# STRONG UNIQUENESS RESULTS FOR FIRST-ORDER PLANAR EQUATIONS

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Abstract. We discuss strong uniqueness results in the forward Cauchy Problem for a class of complex fist-order equations with Hölder coefficients defined in the plane. We use an appropriate variant of the similarity principle in order to reduce the original question to a local version of Riesz's uniqueness theorem for holomorphic functions.

Contents



## 1. Introduction

Let

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$$

be a planar vector field defined on a domain $\Omega \subset \mathbb{R}^2$ that contains the origin, where $a(x,t)$ is a continuous complex function. Consider the initial value problem

$$\begin{cases} Lu = 0 & \text{for } t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

One says that there is uniqueness in the local forward Cauchy problem for (1.1) if $u_0 \equiv 0$ implies that $u(x,t)$ vanishes identically on a neighborhood of the set $\{t = 0\}$. In 1960, P. Cohen [Co] (see also [Hor, Theorem 8.9.2]) gave an example with $a(x,y)$ and $u(x,t)$ smooth in $\mathbb{R}^2$ such that the equation (1.1) is satisfied with $u_0(x) \equiv 0$ and yet the support of $u(x,t)$ is $\mathbb{R} \times [0,\infty)$.

On the other hand, uniqueness in the local one-sided Cauchy problem —which may be regarded also as local boundary uniqueness— is known to hold for smooth locally integrable vector fields in the class of distribution solutions; in this case the

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initial condition $u(x,0)$ must be understood in the sense of the trace (on the subject of locally integrable vector fields we refer to [T] and [BCH]). We recall that a smooth vector field $L$ is said to be locally integrable if for every point of $\mathsf{p}\in\Omega$ there exists a smooth solution $Z(x,t)$ of the equation $LZ=0$ defined on a neighborhood of $\mathsf{p}$ such that $dZ(\mathsf{p})\neq 0$. Any function such as $Z(x,t)$ is called a local first integral of $L$. When $L=\partial_t+i\partial_x$, the solutions of (1.1) are holomorphic functions of the complex variable $z=t+ix$ and a strong uniqueness result is known: if $u$ is bounded and $u_0(x)$ (which is necessarily a bounded function) vanishes on a set of positive measure then $u\equiv 0$. This type of local strong uniqueness for bounded solutions was extended to a class of smooth locally integrable vector fields in [BH].

In this work we will study strong uniqueness results for the first-order equation

$$\begin{cases} Lu = Au + B\overline{u} \text{ for } t>0,\\ u(x,0)=u_0(x),\end{cases} \tag{1.2}$$

where $L$ is a degenerate elliptic operator with Hölder continuous coefficients. It will be assumed that $L$ degenerates at $t=0$ and $A$ and $B$ will be functions in $L^p$ for some $p$ that depends on the degeneracy type of $L$ (*cf.* Definition 3.1). We recall that when $L$ is the Cauchy-Riemann operator, solutions of the homogeneous PDE (1.2) are called generalized analytic functions [V] or pseudoanalytic functions [Bers]. One motivation for considering this problem is that equations of this type when $L$ is an elliptic degenerate vector field whose characteristic set $\Sigma$ is a curve and is such that around any point $\mathsf{p}\in\Sigma$ it has a first integral that —in appropriate coordinates— may be expressed as

$$Z(x,t)=x+i\left(\frac{t|t|^{\sigma}}{\sigma+1}+\beta(x)\right), \tag{1.3}$$

where $\sigma>0$ is real and $\beta(x)$ is real and of class $C^{1+\alpha}$ for some $0<\alpha<1$, have been considered in several recent papers in connection with various topics, including the study of the Riemann-Hilbert problem, the Similarity Principle and the solvability of the equation $Lu=f$ in the torus ([CDM1], [CDM2], [CDM3], [CM], [MZ]).

One of the main results proved in this work is:

**Theorem**. *Suppose that $A,B\in L^p(I\times(0,T))$, $p>2+\tau/(1-\tau)$, and $u_0\in L^1(I)$, where $\tau\in(0,1)$ is the degeneracy type of $L$. Suppose that $u\in L^q(I\times(0,T))$, $q=p/(p-1)$, satisfies the forward Cauchy problem (1.2). If $u_0\in L^1(I)$ satisfies one of the following conditions:*

(1) *$u_0$ vanishes on a subset of $I$ of positive measure, or;*
(2) *$u_0$ has a zero of exponential order in $I$ (i.e., there exists $x_0\in I$, such that $|u_0(x)|=O(e^{-c/|x-x_0|})$, $x\to x_0$, for some $c>0$.*

*Then $u\equiv 0$ in $I\times(0,T)$.*

The organization of the paper is as follows. In Section 2, we begin with the most favorable situation which occurs when $L$ has smooth coefficients and $A=B=0$. In this case the way in which the ellipticity of $L$ degenerates at the characteristic set $\Sigma=\{t=0\}$ is irrelevant. Our arguments give a strong uniqueness local result for holomorphic functions which is one of the key tools in the proofs of all the uniqueness results of this paper. In Section 3, we take up the study of the general equation (1.2), introduce the notion of finite degeneracy type and state and prove the main uniqueness result. In Section 4, we show some applications of the main uniqueness result to solutions of semilinear equations and approximate solutions. In Section 5, we prove the existence of good one-sided first integrals, whose role is essential in Sections 2 and 3, and give some examples of vector fields to which our results apply. In Section 6, we refine the notion of degeneracy type by defining a real valued function $\tau_{\Sigma^+}$ on the characteristic set $\Sigma$ associated to one of the sides locally determined by $\Sigma$. This function is an invariant that only depends on the line bundle spanned by the vector field $L$. Finally, in Section 7, we present an appropriate version of the similarity principle for the class of vector fields considered in Section 3; it is this version which allows us to link the uniqueness results proved for holomorphic functions in Section 2 with uniqueness for the solutions of the problem (1.2).

## 2. Strong Uniqueness for smooth locally integrable vector fields

In this section we will focus on strong uniqueness for the one-sided Cauchy problem for the equation $Lu = 0$ when $L$ is a locally integrable complex vector field which is elliptic off a smooth curve $\Sigma$ transversal to $L$ and the initial condition is given on $\Sigma$. The main result of this section is Theorem 2.1; it follows from an application of Theorem 2.2 which deals with a local boundary problem version of the classic Cauchy-Riemann equation.

After appropriate change of local coordinates (see Section 5) we may assume that we are in the following situation. Let $\Omega$ be an open set where a smooth locally integrable vector field $L$ is defined, let $I \subset \mathbb{R}$ be an bounded open interval and $T > 0$ such that $\overline{I} \times [-T, T] \subset \Omega$. Suppose that there is a smooth function

$$Z(x,t) = x + i\varphi(x,t), \quad (x,t) \in \Omega$$

such that

- $LZ(x,t) = 0$ *on* $\overline{I} \times [0,T]$;
- $\varphi(x,t)$ *is real valued and* $\varphi(\cdot,0) = \varphi_t(\cdot,0) = 0$;
- $\Sigma = \{(x,t) \in \Omega \ ; \ d\mathrm{Re}\, Z \wedge d\mathrm{Im}\, Z(x,t) = 0\} = \{(x,t) \in \Omega \ ; \ t = 0\}$.

Such a function will be called a *good one-sided first integral* for $L$. We will prove in Proposition 5.2 that good one-sided first integrals for $L$ always exist. Since $\varphi_t(x,t)$ does not change sign in $I \times (0,T)$ we may assume without loss of generality that $\varphi_t(x,t) > 0$ if $0 < t < T$ and hence $t \mapsto \varphi(x,t) > 0$ is positive and increasing in $(0,T)$ for $x \in I$.

After multiplying $L$ by a nonvanishing factor, we may write $L$ as

$$L = \frac{\partial}{\partial t} - \frac{Z_t}{Z_x}\frac{\partial}{\partial x} \quad (x,t) \in I \times (0,T),$$

and consider the initial value problem

$$\frac{\partial u}{\partial t} - \frac{Z_t}{Z_x}\frac{\partial u}{\partial x} = 0, \quad (x,t) \in I \times (0,T), \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad x \in I. \tag{2.2}$$

Since $L$ is elliptic for $t > 0$, $u(x,t)$ is smooth on $I \times (0,T)$. Condition (2.2) will be understood in the sense of distributions:

$$\lim_{\varepsilon \searrow 0} \int_I u(x,\varepsilon)\phi(x)\, dx = \int_I u_0(x)\phi(x)\, dx, \quad \phi \in C_c^\infty(I), \tag{2.3}$$

and we will suppose that $u_0(x) \in L^1(I)$. Note that when $u(x,t)$ is continuous on $I \times [0,T)$, (2.3) is equivalent to (2.2) in the usual pointwise sense, that is, $u_0$ is the restriction of $u$ to the set $t = 0$. In other words, the trace function $(0,T) \ni t \mapsto T_t u \in \mathcal{D}'(I)$, which is well known to be continuous, admits a continuous extension to the interval $[0,T)$ and its value at $t = 0$ which we denote by $T_0 u$ is given by $u_0$. When $L$ is the Cauchy-Riemann operator it is customary to denote $T_0 u$ by $bu$.

We now state the main result of this section.

**Theorem 2.1.** *Suppose that $u(x,t)$ satisfies* (2.1), (2.2) *with $u_0 \in L^1(I)$. If $u(x,t)$ satisfies one of the two following properties:*

(1) *$u_0(x)$ vanishes in a set of positive measure of $I$, or*
(2) *$u_0(x)$ has a zero of exponential order at $x_0 \in I$, i.e., $|u_0(x)| = O(e^{-c/|x-x_0|})$, as $x \to x_0$, for some $c > 0$.*

*Then $u \equiv 0$.*

The special case where $L$ is the Cauchy-Riemann operator gives

**Theorem 2.2.** *Let $\mathcal{I} \subset \mathbb{R}$ be bounded interval and $\lambda > 0$. Let $h$ be a holomorphic function defined in $\mathcal{I} + i(0,\lambda)$ and assume that it has a weak boundary value $h_0 \in L^1(\mathcal{I})$, i.e.,*

$$\lim_{\eta \searrow 0} \int_{\mathcal{I}} h(\xi + i\eta)\psi(\xi)d\xi = \int_{\mathcal{I}} h_0(\xi)\psi(\xi)d\xi, \quad \psi \in C_c^\infty(\mathcal{I}).$$

*If one of the following items holds*

(1') *$h_0$ is zero in a set of positive measure in $\mathcal{I}$;*
(2') *$h_0$ has a zero of exponential order in $\mathcal{I}$;*

*then, $h \equiv 0$.*

Conversely, Theorem 2.1 can be deduced from an application of Theorem 2.2 and we will prove it first. We shall need the following lemma.

**Lemma 2.1.** *Let $\mathcal{I}$, $\lambda$ be as in the theorem above. Let $h$ be a holomorphic function defined in $\mathcal{I} + i(0,\lambda)$ and let $h_0 \in L^1(\mathcal{I})$ be such that for each $\psi \in C_c^\infty(\mathcal{I})$ it holds*

$$\lim_{\eta \searrow 0} \int_{\mathcal{I}} h(\xi + i\eta)\psi(\xi)d\xi = \int_{\mathcal{I}} h_0(\xi)\psi(\xi)d\xi. \tag{2.4}$$

*Then, given a compactly contained interval $\mathcal{J} \subset\subset \mathcal{I}$, there exists the non-tangential limit*

$$\lim_{(\xi',\eta') \to (\xi,0)} h(\xi' + i\eta') = h_0(\xi), \tag{2.5}$$

*for almost every $\xi \in \mathcal{J}$, and*

$$\lim_{\delta \searrow 0} \int_{\mathcal{J}} |h(\xi + i\delta) - h_0(\xi)|d\xi = 0. \tag{2.6}$$

Proof: Let $\mathcal{K}$ be an open interval compactly contained in $\mathcal{I}$ such that $\mathcal{J}$ is compactly contained in $\mathcal{K}$. We choose $v(\xi) \in L^1_c(\mathcal{I})$ that coincides with $h_0(\xi)$ in $\overline{\mathcal{K}}$ and let $V : \{(\xi,\eta) \in \mathbb{R}^2 \ ; \ \eta \geq 0\} \to \mathbb{C}$ be its harmonic extension to the upper half-plane given by

$$V(\xi,\eta) = (P_\eta * v)(\xi), \text{ for } \eta > 0 \text{ and } V(\xi,0) = v(\xi),$$

where

$$P_\eta(\xi) = \frac{1}{\pi}\frac{\eta}{\xi^2+\eta^2}, \quad (\xi,\eta) \in \mathbb{R}\times(0,\infty).$$

Since $v \in L^1(\mathbb{R})$ we have that the non-tangential limit

$$\lim_{(\xi',\eta')\to(\xi,0)} V(\xi',\eta') = v(\xi)$$

holds for almost every $\xi \in \mathbb{R}$ (see [S, page 197]) and

$$\lim_{\delta\searrow 0}\int_{\mathbb{R}} |V(\xi,\delta) - v(\xi)|\, d\xi = 0 \tag{2.7}$$

(see [S, page 62]). Let $\psi \in C_c^\infty(\mathcal{K})$. We have from (2.7) that

$$\lim_{\delta\searrow 0}\int_{\mathcal{K}} V(\xi,\delta)\psi(\xi)d\xi = \int_{\mathcal{K}} h_0(\xi)\psi(\xi)d\xi. \tag{2.8}$$

Consider the function $R : \mathcal{K}\times(0,\lambda) \to \mathbb{C}$ given by

$$R(\xi,\eta) = h(\xi+i\eta) - V(\xi,\eta).$$

Then $R$ is harmonic in $\mathcal{K}\times(0,\lambda)$ and (2.4) and (2.8) imply that

$$\lim_{\delta\searrow 0}\int_{\mathcal{K}} R(\xi,\delta)\psi(\xi)d\xi = 0, \quad \psi \in C_c^\infty(\mathcal{K}).$$

It follows from Schwarz's reflection principle (weak boundary value version) that there exists a harmonic function $\tilde{R} : \mathcal{K}\times(-\lambda,\lambda) \to \mathbb{C}$ such that $\tilde{R}(\xi,\eta) = R(\xi,\eta)$ for $(\xi,\eta) \in \mathcal{K}\times(0,\lambda)$. In particular, $R$ has a smooth extension up to $\eta = 0$. Hence,

$$\lim_{(\xi',\eta')\to(\xi,0)} R(\xi',\eta') = 0 \text{ for all } \xi \in \mathcal{K}$$

and

$$\lim_{\delta\searrow 0}\int_{\mathcal{K}} |R(\xi,\delta)|\, d\xi = 0.$$

Since $h(\xi+i\eta) = V(\xi,\eta) + R(\xi,\eta)$, $(\xi,\eta) \in \mathcal{K}\times(0,\lambda)$, the proof is complete. □

We now prove Theorem 2.2.

Proof: It follows from conditions (1') and (2') above that there exists an interval $\mathcal{J} \subset\subset \mathcal{I}$ such that $h_0$ vanishes identically on a subset of positive measure of $\mathcal{J}$ or $h_0$ has a zero of exponential order in $\mathcal{J}$. From Lemma 2.1 there exist $a, b \in \mathcal{I}$ such that $\overline{\mathcal{J}} \subset (a,b)$ and there exist the non-tangential limits

$$\lim_{(\xi,\eta)\to(a,0)} h(\xi+i\eta) \text{ and } \lim_{(\xi,\eta)\to(b,0)} h(\xi+i\eta).$$

It follows that $h(\xi+i\eta)$ is bounded on a pair of truncated cones with vertices at the points $(a,0)$ and $(b,0)$ respectively. Write $\mathcal{Q} := (a,b)+i(0,\lambda/2)$, let $\delta = (b-a)/4$ and consider the trapezoid $\mathcal{T}$ with vertices $\mathsf{p}_1 = a+i0$, $\mathsf{p}_2 = b+i0$, $\mathsf{p}_3 = (b-\delta)+i\lambda/8$, $\mathsf{p}_4 = (a+\delta)+i\lambda/8$.

By the continuity of $h$ in $\mathcal{I}+i(0,\lambda)$ and the boundedness of $h$ on truncated cones with vertices at $\mathsf{p}_1$ and $\mathsf{p}_2$, there exists a constant $C_0 > 0$ such that

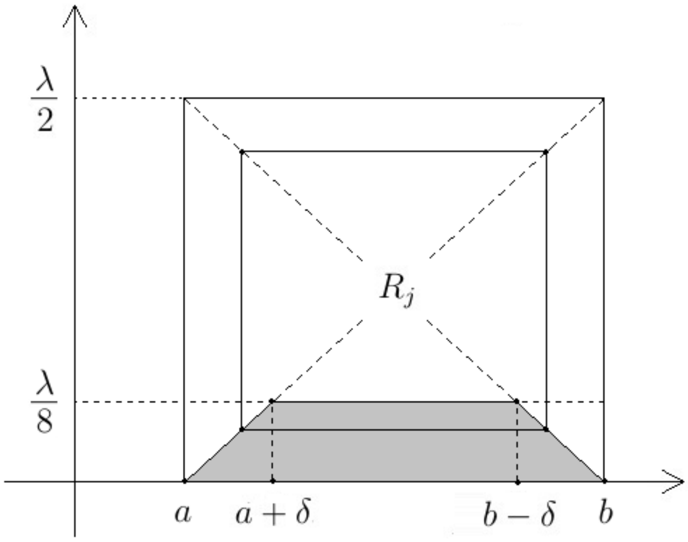
$$|h(\zeta)| \le C_0, \text{ for all } \zeta \in \mathcal{Q}\setminus\mathcal{T}. \tag{2.9}$$

On the other hand, by (2.6) in Lemma 2.1, there exists a constant $M_0 > 0$, such that

$$\int_a^b |h(\xi + i\eta)|\, d\xi \;\le\; M_0, \text{ for all } 0 < \eta < \lambda/2. \tag{2.10}$$

We now claim

(∗) *The restriction of $h$ to $\mathcal{Q}$ belongs to the Hardy space $E^1(\mathcal{Q})$.*




From the definition of $E^1(\mathcal{Q})$ ([Du, page 168]), in order to prove (∗) is enough to find a sequence of rectifiable Jordan closed curves $C_j$ that converges to the boundary in the sense that any compact set of $\mathcal{Q}$ is encompassed by $C_j$ for $j$ big enough and there exists a constant $M > 0$ such that

$$\int_{C_j} |h(\zeta)|\,|d\zeta| \le M, \quad j \to \infty.$$

For each $j \in \mathbb{N}$ consider the rectangle $R_j$ contained in $\mathcal{Q}$ with the same center and similar to $\mathcal{Q}$ whose boundary has distance equal to $\lambda/8j$ of the boundary of $\mathcal{Q}$ and let be $C_j$ the boundary of $R_j$. Then $C_j$ is a piecewise $C^1$ closed curve that converges to the boundary of $\mathcal{Q}$ in the desired sense.

By (2.9) and (2.10) it is easy to check that there exists a constant $M > 0$, such that

$$\int_{C_j} |h(\zeta)|\,|d\zeta| \le M$$

for all $j \in \mathbb{N}$. Hence, the restriction of $h$ to the rectangle $\mathcal{Q}$ belongs to the space $E^1(\mathcal{Q})$. Our next claim is

(∗∗) *The function $h$ restricted to $\mathcal{Q}$ is identically zero.*

Suppose that (1') holds. The function $h$ converges non-tangentially in almost every point of $\partial\mathcal{Q} \cap (\mathcal{J} + i\{0\})$ to $h_0$. Hence $h(\zeta) \equiv 0$ in $\mathcal{Q}$ by [Du, Theorem 10.3] since $h_0$ is zero in a set of positive measure in $\mathcal{J}+i\{0\}$. For the case (2'), let $\Phi(w)$ be a conformal application of the unit disc $\mathbb{D}$ onto $\mathcal{Q}$ and let $\zeta_* \in \mathcal{J}+i\{0\}$ be a zero of infinity order of $h_0$. Let $\mathbf{F}(w) = \Phi'(w)h(\Phi(w))$, $w \in \mathbb{D}$ and $\mathbf{F}_0(w) = \Phi'(w)h_0(\Phi(w))$, $w \in \partial\mathbb{D}$. Hence, $\mathbf{F}(w) \in H^1(\mathbb{D})$ (Corollary to Theorem 10.1 in [Du, page 169]) and $\mathbf{F}(w) \to \mathbf{F}_0(w)$ non-tangentially for almost every $w \in \partial\mathcal{Q}$. In particular, $\mathbf{F}(w)$ belongs to the class $N$ of Nevanlinna. It follows from ([Du, Theorem 2.2]) that

$$\int_{\partial\mathbb{D}} \log|(\mathbf{F}_0(w))|\,|dw| = -\infty \quad \Longrightarrow \quad \mathbf{F}(z) \equiv 0. \tag{2.11}$$

Let $w_* = \Phi^{-1}(\zeta_*) \in \partial\mathbb{D}$. Since $\zeta_*$ belongs to the regular part of $\partial\mathcal{Q}$, $\Phi$ is smooth at $\zeta_*$, $\Phi'(\zeta_*) \neq 0$ and $\Phi|_{\partial\mathbb{D}}$ applies diffeomorphically an arc $S \subset \partial\mathbb{D}$ around $w_*$ over a segment $(\zeta_* - \epsilon, \zeta_* + \epsilon) + i\{0\}$, $\epsilon > 0$. Hence, $\mathbf{F}(w)$ has a zero of exponential order in $\partial\mathbb{D}$ and the integral in (2.11) cannot be finite, implying that $h \equiv 0$ in $\mathcal{Q}$. It follows that $h \equiv 0$ in $\mathcal{I} + i(0, \lambda)$. $\square$

We are almost ready to prove Theorem 2.1 which will be derived from Theorem 2.2 with the help of the next lemma.

**Lemma 2.2.** *Suppose that $u(x,t)$ satisfies $Lu = 0$ in $I \times (0,T)$ and that there exists $u_0 \in L^1(I)$ such that $T_0 u = u_0$ in $\mathcal{D}'(I)$, that is,*

$$\lim_{\varepsilon \searrow 0} \int_I u(x,\varepsilon)\phi(x)\,dx = \int_I u_0(x)\phi(x)\,dx, \quad \phi \in C_c^\infty(I).$$

*Then there exists an holomorphic function $h(x+iy)$ defined in the open set $Z(I \times (0,T))$, with boundary value $bh(x)$, such that $u(x,t) = h(Z(x,t))$, $(x,t) \in I \times (0,T)$ and $bh(x) = u_0(x)$.*

Proof: We observe that $Z(x,t)$ is injective in $I \times (0,T)$ and the chain rule shows that $h \doteq u \circ Z^{-1}$ satisfies the Cauchy-Riemann equation. Hence $u(x,t) = h(x + i\varphi(x,t))$ in $I \times (0,T)$ and it remains to show that (2.4) holds. It follows from (2.3) that

$$\lim_{\varepsilon \searrow 0} \int_I h(x + i\varphi(x,\varepsilon))\phi(x)\,dx = \int_I u_0(x)\phi(x)\,dx, \quad \phi \in C_c^\infty(I).$$

On the other hand, it comes from the definition of $bh$ that

$$\lim_{\varepsilon \searrow 0} \int_I h(x + i\varepsilon)\phi(x)\,dx = \int_I bh(x)\phi(x)\,dx, \quad \phi \in C_c^\infty(I),$$

as soon as we know that $bh$ exists. In fact, since the existence of a boundary value is a local phenomenon, it can be studied around a generic point $x_0 \in I$. In order to demonstrate the existence of $bh$ we can apply [BCH, Theorem V.6.9] to the function $u(x,t)$ and conclude that its trace $u_0$ has the following property: $(x_0, -1) \in T^*(\mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}$ does not belong to the analytic wave front set of $u_0$, $WF_{\mathrm{ha}}(u_0)$. This implies that there exist $\delta > 0$ and an holomorphic function $W(x+iy)$ defined in $\Omega_\delta \doteq (x_0 - \delta, x_0 + \delta) \times (0, \delta)$, of tempered growth when $y \searrow 0$, such that $bW = u_0$ in $\mathcal{D}'((x_0 - \delta, x_0 + \delta))$. If $\phi(x,t) \in C_c^\infty((x_0 - \delta, x_0 + \delta) \times [0, \delta))$, it is known that

$$\lim_{\varepsilon \searrow 0} \int_I W(x + i\varepsilon)\phi(x,t)\,dx = \int_I bW(x)\phi(x,0)\,dx = \int_I u_0(x)\phi(x,0)\,dx.$$

For some $k \in \mathbb{N}$, there exists a holomorphic function $V(x+iy)$ in $\Omega_\delta$ which is continuous up to $y = 0$ and $V^{(k)}(x+iy) = W(x+iy)$ which allow us to write, integrating by parts $k$ times,

$$\lim_{\varepsilon \searrow 0} \int_I W(x + i\varepsilon)\phi(x,t)\,dx = (-1)^k \int_I V(x)\partial_x^k \phi(x,0)\,dx.$$

Analogously, using repeatedly the identity

$$\partial_x V^{(j)}(x+i\varphi(x,\varepsilon)) = V^{(j+1)}(x+i\varphi(x,\varepsilon))Z_x(x,t)$$

to integrate by parts and recalling that $Z_x(x,0)=1$, $x\in I$, we obtain

$$\lim_{\varepsilon\searrow 0}\int_I W(x+i\varphi(x,\varepsilon))\phi(x,t)\,dx = (-1)^k\int_I V(x)\partial_x^k\phi(x,0)\,dx.$$

The last two limits show that, for $\phi\in C_c^\infty((x_0-\delta,x_0+\delta))$, we get

$$\lim_{\varepsilon\searrow 0}\int_I W(x+i\varphi(x,\varepsilon))\phi(x)\,dx = \lim_{\varepsilon\searrow 0}\int_I W(x+i\varepsilon)\phi(x)\,dx = \int_I u_0(x)\phi(x)\,dx. \tag{2.12}$$

Hence, the difference $v(x,t)=u(x,t)-W(Z(x,t))$ satisfies $Lv=0$ in $\Omega_\delta$ and has null trace, therefore $u(x,t)=W(Z(x,t))$ in $\Omega_\delta$, by uniqueness in Cauchy problem for locally integrable vector fields (see, e.g., [BCH, Chapter II]). From the injectivity of $Z(x,t)$ we see that $W$ is simply the restriction of $h$ to $\Omega_\delta$ and we can replace $W$ by $h$ in (2.12), in particular $bh$ exists and $bh=u_0$ in $\mathcal{D}'(I)$. □

Now we are ready to prove Theorem 2.1. Let $u$ be as in the hypothesis of Theorem 2.1. By Lemma 2.2 there exists a holomorphic function $h$ defined in $Z(I\times(0,T))$ such that $u=h\circ Z$ in $I\times(0,T)$ and $bh=u_0$ in $\mathcal{D}'(I)$, that is, for each $\phi\in C_c^\infty(I)$ it holds

$$\lim_{\epsilon\searrow 0}\int_I h(\xi+i\epsilon)\phi(\xi)\,d\xi = \int_I u_0(\xi)\phi(\xi)\,d\xi,$$

where $u_0\in L^1(I)$ satisfies one of the conditions (1) or (2) of Theorem 2.1. It follows from Theorem 2.2 that $h\equiv 0$. Hence $u(x,t)=(h\circ Z)(x,t)\equiv 0$, which concludes the proof of the Theorem 2.1. □

Example 2.1: Theorem 2.1 can be applied to operators of Mizohata type with first integral

$$Z(x,t)=x+i\big(\alpha(x)+\beta(x,t)t^k\big),\quad k\in\mathbb{N},\ \beta(x,t)\neq 0,$$

$\alpha(x),\beta(x,t)$ real (see Examples 5.1 and 5.2).

Example 2.2: Theorem 2.1 can be applied to the operator

$$L=\frac{\partial}{\partial t}-ie^{-\frac{1}{t^2}}\frac{\partial}{\partial x}$$

to prove strong uniqueness. The main point about this example is that strong uniqueness holds although it is not of finite degeneracy type as defined in the next section (Definition 3.1).

## 3. Strong Uniqueness for the equation $Lu=Au+B\overline{u}$

Let

$$\omega = a(x,t)dx+b(x,t)dt$$

be a closed nonvanishing form defined on a domain $\Omega\subset\mathbb{R}^2$ with Hölder complex coefficients of class $C^\alpha$ for some $0<\alpha<1$. We will assume that the set of non-elliptic points of $\omega$

$$\Sigma=\{\mathsf{p}\in\Omega:\ \mathrm{Re}\,\omega(\mathsf{p})\wedge\mathrm{Im}\,\omega(\mathsf{p})=0\}$$

is a curve of class $C^{1+\alpha}$ such that $i^*\omega \neq 0$, where $i : \Sigma \hookrightarrow \Omega$ is the inclusion map. Since $\omega$ is closed, it is locally exact and in an open rectangle around a given point $\mathsf{p} \in \Omega$ it has a primitive $Z(x,t)$ of class $C^{1+\alpha}$, $dZ = \omega$, that can be obtained by integrating $\omega$ along polygonal curves (note that the integral $\int_{\partial R} \omega = 0$ on small rectangles $R$, this follows from Green's theorem if $\omega$ is smooth and by an approximation argument in the general case).

Consider a nonvanishing vector field $L$ orthogonal to $\omega$, i.e., $\langle \omega, L\rangle = 0$. Then $Z$ is a local first integral of $L$, i.e., $LZ = \langle dZ, L\rangle = \langle \omega, L\rangle = 0$ and $L$ is a multiple of the Hamiltonian of $Z$

$$\mathcal{H}_Z = Z_x\partial_t - Z_t\partial_x = a(x,t)\partial_t - b(x,t)\partial_x$$

which is a vector field of class $C^\alpha$ transversal to $\Sigma$.

In this section we will study strong uniqueness results for the first order equation

$$\begin{cases} Lu = Au + B\overline{u} \text{ for } t > 0 \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

where $A$ and $B$ are functions in $L^p$ space for some $p$ that depends on the degeneracy type of $L$. More precisely, given $\mathsf{p} \in \Sigma$, we may choose the coordinates $(x,t)$ centered at $\mathsf{p}$ in which a *one-sided* first integral may be written as

$$Z(x,t) = x + i\varphi(x,t), \quad x \in I \subset \mathbb{R},\ |t| < T,$$

where $Z(x,t)$ is defined on an open subset $\Omega$ that contains $\overline{I} \times [-T,T]$, the characteristic set $\Sigma$ of $L$ is given by $t = 0$ and $LZ = 0$ for $t > 0$.

We will assume without loss of generality that $\varphi(x,t)$ satisfies the following conditions:

- $\varphi(x,t)$ is a real valued function of class $C^{1+\alpha}$, $0 < \alpha < 1$, with $\varphi(\cdot,0) = \varphi_t(\cdot,0) = 0$;
- $LZ(x,t) = 0, \quad x \in \overline{I} \subset \mathbb{R},\ 0 \leq t \leq T$;
- $\Sigma = \{(x,t) \in \Omega\ ;\ d\mathrm{Re}\, Z \wedge d\mathrm{Im}\, Z(x,t) = 0\} = \{(x,t) \in \Omega\ ;\ t = 0\}$.

As in the previous section, we say that $Z$ is a good one-sided first integral for $L$ and it is proved in Proposition 5.1 that good one-sided first integrals for $L$ exist. Since $\varphi_t(x,t)$ does not change sign in $\Omega_0 \doteq I \times (0,T)$ we may assume that $\varphi_t(x,t)$ is positive for $0 < t < T$ and so is $\varphi(x,t)$. We will further assume that

$$\tau \doteq \sup_{\Omega_0} \frac{\log(\varphi_t(x,t))}{\log(\varphi(x,t))} < 1. \tag{†}$$

This condition is motivated by the following example. Suppose that $L$ is smooth and of uniform type $k$ on $\Sigma$, $k \in \mathbb{N}$, so we may assume (see [Me] or [BCH, Proposition III.5.3]) that $\varphi(x,t) = \alpha(x,t)t^{k+1} + \beta(x)$ with $\alpha(x,t)$ bounded away from zero. Thus

$$\frac{\log(\varphi_t(x,t))}{\log(\varphi(x,t))} = \frac{k\log t + r_1(x,t)}{(k+1)\log t + r_2(x,t)}$$

with $r_1, r_2$ bounded, so the quotient converges to $k/(k+1)$ as $t \searrow 0$ and, after shrinking $T$ if necessary, (†) holds.

**Definition 3.1.** *We will say that $L$ has finite degeneracy type in $\Omega_0$ if (†) holds. The number $\tau$ will be called the degeneracy type of $L$ in $\Omega_0$.*

**Remark 3.1.** *We point out that, assuming* $0 < \varphi(x,t) < 1$, (†) *may be rewritten as* $\varphi^\tau \le \varphi_t$ *in* $\Omega_0$. *This is related to the inequalities of Łojasiewicz type mentioned in* [M].

Although condition (†) was defined in terms of a particular good one-sided first integral expressed in particular coordinates it is independent of those choices and depends only on $L$ and the point of $\Sigma$ around which we focus our analysis (this is further discussed in Section 6). Note that for points $(x,t)$ close to $\Sigma$, $x \in I$, $0 < t < T$, we have $0 < \varphi(x,t), \varphi_t(x,t) < 1$, so $\tau > 0$ if $T$ is chosen small. We will always assume that we are in this situation.

From now on, $L$ will be always a vector field of finite degeneracy type $\tau < 1$. Since $\varphi_t(x,t) > 0$ if $0 < t < T$, $t \mapsto \varphi(x,t) > 0$ is positive and increasing in $(0,T)$ for $x \in I$. In particular, $Z(x,t)$ is an injective one-sided first integral of class $C^{1+\alpha}$ of $L$ in $I \times (0,T)$ of

$$L = \mathcal{H}_Z \text{ in } I \times (0,T). \tag{3.2}$$

Example 3.1: The Mizohata vector field

$$M_k = \frac{\partial}{\partial t} - it^k \frac{\partial}{\partial x}, \quad k \in \mathbb{N},$$

corresponds to the first integral $Z(x,t) = x + it^{k+1}/(k+1)$ that satisfies (†) with $\tau = k/(k+1)$. See also Example 5.2.

The main advantage of working with $L = \mathcal{H}_Z$ rather than a more general multiple in (3.2) is the antisymmetric property of the Hamiltonian:

**Lemma 3.1.** *Let* $\psi, \phi \in C^1_c(I \times (0,T))$. *Then,*

$$\int_{I\times(0,T)} (L\psi)(x,t)\phi(x,t)dxdt = -\int_{I\times(0,T)} \psi(x,t)(L\phi)(x,t)dxdt. \tag{3.3}$$

Proof: The identity is obtained by integrating by parts when $Z(x,t)$ is smooth and the general case follows by approximating $Z$ with its regularizations. □

Let $U \subset I \times (0,T)$ be an open set. Let us denote by

$$\langle v, \phi\rangle_U = \int_U v(x,t)\phi(x,t)dxdt, \; v \in L^1_{loc}(U) \text{ and } \phi \in C^\infty_c(U),$$

the usual duality pairing. When $U = I \times (0,T)$ we will simply write $\langle v,\phi\rangle_U = \langle v,\phi\rangle$.
If $u \in \mathcal{D}'_1(U) = \big(C^1_c(U)\big)'$ we may define $Lu$ by

$$\langle Lu, \phi\rangle_U = -\langle u, L\phi\rangle_U, \; \phi \in C^1_c(U),$$

and Lemma 3.1 shows that this extends the classical definition.

Let $A, B \in L^p(I \times (0,T))$, $p > 2 + \tau/(1-\tau)$, and $u_0 \in L^1(I)$ (here $\tau \in (0,1)$ is given by the condition (†)). Suppose that $u \in L^q(I \times (0,T))$, $q = p/(p-1)$, satisfies the following Cauchy problem:

$$\begin{cases} Lu = Au + B\overline{u} & \text{in } \mathcal{D}'_1(I \times (0,T)) \\ T_0 u = u_0 \text{ in } \mathcal{D}'_0(I) \end{cases}, \tag{3.4}$$

where $T_0 u = u_0$ in $\mathcal{D}'_0(I)$ means that the limit

$$\lim_{t \searrow 0} \int_I u(x,t)\phi(x)dx = \int_I u_0(x)\phi(x)dx, \quad \phi \in C^0_c(I), \tag{3.5}$$

holds true. We are interested in establishing conditions on $u_0$ that imply that $u$ is identically zero. The following example is borrowed from [BCH].

EXAMPLE 3.2: We cannot expect in general that $T_0 u \in L^1(I)$, for $L^q$ solutions of $Lu = Au + B\overline{u}$. Assume that $0 \in I$. We will take $A = B = 0$, $Z(x,t) = x + it^2/2$, $L = \partial_t - it\partial_x$ and set $u(x,t) = 1/Z(x,t)$, $(x,t) \in I \times (0,T)$. Then (†) holds with $\tau = 1/2$ and $u \in L^q(I \times (0,T))$ for $1 \leq q < 3/2$. It is simple to verify that $Lu = 0$ in $I \times (0,T)$ and $T_0 u = \mathrm{pv}(1/x) - i\pi\delta(x) \notin L^1(I)$.

The next theorem is the main result of this work.

**Theorem 3.1.** *Suppose that $u_0 \in L^1(I)$ satisfies one of the following conditions:*
(1) *$u_0$ vanishes on a subset of $I$ of positive measure, or;*
(2) *$u_0$ has a zero of exponential order in $I$. This means that there exists $x_0 \in I$, such that $|u_0(x)| = O(e^{-c/|x-x_0|})$, $x \to x_0$, for some $c > 0$.*

*If $u$ is a solution of the Cauchy problem* (3.4)*, then $u \equiv 0$ in $I \times (0,T)$.*

**Corollary 3.1.** *Let $L$ be a vector field defined on a domain $\Omega \subset \mathbb{R}^2$ and let $A, B \in L^p(\Omega)$. Let $\Gamma \subset \Omega$ be a curve of class $C^{1+\alpha}$, assume that $L$ is of finite degeneracy type $0 < \tau < 1$ in a neighborhood of $\Gamma$ and $p > 2 + \tau/(1-\tau)$. Suppose that $u \in C(\Omega)$ satisfies the equation $Lu = Au + B\overline{u}$ in $\Omega$. If $u_0$ vanishes on a set $X \subset \Gamma$ of positive one-dimensional Hausdorff measure then $u \equiv 0$ in $\Omega$.*

PROOF: If $X \setminus \Sigma \subset \Gamma$ has positive measure we are in the elliptic situation and $L$ may be locally transformed at an elliptic point into a multiple of the Cauchy-Riemann vector field by a change of variables so the result follows, on one side of $\Gamma$, from the classical similarity principle by an application of Theorem 2.2. Similarly, if $X \cap \Sigma \subset \Gamma$ has positive measure in $\Gamma$ it will have as well positive measure in $\Sigma$ and one may apply Theorem 3.1 to reach the same conclusion. The argument can be repeated for the other side of $\Gamma$. □

The proof of Theorem 3.1 will take the rest of the section and will depend on several lemmas and preparatory results.

Let $u$ be a solution of the Cauchy problem (3.4). By Theorem 7.1 we have that $u$ is continuous in $I \times (0,T)$ and there is a Hölder continuous function $S : \overline{I} \times [0,T] \to \mathbb{C}$ and a holomorphic function

$$h : Z(I \times (0,T)) \to \mathbb{C} \tag{3.6}$$

such that

$$u(x,t) = h(Z(x,t))e^{S(x,t)}, \text{ for all } (x,t) \in I \times (0,T). \tag{3.7}$$

In view of (3.7), showing that $u$ is identically zero is equivalent to showing that the holomorphic function $h$ is identically zero in an open, nonempty subset of $Z(I \times (0,T))$.

By a standard argument that uses the Uniform Boundedness Principle, (3.5) implies that

**Lemma 3.2.** *Let $J \subset\subset I$ be an open interval. There is a constant $C > 0$ such that*

$$\int_J |u(x,t)|dx \leq C, \text{ for all } t \in (0, T/2]. \tag{3.8}$$

□

In view of Lemma 3.2 and (3.7) we get

**Corollary 3.2.** *Let $J \subset\subset I$ be an open interval. There exists a constant $C > 0$ such that*

$$\int_J |h(Z(x,t))|dx \leq C, \text{ for all } t \in (0, T/2]. \tag{3.9}$$

□

Set $h_0 : I \to \mathbb{R}$

$$h_0(x) = u_0(x)e^{-S(x,0)}. \tag{3.10}$$

Then $h_0 \in L^1(I)$. Furthermore, $\{x \in I \,; h_0(x) = 0\} = \{x \in I \,;\, u_0(x) = 0\}$ and if $x_0 \in I$ is a zero of exponential order of $u_0$ then $x_0$ is a zero of exponential order of $h_0$.

**Corollary 3.3.** *If $\phi \in C_c^0(I)$, then*

$$\lim_{t \searrow 0} \int_I h(Z(x,t))\phi(x)dx = \int_I h_0(x)\phi(x)dx. \tag{3.11}$$

Proof: Let $\phi \in C_c^0(I)$. By the Hölder continuity of $S$ on $\overline{I} \times [0, T]$, there exist a constants $M > 0$ and $0 < \theta < 1$ such that

$$|e^{-S(x,t)} - e^{S(x,0)}| \leq M|t|^\theta \text{ for all } x \in \overline{I} \text{ and } t \in [0, T].$$

On the other hand, let $J$ be an open interval compactly contained in $I$ such that $\text{supp}(\phi) \subset J$. Hence, for $t \in (0, T/2)$, we have

$$\begin{aligned}\Big|\int_I (h(Z(x,t)) - u(x,t)e^{-S(x,0)})\phi(x)dx\Big| &\leq \int_J |u(x,t)|\,|e^{-S(x,t)} - e^{-S(x,0)}|\,|\phi(x)|dx \\ &\leq M||\phi||_\infty |t|^\theta \int_J |u(x,t)|dx.\end{aligned}$$

By the inequality above and Lemma 3.2

$$\lim_{t \searrow 0} \Big|\int_I h(Z(x,t))\phi(x)dx - \int_I u(x,t)e^{-S(x,0)}\phi(x)dx\Big| = 0. \tag{3.12}$$

Since $e^{-S(\cdot,0)}\phi \in C_c^0(I)$ we have from (3.5) that

$$\lim_{t \searrow 0} \int_I u(x,t)e^{-S(x,0)}\phi(x)dx = \int_I u_0(x)e^{-S(x,0)}\phi(x)dx. \tag{3.13}$$

Hence, (3.12) and (3.13) implies (3.11). □

**Lemma 3.3.** *There exists $bh \in \mathcal{D}'(I)$ such that*

$$\langle bh, \phi\rangle = \lim_{\epsilon \searrow 0} \int_I h(Z(x,\epsilon))\phi(x)dx, \quad \phi \in C_c^\infty(I).$$

To prove this lemma, we will first state and prove a proposition about $L^p$ holomorphic functions. Define $\kappa : [1, \infty] \to \{0,\, 1,\, 2\}$ by

$$\kappa(q) = \begin{cases} [2/q] & \text{if } \ 1 \leq q < \infty; \\ 0 & \text{if } \ q = \infty, \end{cases} \tag{3.14}$$

where $[\cdot]$ denotes the integral part function, i.e., $[x]$ is the largest integer $\leq x$.

**Proposition 3.1.** *Let $\mathcal{I} \subset \mathbb{R}$ be bounded interval, $1 \le q \le \infty$, $\lambda > 0$ and $h : \mathcal{I} + i(0,\lambda) \to \mathbb{C}$ a holomorphic function such that $h \in L^q(\mathcal{I} + i(0,\lambda))$. Then, there exists a continuous function $G : \mathcal{I} + i[0,\lambda) \to \mathbb{C}$ that is a holomorphic function on $\mathcal{I} + i(0,\lambda)$ and $G^{(\kappa(q)+1)} = h$ in $\mathcal{I} + i(0,\lambda)$.*

PROOF: Let $\mathcal{J}$ be an interval compactly contained in $\mathcal{I}$. Let $\mathbf{d}$ be the distance from $\mathcal{J}$ to the set $\mathcal{I}^c$. Let $\rho = \min\{\lambda/2, \mathbf{d}\}$. Let $z_0 \in \mathcal{J} + i(0,\rho)$. Consider

$$F_1(z) = \int_{z_0}^{z} h(\zeta)d\zeta, \ z \in \mathcal{J} + i(0,\lambda) \tag{3.15}$$

where the path of integration consists of the horizontal segment $[z_0, z_m]$ followed by the vertical path $[z_m, z]$, with $z_m = \mathrm{Re}\,(z) + i\mathrm{Im}\,(z_0)$. Of course, the integral is independent of the path of integration, $F_1$ is holomorphic in $\mathcal{J} + i(0,\lambda)$ and $F' = h$ in $\mathcal{J} + i(0,\lambda)$. Consider the functions, defined analogously to $F_1$,

$$F_2(z) = \int_{z_0}^{z} F_1(\zeta)d\zeta \text{ and } F_3(z) = \int_{z_0}^{z} F_2(\zeta)d\zeta, \ z \in \mathcal{J} + i(0,\lambda). \tag{3.16}$$

We have that $F_2$ and $F_3$ are holomorphic in $\mathcal{J} + i(0,\lambda)$ and their derivatives satisfy $F_2^{(2)} = h$ and $F_3^{(3)} = h$ in $\mathcal{J} + i(0,\lambda)$.

Let $\zeta = \xi + i\eta \in \mathcal{J} + i(0,\rho)$ and note that the disk centered in $\zeta$ and radius $\eta/2$, denoted by $D(\zeta, \eta/2)$, is contained in $\mathcal{I} + i(0,\lambda)$. The mean value property gives

$$|h(\zeta)| \le \frac{1}{|D(\zeta,\eta/2)|}\int_{D(\zeta,\eta/2)} |h(w)|dw_1dw_2, \ w = w_1 + iw_2. \tag{3.17}$$

If $q = 1$ we have

$$|h(\zeta)| \le ||h||_{L^1(\mathcal{I}\times(0,\lambda))}\Big(\frac{4}{\pi}\Big)\eta^{-2}. \tag{3.18}$$

On the other hand, if $1 < q < \infty$,

$$\begin{aligned}\int_{D(\zeta,\eta/2)} |h(w)|dw_1dw_2 \ &\le \ \Big(\int_{D(\zeta,\eta/2)} |h(w)|^q dw_1dw_2\Big)^{\frac{1}{q}} \cdot \Big(\int_{D(\zeta,\eta/2)} dw_1dw_2\Big)^{\frac{1}{p}} \\ &\le \ ||h||_{L^q(\mathcal{I}\times(0,\lambda))} \cdot |D(\zeta,\eta/2)|^{\frac{1}{p}}, \ p = q/(q-1).\end{aligned}$$

Hence, by (3.17) we have

$$\begin{aligned}|h(\zeta)| \ &\le \ ||h||_{L^q(\mathcal{I}\times(0,\lambda))} \cdot |D(\zeta,\eta/2)|^{\frac{1}{p}-1} \\ &\le \ ||h||_{L^q(\mathcal{I}\times(0,\lambda))}\Big(\frac{4}{\pi}\Big)^{\frac{1}{q}}\eta^{-\frac{2}{q}}.\end{aligned}$$

That is, there exists a constant $C > 0$ such that

$$|h(\zeta)| \le \frac{C}{\mathrm{Im}\,(\zeta)^{\mu(q)}}, \ \text{for } \zeta \in \mathcal{J} + i(0,\rho), \tag{3.19}$$

where $\mu(q) = 2/q$ if $1 \le q < \infty$ and $\mu(\infty) = 0$.

When $q = \infty$ the function $F_1$, defined in (3.15) has a continuous extension in $\mathcal{J} + i[0,\lambda)$. For $1 \le q < \infty$ we can write $2/q = \kappa(q) + \vartheta$, where $\kappa(q) \in \{0, 1, 2\}$ and $0 \le \vartheta < 1$. In the case $\vartheta \ne 0$ and $\kappa(q) = 0$ we have that $F_1$ has a continuous extension in $\mathcal{J} + i[0,\lambda)$.

If $\vartheta = 0$ we have the estimates:

$$
\begin{aligned}
(3.20) \quad & |F_1(z)| \;\leq\; C_1(\mathrm{Im}\,(z))^{-1} + C_1 \text{ if } \kappa(q) = 2,\\
(3.21) \quad & |F_1(z)| \;\leq\; C_1\,|\log(\mathrm{Im}\,(z))| + C_1 \text{ if } \kappa(q) = 1, \text{ for } z \in \mathcal{J} + i(0,\rho).
\end{aligned}
$$

If $\vartheta \neq 0$ and $\kappa(q) = 1$ we have the estimate:

$$
(3.22) \qquad |F_1(z)| \;\leq\; C_1(\mathrm{Im}\,(z))^{-\vartheta} + C_1, \text{ for } z \in \mathcal{J} + i(0,\rho),
$$

where $C_1$ is a positive constant.

In the case $\vartheta = 0$ and $\kappa(q) = 1$ the function $F_2$ has a continuous extension to $\mathcal{J} + i[0,\lambda)$ because $\log(|t|)$ is integrable in a neighborhood of 0. In the case $\vartheta = 0$ and $\kappa(q) = 2$, we have

$$
(3.23) \qquad |F_2(z)| \leq C_2\,|\log(\mathrm{Im}\,(z))| + C_2 \text{ for } z \in \mathcal{J} + i(0,\rho),
$$

where $C_2$ is a positive constant. Hence, reasoning as before, the function $F_3$ has a continuous extension to $\mathcal{J} + i[0,\lambda)$. In the case $\vartheta \neq 0$ and $\kappa(q) = 1$, the function $F_2$ has a continuous extension to $\mathcal{J}+i[0,\lambda)$, because $|t|^{-\vartheta}$ is integrable in a neighborhood of 0. Since $\mathcal{J} \subset\subset \mathcal{I}$ is arbitrary, the result follows. □

We may now prove Lemma 3.3.

PROOF: We have that $h : Z(I \times (0,T)) \to \mathbb{C}$ is a holomorphic function, with $h \in L^q(Z(I \times (0,T)))$, with $1 \leq q < 2 - \tau$, since $2 + \tau/(1-\tau) < p \leq \infty$. Hence, we have that $\kappa(q) = 1$ or 2. Let $\phi \in C_c^{(\kappa(q)+1)}(I)$. Let $J \subset\subset I$ an open interval and $\lambda > 0$ such that $\overline{J} + i(0,\lambda] \subset Z(I \times (0,T))$. By Proposition 3.1 there exists a function $G : J + i[0,\lambda) \to \mathbb{C}$ continuous and holomorphic in $J + i(0,\lambda)$ such that $G^{(\kappa(q)+1)} = h$ in $J + i(0,\lambda)$. Let $\epsilon_0 > 0$ such that $\varphi(x,t) < \lambda$ for $(x,t) \in J \times (0,\epsilon_0)$. For $0 < \epsilon < \epsilon_0$, we may write

$$
(3.24) \qquad \int_I h(Z(x,\epsilon))\phi(x)dx = (-1)^{\kappa(q)+1}\int_I G(Z(x,\epsilon))\phi^{(\kappa(q)+1)}(x)dx + E_{\kappa(q)}(\epsilon).
$$

Integrating by parts and keeping in mind Corollary 3.2 as well as estimates similar to (3.20), (3.21), (3.22) and (3.23), we have $\lim_{\epsilon\searrow 0} E_{\kappa(q)}(\epsilon) = 0$. Hence,

$$
\begin{aligned}
\lim_{\epsilon\searrow 0}\int_I h(Z(x,\epsilon))\phi(x)dx &= \int_I G(x)\phi^{(\kappa(q)+1)}(x)dx\\
&= \lim_{\epsilon\searrow 0}\int_I h(x+i\epsilon)\phi(x)dx = \langle bh,\phi\rangle.
\end{aligned}
$$

This proves the lemma. □

Corollary 3.3 and Lemma 3.3 yield

$$
bh = h_0 \text{ in } \mathcal{D}'(I).
$$

Hence, (3.10) implies that $h_0$ either vanishes identically on a set of positive measure in $I$ or $h_0$ has a zero of exponential order in $I$ and, by Theorem 2.2, $h \equiv 0$. Since $u(x,t) = h(Z(x,t))e^{S(x,t)}$ in $I \times (0,T)$ we see that $u \equiv 0$ which concludes the proof of Theorem 3.1.

## 4. Applications

Let $L$ be as in the previous section, in particular, $\Omega$ is an open set of $\mathbb{R}^2$, $\overline{I}\times[0,T]\subset\Omega$ and $0<\tau<1$ denotes the degeneracy type of $L$ in $I\times(0,T)$. The next application deals with the strong uniqueness property for a class o semilinear equations.

**Theorem 4.1.** *Let $F(x,t,\zeta)$ be a complex function defined in $\overline{\Omega}\times\mathbb{C}$ with the partial derivatives*

$$\frac{\partial F}{\partial\zeta}(x,t,\zeta) \text{ and } \frac{\partial F}{\partial\overline{\zeta}}(x,t,\zeta)$$

*continuous and bounded in $\overline{\Omega}\times\mathbb{C}$. Suppose that $u$, $v\in C(I\times(0,T))$ are such that*

$$Lu=F(x,t,u),\ \ Lv=F(x,t,v)$$

*in $I\times(0,T)$ and there exists $\chi\in L^1(I)$ such that for each $\phi\in C_c^0(I)$ holds*

$$\lim_{t\searrow 0}\int_I(u-v)(x,t)\phi(x)dx=\int_I\chi(x)\phi(x)dx.$$

*If one of the following items holds,*

1. *$\chi$ is zero in a set of positive measure in $I$, or;*
2. *$\chi$ has a zero of exponential order in $I$,*

*then $u=v$ in $I\times(0,T)$.*

Proof: Let

$$A(x,t)=\int_0^1\frac{\partial F}{\partial\zeta}(x,t,su(x,t)+(1-s)v(x,t))ds,\quad (x,t)\in\Omega,$$

and

$$B(x,t)=\int_0^1\frac{\partial F}{\partial\overline{\zeta}}(x,t,su(x,t)+(1-s)v(x,t))ds,\quad (x,t)\in\Omega.$$

We have that $A$, $B\in L^\infty(\Omega)$. Define $w=u-v$. It follows that

$$Lw=A(x,t)w+B(x,t)\overline{w}\text{ in }\Omega.$$

Besides, for each $\phi\in C_c^0(I)$ holds

$$\lim_{t\searrow 0}\int_I w(x,t)\phi(x)dx=\int_I\chi(x)\phi(x)dx.$$

The result follows from Theorem 3.1. □

The next results are concerned with strong uniqueness for approximate solutions and generalizes a key lemma from [Cor] (see to [BCH, page 132] and [BCH, Lemma III.5.12]).

**Theorem 4.2.** *Let $u\in L^q(I\times(0,T))$, $q=p/(p-1)$, where $p>2+\tau/(1-\tau)$. If $Lu\in L^1(I\times(0,T))$ and there exists a function $g\in L^p(I\times(0,T);\mathbb{R}^+)$ such that*

$$|Lu(x,t)|\le g(x,t)|u(x,t)|,\ \text{for all }(x,t)\in I\times(0,T),\tag{4.1}$$

*then $u\in C(I\times(0,T))$. Besides, if $u$ is not identically zero and there exists $u_0\in L^1(I)$ such that for each $\phi\in C_c^0(I)$ holds*

$$\lim_{t\searrow 0}\int_I u(x,t)\phi(x)dx=\int_I u_0(x)\phi(x)dx$$

*then the set $\omega=\{x\in I\ ;\ u_0(x)=0\}$ has null measure in $I$.*

Proof: Suppose that $u$ is not identically zero. Consider the function $A : I \times (0,T) \to \mathbb{C}$ given by

$$A(x,t) = \begin{cases} Lu(x,t)/u(x,t) & \text{if} \quad u(x,t) \neq 0; \\ 0 & \text{if} \quad u(x,t) = 0. \end{cases}$$

The function $A$ is mensurable and by inequality (4.1) the function $A \in L^p(I \times (0,T))$. We have that $u$ satisfies the following equation

$$Lu = Au \text{ in } I \times (0,T).$$

Suppose that the function $u_0$ is identically zero in a set of positive measure in $I$. By Theorem 3.1 the function $u$ is identically zero in $I \times (0,T)$. This is a contradiction. Hence, $\omega$ cannot have positive measure. □

**Corollary 4.1.** *Let $u \in C(I \times [0,T)) \cap C^1(I \times (0,T))$. If $u$ is not identically zero and there exists a constant $M > 0$ such that*

$$|Lu(x,t)| \leq M|u(x,t)| \text{ for all } (x,t) \in I \times (0,T)$$

*then the set $\omega = \{x \in I \ ; \ u(x,0) = 0\}$ has Lebesgue null measure in $I$.* □

## 5. Good one-sided first integrals

In this section we prove the existence results on good first integrals that were used in Sections 2 and 3.

**Proposition 5.1.** *Let $L$ be a nonvanishing vector field defined in an open set $\Omega \subset \mathbb{R}^2$ and assume that for some $0 < \alpha < 1$, $L$ has $C^\alpha$ coefficients. Suppose that $L$ is $C^{1+\alpha}$ locally integrable and that its characteristic set of $L$, denoted by $\Sigma$, is an embedded one-dimensional manifold of class $C^{1+\alpha}$ transversal to $L$. Let $\mathsf{p} \in \Sigma$. Then, there exists a function $Z$ of class $C^{1+\alpha}$ defined on a neighborhood of $\mathsf{p}$ and coordinates $(s,t)$, of class $C^{1+\alpha}$, that take $\mathsf{p}$ to the origin of $\mathbb{R}^2$, such that in these coordinates, in a rectangular neighborhood of the origin $(-\delta,\delta) \times (-T,T)$, the function $Z$ is given by*

$$Z(s,t) = s + i\varphi(s,t), \tag{5.1}$$

*where $\varphi$ is a real valued function, satisfying $\varphi_t(\cdot,0) = \varphi(\cdot,0) = 0$ and*

$$LZ = 0 \text{ in } (-\delta,\delta) \times (0,T).$$

Proof: Let $\mathsf{p} \in \Sigma$ and let $\mathcal{Z}$ be a locally first integral of class $C^{1+\alpha}$ defined in a neighborhood $V$ of $\mathsf{p}$. We can suppose that $\mathcal{Z}(0,0) = 0$. Since $L$ is not tangent to $\Sigma$, replacing $\mathcal{Z}$ by $i\mathcal{Z}$ if necessary and decreasing $V$ around $\mathsf{p}$, there exist coordinates $(x,y)$ of class $C^{1+\alpha}$ about 0, (that is, $x(\mathsf{p}) = y(\mathsf{p}) = 0$) in which

$$\mathcal{Z}(x,y) = x + i\Phi(x,y)$$

where $\Phi$ is real-valued function defined in the rectangle $(-a,a) \times (-Y,Y)$ and $\Phi_y(x,y) = 0$ iff $y = 0$. That is, in the new coordinates the set $\Sigma$ is $(-a,a) \times \{0\}$. For each $x \in (-a,a)$ the function

$$y \ni (0,Y) \mapsto \Phi_y(x,y)$$

is positive. Then, for each $x \in (-a,a)$,

$$y \ni [0,Y) \mapsto \Phi(x,y)$$

is injective. Hence, $\mathcal{Z} : (-a,a) \times [0,Y) \to \mathbb{C}$ is injective. Analogously, we conclude that $\mathcal{Z} : (-a,a) \times (-Y,0) \to \mathbb{C}$ is injective. We can write $\mathcal{Z}((-a,a) \times (-Y,Y))$ as

$$\mathcal{Z}((-a,a) \times (-Y,Y)) = \Gamma \cup U^+ \cup U^-,$$

where

$$\Gamma = \mathcal{Z}((-a,a) \times \{0\}) = \{s + i\Phi(s,0)\ ;\ s \in (-a,a)\} \tag{5.2}$$

is a curve of class $C^{1+\alpha}$ which passes by $0 \in \mathbb{C}$, and $U^+ = \mathcal{Z}((-a,a) \times (0,T))$ and $U^- = \mathcal{Z}((-a,a) \times (-T,0))$ are simply connected open sets. Let $B$ be an open ball centered in 0 such that $\Gamma$ divides $B$ in two simply connected open sets, $B^+ = B \cap U^+$ and $B^- = B \setminus (\Gamma \cup U^+)$. By the Riemann mapping theorem ([Po, Theorem 3.6]), there exists a diffeomorphism

$$h : B \to \mathbb{C} \tag{5.3}$$

of class $C^{1+\alpha}$ such that $h$ is holomorphic in $B^+ = B \cap U^+$, $\mathrm{Im}\,(h)(\Gamma) = 0$, $h(B^+) \subset \{\zeta \in \mathbb{C}\ ;\ \mathrm{Im}\,(\zeta) > 0\}$ and $h(0) = 0$. Shrinking $a$ and $Y > 0$ if necessary, we can consider the composition

$$Z = h \circ \mathcal{Z} \text{ in } (-a,a) \times (-Y,Y).$$

We have that $Z$ is a function of class $C^{1+\alpha}$, $\mathrm{Im}\,(Z)(\cdot,0) = 0$, $LZ = 0$ in $(-a,a) \times (0,Y)$ and $\partial_x \mathrm{Re}\,(Z)(\cdot,0) \neq 0$. Let $s = \mathrm{Re}\,(Z)(x,y)$ and $t = y$. In the coordinates $(s,t)$ the function $Z$ has the desired form (5.1). □

**Proposition 5.2.** *Let $L$ be a $C^\infty$ nonvanishing vector field defined in an open set $\Omega \subset \mathbb{R}^2$. Suppose that $L$ is $C^\infty$ locally integrable and that the characteristic set of $L$, denoted by $\Sigma$, is an one-dimensional smooth embedded manifold transversal to $L$. Let $\mathsf{p} \in \Sigma$. Then, there exists a function $Z$ of class $C^\infty$ defined in a neighborhood of $\mathsf{p}$ and coordinates $(s,t)$, of class $C^\infty$, that take $\mathsf{p}$ in the origin of $\mathbb{R}^2$, such that in these coordinates, in a rectangular neighborhood of the origin $(-\delta,\delta) \times (-T,T)$, the function $Z$ is written as*

$$Z(s,t) = s + i\varphi(s,t),$$

*where $\varphi$ is a real valued function, satisfying $\varphi_t(\cdot,0) = \varphi(\cdot,0) = 0$ and*

$$LZ = 0 \text{ in } (-\delta,\delta) \times (0,T).$$

Proof: The proof is analogous to the proof of the previous proposition. When $Z$ is the class $C^\infty$ the curve $\Gamma$, given by (5.2), is a $C^\infty$ curve. Then, we may find $h$ (in (5.3)) of class $C^\infty$. □

Example 5.1: Let $L$ be nonvanishing vector field defined in an convex neighborhood $U$ of origin in $\mathbb{R}^2$ with $C^\alpha$ coefficients for some $0 < \alpha < 1$. Suppose that there exists a real-valued function $\beta(x)$ of class $C^{1+\alpha}$ and a real constant $\sigma \geq \alpha$ such that the function

$$Z_{\sigma,\beta}(x,y) = x + i\left(\frac{y^{\sigma+1}}{\sigma+1} + \beta(x)\right)$$

satisfies $LZ_{\sigma,\beta} = 0$ in $U \cap \{y > 0\}$. Then we can find coordinates $(s,t)$ of class $C^{1+\alpha}$ around a ball $B(0,r) \subset \mathbb{R}^2$, $r > 0$, in which

$$Z(s,t) = s + i\frac{t^{\sigma+1}}{\sigma+1}$$

and satisfies the equation $LZ = 0$ in $B(0, r) \cap \{t > 0\}$. Vector fields of this type were considered recently in the literature as described in the Introduction. $\square$

**Remark 5.1.** *If $\beta$ is of class $C^\infty$ and $\sigma$ is a non-negative integer, then the coordinates $(s, t)$ may be chosen of class $C^\infty$.*

Example 5.2: Let $L$ be a $C^\infty$ nonvanishing vector field defined near of origin in $\mathbb{R}^2$. Suppose that $L$ is locally integrable and that the characteristic set of $L$, denoted by $\Sigma$, is a one-dimensional manifold. If $L$ is of uniform finite type $k \in \mathbb{N}$ on $\Sigma$, then we can find coordinates $(s, t)$ about the ball $B(0, r) \subset \mathbb{R}^2$, $r > 0$, in which

$$Z(s,t) = s + i\frac{t^{k+1}}{k+1}$$

is a first integral of $L$ in $B(0, r) \cap \{t > 0\}$. Indeed, it is known ([BCH, Proposition III.5.3]) that we can find coordinates $(s, t)$ about 0 in which

$$Z(s,t) = s + i\varphi(s,t)$$

is a first integral of $L$ where $\varphi(s, t)$ is a real-valued and is of the form

$$\varphi(s,t) = \alpha(s,t)t^{k+1} + \beta(s)$$

for some nonvanishing $\alpha(s, t)$ near 0. As in Example 5.1 we obtain the desired expression.

Examples 5.1 and 5.2 show instances of complex vector fields with first integrals $Z$ that satisfy the condition (†) in Section 3.

## 6. The degeneracy type, comments and examples

In this section we will define a function on the characteristic set $\Sigma$ of the vector field $L$ that determines its degeneracy type on a one-sided neighborhood of a given point $\mathsf{p} \in \Sigma$. We keep the notation of Section 3. Let

$$\omega = a(x,t)dx + b(x,t)dt$$

be a closed nonvanishing form defined on a domain $\Omega \subset \mathbb{R}^2$ with Hölder complex coefficients of class $C^\alpha$, for some $0 < \alpha < 1$. We will assume that the set of non-elliptic points of $\omega$ given by

$$\Sigma = \{\mathsf{p} \in \Omega : \operatorname{Re}\omega(\mathsf{p}) \wedge \operatorname{Im}\omega(\mathsf{p}) = 0\}$$

is a curve of class $C^{1+\alpha}$ in $\Omega$ such that $i^*\omega \neq 0$, where $i : \Sigma \hookrightarrow \Omega$ is the inclusion map, and $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$ where $\Omega^\pm$ is a simply connected open set. Consider a nonvanishing vector field $L$ orthogonal to $\omega$, i.e., $\langle \omega, L \rangle = 0$. Then $L$ is locally integrable with first integrals of class $C^{1+\alpha}$.
Fix a point $\mathsf{p} \in \Sigma$. By Proposition 5.1, there exist a simply connected open set $U \subset \Omega$, with $\mathsf{p} \in U$, a diffeomorphism $\chi : U \to \chi(U) \subset \mathbb{R}^2$ of class $C^1$, with $\chi(\mathsf{p}) = (0, 0)$, $\chi(U \cap \Sigma) \subset \mathbb{R} \times \{0\}$, $\chi(U \cap \Omega^+) = \chi(U) \cap \mathbb{R} \times (0, \infty)$ a simply connected open set, and a function

$$Z(x,t) = x + i\varphi(x,t) \text{ in } \chi(U),$$

where $\varphi : \chi(U) \to \mathbb{R}$ is a function of class $C^{1+\alpha}$, $0 < \alpha < 1$, such that

(1) $0 \leq \varphi$, $\varphi_t < 1$ in $\chi(U) \cap \mathbb{R} \times [0, \infty)$;
(2) For $(x, t) \in \chi(U)$, $\varphi(x, t) = 0 \Leftrightarrow t = 0$ and $\varphi_t(x, t) = 0 \Leftrightarrow t = 0$;

(3) $Z \circ \chi : U \to \mathbb{C}$ is a first integral of $L$ in $U \cap \Omega^+$.

We define

$$\tau_k(\mathsf{p}) := \sup\left\{\frac{\log(\varphi_t(x,t))}{\log(\varphi(x,t))}\ ;\ (x,t) \in \Omega_k\right\},$$

where $\Omega_k = (-1/k, 1/k) \times (0, 1/k) \cap \chi(U)$, $k \in \mathbb{N}$. We have that $(\tau_k(\mathsf{p}))_{k\in\mathbb{N}}$ is a monotone nonincreasing sequence. Define $\tau_{\Sigma^+}(\mathsf{p})$ by

$$\tau_{\Sigma^+}(\mathsf{p}) \doteq \lim_{k\to\infty} \tau_k(\mathsf{p}). \tag{6.1}$$

Note that the definition in (6.1) involves the choice of a particular set of local coordinates as well as the choice of a particular one-sided first integral. We will show that $\tau_{\Sigma^+}$ is independent of these choices. Let $\tilde{U}$ be a neighborhood of $\mathsf{p}$, $\tilde{\chi} : \tilde{U} \to \tilde{\chi}(\tilde{U})$ a diffeomorphism of class $C^1$, with $\tilde{\chi}(\mathsf{p}) = (0,0)$, $\tilde{\chi}(\tilde{U} \cap \Sigma) \subset \mathbb{R} \times \{0\}$, $\tilde{\chi}(\tilde{U} \cap \Omega^+) = \tilde{\chi}(\tilde{U}) \cap \mathbb{R} \times (0,\infty)$ a simply connected open set, and let

$$\tilde{Z}(\tilde{x},\tilde{t}) = \tilde{x} + i\tilde{\varphi}(\tilde{x},\tilde{t}) \text{ in } \tilde{\chi}(\tilde{U}),$$

be a function such that $\tilde{\varphi} : \tilde{\chi}(\tilde{U}) \to \mathbb{R}$ is a function of class $C^{1+\alpha}$, $0 < \alpha < 1$, satisfying

(1) $0 \le \tilde{\varphi}$, $\tilde{\varphi}_{\tilde{t}} < 1$ in $\tilde{\chi}(\tilde{U}) \cap \mathbb{R} \times [0,\infty)$;
(2) For $(\tilde{x},\tilde{t}) \in \tilde{\chi}(\tilde{U})$, $\tilde{\varphi}(\tilde{x},\tilde{t}) = 0 \iff \tilde{t} = 0$ and $\tilde{\varphi}_{\tilde{t}}(\tilde{x},\tilde{t}) = 0 \iff \tilde{t} = 0$;
(3) $\tilde{Z} \circ \tilde{\chi} : \tilde{U} \to \mathbb{C}$ is a first integral of $L$ in $\tilde{U} \cap \Omega^+$.

Set

$$\tilde{\tau}_{\Sigma^+}(\mathsf{p}) = \lim_{k\to\infty} \tilde{\tau}_k(\mathsf{p}), \tag{6.2}$$

where $\tilde{\tau}_k(\mathsf{p}) := \sup\{\log(\tilde{\varphi}_{\tilde{t}}(\tilde{x},\tilde{t}))/\log(\tilde{\varphi}(\tilde{x},\tilde{t}))\ ;\ (\tilde{x},\tilde{t}) \in \tilde{\Omega}_k\}$, and $\tilde{\Omega}_k = (-1/k, 1/k) \times (0, 1/k) \cap \tilde{\chi}(\tilde{U})$, $k \in \mathbb{N}$. The following proposition will show that $\tau_{\Sigma^+}(\mathsf{p})$ is well defined on $\Sigma$.

**Proposition 6.1.** $\tau_{\Sigma^+}(\mathsf{p}) = \tilde{\tau}_{\Sigma^+}(\mathsf{p})$.

Suppose that $\tau_{\Sigma^+}(\mathsf{p}) < \infty$. Then there exist $\tau, T > 0$ and $I \subset \mathbb{R}$ a bounded open interval, with $0 \in I$, such that $\overline{I} \times [-T,T] \subset \chi(U)$ and

$$\sup_{I\times(0,T)} \frac{\log(\varphi_t(x,t))}{\log(\varphi(x,t))} < \tau < \infty. \tag{6.3}$$

We shall state some relations between $Z$ and $\tilde{Z}$. Let $F : \tilde{\chi}(\tilde{U}) \to \chi(U)$ be the diffeomorphism of class $C^1$, given by $F = \chi \circ \tilde{\chi}^{-1}$, with

- $F(0,0) = (0,0)$;
- $\overline{J} \times [0,P] \subset \tilde{\chi}(\tilde{U})$ and $F(\overline{J} \times [0,P]) \subset I \times [0,T)$, where $J$ is an open bounded interval, $0 \in J$, and $P > 0$;
- $F(\overline{J} \times \{0\}) \subset I \times \{0\}$;

such that

$$\tilde{Z} = h \circ Z \circ F \text{ in } \tilde{\chi}(\tilde{U}), \tag{6.4}$$

where $h$ is a function of class $C^1$ in $Z(F(\tilde{\chi}(\tilde{U})))$ and holomorphic in $Z(F(J\times(0,P)))$.

Consider the functions $g : \overline{I} \times (0,T] \to \mathbb{R}^+$ and $\tilde{g} : \overline{J} \times (0,P] \to \mathbb{R}^+$ given by

$$g(x,t) = \frac{\log(\varphi_t(x,t))}{\log(\varphi(x,t))}, \quad \tilde{g}(\tilde{x},\tilde{t}) = \frac{\log(\tilde{\varphi}_{\tilde{t}}(\tilde{x},\tilde{t}))}{\log(\tilde{\varphi}(\tilde{x},\tilde{t}))}.$$

By (6.3) we have that $g(x,t) \le \tau$ for all $(x,t) \in \overline{I} \times (0,T]$. Consequently,

$$g(F(\tilde{x},\tilde{t})) \le \tau \text{ for all } (\tilde{x},\tilde{t}) \in \overline{J} \times (0,P]. \tag{6.5}$$

**Lemma 6.1.** *There exist constants $K, K'$ such that*

$$\tilde{g}(\tilde{x},\tilde{t}) \le \frac{K}{\log(\tilde{\varphi}(\tilde{x},\tilde{t}))} + g(F(\tilde{x},\tilde{t})),\ (\tilde{x},\tilde{t}) \in \overline{J} \times (0,P], \tag{6.6}$$

*and*

$$\tilde{g}(\tilde{x},\tilde{t}) \ge \frac{K'}{\log(\tilde{\varphi}(\tilde{x},\tilde{t}))} + g(F(\tilde{x},\tilde{t})),\ (\tilde{x},\tilde{t}) \in \overline{J} \times (0,P]. \tag{6.7}$$

Proof: By (6.4) and the Lipschitz continuity of $h$ and $h^{-1}$ there exist constants $C$, $C' > 0$, $M > 1$ and $M' < 1$ such that

(1) $C\varphi_t(F(\tilde{x},\tilde{t})) \le \tilde{\varphi}_{\tilde{t}}(\tilde{x},\tilde{t}) \le C'\varphi_t(F(\tilde{x},\tilde{t}))$ for $(\tilde{x},\tilde{t}) \in \overline{J} \times [0,P]$;

(2) $M\varphi(F(\tilde{x},\tilde{t})) \ge \tilde{\varphi}(\tilde{x},\tilde{t}) \ge M'\,\varphi(F(\tilde{x},\tilde{t}))$ for $(\tilde{x},\tilde{t}) \in \overline{J} \times [0,P]$.

Inequalities (1) and (2) above and (6.3) give the result with $K = \log\Big(C/M^{\tau}\Big)$ and $K' = \log\Big(C'/M'^{\tau}\Big)$. □

By (6.6), there exists a nonnegative integer $k_0$, such that $\tilde{\Omega}_{k_0} \subset J \times (0,P)$ and $\tilde{\tau}_k(\mathsf{p}) \le \tau$ for all $k \ge k_0$. It follows that

$$\tilde{\tau}_{\Sigma^+}(\mathsf{p}) \le \tau.$$

For each positive integer $k \ge k_0$ there exists a $l_k \ge k$ such that

$$F(\tilde{\Omega}_{l_k}) \subset \Omega_k,\ k = k_0, k_0+1, k_0+2, ..., \tag{6.8}$$

which allows us to compare $\tilde{g}$ and $g$ pointwise. By (6.6) and (6.8), for $k \ge k_0$,

$$\begin{aligned}\sup_{\tilde{\Omega}_{l_k}} \tilde{g}(\tilde{x},\tilde{t}) &\le \sup_{\tilde{\Omega}_{l_k}} \frac{K}{\log(\tilde{\varphi}(\tilde{x},\tilde{t}))} + \sup_{\tilde{\Omega}_{l_k}} g(F(\tilde{x},\tilde{t})) \\ &\le \sup_{\tilde{\Omega}_{l_k}} \frac{K}{\log(\tilde{\varphi}(\tilde{x},\tilde{t}))} + \sup_{\Omega_k} g(x,t).\end{aligned}$$

Thus

$$\tilde{\tau}_{\Sigma^+}(\mathsf{p}) = \lim_{k\to\infty} \sup_{\tilde{\Omega}_{l_k}} \tilde{g}(\tilde{x},\tilde{t}) \le \lim_{k\to\infty} \sup_{\Omega_k} g(x,t) = \tau_{\Sigma^+}(\mathsf{p}). \tag{6.9}$$

For each integer $k \ge k_0$ there exists a $m_k \ge k$ such that

$$\Omega_{m_k} \subset F(\tilde{\Omega}_k). \tag{6.10}$$

Using (6.10) and (6.7) we have

$$\tilde{\tau}_{\Sigma^+}(\mathsf{p}) = \lim_{k\to\infty} \sup_{\tilde{\Omega}_k} \tilde{g}(\tilde{x},\tilde{t}) \ \ge\ \lim_{k\to\infty} \sup_{\Omega_{m_k}} g(x,t) = \tau_{\Sigma^+}(\mathsf{p}). \tag{6.11}$$

Now (6.9) and (6.11) imply $\tau_{\Sigma^+}(\mathsf{p}) = \tilde{\tau}_{\Sigma^+}(\mathsf{p})$. Note that the argument above shows that $\tilde{\tau}_{\Sigma^+}(\mathsf{p}) = \infty$ implies $\tau_{\Sigma^+}(\mathsf{p}) = \infty$. This completes the proof of Proposition 6.1. □

**Definition 6.1.** *We will say that $L$ has one-sided finite degeneracy type at $\mathsf{p} \in \Sigma$ if $\tau_{\Sigma^+}(\mathsf{p}) < 1$. The number $\tau_{\Sigma^+}(\mathsf{p})$ will be called the one-sided degeneracy type of $L$ at $\mathsf{p}$.*

Note that given any $\mathsf{p} \in \Sigma$ with $\tau_{\Sigma^+}(\mathsf{p}) < 1$ we are able to find a small one-sided neighborhood of $\mathsf{p}$ on which $L$ has finite degeneracy type where we can apply Theorem 3.1 and get one-sided strong uniqueness holds for solutions defined on that neighborhood.

Example 6.1: Let $L$ be a vector field of class $C^\alpha$, $0 < \alpha < 1$, define in $[-1,1] \times [-1/2, 1/2]$. Consider $Z(x,t) = x + i\varphi(x,t)$ in $[-1,1] \times [-1/2,1/2]$, where $\varphi : [-1,1] \times [-1/2,1/2] \to \mathbb{R}$ is a function of class $C^{1+\alpha}$ such that

(1) $\varphi(x,t) = t^{\sigma(x)+1}$ in $[-1,1] \times [0,1/2]$, with $\sigma : [-1,1] \to \mathbb{R}$ a function of class $C^2$ with $\sigma(x) > 1$ for $x \in [-1,1]$;
(2) $LZ = 0$ in $[-1,1] \times [0,1/2]$ and $dZ \neq 0$ in $[-1,1] \times [0,1/2]$.

Let $g : [-1,1] \times [0,1/2] \to \mathbb{R}$ given by

$$g(x,t) = \begin{cases} \log(\varphi_t(x,t))/\log(\varphi(x,t)), & \text{if } (x,t) \in [-1,1] \times (0,1/2], \\ \sigma(x)/(\sigma(x)+1), & \text{if } x \in [-1,1] \text{ and } t = 0. \end{cases} \tag{6.12}$$

Note that, for $(x,t) \in [-1,1] \times (0,1/2]$,

$$g(x,t) = \frac{\log(\sigma(x)+1)}{(\sigma(x)+1)} \frac{1}{\log(t)} + \frac{\sigma(x)}{\sigma(x)+1}.$$

It follows that $g$ is a continuous function in $[-1,1] \times [0,1/2]$ and of class $C^1$ in $(-1,1) \times (0,1/2)$. Besides, for each $x \in [-1,1]$ we have that the function $(0,1/2] \ni t \mapsto g(x,t)$ is decreasing. Let $\mathsf{p} = (x_0, 0) \in (-1,1) \times \{0\}$. It follows that

$$\begin{aligned}
\tau_{\Sigma^+}(\mathsf{p}) &= \lim_{k\to\infty} \sup\{g(x+x_0,t); x \in (-1/k,1/k) \times (0,1/k) \cap (-1-x_0, 1-x_0) \times (0,1/2)\} \\
&= \lim_{k\to\infty} \sup\{g(x,t); x \in (x_0 - 1/k, x_0 + 1/k) \times (0,1/k) \cap (-1,1) \times (0,1/2)\} \\
&= \frac{\sigma(x_0)}{\sigma(x_0)+1}.
\end{aligned}$$

**Proposition 6.2.** *Let $Z$ be an one-sided first integral of $L$. Assume that there exists a change of local coordinates around a point $\mathsf{p} \in \Sigma$ such that $Z$ can be written as $\tilde{Z}(\tilde{x},\tilde{t}) = \tilde{x} + i\tilde{\varphi}(\tilde{x},\tilde{t})$ with $\tilde{\varphi}(\tilde{x},\tilde{t}) = \tilde{t}^{k+1}$ for some positive integer $k$. Then*

$$\tau_{\Sigma^+}(\mathsf{q}) \equiv k/(k+1) \tag{6.13}$$

*for all $\mathsf{q} \in \Sigma$ close to $\mathsf{p}$.*

Proof: It is enough to compute $\tau_{\Sigma^+}(\mathsf{q})$ in the coordinates $(\tilde{x},\tilde{t})$ to reach the conclusion. $\square$

## 7. Appendix - One-sided Similarity Principle

In this section we keep the notation of Section 3 and deal with a vector field of finite degeneracy type $\tau < 1$. In particular, we assume that (†) holds throughout.

Let $u$ be a homogeneous solution of the first order equation $Lu = Au + B\overline{u}$. In order to obtain an expression of $u$ that involves a holomorphic function of $Z$, we will need an appropriate version of the similarity principle for $L$. An important tool for that will be the integral operator $T_Z$ given by (7.4).

**Theorem 7.1.** *(One-sided Similarity Principle). Let $A, B \in L^p(I \times (0,T))$, $p > 2 + \tau/(1-\tau)$. Then for $u \in L^q(I \times (0,T))$, $q = p/(p-1)$, solution of the equation*

$$Lu = Au + B\overline{u} \text{ in } \mathcal{D}_1'(I \times (0,T)) \tag{7.1}$$

*there exists an holomorphic function $h$ defined in $Z(I \times (0,T))$ and a function $S \in C^\theta(\overline{I} \times [0,T])$, $0 < \theta < 1$, given by $\theta = (2-q-\tau)/q$, such that*

$$u(x,t) = h(Z(x,t))e^{S(x,t)}, \quad \forall (x,t) \in I \times (0,T). \tag{7.2}$$

The proof of Theorem 7.1 will be split in several lemmas and propositions that we state and prove below.

The first lemma involves no more than a change of variables and may be easily checked.

**Lemma 7.1.** *If $u \in L^p(I \times (0,T))$, $1 \le p \le \infty$, then $h \doteq u \circ Z^{-1} : Z(I \times (0,T)) \to \mathbb{C}$ is a measurable function and $h \in L^q(Z(I \times (0,T)))$.* □

The next result will allow the representation of a $L^p$ homogeneous solution of the equation $Lu = 0$ as a holomorphic function of the one-sided first integral $Z(x,t)$. When $Z(x,t)$ is smooth this would be a local standard consequence of the Baouendi-Treves approximation theorem ([BCH],[T]) but in the present simple situation in which $Z(x,t)$ is injective no approximation argument is needed.

**Lemma 7.2.** *Let $u \in L^p(I \times (0,T))$, $1 \le p \le \infty$, such that $Lu = 0$ in $\mathcal{D}_1'(I \times (0,T))$. Then, there exists a holomorphic function $h : Z(I \times (0,T)) \to \mathbb{C}$ such that*

$$u = h \circ Z \text{ in } I \times (0,T).$$

Proof: Let $h \doteq u \circ Z^{-1} : Z(I \times (0,T)) \to \mathbb{C}$. By the Lemma 7.1 $h \in L^p(Z(I \times (0,T)))$. Let $\psi \in C_c^1(Z(I \times (0,T)))$. Define $\phi = (\psi \circ Z) : I \times (0,T) \to \mathbb{C}$. We have $\phi \in C_c^1(I \times (0,T))$ and

$$uL\phi = 2i\, h(Z)\,\frac{\partial \psi}{\partial \overline{\zeta}}(Z)\det(DZ) \text{ in } I \times (0,T),\ \zeta = \xi + i\eta \in Z(I \times (0,T)). \tag{7.3}$$

By changing of variables in the integral and using (7.3) we have

$$\begin{aligned}
\left\langle \frac{\partial h}{\partial \overline{\zeta}}, \psi \right\rangle_{Z(I\times(0,T))} &= -\int_{Z(I\times(0,T))} h(\zeta)\frac{\partial \psi}{\partial \overline{\zeta}}(\zeta)d\xi d\eta,\ \zeta = \xi + i\eta, \\
&= -\frac{1}{2i}\int_{I\times(0,T)} u(x,t)L\phi(x,t)dxdt = 0.
\end{aligned}$$

It follows that $\partial h/\partial\overline{\zeta} = 0$ em $\mathcal{D}_1'(Z(I \times (0,T)))$. □

For $f \in L^1(I \times (0,T))$, we set

$$T_Z f(x,t) = \frac{1}{2\pi i} \int_{I\times(0,T)} \frac{f(\xi,\eta)}{Z(\xi,\eta) - Z(x,t)}\, d\xi\, d\eta,\ (x,t) \in \overline{I} \times [0,T]. \tag{7.4}$$

The theorem below lists some properties of $T_Z$ that will be useful in the sequel.

**Theorem 7.2.** *Let $T_Z$ be defined by* (7.4). *Then, if $p > 2 + \tau/(1-\tau)$,*

(1) *(Continuity and Hölder regularity). There is a constant $M = M(p,\tau,I,T) > 0$ such that*

$$(\clubsuit) \qquad ||T_Z f||_{L^\infty(I\times(0,T))} \le M\, ||f||_{L^p(I\times(0,T))}, \quad f \in L^p(I \times (0,T)),$$

*and*

$$(\spadesuit) \qquad |T_Z f(\mathsf{p}_1) - T_Z f(\mathsf{p}_2)| \le M\, ||f||_{L^p(I\times(0,T))}\, |Z(\mathsf{p}_1) - Z(\mathsf{p}_2)|^\theta, \quad \forall \mathsf{p}_1, \mathsf{p}_2 \in \overline{I} \times [-T,T],$$

*where $0 < \theta < 1$ is given by $\theta = (2-q-\tau)/q$, with $q = p/(p-1)$;*

(2) *(Antisymmetry). If $f \in L^p(I\times(0,T))$ and $\psi \in C_c^0(I\times(0,T))$, then $\langle T_Z f, \psi\rangle = \langle f, -T_Z\psi\rangle$.*

The proof of this theorem adapts some arguments from [CDM1, Theorem 9 and 16].

Proof: Let $f \in L^p(I\times(0,T))$, $p > 2+\tau/(1-\tau)$. For fixed $(x,t) \in \mathbb{R}^2$, use Hölder's inequality and the change of variables $\zeta = \xi + i\eta = Z(\omega_1,\omega_2)$, to get

$$|T_Z f(x,t)| \;\le\; \frac{||f||_{L^p(I\times(0,T))}}{2\pi} \Big( \int_{Z(I\times(0,T))} \frac{d\xi d\eta}{\varphi_t(Z^{-1}(\zeta))|\zeta - Z(x,t)|^q} \Big)^{\frac{1}{q}}, \tag{7.5}$$

where $q = p/(p-1)$. By the inequality (7.5) and condition (†) we have

$$|T_Z f(x,t)| \le \frac{||f||_{L^p(I\times(0,T))}}{2\pi}\, I(x,t)^{\frac{1}{q}} \tag{7.6}$$

where

$$I(x,t) = \int_{Z(I\times(0,T))} \frac{d\xi d\eta}{\eta^\tau |\zeta - Z(x,t)|^q}. \tag{7.7}$$

Since $1 \le q < 2-\tau$ it follows from [CDM1, Lemma 7] that there exists a constant $M = M(p,\tau,I,T) > 0$ such that $I(x,t) \le M$ for all $(x,t) \in \overline{I}\times[0,T]$. This implies ($\clubsuit$).

Let $(x_0,y_0), (x_1,y_1) \in \overline{I}\times[0,T]$. Set $z_1 = Z(x_1,y_1)$ and $z_0 = Z(x_0,y_0)$. Then, analogously above, using again Hölder's inequality, the change of variables $\zeta = \xi + i\eta = Z(\omega_1,\omega_2)$ and (†) we obtain

$$|T_Z f(x_1,y_1) - T_Z f(x_0,y_0)| \le \frac{|z_1 - z_0|}{2\pi} ||f||_{L^p(I\times(0,T))} J(z_1,z_0)^{\frac{1}{q}}, \tag{7.8}$$

where

$$J(z_1,z_0) = \int_{Z(I\times(0,T))} \frac{d\xi d\eta}{\eta^\tau |\zeta - z_1|^q |\zeta - z_0|^q}. \tag{7.9}$$

Since $1 \le q < 2-\tau$, according to [CDM1, Lemma 8], there exists a constant $N = N(p,\tau,I,T) > 0$ such that

$$J(z_1,z_0) \le N|z_1 - z_0|^{2-2q-\tau} \text{ for all } (x,t) \in \overline{I}\times[0,T]. \tag{7.10}$$

Then (7.8) and (7.10) imply (♠) and the first part of the theorem is proved. The second part follows by changing the order of integration to get

$$\begin{aligned}\langle T_Z f, \psi\rangle &= \int_{I\times(0,T)}\left(\frac{1}{2\pi i}\int_{I\times(0,T)}\frac{f(\xi,\eta)d\xi d\eta}{Z(\xi,\eta)-Z(x,y)}\right)\psi(x,y)dxdy \\ &= \int_{I\times(0,T)} f(\xi,\eta)\left(-\frac{1}{2\pi i}\int_{I\times(0,T)}\frac{\psi(x,y)dxdy}{Z(x,y)-Z(\xi,\eta)}\right)d\xi d\eta \\ &= \langle f, -T_Z\psi\rangle. \quad \square\end{aligned}$$

**Lemma 7.3.** *Let $U \subset\subset I\times(0,T)$ be a connected open set with piecewise smooth boundary. Let $w \in C^1(\overline{U})$. Then*

$$\int_U Lw(x,y)dxdy = \int_{\partial U} w(x,y)dZ(x,y). \tag{7.11}$$

Proof: We have that $Z : I\times(0,T) \to Z(I\times(0,T))$ is a diffeomorphism and $\det(DZ) = \varphi_t > 0$ in $\overline{U}$. Let $\tilde{w} \doteq w\circ Z^{-1}$ and $\zeta = Z(x,t)$. Note that $\tilde{w}$ is of class $C^1$ in $\overline{U}$. We have

$$Lw(x,y) = 2i\frac{\partial\tilde{w}}{\partial\bar{\zeta}}(Z(x,y))\det(DZ)(x,y),\ (x,y)\in\overline{U}. \tag{7.12}$$

Therefore, by (7.12) and by Green's Theorem,

$$\begin{aligned}\int_U Lw(x,y)dxdy &= 2i\int_{Z(U)}\frac{\partial\tilde{w}}{\partial\bar{\zeta}}(\zeta)d\xi d\eta,\ \zeta=\xi+i\eta, \\ &= \int_{\partial Z(U)}\tilde{w}(\zeta)d\zeta = \int_{\partial U} w(x,y)dZ(x,y).\end{aligned}$$

□

**Proposition 7.1.** *(Representation formula). Let $\phi\in C^1_c(I\times(0,T))$. Then, for all $(x,t)\in I\times(0,T)$, we have*

$$\phi(x,t) = T_Z(L\phi)(x,t),\ (x,t)\in I\times(0,T). \tag{7.13}$$

Proof: Let $\phi\in C^1_c(I\times(0,T))$ and let $(x_0,t_0)\in I\times(0,T)$ be fixed. Let $U\subset\subset I\times(0,T)$ be an simply connected open set with piecewise smooth boundary such that $\operatorname{supp}(\phi)\cup\{(x_0,t_0)\}\subset U$. Set $z_0 = Z(x_0,t_0)$ and let $\epsilon>0$ be such that $\overline{D_\epsilon}\subset Z(U)$, where $D_\epsilon = \{\zeta\in\mathbb{C}\ ;\ |\zeta-z_0|<\epsilon\}$. Define $K_\epsilon = Z^{-1}(\overline{D_\epsilon})$ and $U_\epsilon = U\setminus K_\epsilon$. By Lemma 7.3,

$$\int_{U_\epsilon}\frac{L\phi(x,y)dxdy}{Z(x,y)-z_0} = \int_{U_\epsilon} L\Big(\frac{\phi(x,y)}{Z(x,y)-z_0}\Big)dxdy = \int_{\partial K_\epsilon}\frac{\phi(x,y)dZ(x,y)}{Z(x,y)-z_0}. \tag{7.14}$$

Changing the variables in the integral we have that

$$\lim_{\epsilon\to0}\int_{\partial K_\epsilon}\frac{\phi(x,y)}{Z(x,y)-z_0}dZ(x,y) = \lim_{\epsilon\to0}\int_{\partial D_\epsilon}\frac{\phi(Z^{-1}(\zeta))}{\zeta-z_0}d\zeta = 2\pi i\,\phi(x_0,t_0). \tag{7.15}$$

On the other hand, by Theorem 7.2,

$$||T_Z(\chi_{K_\epsilon}L\phi)||_{L^\infty(I\times(0,T))} \le M||\chi_{K_\epsilon}L\phi||_{L^p(I\times(0,T))},$$

where $\chi_{K_\epsilon}$ are the characteristic functions of the sets $K_\epsilon$ and $M > 0$ is a constant. Hence,

$$\lim_{\epsilon\to 0}\int_{U_\epsilon}\frac{L\phi(x,y)dxdy}{Z(x,y)-z_0} = \int_U \frac{L\phi(x,y)dxdy}{Z(x,y)-z_0} = \int_{I\times(0,T)}\frac{L\phi(x,y)dxdy}{Z(x,y)-z_0}. \tag{7.16}$$

By (7.14), (7.15) and (7.16) the result follows. □

**Theorem 7.3.** *Let $f \in L^p(I\times(0,T))$, $p > 2+\tau/(1-\tau)$. Then $L(T_Z f) = f$ in $\mathcal{D}'_1(I\times(0,T))$.*

Proof: Let $\phi \in C^1_c(I\times(0,T))$. By the antisymmetry of $T_Z$ and Lemma 7.1 we have

$$\begin{aligned}\langle L(T_Z f),\phi\rangle &= \langle T_Z f, -L\phi\rangle\\ &= \langle f, T_Z(L\phi)\rangle = \langle f,\phi\rangle.\end{aligned}$$

Therefore, $L(T_Z f) = f$ in $\mathcal{D}'_1(I\times(0,T))$. □

**Lemma 7.4.** *Let $v \in C(I\times(0,T))$ be such that $Lv \in L^p(I\times(0,T))$, with $p > 2+\tau/(1-\tau)$. Let $U \subset\subset I\times(0,T)$ be a simply connected open set with smooth boundary. Then, there exists a sequence $(v_\epsilon)_{\epsilon>0}$, with $v_\epsilon \in C^1(U)\cap C(\overline{U})$, such that*

$$||v_\epsilon - v||_{L^\infty(U)} \to 0 \text{ as } \epsilon \searrow 0 \text{ and } ||L(v_\epsilon)-L(v)||_{L^p(U)} \to 0 \text{ as } \epsilon\searrow 0.$$

Proof: There exists $\Phi \in C^1(U)\cap C(\overline{U})$ with $L\Phi = 0$ in $U$ such that

$$v = T_{Z,U}(Lv) + \Phi \text{ in } \overline{U},$$

with $T_{Z,U}(Lv)(x,t) \doteq T_Z(\chi_U(Lv))(x,t)$, $(x,t)\in\overline{U}$, where $\chi_U$ is a characteristic function of set $U$ and $T_Z$ is given by (7.4). Let $\rho \in C_c^\infty(\mathbb{R}^2)$, with $\mathrm{supp}(\rho)\subset\{\mathsf{p}\in\mathbb{R}^2\ ;\ |\mathsf{p}|\le 1\}$, $\rho\ge 0$ and $\int_{\mathbb{R}^2}\rho(\mathsf{p})d\mathsf{p} = 1$. For $\varepsilon>0$, set $\rho_\varepsilon(\mathsf{p}) = \rho(\mathsf{p}/\varepsilon)\varepsilon^{-2}$ and define

$$v_\epsilon \doteq T_{Z,U}((Lv)*\rho_\epsilon) + \Phi \text{ in } \overline{U}. \tag{7.17}$$

Note that $(Lv)*\rho_\epsilon$ is a function of class $C^\infty$ in $\mathbb{R}^2$. Besides,

$$||(Lv)*\rho_\epsilon - Lv||_{L^p(U)} \to 0 \text{ as } \epsilon\searrow 0.$$

We have

$$v_\epsilon(x,t) = \frac{1}{2\pi i}\int_{Z(U)}\frac{w(\zeta)}{\zeta - Z(x,t)}d\xi d\eta + \Phi(x,t),\ \zeta = \xi + i\eta,$$

where $w(\zeta) = ((Lv)*\rho_\epsilon)/\det(DZ))(Z^{-1}(\zeta))$, $\zeta\in\overline{Z(U)}$. Note that $w\in C^\alpha(\overline{Z(U)})$. Hence, by Theorem 1.32 in [V, page 56], we have that $v_\epsilon \in C^{1+\alpha}(U)$. This proves the desired regularity of the approximating sequence $v_\epsilon$. On the other hand, since $Lv_\epsilon = (Lv)*\rho_\epsilon$,

$$||L(v_\epsilon) - Lv||_{L^p(U)} = ||(Lv)*\rho_\epsilon - Lv||_{L^p(U)} \to 0 \text{ as } \epsilon\searrow 0. \tag{7.18}$$

Also, by Theorem 7.2 we have

$$||v_\epsilon - v||_{L^\infty(U)} \le M||(Lv)*\rho_\epsilon - Lv||_{L^p(U)},$$

where $M = M(p,\tau,U) > 0$. Hence, (7.18) shows that

$$||v_\epsilon - v||_{L^\infty(U)} \ \to\ 0 \text{ as } \epsilon\searrow 0.$$

□

**Lemma 7.5.** *Let $u \in L^q(I \times (0,T))$, $1 \le q < 2-\tau$, and $v \in C(I \times (0,T))$ be such that $Lu \in L^1(I \times (0,T))$ and $Lv \in L^p(I \times (0,T))$, $1/p + 1/q = 1$. Then,*

$$L(uv) = L(u)v + uL(v) \text{ in } \mathcal{D}'_1(I \times (0,T)).$$

PROOF: Let $\phi \in C^1_c(I \times (0,T))$. Let $U \subset\subset I \times (0,T)$ by a simply connected open with smooth boundary and $\operatorname{supp}(\phi) \subset U$. Let $v_\epsilon$ given by Lemma 7.4. We have by the classical Leibniz rule, applied to $\phi v_\epsilon$,

$$\begin{aligned}
\langle L(uv_\epsilon), \phi\rangle_U &= \langle uv_\epsilon, -L\phi\rangle_U \\
&= -\int_U (uv_\epsilon L\phi)(\xi,\eta)d\xi d\eta = -\int_U (uL(v_\epsilon\phi) - uL(v_\epsilon)\phi)(\xi,\eta)d\xi d\eta \\
&= \langle u, -L(\phi v_\epsilon)\rangle_U + \langle uL(v_\epsilon), \phi\rangle_U \\
&= \langle L(u)v_\epsilon, \phi\rangle_U + \langle uL(v_\epsilon), \phi\rangle_U.
\end{aligned}$$

Taking advantage of Lemma (7.4) we get

$$\begin{aligned}
|\langle L(uv_\epsilon), \phi\rangle_U - \langle L(uv), \phi\rangle_U| &\le ||uL(\phi)||_{L^1(U)}||v_\epsilon - v||_{L^\infty(U)} \;\to\; 0 \text{ as } \epsilon \searrow 0, \\
|\langle L(u)v_\epsilon, \phi\rangle_U - \langle L(u)v, \phi\rangle_U| &\le ||L(u)\phi||_{L^1(U)}||v_\epsilon - v||_{L^\infty(U)} \;\to\; 0 \text{ as } \epsilon \searrow 0, \\
|\langle uL(v_\epsilon), \phi\rangle_U - \langle uL(v), \phi\rangle_U| &\le ||u\phi||_{L^q(U)}||L(v_\epsilon) - L(v)||_{L^p(U)} \;\to\; 0 \text{ as } \epsilon \searrow 0.
\end{aligned}$$

Thus,

$$\langle L(uv), \phi\rangle_U = \langle uL(v), \phi\rangle_U + \langle L(u)v, \phi\rangle_U.$$

□

**Lemma 7.6.** *Let $g \in C(I\times(0,T))$ be such that $Lg \in L^p(I\times(0,T))$, $p > 2+\tau/(1-\tau)$. Then,*

$$L(e^g) = e^g Lg \text{ in } \mathcal{D}'_1(I \times (0,T)). \tag{7.19}$$

PROOF: Let $\phi \in C^1_c(I \times (0,T))$. Let $U \subset\subset I \times (0,T)$ be a simply connected open set with smooth boundary and $\operatorname{supp}(\phi) \subset U$. By Lemma 7.4 there exists $g_\epsilon \in C^1(U)\cap C(\overline{U})$ such that $||g_\epsilon - g||_{L^\infty(U)} \to 0$ as $\epsilon \searrow 0$ and $||L(g_\epsilon) - L(g)||_{L^p(U)} \to 0$ as $\epsilon \searrow 0$. Since $g_\epsilon$ is of class $C^1$ we have by classical chain rule

$$L(e^{g_\epsilon}) = e^{g_\epsilon} Lg_\epsilon \text{ in } U. \tag{7.20}$$

Let $C > ||g||_{L^\infty(U)}$ be a constant such that $||g_\epsilon||_{L^\infty(U)} < C$ for all $\epsilon > 0$. Since $e^z$ is a Lipschitz function on the bounded set $\{z \in \mathbb{C}\ ;\ |z| \le 2C\}$, there exists a constant $K > 0$ such that

$$||e^g - e^{g_\epsilon}||_{L^\infty(U)} \le K||g - g_\epsilon||_{L^\infty(U)} \text{ for all } \epsilon > 0. \tag{7.21}$$

Hence, we have that following estimate:

$$||e^g L(g) - e^{g_\epsilon}L(g_\epsilon)||_{L^p(U)} \le ||L(g)||_{L^p(U)}K||g - g_\epsilon||_{L^\infty(U)} + e^C||L(g) - L(g_\epsilon)||_{L^p(U)}.$$

It follows that,

$$\langle e^{g_\epsilon}L(g_\epsilon) - e^g L(g), \phi\rangle_U \;\to 0. \tag{7.22}$$

On the other hand, by (7.21),

$$|\langle L(e^{g_\epsilon}) - L(e^g), \phi\rangle_U| \le K||L\varphi||_{L^1(U)}||g_\varepsilon - g||_{L^\infty(U)}.$$

It follows that,

$$\langle L(e^{g_\epsilon}) - L(e^g), \phi\rangle_U \;\to\; 0. \tag{7.23}$$

Therefore, (7.20), (7.22) and (7.23) imply (7.19). □

We may now prove Theorem 7.1

PROOF: Suppose that $u \in L^q(I \times (0,T))$ and that $u$ is not identically zero. Define the function $\chi$ in $I \times (0,T)$ by $\chi = \overline{u}/u$ on the points where $u$ is not zero and by $\chi = 0$ on the points that $u = 0$. Note that $\chi \in L^\infty(I \times (0,T))$. It follows that $A + B\chi \in L^p(I \times (0,T))$. Consider the equation

$$LS = A + B\chi \text{ in } I \times (0,T). \tag{7.24}$$

By Theorem 7.2 there is a solution $S \in C^\theta(\overline{I} \times [0,T])$, with $\theta = (2-q-\tau)/q$. Define

$$H = ue^{-S} \text{ in } I \times (0,T).$$

In view of Lemmas 7.5 and 7.6 we have $LH = 0$ in $\mathcal{D}'_1(I \times (0,T))$. We may now invoke Lemma 7.2 to write $H = h \circ Z$ in $I \times (0,T)$, with $h$ holomorphic in $Z(I \times (0,T))$. Then (7.2) follows from the definition of $H$. □

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