$\sup_{\xi\in\mathbb{R}^n} \frac{|\mathscr{F} g_\varepsilon(\xi)|}{\langle\xi\rangle^{N(q^{-1}-1)}} \to \infty \quad \text{as } \varepsilon \searrow 0. \tag{3.8}$$

This shows that J_m cannot be continuous from $h^p(\mathbb{R}^N)$ to $h^q(\mathbb{R}^N)$ in view of the Fourier transform decay (see [12, Proposition 5.1]).

Similarly, this example can be extended for $0 < p \leq N/(N+1)$. In order to do so, we will first need to extend the construction of the function α given in (3.6). More precisely, we have the following result.

Lemma 3.1. *Let k be a nonnegative integer, then there exists a compact supported function α_k such that*

$$\int \alpha_k(t) t^\ell dt = 0, \quad \text{for every} \quad 0 \leq \ell \leq k-1. \tag{3.9}$$

Proof: In fact, when $k = 1$, the construction is already given in (3.6) with $\alpha_1(t) = \chi_{(0,1]}(t) - \chi_{(0,1]}(-t)$ which is odd and easily satisfies (3.9). Suppose now that $k = 2$, and note that the function $\alpha_1(t-1)$ is supported in $\mathbb{R}^+$ and satisfies (3.9) for $\ell = 0$ thus if we extend it in a even fashion to $\mathbb{R}^-$ by $\alpha_2(t) = \alpha_1(t-1) + \alpha_1(-t+1)$ we will also have $\int \alpha_2(t) t dt = 0$.

This procedure can be further continued as we will now show. If α_k is constructed to satisfy (3.9) and is supported in $[-2^{k-1}, 2^{k-1}]$ then $\alpha_k(t - 2^{k-1})$ will be supported in $[0, 2^k]$. If we extend it either in an odd fashion or in an even fashion accordingly with the parity of k we obtain a function α_{k+1} that will satisfy (3.9) for $k+1$. To be more precise, define

$$\alpha_{k+1} = \begin{cases} \alpha_k(t-2^k) - \alpha_k(-t+2^k) & \text{if} \quad k \text{ is even}, \\ \alpha_k(t-2^k) + \alpha_k(-t+2^k) & \text{if} \quad k \text{ is odd}. \end{cases} \tag{3.10}$$

This concludes the proof of the lemma. □

In order to extend the previous example for smaller values of p, we need to analyse the Fourier transform of the functions α_k given in Lemma 3.1.

Lemma 3.2. *For every $k \in \mathbb{N}$, the Fourier transform of α_k defined at* (3.10) *is given by*

$$\mathscr{F}\alpha_k(\tau) = (-i2)^{k-1} \mathscr{F}\alpha_1(\tau) \prod_{j=1}^{k-1} \sin(j2\pi\tau). \tag{3.11}$$

PROOF: Assume first $k = 2$. The Fourier transform of $\alpha_1(t-a) := (\alpha_1 \circ T_a)(t)$ is given by
$\mathscr{F}(\alpha_1 \circ T_a)(\tau) = e^{-2\pi i a\tau}\mathscr{F}\alpha_1(\tau)$. Thus

$$\begin{aligned}\mathscr{F}\alpha_2(\tau) &= e^{-2\pi i\tau}\mathscr{F}\alpha_1(\tau) + e^{2\pi i\tau}\mathscr{F}\alpha_1(-\tau)\\ &= e^{-2\pi i\tau}\mathscr{F}\alpha_1(\tau) - e^{2\pi i\tau}\mathscr{F}\alpha_1(\tau)\\ &= -2i(\sin 2\pi\tau)\cdot\mathscr{F}\alpha_1(\tau)\end{aligned}$$

since $\mathscr{F}\alpha_1$ is an odd function, see (3.7).

The general case can be done by induction. In fact, assume that (3.11) has already been proved for some k and suppose without loss of generality that k is even (if k is odd the argument is similar). Consequently $\mathscr{F}\alpha_k$ is an even function in view of (3.11). Thus, using (3.10) with the same notation as before, one can write

$$\begin{aligned}\mathscr{F}\alpha_{k+1}(\tau) &= e^{-2\pi ik\tau}\mathscr{F}\alpha_k(\tau) - e^{2\pi ik\tau}\mathscr{F}\alpha_k(-\tau)\\ &= \left(e^{-2\pi ik\tau} - e^{2\pi ik\tau}\right)(-2i)^{k-1}\mathscr{F}\alpha_1(\tau)\prod_{j=1}^{k-1}\sin(j2\pi\tau)\\ &= (-i2)^k\mathscr{F}\alpha_1(\tau)\prod_{j=1}^{k}\sin(j2\pi\tau)\end{aligned}$$

as we wished to prove. □

Now we are ready to present a counter example for small values of p. To do so, fix $k \geq 2$ integer along with an integrability exponent $p \in]N/(N+k), N/(N+k-1)]$. Let $\alpha := \alpha_k$ the function given by Lemma 3.1, supported in $[-2^{k-1}, 2^{k-1}]$ and choose $\psi(x') \in C_c^\infty(\mathbb{R}^{N-1})$ supported in the unit cube Q' in $\mathbb{R}^{N-1}$ with $\mathscr{F}(\partial_{x'}^{\beta'}\psi)(0) = 1$, for every $\beta' \in \mathbb{N}_0^{N-1}$ with $|\beta'| \leq k$.

Define $f_\varepsilon(x) = 2^{-k/p}\varepsilon^{-N/p}\alpha(x_1/\varepsilon)\psi(x'/\varepsilon)$. Then $f_\varepsilon(x)/\|\psi\|_{L^\infty}$ is an h^p-atom and we see that $\|f_\varepsilon\|_{h^p} \simeq \|f_\varepsilon\|_{H^p} \leq C\|\psi\|_{L^\infty} \leq C'$ for $0 < \varepsilon \leq 1$. If $g_\varepsilon(x) = J_m f_\varepsilon(x)$, then one can use Lemma 3.2 to compute its Fourier transform

$$\begin{aligned}\mathscr{F}g_\varepsilon(\xi_1,\xi') &= \langle\xi\rangle^m\, 2^{-\frac{k}{p}}\,\varepsilon^{N(1-\frac{1}{p})}\,\widehat{\psi}(\varepsilon\xi_1')\widehat{\alpha}(\varepsilon\xi_1)\\ &= \langle\xi\rangle^m\, 2^{-\frac{k}{p}}\,\varepsilon^{N(1-\frac{1}{p})}\,\widehat{\psi}(\varepsilon\xi')\left(\frac{1-\cos(2\pi\varepsilon\xi_1)}{i\pi\varepsilon\xi_1}\right)(-i2)^k\prod_{j=1}^{k}\sin(j\varepsilon 2\pi\xi_1).\end{aligned}$$

Hence, if we choose $\xi_1 = 1/4k\varepsilon$ then we have $|\mathscr{F}g_\varepsilon(1/4k\varepsilon, 0')| \simeq \varepsilon^{N(1-1/p)-m}$ and we reach the same conclusion as in (3.8) and the proof follows similarly.

3.2. **Generalizations of Proposition 1.1.** We shall now discuss another aspect of the sharpness of the Proposition 1.1 in the following sense: since $OpS^{-\alpha}_{1,0}(\mathbb{R}^N) \subset OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ one could ask ourselves whether the operators in $OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ $\alpha \in [0, N[$, are $(h^p, h^{p^*_\alpha})$ bounded.

To answer this question, fix $\alpha \in [0, N[$, $0 < \rho < 1$ and suppose that every $b(x, D) \in OpS^{-\alpha}_{\rho,0}(\mathbb{R}^N)$ maps continuously $h^p(\mathbb{R}^N)$ to $h^{p^*_\alpha}(\mathbb{R}^N)$. By abuse of notation we write $J_{-\alpha} \in OpS^{+\alpha}_{1,0}(\mathbb{R}^N)$ to denote the pseudodifferential defined in (1.1) replacing α by $-\alpha$. Thus, $a(x, D) := J_{-\alpha} \circ b(x, D) \in OpS^0_{\rho,0}(\mathbb{R}^N)$ maps continuously $h^p(\mathbb{R}^N)$ to itself for all $b(x, D) \in OpS^{-\alpha}_{\rho,0}(\mathbb{R}^N)$ that yields a contradiction. To see this, let $\psi \in C^\infty(\mathbb{R}^N)$ satisfy $\psi(\xi) \equiv 0$ for $|\xi| \leq 1$ and $\psi \equiv 1$ for $|\xi| \geq 2$ then the pseudodifferential operator $a(D) = \psi(\xi)\exp(i|\xi|(1-\rho))$ belongs to $OpS^0_{\rho,0}(\mathbb{R}^N)$ and it is not bounded on $h^p(\mathbb{R}^N)$ for $0 < p \leq 1$, (see [12, Example 5.2]).

It is therefore natural to investigate a version of Proposition 1.1 for operators in the class $OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ when $0 \leq \delta < 1$, $0 < \rho \leq 1$ and $\delta \leq \rho$. To do so, we recall what we have learnt from Päivärinta and Somersalo [18], in the critical case $\delta = \rho$.

Theorem 3.1 (Theorem 4.1 in [18])**.** *Let* $0 < q \leq 1$, $0 \leq \rho < 1$ *and* $\mu = (1-\rho)N\left(\frac{1}{q} - \frac{1}{2}\right)$. *Then* $b(x, D) \in OpS^{-\mu}_{\rho,\rho}(\mathbb{R}^N)$ *maps continuously* $h^q(\mathbb{R}^N)$ *to itself.*

The next result follows from a combination of Proposition 1.1 with Theorem 3.1 keeping in mind that, in this scenario, the composition works in a similar fashion as described in Proposition 2.1 property (2.2) (see [11] for details).

Corollary 3.1. *Let* $0 < p \leq 1$ *and* $\alpha \in [0, N[$. *If* $b(x, D) \in OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ *for* $0 < \delta \leq \rho < 1$ *then* $b(x, D)$ *is* (h^p, h^q) *bounded where* q *is defined by*

$$\alpha = N\left[\frac{1}{p} - \frac{1}{q} + (1-\rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right].$$

Proof: Let $b(x, D) \in OpS^{-\alpha}_{\rho,\delta}(\mathbb{R}^N)$ for $0 < \delta \leq \rho < 1$ and write $b(x, D) = (b(x, D) \circ J_{-\mu}) \circ J_\mu$ for $\mu = N\left(\frac{1}{p} - \frac{1}{q}\right)$. First, note that $J_\mu \in OpS^{-\mu}_{1,0}(\mathbb{R}^N)$ is (h^p, h^q) bounded in view of Proposition 1.1. Second, since $\alpha-\mu = N(1-\rho)\left(\frac{1}{q}-\frac{1}{2}\right)$ one can apply Theorem 3.1 to conclude that $b(x,D)\circ J_{-\mu} \in OpS^{-\alpha+\mu}_{\rho,\delta}(\mathbb{R}^N) \subset OpS^{-\alpha+\mu}_{\rho,\rho}(\mathbb{R}^N)$ is bounded from $h^q(\mathbb{R}^N)$ to itself and then the proof follows. □

Pseudodifferential operators with symbols in the exotic class $S^m_{0,0}(\mathbb{R}^N)$ lack from a peculiar property common to operators in the class $S^m_{\rho,\delta}(\mathbb{R}^N)$ when $0<\delta\leq\rho<1$, the so-called pseudo-local property i.e., the operators in this class $S^m_{0,0}(\mathbb{R}^N)$ cannot be represented by smooth kernels away from the diagonal (see [20, p.323]). We point out this property was fundamental in the proof of Proposition 1.1. Assume, for the moment, that Corollary 1.1 has been proved (see Subsection 3.3) and let us see how to extend Corollary 3.1 for exotic symbols, i.e., $\delta=\rho=0$.

Corollary 3.2. *Let $0<p\leq 1$ and $\alpha = N\left(\frac{1}{p}-\frac{1}{2}\right)$. Then any operator $b(x,D)\in OpS^{-\alpha}_{0,0}(\mathbb{R}^N)$ maps $h^p(\mathbb{R}^N)$ continuously to $L^2(\mathbb{R}^N)$.*

Proof: Let $b(x,D)\in OpS^{-\alpha}_{0,0}(\mathbb{R}^N)$ and write

$$b(x,D) = (b(x,D)\circ J_{-\alpha})\circ J_{\alpha}.$$

Then one can use the symbolic calculus of composition ([11, Proposition 11.2]) to conclude that $b(x,D)\circ J_{-\alpha}\in OpS^{0}_{0,0}(\mathbb{R}^N)$ is bounded from $L^2(\mathbb{R}^n)$ to itself (see [20, p.282]). The conclusion follows from Corollary 1.1 item (i) since $J_\alpha \in OpS^{-\alpha}_{1,0}(\mathbb{R}^N)$ maps continuously $h^p(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. □

Note that, in general, we cannot expect boundedness on $L^2(\mathbb{R}^N)$ of pseudodifferential operators in the class $OpS^{-\alpha}_{1,1}(\mathbb{R}^N)\subset OpS^{0}_{1,1}(\mathbb{R}^N)$, which was crucial in the proof of Proposition 1.1 when $\alpha=0$ (see also [12, Section 3.1]). However, we may consider special subclasses of $OpS^{-\alpha}_{1,1}(\mathbb{R}^N)$ namely the Bony classes $\mathscr{B}S^{-\alpha}_{1,1}$, and Hörmander classes $\tilde{S}^{-\alpha}_{1,1}$ (see [10, 13]). These classes are stable under composition of symbols and their associated kernels possesses pointwise estimates yielding boundedness in Lebesgue spaces and consequently the same conclusions as stated in Proposition 1.1. The proof of Proposition 1.1 for $\mathscr{B}S^{-\alpha}_{1,1}$ and $\tilde{S}^{-\alpha}_{1,1}$ follows by composing with J_α and [12, Theorem 4.3].

3.3. **Proof of Corollary 1.1.** In this subsection we will assume that $b(x,D)\in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ for $\alpha>0$ and $0\leq\delta<1$ as in the statement of Corollary 1.1.

Proof of item (i). Suppose that $p<N/\alpha$. If $\alpha\geq N$ choose $k\in\mathbb{N}$ such that $\beta := \alpha/k < N$ and consider the semigroup decomposition of

J_α given by
$$J_\alpha = \underbrace{J_\beta \circ J_\beta \circ \cdots \circ J_\beta}_{k-\text{times}}.$$
For each $\ell = 0, 1, ..., k$ let $p_\ell \in]0, \infty[$ given by
$$\frac{1}{p_\ell} := \frac{1}{p} - \ell \frac{\beta}{N}.$$
Since
$$b(x, D) = \big(b(x, D) \circ J_{-\alpha}\big) \circ J_\alpha \quad \text{and} \quad b(x, D) \circ J_{-\alpha} \in OpS^0_{1,\delta}(\mathbb{R}^N)$$
to conclude the result, it will be enough to show that J_β maps continuously $h^{p_\ell}(\mathbb{R}^N)$ to $h^{p_{\ell+1}}(\mathbb{R}^N)$ for every $\ell = 0, ..., k-1$. The latter is a direct consequence of Proposition 1.1 and Proposition 2.3, since
$$\frac{1}{p_{\ell+1}} = \frac{1}{p_\ell} - \frac{\beta}{N}$$
and $\beta \in]0, N[$. □

Proof of item (ii). Suppose now that $p = N/\alpha$. To show that $T := b(x, D) : h^p(\mathbb{R}^N) \to \mathrm{bmo}(\mathbb{R}^N)$ is bounded, since $[h^1(\mathbb{R}^N)]^* = bmo(\mathbb{R}^N)$ (see Section 2.2 or [[8], Corollary 1]) it will be enough to show that for every $f \in h^p(\mathbb{R}^N)$, Tf defines a bounded linear functional in $h^1(\mathbb{R}^N)$ and satisfies
$$|\langle Tf, \eta\rangle| \leq C\|f\|_{h^p}\|\eta\|_{h^1}, \qquad \text{for all } \eta \in h^1(\mathbb{R}^N),$$
for some constant $C > 0$ independent of f and η.

Let $0 < \tilde{\alpha} < \alpha$ so that $0 < \beta := \alpha - \tilde{\alpha} < N$. Thus $p < N/\widetilde{\alpha}$ and by Proposition 1.1 (or Corollary 1.1, item (i)) implies that
$$OpS^{-\tilde{\alpha}}_{1,\delta}(\mathbb{R}^N) : h^p(\mathbb{R}^N) \to h^{p^*_{\tilde{\alpha}}}(\mathbb{R}^N) = L^{p^*_{\tilde{\alpha}}}(\mathbb{R}^N), \quad \frac{1}{p^*_{\tilde{\alpha}}} = \frac{1}{p} - \frac{\tilde{\alpha}}{N} = \frac{\beta}{N} < 1,$$
and
$$OpS^{-\beta}_{1,\delta}(\mathbb{R}^N) : h^1(\mathbb{R}^N) \to h^{1^*_\beta}(\mathbb{R}^N) = L^{1^*_\beta}(\mathbb{R}^N), \quad \frac{1}{1^*_\beta} = 1 - \frac{\beta}{N} < 1.$$
Since $(J_\beta)^* = J_\beta$ see, for instance, [23, Proposition 3.2], we can write
$$\langle Tf, \eta\rangle = \langle J_{-\beta}Tf, (J_\beta)^*\eta\rangle = \langle J_{-\beta}Tf, J_\beta\eta\rangle. \tag{3.12}$$
The proof is complete since $(p^*_{\tilde{\alpha}}, 1^*_\beta)$ are Hölder conjugate. □

Proof of item (iii). Suppose that $p > N/\alpha$. The boundedness of $T := b(x, D) : h^p(\mathbb{R}^N) \to \Lambda^{\alpha - N/p}(\mathbb{R}^N)$ is analogous to the one given on (ii) and it will be included here for the sake of completeness.

In fact, it will be enough to show that for every $f \in h^p(\mathbb{R}^N)$, Tf defines a bounded linear functional in $h^q(\mathbb{R}^N)$ with $\dfrac{1}{q} := \dfrac{\alpha}{N} - \dfrac{1}{p} + 1$. Indeed, since $[h^q(\mathbb{R}^N)]^* = \Lambda^{\alpha - N/p}(\mathbb{R}^N)$ (see Section 2.2 or [[8], Theorem 5]) this will be achieved once we show that

$$|\langle Tf, \eta\rangle| \leq C\|f\|_{h^p}\|\eta\|_{h^q}, \qquad \text{for all } \eta \in h^q(\mathbb{R}^N) \ \text{ and } \ f \in h^p(\mathbb{R}^N),$$

for some positive constant C independent of f and η.

Let $0 < \tilde{\alpha} := \frac{N}{p} < \alpha$ so that $0 < \beta := \alpha - \tilde{\alpha}$. Thus $p = N/\widetilde{\alpha}$. Therefore, Proposition 1.1 (or Corollary 1.1, item (i)) and Corollary 1.1 item (ii), implies that

$$OpS^{-\tilde{\alpha}}_{1,\delta}(\mathbb{R}^N) : h^p(\mathbb{R}^N) \to \text{bmo}(\mathbb{R}^N)$$

and

$$OpS^{-\beta}_{1,\delta}(\mathbb{R}^N) : h^q(\mathbb{R}^N) \to h^{q^*_\beta}(\mathbb{R}^N), \quad \frac{1}{q^*_\beta} = \frac{1}{q} - \frac{\beta}{N} = 1.$$

The conclusion now follows from (3.12), and the duality between $h^1(\mathbb{R}^N)$ and $\text{bmo}(\mathbb{R}^N)$. □

4. Proof of the main result: Theorem 1.1

According with Proposition 1.1 it will be enough to verify the compactness. Since the proof does not depend on $\delta \in [0, 1[$, we may assume, without loss of generality, that $\delta = 0$. Fix $p \leq q < p^*_\alpha$ and $b(x, D) \in OpS^{-\alpha}_{1,0}(\mathbb{R}^N)$ such that $b(x, \xi) = 0$ for $|x| > R$. We will show that if $\{u_m\}_m$ is a bounded sequence in $h^p(\mathbb{R}^N)$ then there exist a subsequence $\{u_{m_j}\}_{m_j}$ such that $\{b(x, D)u_{m_j}\}_j$ converges in $h^q(\mathbb{R}^N)$. Consider the regularizations $b(x, D)^\varepsilon u_m := \eta_\varepsilon * b(x, D)u_m$ where $\eta \in C^\infty_c(B^1_0)$, $\int \eta(x)\, dx = 1$, $\eta_\varepsilon(x) = \varepsilon^{-N}\eta(x/\varepsilon)$ and $0 < \varepsilon \leq 1$. It is enough to show that the double sequence $\{b(x, D)^\varepsilon u_m\}_{\varepsilon,m}$ has the following two properties:

(i) For any fixed $0 < \varepsilon < 1$, the sequence $\{b(x, D)^\varepsilon u_m\}_{m\in\mathbb{N}}$ is a precompact subset of $h^q_c(B')$;

(ii) $b(x, D)^\varepsilon u_m \to b(x, D)u_m$ in $h^q_c(B')$ *uniformly* in m as $\varepsilon \searrow 0$.

Here B' is a closed ball that contains the support of all $b(x, D)^\varepsilon u_m$ and $h^q_c(B') := h^q(\mathbb{R}^N) \cap \mathscr{E}'(B')$. Since the inclusion $C_c(B') \subset h^q_c(B')$ is continuous, the property (i) will follow once we proof that $\{b(x, D)^\varepsilon u_m\}_m$ is a precompact subset of $C_c(B')$. We claim that for each $\varepsilon > 0$, $\{b(x, D)^\varepsilon u_m\}$ is uniformly bounded and equicontinuous. In fact, from the fact that the dual of the localizable Hardy space $h^p(\mathbb{R}^N)$ is the

Hölder space Λ^γ for $\gamma = N\left(\frac{1}{p} - 1\right)$ — or $(h^1(\mathbb{R}^N))^* = \mathrm{bmo}(\mathbb{R}^N)$ — we have

$$\begin{aligned} |b(x,D)^\varepsilon u_m(x)| &= |\langle b(x,D)u_m, \eta_\varepsilon(x-\cdot)\rangle| \\ &\leq \|b(x,D)u_m\|_{h^p(\mathbb{R}^N)}\|\eta_\varepsilon\|_{\Lambda^\gamma(\mathbb{R}^N)} \\ &\leq C_1(p,N,\alpha)\|u_m\|_{h^p(\mathbb{R}^N)}\|\eta_\varepsilon\|_{\Lambda^\gamma(\mathbb{R}^N)} \\ &\leq C_2(p,N,\alpha,\eta)\varepsilon^{-(N+\gamma)}\|u_m\|_{h^p(\mathbb{R}^N)} \end{aligned}$$

and analogously

$$\begin{aligned} |\nabla b(x,D)^\varepsilon u_m(x)| &\leq \|b(x,D)u_m\|_{h^p(\mathbb{R}^N)}\|\nabla\eta_\varepsilon\|_{\Lambda^\gamma(\mathbb{R}^N)} \\ &\leq C_3(p,N,\alpha,\eta)\varepsilon^{-(N+1+\gamma)}\|u_m\|_{h^p(\mathbb{R}^N)}. \end{aligned}$$

The conclusion follows from Arzelà-Ascoli theorem. To prove (ii) we will first consider the identity

$$\begin{aligned} b(x,D)^\varepsilon u_m(x) - b(x,D)u_m(x) &= \int_0^\varepsilon \frac{\partial}{\partial s}\left(b(x,D)u_m * \eta_s\right)(x)\,ds \\ &= -\int_0^\varepsilon \left\{b(x,D)u_m * \left(\nabla[x\cdot\eta]_s\right)\right\}(x)\,ds. \end{aligned}$$

In fact

$$\begin{aligned} \frac{\partial}{\partial s}\big(b(x,D)u_m * \eta_s\big)(x) &= \frac{\partial}{\partial s}\int b(x,D)u_m(x-y)\eta_s(y)\,dy \\ &= \int b(x,D)u_m(x-y)\left\{\frac{\partial}{\partial s}\left(\frac{1}{s^N}\eta\left(\frac{y}{s}\right)\right)\right\}dy \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{1}{s^N}\eta\left(\frac{y}{s}\right)\right) &= -N\frac{1}{s^{N+1}}\eta\left(\frac{y}{s}\right) - \frac{1}{s^N}\sum_k(\partial_k\eta)\left(\frac{y}{s}\right)\frac{y_k}{s^2} \\ &= -\frac{1}{s}\left(N\frac{1}{s^N}\eta\left(\frac{y}{s}\right) + \frac{1}{s^N}\sum_k(\partial_k\eta)\left(\frac{y}{s}\right)\frac{y_k}{s}\right) \\ &= -\frac{1}{s}\sum_k\left(\frac{1}{s^N}\eta\left(\frac{y}{s}\right) + \frac{1}{s^N}(\partial_k\eta)\left(\frac{y}{s}\right)\frac{y_k}{s}\right) \\ &= -\frac{1}{s}\sum_k\frac{1}{s^N}\partial_k\left[\eta\left(\frac{y}{s}\right)y_k\right] \\ &=: -\nabla[y\cdot\eta]_s(y). \end{aligned}$$

Now, define the function $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ by

$$h(\beta,t,s,x) := t^{\beta-1}[\Gamma_\beta(t,D)(x\cdot\eta)]_s, \tag{4.1}$$

where $\Gamma_\beta(t,\xi) := (2\pi i)^{-1}\sum_{k=1}^N \xi_k(t^2+4\pi^2|\xi|^2)^{-\frac{\beta}{2}}$ and

$$\begin{aligned}(J_\beta\circ\nabla)g_t(x) &:= \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\,\Gamma_\beta(1,\xi)\,\widehat{g_t}(\xi)\,d\xi\\ &= \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\,\Gamma_\beta(1,\xi)\,\widehat{g}(t\xi)\,d\xi\\ &= t^{-N}\int_{\mathbb{R}^N} e^{2\pi i x\cdot\frac{\xi}{t}}\,\Gamma_\beta(1,\xi/t)\,\widehat{g}(\xi)\,d\xi\\ &= t^{\beta-1-N}\int_{\mathbb{R}^N} e^{2\pi i \frac{x}{t}\cdot\xi}\,\Gamma_\beta(t,\xi)\,\widehat{g}(\xi)\,d\xi\\ &= t^{\beta-1}[\Gamma_\beta(t,D)g]_t(x). \qquad (4.2)\end{aligned}$$

Thus, it follows first from (4.2) and then from (4.1), that

$$\begin{aligned}\big(b(x,D)^\varepsilon u_m - b(x,D)u_m\big)(x) &= -\int_0^\varepsilon \{b(x,D)u_m * \nabla[x\cdot\eta]_s\}(x)\,ds\\ &= -\int_0^\varepsilon \big\{(J_{-\beta}\circ b(x,D))u_m * \big((J_\beta\circ\nabla)\{[x\cdot\eta]_s\}\big)\big\}(x)\,ds\\ &= -\int_0^\varepsilon \{b_{\beta-\alpha}(x,D)u_m * h(\beta,s,s,x)\}(x)ds, \qquad (4.3)\end{aligned}$$

where $b_{\beta-\alpha}(x,D) := J_{-\beta}\circ b(x,D) \in OpS^{\beta-\alpha}_{1,0}(\mathbb{R}^N)$ vanishes identically for $|x|>R$ and for any $\beta>0$ (that will be chosen later). Note that the convolution in the last integral in (4.3) is given by

$$s^{\beta-1}\int_{\mathbb{R}^n} \Gamma_\beta(s,D)\{y\cdot\eta\}(y)\cdot b_{\beta-\alpha}(x,D)u_m(x-sy)\,dy. \qquad (4.4)$$

Convolving both sides of (4.3) with $\varphi_\rho(x)=\rho^{-N}\varphi(x/\rho)$, $0<\rho<1$, keeping in mind (4.4), we have

$$\begin{aligned}(4.5)\quad &\big(\varphi_\rho * [b(\cdot,D)^\varepsilon u_m - b(\cdot,D)u_m]\big)(x) =\\ &= -\int_0^\varepsilon\int_{K\subset\mathbb{R}^N} s^{\alpha-1}\Big\{[\Gamma_\alpha(s,D)(y\cdot\eta)](y)\times\\ &\qquad\times[\varphi_\rho * b_{\beta-\alpha}(\cdot,D)u_m](x-sy)\Big\}dy\,ds\end{aligned}$$

where K is a compact set that contains the support of the function $(\varphi_\rho * b_{\beta-\alpha}(\cdot,D)u_m)(x-sy)$, for all $|x|<R$ and $0<s<1$. Write

$$(\varphi_\rho * b_{\beta-\alpha}(\cdot,D)u_m)(x-sy) = \int \varphi_\rho(x-x'-sy)b_{\beta-\alpha}(\cdot,D)u_m(x')\,dx'.$$

Using the integral estimate (see the proof in Subsection 4.1 bellow)

$$\int_0^\varepsilon \int_{K\subset\mathbb{R}^N} s^{\beta-1}\left[\Gamma_\beta(s,D)(y\cdot\eta)\right] dy\, ds < C\varepsilon^\gamma, \tag{4.6}$$

where $C = C(\eta, K) > 0$, $\gamma = \gamma(N, \alpha) > 0$ and $\{\varphi(\cdot - sy) : |s|, |y| \leq 1\}$ is a bounded subset of $C_c^\infty(\mathbb{R}^N)$, we obtain

$$|\varphi_\rho * (b(x,D)^\varepsilon u_m - b(x,D)u_m)(x)| \;\leq\; C\,\varepsilon^\gamma\, \mathfrak{m}[b_{\beta-\alpha}(x,D)u_m](x)$$

and taking the supremum for $0 < \rho < 1$, we obtain

$$m_\varphi[b(x,D)^\varepsilon u_m - b(x,D)u_m](x) \;\leq\; C\,\varepsilon^\gamma\, \mathfrak{m}[b_{\beta-\alpha}(x,D)u_m](x).$$

This allow us to estimate

$$\begin{aligned} \|b(x,D)^\varepsilon u_m - b(x,D)u_m\|_{h^q} &= \|m_\varphi\,[b(x,D)^\varepsilon u_m - b(x,D)u_m]\,\|_{L^q} \\ &\lesssim \varepsilon^\gamma\,\|\mathfrak{m}[b_{\beta-\alpha}(x,D)u_m]\|_{L^q} \\ &\lesssim \varepsilon^\gamma \|b_{\beta-\alpha}(x,D)u_m\|_{h^q}. \end{aligned} \tag{4.7}$$

It is at this point that we choose β. In fact, let $\beta := \alpha - \mu$ with $\mu := N(1/p - 1/q)$ thus $\mu \in [0, \alpha[$ and $p^*_\mu = q$. Then it follows that $b_{\beta-\alpha}(x,D) \in OpS^{-\mu}_{1,0}(\mathbb{R}^N)$ and from Proposition 1.1 we conclude that $b_{\beta-\alpha}(x,D)$ is (h^p, h^q) bounded. In particular, we can further estimate the expression in (4.7) by

$$\|b(x,D)^\varepsilon u_m - b(x,D)u_m\|_{h^q} \lesssim \varepsilon^\gamma\,\|u_m\|_{h^p} \tag{4.8}$$

and this concludes the proof of (ii).

To finish the proof, we claim that for a given $\delta > 0$, there exist a subsequence $\left\{b(x,D)u_{m_j}\right\}_j \subset \{b(x,D)u_m\}_m$ such that

$$\limsup_{j,k\to\infty} \|b(x,D)u_{m_j} - b(x,D)u_{m_k}\|_{h^q} \leq \delta. \tag{4.9}$$

Indeed, for $\varepsilon > 0$ sufficiently small, we have

$$\|b(x,D)^\varepsilon u_m - b(x,D)u_m\|_{h^q} \leq \delta/2 \tag{4.10}$$

uniformly in m. Since $\{b(x,D)u_m\}_m$ and $\{b(x,D)^\varepsilon u_m\}_m$ are supported in a closed ball B', by Arzelà-Ascoli theorem there exist a subsequence $\left\{b(x,D)^\varepsilon u_{m_j}\right\}_j$ which converges uniformly in B'. In particular,

$$\limsup_{j,k\to\infty} \|b(x,D)^\varepsilon u_{m_j} - b(x,D)^\varepsilon u_{m_k}\|_{h^q} = 0. \tag{4.11}$$

Note that (4.9) is a consequence of (4.10) and (4.11). Thus, we can use (4.9) for $\delta = 1/n$ for $n = 1, 2, 3, ...$ and the diagonal process to extract a convergent subsequence $\{b(x,D)u_{m_l}\}_l$, as we wished to prove. □

4.1. **Integral estimate.** Let us now verify the validity of the estimate in (4.6). Write $g(y) := y \cdot \eta(y)$. Suppose first that $\beta \geq 1$, in this case $\Gamma_\beta(1, D) \in OpS^{1-\beta}_{1,0}(\mathbb{R}^N) \subset OpS^0_{1,0}(\mathbb{R}^N)$ is (L^r, L^r) bounded for every $1 < r < \infty$. Choosing $1 < r < N/(N-1)$, we have

$$\begin{aligned}
&\left| \int_0^\varepsilon \int_{K\subset\mathbb{R}^N} s^{\beta-1}[\Gamma_\beta(s,D)g](y)dyds \right| \\
&\qquad = \left| \int_0^\varepsilon \int_{K\subset\mathbb{R}^N} s^N[\Gamma_\beta(1,D)g_s](sy)dyds \right| \\
&\qquad = \left| \int_0^\varepsilon \left(\int_{\tilde{K}\subset\mathbb{R}^N} [\Gamma_\beta(1,D)g_s](y)dy \right) ds \right| \\
&\qquad \leq c(\tilde{K}) \int_0^\varepsilon \|\Gamma_\beta(1,D)g_s\|_{L^r} ds \\
&\qquad \lesssim \int_0^\varepsilon \|g_s\|_{L^r} ds \\
&\qquad \lesssim \|g\|_{L^r} \int_0^\varepsilon s^{\frac{N}{r}-N} ds \\
&\qquad \lesssim \varepsilon^\gamma
\end{aligned}$$

for $\gamma := \frac{N}{r} - N + 1 > 0$. We note that if $\beta > 1$ then $\Gamma_\beta(1, D)$ has negative order thus we may simply choose $r = 1$ (see [17, Theorem 6.1]).

Now consider the case where $0 < \beta < 1$. Note that the multiplier operator $(-\partial_k)^\sigma$ defined by $\mathscr{F}[(-\partial_k)^\sigma g](\xi) := (\xi_k)^\sigma\, \widehat{g}(\xi)$ for $g \in \mathscr{S}(\mathbb{R}^N)$ satisfies the following scaling property

$$\begin{aligned}
\|(-\partial_k)^\sigma g_t\|_{L^r} &= \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}(\xi_k)^\sigma \widehat{g}(t\xi)d\xi \right|^r dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi t^{-1}}(t^{-1}\xi_k)^\sigma \widehat{g}(\xi)t^{-N}d\xi \right|^r dx \right)^{1/r} \\
&= t^{-\sigma-N} \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi t^{-1}}(\xi_k)^\sigma \widehat{g}(\xi)d\xi \right|^r dx \right)^{1/r} \\
&= t^{-\sigma} \|[(-\partial_k)^\sigma g]_t\|_{L^r} \\
&= t^{-\sigma+\frac{N}{r}-N} \|(-\partial_k)^\sigma g\|_{L^r},
\end{aligned}$$

for any $1 \leq r < \infty$. In addition, we may write

$$\Gamma_\beta(1, D) = \sum_{k=1}^N (J_\beta \circ (-\partial_k)^\beta) \circ (-\partial_k)^{1-\beta}.$$

where the symbol $J_\beta \circ (-\partial_k)^\beta(\xi) := (2\pi i)^{-1}\xi_k^\beta(1+4\pi^2|\xi|^2)^{-\beta/2}$ defines a multiplier operator that is bounded from $L^r(\mathbb{R}^N)$ to itself for any $1 < r < \infty$ (Mikhlin-Hörmander theorem see [19, Theorem 3]).

Therefore, if $1 < r < N/(N-\beta)$, we have

$$\begin{aligned}
\int_0^\varepsilon \|\Gamma_\beta(1,D)g_s\|_{L^r}\,ds &\leq \sum_{k=1}^N \int_0^\varepsilon \left\|(J_\beta \circ (-\partial_k)^\beta)\circ(-\partial_k)^{1-\beta}g_s\right\|_{L^r} ds \\
&\lesssim \sum_{k=1}^N \int_0^\varepsilon \left\|(-\partial_k)^{1-\beta}g_s\right\|_{L^r} ds \\
&= \sum_{k=1}^N \left\|(-\partial_k)^{1-\beta}g\right\|_{L^r} \int_0^\varepsilon s^{\frac{N}{r}-N-(1-\beta)}ds \\
&\lesssim \varepsilon^\gamma,
\end{aligned}$$

with $\gamma := \frac{N}{r} - N + \beta > 0$. This finishes the proof of (4.6).

5. Localizable Hardy-Sobolev spaces

Let $\alpha \geq 0$ and $0 < p < \infty$. We say that a tempered distribution $u \in h^p(\mathbb{R}^N)$ belongs to nonhomogeneous localizable Hardy-Sobolev space $h^{\alpha,p}(\mathbb{R}^N)$ if $u = J_\alpha g$ with for some $g \in h^p(\mathbb{R}^N)$. The space is endowed with the semi norm $\|u\|_{h^{\alpha,p}} := \|g\|_{h^p(\mathbb{R}^N)}$. Clearly it is well defined since $J_\alpha g_1 = J_\alpha g_2$ if only if $g_1 = g_2$ in $\mathscr{S}'(\mathbb{R}^N)$ (see [19, p.135]). Again, using the notation $J_{-\alpha} \in OpS^\alpha_{1,\delta}(\mathbb{R}^N)$ (the inverse of Bessel potential operator), we may write formally $g = J_{-\alpha}u$. When $\alpha = k \in \mathbb{Z}^+$ we have the equivalence

$$\|u\|_{h^{k,p}} \cong \sum_{|\beta|\leq k} \|\partial^\beta u\|_{h^p}, \tag{5.1}$$

and we naturally recover the Sobolev spaces $W^{k,p}(\mathbb{R}^N)$ for $1 < p < \infty$ (see [20, Section 5.2.1] for details). We point out that the spaces $h^{\alpha,p}(\mathbb{R}^N)$ can be characterized via general elliptic pseudodifferential operators in the class $OpS^\alpha_{1,\delta}(\mathbb{R}^N)$. Recall that $a(x,D) \in OpS^m_{1,\delta}(\mathbb{R}^N)$ is said to be elliptic if, for some $0 < \gamma < \infty$ the symbol $a(x,\xi)$ satisfies $|a(x,\xi)| \geq C\langle\xi\rangle^m$ for $|\xi| \geq \gamma$ and for all $x \in \mathbb{R}^N$.

Proposition 5.1. *Let $\alpha \geq 0$ and $a(x,D) \in OpS^\alpha_{1,\delta}(\mathbb{R}^N)$ be an elliptic pseudodifferential operator with $0 \leq \delta < 1$. Then*

$$h^{\alpha,p}(\mathbb{R}^N) = \left\{u \in \mathscr{S}'(\mathbb{R}^N) \;:\; u,\; a(x,D)u \in h^p(\mathbb{R}^N)\right\}.$$

Proof: ($\subset$) Since J_α is invertible and $a(x,D)\circ J_\alpha$ has order zero (therefore maps continuously $h^p(\mathbb{R}^N)$ to itself), we have

$$\|a(x,D)u\|_{h^p} \leq C\|u\|_{h^{\alpha,p}}.$$

($\supset$) Conversely, let $Q(x,D)\in OpS^{-\alpha}_{1,\delta}(\mathbb{R}^N)$ be a parametrix of $a(x,D)$, i.e. there exist a pseudodifferential operator $R(x,D)\in OpS^{-\infty}(\mathbb{R}^N)$ such that

$$u = Q(x,D)a(x,D)u + R(x,D)u, \quad \forall\, u\in C^\infty(\mathbb{R}^N).$$

Since $J_{-\alpha}\circ Q(x,D)$ has order zero and $R(x,D)$ is a regularizing operator, we obtain

$$\|J_{-\alpha}u\|_{h^p} \lesssim \|a(x,D)u\|_{h^p} + \|u\|_{h^p},$$

concluding the proof. □

Corollary 5.1. *Let $N\geq 2$, $k\in\mathbb{Z}^*_+$ such that $0\leq k<N$ and $p^*_k = pN/(N-kp)$ for $0<p\leq 1$. Then there exists a constant $C>0$ such that*

$$\|u\|_{h^{p^*_k}} \leq C\sum_{|\beta|\leq k}\|\partial^\beta u\|_{h^p}. \tag{5.2}$$

Proof: Follows from the equivalence (5.1) and Proposition 1.1. □

Let us continue to explore the case where $\alpha = k$ is a positive integer. Inspired in the homogeneous Sobolev-Gagliardo-Nirenberg inequality we can ask about the validity of the following estimate

$$\|u\|_{h^{p^*}} \leq C\sum_{|\beta|= k}\|\partial^\beta u\|_{h^p}. \tag{5.3}$$

Although this estimate holds for $H^p(\mathbb{R}^N)$ it was shown (see [15, Remark 2.7]) that it fails for $h^p(\mathbb{R}^N)$. On the other hand, in [9], the authors proved that for $k=1$, estimate (5.3) holds if we restrict to the subspace $h^p_c(B)\subset h^p(\mathbb{R}^N)$, of tempered distributions supported in a fix ball B. For B a generic ball and $k\in\mathbb{Z}^*_+$ we denote $\dot{h}^{k,p}_c(B)$ as the subspace of all $u\in\mathscr{E}'(B)$ such that $\partial^\beta u\in h^p(\mathbb{R}^N)$ for $|\beta|=k$. Consider $0\leq k<N$ to be an integer, $0<p\leq 1$ and p^*_k as before, then by induction on k we have:

(i) there is a continuous embedding $\dot{h}^{k,p}_c(B)\subset h^{p^*_k}_c(B)$;
(ii) for $p\leq q<p^*_k$, the embedding $\dot{h}^{k,p}_c(B)\subset\subset h^q_c(B)$ is compact.

As an application of Theorem 1.1 we present a natural extension of compactness embedding results for homogeneous localizable Hardy-Sobolev space $\dot{h}^{k,p}_c(B)$.

5.1. **Rellich-Kondrachov compactness theorem for $h_c^{\alpha,p}(B)$.** For $B = B(x_0, \ell)$ a fixed ball let $\widetilde{B} = B(x_0, 2\ell)$ the ball with same center as B but twice the radius. Let $\psi(x) \in C_c^\infty(\widetilde{B})$ satisfying $\psi(x) \equiv 1$ on B and define $\Lambda_\alpha := \Lambda_\alpha(x, D)$ the pseudodifferential operator with symbol $\lambda_\alpha(x,\xi) = \psi(x)\langle\xi\rangle^\alpha$. Denote by $h_c^{\alpha,p}(B)$ the set of distributions $u \in \mathscr{E}'(B)$ such that $\Lambda_\alpha u \in h^p(\mathbb{R}^N)$ equipped with the semi-norm $\|u\|_{h_c^{\alpha,p}(B)} := \|\Lambda_\alpha u\|_{h^p(\mathbb{R}^N)}$. Note that the space $h_c^{\alpha,p}(B)$ is independent of the choice of $\psi(x)$; i.e, if $\psi_2(x), \psi_1(x) \in C_c^\infty(\widetilde{B})$ satisfies $\psi_1(x) = \psi_2(x) \equiv 1$ on B then $\|\Lambda_{\alpha,\psi_1} u\|_{h^p} \cong \|\Lambda_{\alpha,\psi_2} u\|_{h^p}$. In view of the previous comment, we have the inclusion $h_c^{\alpha,p}(B) \subset h_c^p(\tilde{B})$.

Proof of Corollary 1.2: The first item is a direct consequence of the Proposition 1.1 while the second follows from Theorem 1.1 applied to the pseudodifferential operator with symbol $\lambda_\alpha(x,\xi) = \psi(x)\langle\xi\rangle^\alpha$. □

Under the compact support assumption, we have the following consequence.

Corollary 5.2. *Let $N \geq 2$, $0 < \alpha < N$ and $\frac{N}{N+\alpha} < p \leq 1$. Then for each $\epsilon > 0$ there exists a ball $B = B_0^s$ with $s = s(\epsilon) > 0$ such that*

$$\|u\|_{h^p(\mathbb{R}^N)} \leq \epsilon \|u\|_{h_c^{\alpha,p}(B)}, \quad \forall\ u \in C_c^\infty(B). \tag{5.4}$$

Proof: The proof is by contradiction. Suppose that there exists $\varepsilon_0 > 0$ such that, for each $0 < s \leq 1$, there exists $\phi_s \in C_c^\infty(B_0^s)$ satisfying

$$\|\phi_s\|_{h^p(\mathbb{R}^N)} \geq \varepsilon_0 \|\Lambda_\alpha \phi_s\|_{h^p(\mathbb{R}^N)}.$$

Now, we can use the fact that $\frac{N}{N+\alpha} < p$, which implies $p_\alpha^* > 1$, and choose $1 < q < p_\alpha^*$, to obtain

$$C\|\phi_s\|_{L^q} \geq \|\Lambda_\alpha \phi_s\|_{h^p}.$$

Taking $s = 1/n$, $n = 1, 2, 3, \ldots$, we get a sequence $\{\phi_n\} \subset C_c^\infty(B_0^{1/n})$ such that, after renormalization, it will satisfy

$$\|\phi_n\|_{L^q} = 1 \quad \text{and} \quad \|\Lambda_\alpha \phi_n\|_{h^p} \leq C.$$

By the Rellich-Kondrachov theorem for $h_c^{\alpha,p}(B)$, one can find a subsequence $\{\phi_n\}$ and $\phi \in h^q(\mathbb{R}^N) = L^q(\mathbb{R}^N)$ such that $\phi_n \to \phi$ in $L^q(\mathbb{R}^N)$. Hence $\|\phi\|_{L^q} = 1$ although $\operatorname{supp}\phi \subset \{0\}$, a contradiction. □

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Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brasil
E-mail address: hoepfner@dm.ufscar.br

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brasil
E-mail address: rkapp@dm.ufscar.br

Departamento de Computação e Matemática, Universidade de São Paulo, Ribeirão Preto, SP, 14040-901, Brasil
E-mail address: picon@ffclrp.usp.br