UFSCar Summer Webinar on PDEs
17 - 19 February 2021

Speakers
Andrzej Szulkin (at Stockholm)
Augusto Ponce (at Louvain-la-Neuve)
Boyan Sirakov (at Rio de Janeiro)
Claudianor Alves (at Campina Grande)
David Arcoya (at Granada)
Denis Bonheure (at Bruxelles)
Djairo de Figueiredo (at Campinas)
Eduardo Teixeira (at Orlando)
Giovany Figueiredo (at Brasilia)
João Marcos do Ó (at João Pessoa)
Lucio Boccardo (at Roma)
Manuel Del Pino (at Bath)
Patrizia Pucci (at Perugia)

Scientific Committee
Andrzej Szulkin
Boyan Sirakov
Djairo de Figueiredo
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João Marcos do Ó
Manuel Del Pino

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# UFSCar Summer Webinar on PDEs
São Carlos, Brazil, 17 - 19 February 2021

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More information at: [www.dm.ufscar.br/webinarpdes](http://www.dm.ufscar.br/webinarpdes)
PRESENTATION

It is a pleasure to welcome you to the UFSCar Summer Webinar on PDEs.

During the XXXIV Summer Program of the PPGM-UFSCar Graduate Program in Mathematics there will be the UFSCar Summer Webinar on PDEs. The webinar is devoted to disseminate research related to Partial Differential Equations, PDEs for short, along a series of lectures given by specialists on the field. We believe that although the pandemic is a period where many difficulties must be faced effort to gain inspiration is possible to be done. That is our main motivation about this webinar.

We wish you enjoy.

The local committee.
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1 A SOBOLEV-TYPE INEQUALITY FOR THE CURL OPERATOR AND GROUND STATES FOR THE CURL-CURL EQUATION WITH CRITICAL SOBOLEV EXPONENT

Andrzej Szulkin
Stockholm

Abstract

Let \( \Omega \) be a domain in \( \mathbb{R}^3 \) and let

\[
S(\Omega) := \inf \{ |\nabla u|^2_2 / |u|^2_6 : u \in C_0^\infty (\Omega) \setminus \{0\} \}
\]

be the Sobolev constant with respect to the embedding \( D^{1,2}_0(\Omega) \hookrightarrow L^6(\Omega) \).

As is well known, \( S(\Omega) \) is independent of \( \Omega \), it is attained if and only if \( \Omega = \mathbb{R}^3 \) and the infimum is taken by ground state solutions for the equation \( -\Delta u = |u|^4 u \) in \( D^{1,2}(\mathbb{R}^3) \) (the Aubin-Talenti instantons).

In this talk we will be concerned with the curl operator \( \nabla \times \cdot \). In order to define a Sobolev-type constant it seems natural to replace \( S(\Omega) \) by

\[
\overline{S}(\Omega) := \inf \{ |\nabla \times u|^2_2 / |u|^2_6 : u \in C_0^\infty (\Omega, \mathbb{R}^3) \setminus \{0\} \}.
\]

However, since the kernel of curl is nontrivial (\( \nabla \times u = 0 \forall u = \nabla \varphi \)), this constant would always be 0.

After discussing the physical background we will define another constant, \( S_{\text{curl}}(\Omega) \), as a certain infimum. It has the following properties:

\( S_{\text{curl}}(\Omega) > S(\Omega) \); \( S_{\text{curl}}(\Omega) \) is independent of \( \Omega \); the infimum is attained when \( \Omega = \mathbb{R}^3 \) and is taken by a ground state solution to the equation \( \nabla \times (\nabla \times u) = |u|^4 u \) (which is related to Maxwell’s equations). The problem of (non)existence of the minimum for \( \Omega \neq \mathbb{R}^3 \) remains open.

If time permits, we shall briefly discuss the Brezis-Nirenberg problem for the curl-curl operator on bounded domains.

This is joint work with Jarosław Mederski.
2 THE UNCHARTED TERRITORY OF $W^{\alpha,1}$ SOBOLEV SPACES

Augusto Ponce
Louvain la Neuve

Abstract

I will address properties of the fractional Sobolev space $W^{\alpha,1}(\mathbb{R}^N)$ which cannot be answered by classical representation formulas from Harmonic Analysis, for any exponent $\alpha > 0$. They can be handled instead in terms of a strong capacitory inequality which is based itself on a new geometric boxing inequality that connects the Hausdorff content of dimension $N - \alpha$ and the fractional perimeter of order $0 < \alpha < 1$. These results have been obtained in collaboration with D. Spector (Okinawa, Japan).
3 THE VÁZQUEZ MAXIMUM PRINCIPLE AND THE LANDIS CONJECTURE FOR ELLIPTIC PDE WITH UNBOUNDED COEFFICIENTS

Boyan Sirakov
Pontifícia Universidade Católica do Rio de Janeiro

Abstract

In this joint work with P. Souplet we develop a new, unified approach to the following two classical questions on elliptic PDE: (i) the strong maximum principle for equations with non-Lipschitz nonlinearities; and (ii) the at most exponential decay of solutions in the whole space or exterior domains. Our results apply to divergence and nondivergence operators with locally unbounded lower-order coefficients, in a number of situations where all previous results required bounded ingredients. Our approach, which allows for relatively simple and short proofs, is based on a (weak) Harnack inequality with optimal dependence of the constants in the lower-order terms of the equation and the size of the domain, which we establish.
4 NORMALIZED SOLUTIONS FOR SCHRÖDINGER EQUATION WITH CRITICAL GROWTH IN $\mathbb{R}^N$.

Claudianor Alves
Universidade Federal de Campina Grande

Abstract

In this lecture I will talk about the existence of normalized solutions to the following nonlinear Schrödinger equation with critical growth

$$\begin{cases}
-\Delta u + \lambda u = f(u), & \text{in } \mathbb{R}^N, \\
u > 0, \int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}$$

where $a > 0$, $\lambda < 0$ and $f$ has an exponential critical growth when $N = 2$, and $f(t) = \mu|t|^{q-2}t + |t|^{2^*-2}t$ with $q \in (2 + \frac{4}{N}, 2^*)$, $\mu > 0$ and $2^* = \frac{2N}{N-2}$ when $N \geq 3$. Our main results complement some recent results for $N \geq 3$ and it is totally new for $N = 2$. This is a joint work with Olimpio Miyagaki and Chao Ji.
ON THE INTERPLAY BETWEEN COEFFICIENTS IN SOME NONLINEAR DIRICHLET PROBLEMS WITH DISTRIBUTIONAL DATA

David Arcoya
Universidad de Granada

Abstract
We present the main results of the joint paper [1] with Lucio Boccardo and Luigi Orsina. Specifically, we prove the exponential summability of the solution $u$ of the Dirichlet problem

$$u \in W^{1,2}_0(\Omega) : -\text{div}(M(x,\nabla u)) + a(x)u = -\text{div}(F).$$

where $\Omega$ is a bounded set in $\mathbb{R}^N$ and $-\text{div}(M(x,\nabla u))$ is a classical nonlinear differential operator, defined by a Carathéodory function $M(x,\xi)$ satisfying, for some $0 < \alpha \leq \beta$, and for almost every $x \in \Omega$,

$$M(x,\xi)\xi \geq \alpha|\xi|^2, \quad |M(x,\xi)| \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^N,$$

$$[M(x,\xi) - M(x,\eta)](\xi - \eta) > 0, \quad \forall \xi, \eta \in \mathbb{R}^N, \quad \xi \neq \eta.$$

Our key assumption is that the function $0 \leq a(x) \in L^1(\Omega)$ and the vector-valued function $F(x)$ are such that

$$\exists R > 0 \text{ such that } |F(x)|^2 \leq Ra(x).$$

In addition, we prove the boundedness of $u$ under the slightly stronger assumption that

$$\exists R > 0 \text{ and } \exists p > 2 \text{ such that } |F(x)|^p \leq Ra(x).$$

References
6 EQUILIBRIUM CONFIGURATION OF A RECTANGULAR
OBSTACLE IMMERSED IN A CHANNEL FLOW

Denis Bonheure
Université Libre de Bruxelles

Abstract

Fluid flows around an obstacle generate vortices which, in turn, generate lift forces on the obstacle. Therefore, even in a perfectly symmetric framework equilibrium positions may be asymmetric. We show that this is not the case for a Poiseuille flow in an unbounded 2D channel, at least for small Reynolds number and flow rate. We consider both the cases of vertically moving obstacles and obstacles rotating around a fixed pin.
7 p-LAPLACIAN WITH NONLINEARITIES ON THE GRADIENT WITH NATURAL GROWTH

Djairo Guedes de Figueiredo
Universidade Estadual de Campinas - UNICAMP

Abstract

This talk is concerned with existence of solutions for the problem

\[
\begin{align*}
-\Delta_p u &= g(u)|\nabla u|^p + f(x, u) \quad \text{in } \Omega, \\
u &> 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

Here \( -\Delta_p u := \text{div}(|\nabla u|^p \nabla u) \) is the \( p \)-Laplacian operator with \( 1 < p < \infty \) and \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain.

This kind of problem is known as "...gradient term with natural growth" to contrast with problems where the power on the gradient is strictly smaller than \( p \).

In this last case there is a very extensive literature.

In the case of natural growth there are several works using topological techniques. Here we use variational methods.
ON THE CRITICAL POINT REGULARITY FOR QUASILINEAR PROBLEMS

Eduardo V. Teixeira
University of Central Florida

Abstract

The $C^\alpha$ regularity conjecture claims that any solution to $-\Delta_p = f(x)$, with $f$ bounded and $p > 2$, is locally of class $C^\alpha = C^{1, \frac{1}{p-1}}$. That the sharp estimate is attained by a naive looking, radially symmetric example whose $p$-laplacian is actually constant, begs the question: assuming the $C^\alpha$ regularity conjecture true, is there any further regularity left for non-homogeneous problems driven by the $p$-laplacian? In this talk I shall discuss this question through a new prism.
POSITIVE MAXIMAL AND MINIMAL SOLUTIONS FOR NON-HOMOGENEOUS ELLIPTIC DEPENDING ON THE GRADIENT

Giovany M. Figueiredo
Universidade de Brasília

Abstract

We are concerned with positive maximal and minimal solutions for non-homogeneous elliptic equations of the form

$$- \text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u) \quad \text{in } \Omega,$$

supplied with Dirichlet boundary conditions. First we localize maximal and minimal solutions between not necessarily bounded sub-super solutions. Then using a uniform gradient estimate, which seems of independent interest, we show the existence of positive maximal and minimal solutions in some situations. More precisely, we obtain positive maximal and minimal solution to some classes of non-homogeneous equations depending on the gradient which may be perturbed by unbounded, singular or logistic sources.
10 k-HESSIAN INEQUALITY OF TRUDINGER-MOSER TYPE

João Marcos Bezerra do Ó
Universidade Federal da Paraíba

Abstract

We present some recent Trudinger-Moser type inequalities for a class of fully nonlinear partial differential operators of divergence form. Trudinger-Moser inequalities yield exponential-type embeddings of Sobolev functions in a critical dimension and play an essential role in studying partial differential equations and geometric problems.
11 RECENT DEVELOPMENTS ON DIRICHLET PROBLEMS WITH SINGULAR CONVECTION/DRIFT TERMS

Lucio Boccardo
Sapienza Università di Roma - Istituto Lombardo

Abstract

In this talk we study existence and properties of weak/distributional solutions of the Dirichlet problems

\[-\frac{\partial}{\partial x} (M(x)\nabla u) + a(x)u = -\frac{\partial}{\partial x} (u E(x)) + f(x),\]

\[-\frac{\partial}{\partial x} (M(x)\nabla \psi) + a(x)\psi = E(x)\nabla \psi + g(x).\]

In particular we present the Calderon-Zygmund-Stampacchia theory, weak maximum principle, nonlinear problems, extra regularity properties if \(|f(x)| \leq Q a(x), |g(x)| \leq Q a(x), Q \in \mathbb{R}^+.\)
SINGULARITY FORMATION FOR THE KELLER-SEGEL SYSTEM IN THE PLANE

Manuel del Pino
University of Bath

Abstract

The classical model for chemotaxis is the planar Keller-Segel system

\[ u_t = \Delta u - \nabla \cdot (u \nabla v), \quad v(\cdot, t) = \frac{1}{2\pi} \log 1|\cdot| \ast u(\cdot, t). \]

in \( \mathbb{R}^2 \times (0, \infty) \). Blow-up of finite mass solutions is expected to take place by aggregation, which is a concentration of bubbling type, common to many geometric flows. We build with precise profiles solutions in the critical-mass case \( 8\pi \), in which blow-up in infinite time takes place. We establish stability of the phenomenon detected under arbitrary mass-preserving small perturbations and present new constructions in the finite time blow-up scenario.
NONLINEAR SUBELLIPTIC PROBLEMS INVOLVING OPERATORS WITH NON STANDARD GROWTH

Patrizia Pucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia

Abstract

Recently, the study of nonlinear problems involving critical nonlinearities has attracted increasing attention in the context of stratified groups: the geometric analysis in the Heisenberg group and, more generally, in the sub-Riemannian manifolds is currently one of the most active and exciting mathematical areas. We present existence results for nonlinear problems involving subelliptic operators of Marcellini’s type, with critical nonlinearities. As far as we know, the existence results represent one of the first attempts in the literature to set such general critical elliptic equations in the context of stratified groups.

In the first part of the talk, we consider operators with \((p, q)\) growth, when both the exponents are strictly less than the homogeneous dimension \(Q\). Then, we also show existence results for problems involving critical exponential nonlinearities when \(q = Q\). In this case, a “double absence of compactness” appears in the critical nonlinearities, which now involve also singular coefficients. This study is strongly motivated by physical applications in molecular physics, quantum cosmology and linearization of combustion models. However, the extension to this general context and the presence of critical nonlinearities require several delicate estimates and techniques to deal with, in order to overcome the lack of compactness.

The results of the talk are obtained in papers, written jointly with L. Temperini.