Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

STABLE TRANSITION LAYER INDUCED BY DEGENERACY OF THE SPATIAL INHOMOGENEITIES IN THE ALLEN-CAHN PROBLEM

Maicon Sônego

Instituto de Matemática e Computação,
Universidade Federal de Itajubá, MG, Brazil

Arnaldo Simal do Nascimento

Departamento de Matemática - DM,
Universidade Federal de São Carlos, SP, Brazil

(Communicated by the associate editor name)

Abstract. In this article we consider a singularly perturbed Allen-Cahn problem $u_t = \epsilon^2(a^2u_x)_x + b^2(u-u^3)$, for $(x,t) \in (0,1)\times\mathbb{R}^+$, supplied with no-flux boundary condition. The novelty here lies in the fact that the nonnegative spatial inhomogeneities $a(\cdot)$ and $b(\cdot)$ are allowed to vanish at some points in $(0,1)$. Using the variational concept of Γ-convergence we prove that, for ϵ small, such degeneracy of $a(\cdot)$ and $b(\cdot)$ induces the existence of stable stationary solutions which develop internal transition layer as $\epsilon \to 0$.

1. **Introduction.** Consider the following singularly perturbed semilinear problem

$$\begin{cases} u_t(x,t) = \epsilon^2(a^2(x)u_x(x,t))_x + b^2(x)f(u(x,t)), & (x,t)\in I\times\mathbb{R}^+, \\ u_x(0,t) = u_x(1,t) = 0, & t\in\mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $I := (0,1)$; $\epsilon > 0$ is a positive parameter; $f(u) = u - u^3$ and $a(\cdot), b(\cdot) \in C(I)$ are nonnegative functions. Our concern in this paper is to study the role of the spatial inhomogeneities $a(\cdot)$ and $b(\cdot)$ in relation to the existence of nonconstant stable stationary solution to (1) and its asymptotic profile as $\epsilon \to 0$. We recall that a stationary solution u_ϵ of (1), i.e. a solution of problem

$$\begin{cases} \epsilon^2(a^2(x)u'(x))' + b^2(x)f(u) = 0, & x\in I \\ u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

is called *stable* if for every $\eta > 0$, $\exists\,\delta > 0$ such that for every solution v_ϵ to (1) satisfying $\|v_\epsilon(\cdot\,;0) - u_\epsilon(\cdot)\|_{L^\infty} < \delta$ it holds that $\|v_\epsilon(\cdot\,;t) - u_\epsilon(\cdot)\|_{L^\infty} < \eta$, $\forall\, t > 0$. Roughly speaking, the study of existence of stable solutions can give us the whole dynamics of the parabolic problem. We refer to the monograph [10] for a complete presentation of the main results available on stable solutions.

The novelty here lies in the fact that $a(\cdot)$ and $b(\cdot)$ are allowed to vanish at some points in I as described below. In this sense we say that these spatial inhomogeneities degenerate in I.

2010 *Mathematics Subject Classification.* Primary: 35K65; Secondary: 35K57, 35B25, 35B35.
Key words and phrases. stable transition layer, degenerate problem, Γ-convergence, Allen-Cahn equation .

We assume that the diffusivity coefficient $a(\cdot)$ satisfies either

(a_1) $a(x) > 0$, $\forall\, x \in \overline{I}$,

or

(a_2) $a(x_0) = 0$ at some $x_0 \in I$;
(a_3) $a(x) > 0$ for $x \in \overline{I} \setminus \{x_0\}$ and
(a_4) $1/a^2 \in L^1(I)$.

A typical example for $a(\cdot)$ satisfying the conditions (a_2)–(a_4) is given by

$$a(x) = |x - x_0|^{\alpha/2} \quad (0 < \alpha < 1). \tag{3}$$

Regarding $b(\cdot)$ we assume that

(b_1) $\mathcal{B} := \{x \in I;\ b(x) = 0\}$ is a finite set.

We prove the existence of families of nonconstant stable solutions that develop transition layer as $\epsilon \to 0$ (usually referred to in the literature as *stable transition layer*). This result is obtained due to the degeneracy conditions on $a(\cdot)$ and $b(\cdot)$ assumed above. All the details can be seen in Theorem 1.1 and Remark 1.

The study of degenerate parabolic problems is the subject of numerous articles under various aspects. Indeed, many problems coming from physics, biology, chemistry and economics are described by degenerate parabolic problems, see [16, 7, 13, 22, 2, 6] and references therein. Here we consider the inhomogeneous Allen-Cahn problem which has its origin in the theory of phase transitions [3] and it is used as a model for some nonlinear reaction-diffusion processes. For instance, in a process of competition between two species, the vanishing at some point of spatial inhomogeneity $a(\cdot)$ may indicate an interruption of the migration and/or interaction of the species. A similar phenomenon occurs in the interaction between substances in a chemical reaction or in the heat propagation in heterogeneous materials. In its turn, $b(x) = 0$ at some $x \in I$ means that, at this point, the action of the reaction term on the diffusion process is null. Of course, it seems natural and of great importance to study such phenomena, however, allowing the vanishing of these spatial inhomogeneities brings with it several technical difficulties. For instance, if $a(\cdot)$ satisfies (a_2)–(a_4) then the operator $Au := (a^2 u_x)_x$ is no longer elliptic (sometimes called *degenerate elliptic*) and many basic analysis tools can not be used directly.

The main result of this work is stated below. In what follows χ_A denotes the characteristic function of the set A.

Theorem 1.1. *Suppose that $a(\cdot)$ satisfies either (a_1) or (a_2)–(a_4) and $b(\cdot)$ satisfies (b_1). If the functions $c(x) := b(x)/a(x)$ and $\overline{c}(x) := a(x)b(x)$ are Lipschitz continuous in I then for each $\overline{x} \in \mathcal{B}$ satisfying either $a(\overline{x}) = 0$ or $a(\overline{x}) > 0$, there exists $\epsilon_0 > 0$ and two families $\{u_{\epsilon,i}\}_{0<\epsilon<\epsilon_0}$ $(i = 1, 2)$ of stable stationary solutions of (1) such that*

(i) $||u_{\epsilon,1} - u_{0,1}||_{L^1(I)} \to 0$ *as* $\epsilon \to 0$ *where* $u_{0,1} := \chi_{(0,\overline{x})} - \chi_{(\overline{x},1)}$ *and*
(ii) $||u_{\epsilon,2} - u_{0,2}||_{L^1(I)} \to 0$ *as* $\epsilon \to 0$ *where* $u_{0,2} := -\chi_{(0,\overline{x})} + \chi_{(\overline{x},1)}$.

Remark 1. The regularity assumptions on $c(\cdot)$ and $\overline{c}(\cdot)$ are necessary to our method because somewhere in the computation of the Γ-limit of the corresponding energy functional it is required that both c_x and $\overline{c}_x$ be L^∞-bounded. In particular it implies that if $a(\cdot)$ satisfies $(a_2) - (a_4)$ with $a(\overline{x}) = 0$ then necessarily $b(\overline{x}) = 0$ as well. For instance, for $a(\cdot)$ defined as in (3) with $\alpha = 1/2$, if we take $b(x) = |x - x_0|^{7/4}$ then $c(\cdot)$ and $\overline{c}(\cdot)$ will be Lipschitz continuous in $(0, 1)$.

Indeed these regularity assumptions account for the adaptability of the Γ- convergence approach utilized in [23] for the regular case, meaning that $a > 0$ and $b > 0$ in I, to the singular one tackled in the present work.

More specifically this issue comes about when in the proof of Theorem 1.1 we construct, through a solution of an ordinary differential equation, the transition layer connecting the two stable constant solutions given in the present case by $u_1 = \pm 1$ and $u_2 = \mp 1$.

Regarding this matter we should also call attention to the key role that the weighted spaces $H^1_a(I)$ and $H^2_a(I)$ (see Section 2) have while assuring existence of solution to (1) as well as the fact they generate an analytic contraction semi-group which is need to perform stability analysis.

In the first case – $a(\cdot)$ satisfying (a_1) – we have the usual diffusion operator with variable diffusivity. For this case there is a vast literature when the concern is to study the role of $a(\cdot)$ and/or $b(\cdot)$ in the search for stable solutions or just solutions developing transition layer. Only for the one-dimensional case we cite [1, 19, 20, 21, 12, 8, 4, 24] (in the higher dimensional setting, we refer to [18, 11] and their references). In all these previous articles it is assumed that $b(\cdot)$ is a function strictly positive too. To the best of our knowledge, the present article is the first to study the role of the isolated zeros of $b(\cdot)$ regarding the existence of nonconstant stable solution and transition layer.

Under conditions (a_2)–(a_4), problem (1) is called *weakly degenerate with degeneracy in the interior of the space domain* (see [13], for instance). Recently many authors have considered degenerate problems in this regard, see [6, 13, 7, 2, 16] and references therein. These references address the well-posedness of the problem and issues related to control theory and inverse problems. As far as we know, the first article to consider stability issues for a weakly degenerate problem was [22]. Indeed, for $b(x) \equiv 1$ and $\epsilon = 1$, the author in [22] provided a condition on the behavior of $a(\cdot)$ in a neighborhood of its zero to obtain the existence of a stable solution (in a weaker sense).

In this article we use a variational procedure based on Γ-convergence. It provides sufficient conditions under which the problem of finding local minimizers of the family of corresponding energy functionals is reduced to finding local minimizers of a more tractable geometric problem in the space of functions of bounded variation. The computation of Γ-convergence was inspired in some results from [23]. However, due to the assumptions on $a(\cdot)$ and $b(\cdot)$, several modifications were necessary. Therefore, the computation of the Γ-convergence is another contribution of this work and it may have interest of its own.

This article is divided as follows. In Section 2 we establish the well-posedness of the problem (1) and present some material on stability of solutions when $a(\cdot)$ satisfies (a_2)–(a_4). Moreover, we recall some notations and results from theory of functions of bounded variation and Γ-convergence. In Section 3 we present the complete computation of Γ-convergence and in Section 4 we prove Theorem 1.1.

2. **Preliminaries.** The next two subsections are devoted to when $a(\cdot)$ satisfies (a_2)–(a_4). Indeed, in the other case we have the elliptic operator for which such a theory is very well known.

2.1. **Function space setting.** Due to the degeneracy of the problem when $a(\cdot)$ satisfies (a_2)–(a_4), even the well-posedness of parabolic problem (1) is unknown.

Here we establish the appropriate function space for problem (1) and the results that guarantee the existence of solution and its regularity.

Firstly, we should note that $Au = (a^2 u_x)_x$, in a suitable domain, generates an analytic semigroup. For this purpose, we introduce the following weighted spaces (sometimes, we use $'$ to denote the derivative with respect to x):

$$H^1_a(I) := \{u \in L^2;\ u \text{ absolutely continuous in } \overline{I} \text{ and } \ au' \in L^2(I)\} \tag{4}$$

with the norm

$$||u||^2_{H^1_a(I)} := ||u||^2_{L^2(I)} + ||au'||^2_{L^2(I)} \tag{5}$$

and

$$H^2_a(I) := \{u \in H^1_a(I); a^2 u' \in H^1(I)\} \tag{6}$$

with

$$||u||^2_{H^2_a(I)} := ||u||^2_{H^1_a(I)} + ||(au')'||^2_{L^2(I)}.$$

Definition 2.1. If $u_0 \in L^2(I)$, a function u is said to be a weak solution of (1) if

$$u \in C([0,T]; L^2(I)) \cap L^2(0,T; H^1_a(I))$$

and

$$\int_0^1 u(T,x)\phi(T,x)dx - \int_0^1 u_0(x)\phi(0,x)dx - \int_{(0,T)\times I} u\phi_t dxdt =$$

$$- \int_{(0,T)\times I} a^2 u_x \phi_x dxdt + \int_{(0,T)\times I} b^2 f(u)\phi dxdt$$

for all $\phi \in H^1(0,T; L^2(I)) \cap L^2(0,T; H^1_a(I))$.

Now we define the operator A by $D(A) := \{u \in H^2_a(I); u'(0) = u'(1) = 0\}$ and for any $u \in D(A)$, $Au = (a^2 u')'$.

The next two results can be found in [6, Theorem 2.1] and [22, Theorem 2.6], respectively. In [22] the problem is considered with a homogeneous reaction term, however the presence of the nonnegative function $b(\cdot)$ adds no additional difficulty.

Theorem 2.2. *The operator $A : D(A) \to L^2(I)$ is self-adjoint, nonpositive on $L^2(I)$ and it generates an analytic contraction semigroup.*

Theorem 2.3. *If $u_0(x) \in H^1_a(I)$ then (1) has a solution*

$$u \in H^1(0,T; L^2(I)) \cap L^2(0,T; H^2_a(I)).$$

2.2. **Stability analysis.** Again, we restrict this subsection to the case where $a(\cdot)$ satisfies (a_2)–(a_4). As we will see, the results presented here for the degenerate case are the same as those well known for the non-degenerate case. This mainly occurs because of the condition (a_4).

The eigenvalue problem associate to linearisation of problem (2) around a stationary solution u_0 is

$$\begin{cases} \epsilon^2 (a^2(x)\phi'(x))' + b^2(x) f'(u_0)\phi = \lambda\phi, & x \in (0,1) \\ \phi'(0) = \phi'(1) = 0. \end{cases} \tag{7}$$

Our hypotheses on $a(\cdot)$, $b(\cdot)$ and $f(\cdot)$ – here, we note that hypothesis (a_4) is fundamental – ensure that (7) is a Sturm-Liouville problem of the form discussed in [5, Chapter 8] and it has the usual spectral properties of such problems as described in [5, Theorem 8.4.5]. We have that all eigenvalues are real and there is a unique *principal eigenvalue* λ_0 such that any eigenvalue $\lambda \neq \lambda_0$ satisfies $\lambda < \lambda_0$.

The well known result below (at least for the non-degenerate case) will be helpful in the proof of the main result of this article. The proof can be made following rigorously the steps of [15] since by virtue of Theorems 2.2 and 2.3 it makes sense to perform the very same analysis used in the regular case.

Lemma 2.4. *Let λ_0 be the principal eigenvalue of (7) and ϕ_0 its eigenfunction. Then λ_0 is a simple eigenvalue and we have the following variational characterization for it:*

$$\lambda_0 = \inf\{R_{u_0}[\phi]; \quad \phi \in H^1_a(I), \quad ||\phi||_{L^2(I)} = 1\} \tag{8}$$

where

$$R_{u_0}[\phi] := \int_0^1 \left[\epsilon^2 a^2(x)(\phi'(x))^2 - b^2(x) f'(u_0)\phi^2\right] dx.$$

We have that the sign of the principal eigenvalue λ_0 indicates the stability of u_0; that is, if $\lambda_0 > 0$ then u_0 is stable and if $\lambda_0 < 0$ then u_0 is unstable. If $\lambda_0 = 0$ then stability or instability can occur. These results can be seen in [15] (note that the generality of the results in [15] covers the degenerate case studied here).

2.3. **Functions of bounded variation and Γ -convergence.** In the sequel some definitions, notations and results about functions of bounded variations and Γ-convergence are presented. The interested reader is referred to [14, 9], for instance, for more on this matter.

We say that u is a function of *essential bounded variation* in I (and write $u \in BV(I)$) if its derivative in the sense of distributions is a measure with finite total variation in I. In the sense of distributions, Du is a vector valued Radon measure with finite total variation in I given by

$$|Du| = \sup\left\{\int_0^1 u\sigma' dx; \quad \sigma \in C_0^\infty(I), \quad |\sigma| \le 1\right\}.$$

The total variation $|Du|$ is a Radon measure itself. We denote by $BV(I, \{\alpha, \beta\})$, $\alpha, \beta \in \mathbb{R}$, the class of all $u \in BV(I)$ which take values α and β only. If $u \in BV(I)$, the integral of any positive continuous function h with respect to the measure $|Du|$ can be expressed as

$$\int_0^1 h|Du| = \sup\left\{\int_0^1 u\sigma' dx; \quad \sigma \in C_0^\infty(0,1), \quad |\sigma| \le h\right\}. \tag{9}$$

Given $u \in L^1_{loc}(I)$, the *jump set of* u, denoted by S_u, is the complement of the set of Lebesgue points of u, i.e., the set of points where the upper and lower approximate limits of u differ or are not finite. If $u \in BV(I, \{\alpha, \beta\})$, α and β constants, then $\mathcal{H}^0(S_u) < \infty$ (here $\mathcal{H}^0$ stands for the Hausdorff counting measure). For details the reader is referred to [14], for instance.

The following version of the *co-area formula* will be used:

$$\int_0^1 f(h(x))|h'(x)|dx = \int_{\mathbb{R}} \int_{\{x \in I;\ h(x)=s\}} f(s) d\mathcal{H}^0 ds, \tag{10}$$

which holds for any Lebesgue measurable f and Lipschitz-continuous h.

Next, the definition of the Γ-convergence of a family of functionals with respect to the L^1-topology is given.

Definition 2.5. A family $\{E_\epsilon\}_{\epsilon>0}$ of real-extended functionals defined in $L^1(I)$ is said to Γ-converge, as $\epsilon \to 0$, to a functional E_0 if:

(i) $\forall\, v \in L^1(I)$ and $\forall\ \{v_\epsilon\} \subset L^1(I)$ such that $v_\epsilon \to v$ in $L^1(I)$, as $\epsilon \to 0$,

$$E_0(v) \leq \lim_{\epsilon\to 0}\inf E_\epsilon(v_\epsilon).$$

(ii) $\forall\, v \in L^1(I)$, $\exists\ \{v_\epsilon\}$ in $L^1(I)$ such that $v_\epsilon \to v$ in $L^1(I)$, as $\epsilon \to 0$, and

$$E_0(v) \geq \lim_{\epsilon\to 0}\sup E_\epsilon(v_\epsilon).$$

The theorem below, which can be found in [17], is essential to our analysis.

Theorem 2.6. *Suppose that the following hypotheses are satisfied:*

(i) *E_ϵ Γ-converges to E_0 in $L^1(I)$, as $\epsilon \to 0$;*
(ii) *$\forall\ \{u_\epsilon\}_{\epsilon>0} : E_\epsilon(u_\epsilon) \leq C < \infty$, $\forall\, \epsilon > 0$, is compact in $L^1(I)$ and*
(iii) *there exists an isolated L^1-local minimizer u_0 of E_0.*

Then there exists $\epsilon_0 > 0$ and a family $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ such that:

- *u_ϵ is an L^1-local minimizer of E_ϵ and*
- *$||u_\epsilon - u_0||_{L^1(I)} \to 0$ as $\epsilon \to 0$.*

3. **Computation of the Γ-limit.** Our goal in this section is to find the Γ-limit E_0 of the family of functionals E_ϵ whose critical points are stationary solutions to (1). We note that, by the assumptions on c and $\bar{c}$, the arguments used in this section do not distinguish whether $a(\cdot)$ satisfies (a_1) or (a_2)–(a_4). In fact, when $a(\cdot)$ satisfies (a_1) just replace $H^1_a(I)$ with $H^1(I)$.

We define $E_\epsilon : L^1(I) \to \mathbb{R} \cup \{\infty\}$ by

$$E_\epsilon(v) = \begin{cases} \displaystyle\int_0^1 \left[\frac{\epsilon a^2(x)}{2}\left|v'\right|^2 + \frac{1}{\epsilon} b^2(x) F(v)\right] dx, & v \in H^1_a(I) \\ \infty, & \text{otherwise,} \end{cases} \tag{11}$$

where F is given by

$$F(v) = -\int_{-1}^{v} f(s)ds = \frac{-v^2}{2} + \frac{v^4}{4} + \frac{1}{4}. \tag{12}$$

It is routine to prove that E_ϵ is of class C^2 and the next theorem shows us its Γ-limit (here we denote by $\chi_{\{v=1\}}$ the characteristic function related to set $\{x \in [0,1]; v(x) = 1\}$).

Theorem 3.1. *Consider $E_0 : L^1(I) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$E_0(v) = \begin{cases} \displaystyle\int_0^1 \mathcal{P}(x)\left|D\chi_{\{v=1\}}\right|, & v \in BV(I, \{-1,1\}) \\ \infty, & \textit{otherwise,} \end{cases}$$

where

$$\mathcal{P}(x) = \sqrt{2}\bar{c}(x)\int_{-1}^{1}\sqrt{F(s)}ds. \tag{13}$$

Then E_ϵ Γ-converges to E_0 in $L^1(I)$, as $\epsilon \to 0$.

Remark 2. (i) Note that due to the fact that $\mathcal{P}$ is degenerate – in the sense that $\mathcal{P}(x) = 0$ for all $x \in \mathcal{B}$ – and the definition of E_ϵ depends on the weighted Sobolev space $H^1_a(I)$, the computation of the Γ-limit provided in [23] does not apply directly.

(ii) If S_v is the jump set of $v \in BV(I, \{-1,1\})$, then we can write

$$E_0(v) = \sum_{x \in S_v} \mathcal{P}(x).$$

In the sequel, we verify the requirements of Definition 2.5 of Γ-convergence.

3.1. **Condition** (i) **of Definition 2.5.** Let $v \in L^1(I)$ and a sequence $\{v_\epsilon\} \subset L^1(I)$ such that $v_\epsilon \to v$ in $L^1(I)$. We can suppose that $\liminf_{\epsilon \to 0} E_\epsilon(v_\epsilon)$ is finite (in particular, we suppose $v_\epsilon \in H^1_a(I)$) and then there is $\{v_j\}_{j\in\mathbb{N}}$ such that $v_j = v_{\epsilon_j}$ and

$$\lim_{j\to\infty} E_{\epsilon_j}(v_j) = \liminf_{\epsilon\to 0} E_\epsilon(v_\epsilon) \in \mathbb{R}.$$

As

$$\int_0^1 b^2(x)F(v_j)dx \le \epsilon_j E_{\epsilon_j}(v_j)$$

we use the Fatou's lemma to conclude that $v(x) \in \{-1,1\}$ a.e. $x \in I$. In this case we can suppose that $-1 \le v_j(x) \le 1$ since, otherwise, we consider the truncated sequence $\{v_j^*\}$ defined by

$$v_j^* = \begin{cases} -1, & v_j < -1 \\ v_j, & -1 \le v_j \le 1 \\ 1, & v_j > 1. \end{cases}$$

It not difficult to see that $v_j^* \to v$ in $L^1(I)$ and $E_{\epsilon_j}(v_j) \ge E_{\epsilon_j}(v_j^*)$.

Using that $a^2 + b^2 \ge 2ab$ for any $a, b \in \mathbb{R}$ (Young's inequality) and setting

$$\mathcal{A} := \{\sigma \in C_0^1(I); \quad |\sigma(x)| \le 1\},$$

we have

$$\begin{aligned} E_{\epsilon_j}(v_j) &= \int_0^1 \left[\frac{\epsilon_j a^2(x)}{2}|v_j'|^2 + \frac{1}{\epsilon_j} b^2(x)F(v_j)\right] dx \\ &= \int_0^1 \left[\left(\frac{\sqrt{\epsilon_j}a(x)}{\sqrt{2}}|v_j'|\right)^2 + \left(\frac{1}{\sqrt{\epsilon_j}} b(x)\sqrt{F(v_j)}\right)^2\right] dx \\ &\ge \sqrt{2}\int_0^1 a(x)b(x)|v_j'|\sqrt{F(v_j)}dx \\ &\ge \sup_{\sigma\in\mathcal{A}}\left\{\sqrt{2}\int_\Omega a(x)b(x)\sqrt{F(v_j)}\; v_j'\sigma \; dx\right\}. \end{aligned} \tag{14}$$

We note that if

$$\eta(t,x) := \int_{-1}^t a(x)b(x)\sqrt{F(s)}ds$$

then (recall that $F(-1) = 0$)

$$\frac{d}{dx}\eta(v_j(x),x) = \int_{-1}^{v_j(x)} \frac{d}{dx}\left(a(x)b(x)\sqrt{F(s)}\right) ds + a(x)b(x)\sqrt{F(v_j(x))}v_j'(x).$$

Hence,

$$\int_0^1 a(x)b(x)\sqrt{F(v_j)}v_j'\sigma\ dx = -\int_0^1\int_{-1}^{v_j(x)} a(x)b(x)\sqrt{F(s)}ds\sigma' dx - \int_0^1\int_{-1}^{v_j(x)} (a(x)b(x))'\sqrt{F(s)}ds\sigma\ dx.$$

By (14),

$$E_{\epsilon_j}(v_j) \geq \sup_{\sigma\in\mathcal{A}}\left\{-\sqrt{2}\int_0^1\int_{-1}^{v_j(x)} a(x)b(x)\sqrt{F(s)}ds\sigma' dx -\sqrt{2}\int_0^1\int_{-1}^{v_j(x)} (a(x)b(x))'\sqrt{F(s)}ds\sigma\ dx\right\}.$$

Using that $v_j \to v$ in $L^1(I)$, the regularity of $a(\cdot)b(\cdot)\sqrt{F(\cdot)}$ and the L^∞-bound of v_j, $(ab)'$, σ, σ', we pass to the limit as $j\to\infty$ to get

$$\lim_{j\to\infty} E_{\epsilon_j}(v_j) \geq \sup_{\sigma\in\mathcal{A}}\left\{-\sqrt{2}\int_0^1\int_{-1}^{v(x)} a(x)b(x)\sqrt{F(s)}ds\sigma' dx -\sqrt{2}\int_0^1\int_{-1}^{v(x)} (a(x)b(x))'\sqrt{F(s)}ds\sigma\ dx\right\}.$$

As $v(x)\in\{-1,1\}$ a.e. $x\in I$,

$$\begin{aligned}\lim_{j\to\infty} E_{\epsilon_j}(v_j) &\geq \sup_{\sigma\in\mathcal{A}}\left\{-\sqrt{2}\int_0^1 \chi_{\{v=1\}}\int_{-1}^{1} a(x)b(x)\sqrt{F(s)}ds\sigma' dx -\sqrt{2}\int_0^1 \chi_{\{v=1\}}\int_{-1}^{1} (a(x)b(x))'\sqrt{F(s)}ds\sigma\ dx\right\}\\ &= \sup_{\sigma\in\mathcal{A}}\left\{-\sqrt{2}\int_0^1 \chi_{\{v=1\}}\left(\int_{-1}^{1} a(x)b(x)\sqrt{F(s)}ds\sigma(x)\right)' dx\right\}\\ &= \sqrt{2}\int_0^1\int_{-1}^1 a(x)b(x)\sqrt{F(s)}ds\left|D_{\chi_{\{v=1\}}}\right|\\ &= \int_0^1 \mathcal{P}(x)\left|D\chi_{\{v=1\}}\right| = E_0(v).\end{aligned}$$

3.2. **Condition** (ii) **of Definition 2.5.** Now we will prove that given $v\in L^1(I)$ there exists a sequence $\{\rho_\epsilon\}$ such that $\rho_\epsilon \overset{\epsilon\to 0}{\longrightarrow} v$ in $L^1(I)$ and $\lim_{\epsilon\to 0} E_\epsilon(\rho_\epsilon)\leq E_0(v)$.

Here we can suppose that $v\in BV(I,\{-1,1\})$ and we set

$$A := \{x\in I; v(x)=1\}.$$

The lemma below is used in the construction of sequence $\{\rho_\epsilon\}$.

Lemma 3.2. *The following initial value problem*

$$\begin{cases} \dfrac{\partial Z(s,x)}{\partial s} = \dfrac{b(x)}{a(x)}\sqrt{2F(Z(s,x))}, & (s,x)\in\mathbb{R}\times I\\ Z(0,x) = 0, & x\in I\end{cases}\tag{15}$$

has a unique solution in $\mathbb{R}\times I$ *satisfying*

(Z_1) $-1 < Z(s,x) < 1$ *for all* $(s,x) \in \mathbb{R} \times I$;
(Z_2) $|-1 - Z(s,x)| \leq c_1 e^{c_2 sb(x)/a(x)}$ *as* $s \to -\infty$, $|1 - Z(s,x)| \leq c_3 e^{-c_4 sb(x)/a(x)}$ *as* $s \to \infty$ *for all* $x \in I \backslash \mathcal{B}$ *and for some positive constants* c_1, c_2, c_3, c_4 *dependent on* F;
(Z_3) $|\partial_x Z(s,x)|$ *is* $L^\infty(\mathbb{R} \times I)$*-bounded.*

Proof. The local existence of a solution is clear because $\sqrt{F(Z)}$ is continuous and if we write

$$\int_0^{Z(s,x)} \frac{1}{\sqrt{F(\eta)}} d\eta = s\frac{b(x)}{a(x)} = sc(x) \tag{16}$$

then we can conclude that Z may be extended to $\mathbb{R}$, $-1 < Z(s,x) < 1$ and $\lim_{s\to\infty} Z(s,x) = 1$, $\lim_{s\to-\infty} Z(s,x) = -1$ for all $x \in I\backslash\mathcal{B}$. As $F_{ss}(-1) > 0$ and $F_{ss}(1) > 0$, it follows from Taylor's Theorem that

$$\frac{1}{\sqrt{F(\eta)}} \leq \frac{d_1}{|-1-\eta|}, \text{ for } |-1-\eta| \text{ small;}$$

$$\frac{1}{\sqrt{F(\eta)}} \leq \frac{d_2}{|1-\eta|}, \text{ for } |1-\eta| \text{ small}$$

and some positive constants d_1, d_2 dependent on F. Hence,

$$|-1 - Z(s,x)| \leq c_1 e^{c_2 sc(x)} \text{ as } s \to -\infty$$

and

$$|1 - Z(s,x)| \leq c_3 e^{-c_4 sc(x)} \text{ as } s \to \infty$$

where c_1, c_2, c_3, c_4 are positive constants dependent on F.

In order to prove (Z_3) we derive (16) with respect to x and solve the resulting equation for $\partial_x Z$ (here we note that $\sqrt{F(\eta)} = (\eta+1)(1-\eta)$),

$$\partial_x Z(s,x) = sc'(x)(Z(s,x)+1)(1-Z(s,x)).$$

Note that $\partial_x Z(s,\cdot)$ is $L^\infty(I)$-bounded for any finite s and by (Z_2),

$$\lim_{s\to\pm\infty} \partial_x Z(s,x) = 0, \ \forall\ x \in I$$

since $c'(x) = 0$ in $\mathcal{B}$. This proves (Z_3).

□

Remark 3. Let us emphasize here the key issue that has already been called attention to in the Introduction. The sequence $\{\rho_\epsilon\}$ must efficiently cross the boundary layer of v – in this case ∂A – connecting the values 1 and -1. As we will see below, the function Z obtained in Lemma 3.2 plays a fundamental role in this task. This lemma appears in [23] with $a \equiv b \equiv 1$ and the only novelty in the case considered here is that the coefficient b/a vanishes in the set of zero measure $\mathcal{B}$. We note that for $y \in \mathcal{B}$ we have $Z(s,y) \equiv 0$ and, because of the regularity of function b/a, this will not affect the construction and behavior of the sequence $\{\rho_\epsilon\}$.

Now, we set

$$g_\epsilon(s,x) = \begin{cases} 1, & \text{if } s > 2\sqrt{\epsilon} \\ [1 - Z(1/\sqrt{\epsilon}, x)]\, \frac{s-2\sqrt{\epsilon}}{\sqrt{\epsilon}} + 1, & \text{if } \sqrt{\epsilon} \le s \le 2\sqrt{\epsilon} \\ Z(s/\epsilon, x), & \text{if } -\sqrt{\epsilon} < s < \sqrt{\epsilon} \\ [Z(-1/\sqrt{\epsilon}, x) + 1]\, \frac{s+2\sqrt{\epsilon}}{\sqrt{\epsilon}} - 1, & \text{if } -2\sqrt{\epsilon} \le s \le -\sqrt{\epsilon} \\ -1, & \text{if } s < -2\sqrt{\epsilon}. \end{cases}$$

We have that $A \subset I$ is a set consisting of isolated points and intervals. Since v has bounded variation we conclude that A has a finite number of intervals. Obviously, to assume that A has not isolated points represents no loss of generality. Let d be the function defined by

$$d(x) := \begin{cases} \operatorname{dist}(x, \partial A \cap I), & x \in A \\ -\operatorname{dist}(x, \partial A \cap I), & x \in I \backslash A. \end{cases} \tag{17}$$

Finally, we define $\rho_\epsilon(x) = g_\epsilon(d(x), x)$. It is easy see that $\rho_\epsilon \in H^1_a(I)$ and, using properties (Z_1)–(Z_3), and with some routine computations (see [23], for instance), we obtain

$$|\rho_\epsilon - v|_{L^1(I)} \to 0 \text{ as } \epsilon \to 0. \tag{18}$$

Our next goal is to show that ρ_ϵ satisfies $\lim_{\epsilon\to 0} E_\epsilon(\rho_\epsilon) \le E_0(v)$. Writing $I = \{|d| > 2\sqrt{\epsilon}\} \cup \{|d| \le 2\sqrt{\epsilon}\}$ is easy to see that

$$\int_{|d|>2\sqrt{\epsilon}} \left[\frac{\epsilon a^2(x)}{2} |\rho_\epsilon'|^2 + \frac{b^2(x)}{\epsilon} F(\rho_\epsilon) \right] dx = 0$$

and then

$$E_\epsilon(\rho_\epsilon) = \int_{|d|\le 2\sqrt{\epsilon}} \left[\frac{\epsilon a^2(x)}{2} |\rho_\epsilon'|^2 + \frac{b^2(x)}{\epsilon} F(\rho_\epsilon) \right] dx. \tag{19}$$

If $\sqrt{\epsilon} < d(x) < 2\sqrt{\epsilon}$,

$$\rho_\epsilon' = \left[-\partial_x Z(1/\sqrt{\epsilon}, x) \right] \frac{d(x) - 2\sqrt{\epsilon}}{\sqrt{\epsilon}} + \frac{d'(x)}{\sqrt{\epsilon}} \left[1 - Z(1/\sqrt{\epsilon}, x) \right] = I_1 + I_2.$$

In this case $|d(x) - 2\sqrt{\epsilon}|/\sqrt{\epsilon} \le 1$ and then, by (Z_3), we can conclude that I_1 is L^∞-bounded. We use (Z_2) to get

$$\int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \frac{\epsilon a^2(x)}{2} |\rho_\epsilon'(x)|^2 dx \to 0 \text{ as } \epsilon \to 0. \tag{20}$$

Now, we analyze

$$\int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \frac{b^2(x)}{\epsilon} F(\rho_\epsilon) dx$$

$$= \int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \frac{b^2(x)}{\epsilon} F\left(\left[1 - Z(1/\sqrt{\epsilon}) \right] \frac{d(x) - 2\sqrt{\epsilon}}{\sqrt{\epsilon}} + 1 \right) dx$$

$$= \int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \int_{\{x\in I; d(x)=t\}} \frac{b^2(x)}{\epsilon} F\left(\left[1 - Z(1/\sqrt{\epsilon}, x) \right] \frac{t - 2\sqrt{\epsilon}}{\sqrt{\epsilon}} + 1 \right) d\mathcal{H}^0 dt.$$

We define

$$k := \sup\left\{F'(s);\quad 1 \le s \le 1 + c_3 e^{-c_4 \frac{c(x)}{\sqrt{\epsilon}}}\right\}$$

with c_3 and c_4 defined in (3.2). Moreover we have that

$$\frac{b^2(x)}{\epsilon} e^{-c_4 \frac{c(x)}{\sqrt{\epsilon}}}$$

is uniformly bounded for $x \in I$ and $\epsilon > 0$. As

$$\sup_{\sqrt{\epsilon} \le t \le 2\sqrt{\epsilon}} \{\mathcal{H}^0(\{x; d(x) = t\})\}$$

is finite, we get

$$\begin{aligned}
&\int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \frac{b^2(x)}{\epsilon} F(\rho_\epsilon) dx \\
&\le \int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \int_{d=t} \frac{b^2(x)}{\epsilon} F\left(1 + c_3 e^{-c_4 \frac{c(x)}{\sqrt{\epsilon}}}\right) d\mathcal{H}^0 dt \\
&\le \int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \int_{d=t} \frac{b^2(x)}{\epsilon} k c_3 e^{-c_4 \frac{c(x)}{\sqrt{\epsilon}}} d\mathcal{H}^0 dt.
\end{aligned}$$

Thus,

$$\int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \frac{b^2(x)}{\epsilon} F(\rho_\epsilon) dx \to 0 \text{ as } \epsilon \to 0. \tag{21}$$

By (20) and (21),

$$\int_{\sqrt{\epsilon}<d<2\sqrt{\epsilon}} \left[\frac{\epsilon a^2(x)}{2} |\rho_\epsilon'|^2 + \frac{b^2(x)}{\epsilon} F(\rho_\epsilon)\right] dx \to 0 \text{ as } \epsilon \to 0$$

and the same occurs if $-\sqrt{\epsilon} > d > -2\sqrt{\epsilon}$. We recall (19) to write,

$$E_\epsilon(\rho_\epsilon) = \int_{|d| \le \sqrt{\epsilon}} \left[\frac{\epsilon a^2(x)}{2} |\rho_\epsilon'|^2 + \frac{b^2(x)}{\epsilon} F(\rho_\epsilon)\right] dx + o(\epsilon). \tag{22}$$

Now, to analyze the last case $-\sqrt{\epsilon} \le d \le \sqrt{\epsilon}$, we will do a local analysis at each point of the jump set S_v. That is, we will look at the neighborhood of a point $\tilde{x} \in S_v$ and, since S_v is a finite set, the argument can be repeated at each of the other points of S_v. The purpose of such a choice is to facilitate the notation and understanding of the argument.

We note that if $\tilde{x} \in S_v$ then $\tilde{x}$ is an endpoint of an interval I contained in A. Without loss of generality, we can assume that $\tilde{x} = \sup\{x;\quad x \in I\}$. In this case, the function d defined in (17), can be written as $d(x) = \tilde{x} - x$ in a neighborhood of $\tilde{x}$.

Hence, we study the case $x \in [\tilde{x} - \sqrt{\epsilon}, \tilde{x} + \sqrt{\epsilon}]$ and we use the following notation

$$E_\epsilon(\rho_\epsilon, [\tilde{x} - \sqrt{\epsilon}, \tilde{x} + \sqrt{\epsilon}]) := \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} \left[\frac{\epsilon a^2(x)}{2} |\rho_\epsilon'|^2 + \frac{b^2(x)}{\epsilon} F(\rho_\epsilon)\right] dx.$$

For ϵ small enough, $\rho_\epsilon(x) = Z(x, (\tilde{x} - x)/\epsilon)$ and then

$$\begin{aligned}
&E_\epsilon(\rho_\epsilon, [\tilde{x} - \sqrt{\epsilon}, \tilde{x} + \sqrt{\epsilon}]) \\
&= \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} \frac{\epsilon}{2} a^2(x) \left|\frac{d}{dx} Z((\tilde{x} - x)/\epsilon, x)\right|^2 + \frac{b^2(x)}{\epsilon} F(Z((\tilde{x} - x)/\epsilon, x)) dx
\end{aligned}$$

$$= \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} \frac{\epsilon}{2} a^2(x) \left| -\frac{1}{\epsilon} \partial_1 Z((\tilde{x}-x)/\epsilon, x) + \partial_2 Z((\tilde{x}-x)/\epsilon, x) \right|^2$$

$$+ \frac{b^2(x)}{\epsilon} F(Z((\tilde{x}-x)/\epsilon, x)) dx$$

$$= \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} \frac{\epsilon}{2} a^2(x) \left| \partial_2 Z((\tilde{x}-x)/\epsilon, x) - \frac{1}{\epsilon} \frac{b(x)}{a(x)} \sqrt{2F(Z((\tilde{x}-x)/\epsilon, x))} \right|^2$$

$$+ \frac{b^2(x)}{\epsilon} F(Z((\tilde{x}-x)/\epsilon, x)) dx$$

$$= \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} \left[\frac{1}{2} \epsilon a^2(x) (\partial_2 Z((\tilde{x}-x)/\epsilon, x))^2 \right.$$

$$\left. - a(x) b(x) \partial_2 Z((\tilde{x}-x)/\epsilon, x) \sqrt{2F(Z((\tilde{x}-x)/\epsilon, x))} + \frac{1}{\epsilon} b^2(x) F(Z((\tilde{x}-x)/\epsilon, x)) \right]$$

$$+ \frac{b^2(x)}{\epsilon} F(Z((\tilde{x}-x)/\epsilon, x)) dx.$$

Note that,

$$\frac{1}{2} \epsilon a^2(x) (\partial_2 Z((\tilde{x}-x)/\epsilon, x))^2 - a(x) b(x) \partial_2 Z((\tilde{x}-x)/\epsilon, x) \sqrt{2F(Z((\tilde{x}-x)/\epsilon, x))}$$

is bounded in $[\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}]$ and then

$$E_\epsilon(\rho_\epsilon, [\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}]) = \int_{\tilde{x}-\sqrt{\epsilon}}^{\tilde{x}+\sqrt{\epsilon}} 2 \frac{b^2(x)}{\epsilon} F(Z((\tilde{x}-x)/\epsilon, x)) dx + o(\epsilon).$$

For $t = \tilde{x} - x$ we consider $s = t/\epsilon = (\tilde{x}-x)/\epsilon$ and so $x = x(s) = \tilde{x} - \epsilon s$ (for $x \in [\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}]$). Thus

$$E_\epsilon(\rho_\epsilon, [\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}]) = \int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} 2b^2(x(s)) F(Z(s, x(s))) ds + o(\epsilon). \tag{23}$$

Now we need to do some calculations before we return to (23). First, we note that

$$\frac{d}{ds} \int_{-1}^{Z(s,x(s))} a(x(s)) b(x(s)) \sqrt{F(t)/2} dt = \int_{-1}^{Z(s,x(s))} \frac{d}{ds} [a(x(s)) b(x(s)) \sqrt{F(t)/2}] dt$$

$$+ a(x(s)) b(x(s)) \sqrt{F(Z(s, x(s)))/2} \left[\partial_1 Z(s, x(s)) - \epsilon \partial_2 Z(s, x(s)) \right].$$

As

$$b^2(x(s)) F(Z(s, x(s))) = a(x(s)) b(x(s)) \sqrt{F(Z(s, x(s)))/2} \partial_1 Z(s, x(s))$$

it follows that

$$b^2(x(s)) F(Z(s, x(s))) = \frac{d}{ds} \int_{-1}^{Z(s,x(s))} a(x(s)) b(x(s)) \sqrt{F(t)/2} dt + I_{1,\epsilon} + I_{2,\epsilon}$$

where

$$I_{1,\epsilon} = \int_{-1}^{Z(s,x(s))} \frac{d}{ds} [a(x(s)) b(x(s)) \sqrt{F(t)/2}] dt$$

and

$$I_{2,\epsilon} = a(x(s)) b(x(s)) \sqrt{F(Z(s, x(s)))/2} \left[\partial_2 Z(s, x(s)) (-\epsilon) \right].$$

Hence,

$$\begin{aligned}&\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} 2b^2(x(s))F(Z(s,x(s)))ds\\ =&\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \frac{d}{ds}\int_{-1}^{Z(s,x(s))} 2a(x(s))b(x(s))\sqrt{F(t)/2}dtds\\ &-\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{1,\epsilon}ds-\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{2,\epsilon}ds.\end{aligned} \tag{24}$$

We claim that the last two integrals above go to zero when $\epsilon \to 0$. Indeed, it is true to the latter one because $I_{2,\epsilon}/\epsilon$ is bounded. As

$$\frac{d}{ds}[a(x(s))b(x(s))\sqrt{F(t)/2}] = \partial_x[a(x)b(x)\sqrt{F(t)/2}](-\epsilon)$$

we conclude that $\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} I_{1,\epsilon}ds = o(\epsilon)$ because $\partial_x[a(x)b(x)\sqrt{F(t)/2}]$ is L^∞-bounded for $(t,x) \in (-1,1) \times I$.

Therefore by (23), (24) and remembering that $x(s) = \tilde{x} - \epsilon s$

$$\begin{aligned}&E_\epsilon(\rho_\epsilon, [\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}])\\ =&\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \frac{d}{ds}\int_{-1}^{Z(s,x(s))} a(x(s))b(x(s))\sqrt{2F(t)}dtds + o(\epsilon)\\ =&\int_{-1}^{Z(1/\sqrt{\epsilon},x(1/\sqrt{\epsilon}))} a(x(1/\sqrt{\epsilon}))b(x(1/\sqrt{\epsilon}))\sqrt{2F(t)}dt + o(\epsilon)\\ &-\int_{-1}^{Z(-1/\sqrt{\epsilon},x(-1/\sqrt{\epsilon}))} a(x(-1/\sqrt{\epsilon}))b(x(-1/\sqrt{\epsilon}))\sqrt{2F(t)}dt + o(\epsilon)\\ =&\int_{-1}^{Z(1/\sqrt{\epsilon},\tilde{x}-\sqrt{\epsilon})} a(\tilde{x}-\sqrt{\epsilon})b(\tilde{x}-\sqrt{\epsilon})\sqrt{2F(t)}dt + o(\epsilon)\\ &-\int_{-1}^{Z(-1/\sqrt{\epsilon},\tilde{x}+\sqrt{\epsilon})} a(\tilde{x}+\sqrt{\epsilon})b(\tilde{x}+\sqrt{\epsilon})\sqrt{2F(t)}dt + o(\epsilon).\end{aligned}$$

By properties of Z (see (Z_1)), we can conclude that

$$\begin{aligned}\lim_{\epsilon\to 0} E_\epsilon(\rho_\epsilon, [\tilde{x}-\sqrt{\epsilon}, \tilde{x}+\sqrt{\epsilon}]) &= \int_{-1}^{1} a(\tilde{x})b(\tilde{x})\sqrt{2F(t)}dt\\ &= \sqrt{2}\overline{c}(\tilde{x})\int_{-1}^{1}\sqrt{F(t)}dt = \mathcal{P}(\tilde{x}).\end{aligned} \tag{25}$$

Finally, proceeding in the same way for each point in S_v we obtain

$$\lim_{\epsilon\to 0} E_\epsilon(\rho_\epsilon) = \sum_{x\in S_v} \mathcal{P}(x) = E_0(v)$$

and it is proven that E_ϵ Γ-converges to E_0 in $L^1(I)$, as $\epsilon \to 0$.

4. **Proof of Theorem 1.1.** The proof is accomplished by a direct application of Theorem 2.6. The item (i) of Theorem 2.6, that is, the Γ-convergence, was proved in the previous section; the item (ii) occurs because of the polynomial growth of F (see (12)). A proof can be made following the steps of [23, Proposition 3] where, essentially, was used the compactness of BV in L^1.

Finally, for (iii), we note that $u_{0,1}$ and $u_{0,2}$ (see Theorem 1.1) are isolated L^1-local minimizers of E_0 (actually, they are isolated L^1-global minimizers of E_0). We render the proof for $u_{0,1}$ only since the other case is similar.

We have that $E_0(u_{0,1}) = 0$ because $\mathcal{P}(\overline{x}) = 0$. By hypothesis there is $\delta_1 > 0$ such that $b(x) > 0$ if $0 < |x - \overline{x}| < \delta_1$. If we take

$$\delta := \min\{\overline{x}, 1 - \overline{x}, \delta_1/2\}$$

then $E_0(u) > 0$ for any u such that

$$0 < ||u - u_{0,1}||_{L^1(I)} < \delta, \tag{26}$$

i.e., $u_{0,1}$ is an isolated L^1-local minimizer of E_0. As stated above the argument for $u_{0,2}$ is similar. Hence, there exists $\epsilon_0 > 0$ and two families $\{u_{\epsilon,i}\}_{0<\epsilon<\epsilon_0}$ $(i = 1, 2)$ of L^1-local minimizers of E_ϵ such that $||u_{\epsilon,1} - u_{0,i}||_{L^1(I)} \to 0$ $(i = 1, 2)$ as $\epsilon \to 0$.

Each $u_{\epsilon,i}$ $(i = 1, 2)$ is a stationary solution of (1) in $H^1_a(I)$ and it remains to verify its stability. The second variation of the energy functional E_ϵ at $u_{\epsilon,i}$ $(0 < \epsilon < \epsilon_0$ and $i = 1, 2)$ is nonnegative, i.e. for all $\phi \in H^1_a(I)$

$$\int_0^1 \left[\epsilon^2 a^2(x)(\phi'(x))^2 - b^2(x)f'(u_{\epsilon,i})\phi^2\right] dx \geq 0.$$

By results from Subsection 2.2 we conclude that $\lambda_{0,i} \geq 0$ and if $\lambda_{0,i} > 0$ then $u_{\epsilon,i}$ is stable. If $\lambda_{0,i} = 0$, since $\lambda_{0,i}$ is a simple eigenvalue, there is a local one-dimensional critical manifold $W(u_{\epsilon,i})$, tangent to the eigenspace spanned by the principal eigenfunction $\phi_{0,i}$, at $u_{\epsilon,i}$, such that if $u_{\epsilon,i}$ is stable in $W(u_{\epsilon,i})$ then it is also stable in $H^1_a(I)$. For this matter we refer to [15, Theorem 6.2.1] which proof can be adapted to fit our case. Now, the stability of $u_{\epsilon,i}$ in $W(u_{\epsilon,i})$ follows from the fact that the semigroup generated by (1) defines a gradient flow in $H^1_a(I)$. The proof of Theorem 1.1 is complete.

The following corollary is a direct consequence of the above results.

Corollary 1. *Suppose that $y_0 < y_1 < \ldots < y_n$ are isolated zeros of $b(\cdot)$ in I. Under the same hypotheses of Theorem 1.1, there is $\epsilon_0 > 0$ and a family $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ of stable stationary solutions of (1) such that $||u_\epsilon - u_0||_{L^1(I)} \to 0$ as $\epsilon \to 0$ where $u_0 := \chi_{(0,y_0)} - \chi_{(y_0,y_1)} + \chi_{(y_1,y_2)} - \ldots + \chi_{(y_n,1)}$.*

REFERENCES

[1] S. Ai, X. Chen and S. P. Hastings, Layers and Spikes in Non-Homogeneous Bistable Reaction-Diffusion Equations, Transactions of the American Mathematical Society, 358 (2006), 7, pp. 3169–3206.

[2] F. Alabau-Boussouira, P. Cannarsa and G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, Journal of Evolution Equations, 2 (2006), 6, pp. 161–204.

[3] S. M. Allen and J. W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metallurgica, 6 (1979), 27, pp. 1085–1095.

[4] S. B. Angenent, J. Mallet-Paret and L. A. Peletier, Stable transition layers in a semi-linear boundary value problem, Journal of Differential Equations, 67 (1987), pp. 212–24.

[5] F. V. Atkinson, Discrete and continuous boundary problems, Academic Press, New York, 1964.

[6] I. Boutaayamou, G. Fragnelli and L. Maniar, Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions, Journal d'Analyse Mathématique. **(1)**, 135 (2018), pp. 1–35.

[7] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem, Nonlinear Differential Equations and Applications NoDEA, 2 (2000), 7, pp. 187–199.

[8] M. Chipot and J. K. Hale, Stable equilibria with variable diffusion, Contemp. Math., 17 (1983), pp. 209–213.

[9] G. Dal Maso, An Introduction to Γ-convergence, Progress in nonlinear differential equations and their applications, Birkhäuser, 8, 1993.

[10] L. Dupaigne, Stable Solutions of Elliptic Partial Differential Equations, CRC press, 2011.

[11] A. S. do Nascimento, Local minimizers induced by spatial inhomogeneity with inner transition layer, J. Differ. Equ., 133 (1997), pp. 203–223.

[12] G. Fusco and J. K. Hale, Stable equilibria in a scalar parabolic equation with variable diffusion, SIAM Journal on Mathematical Analysis,. 6 (1985), 16, 1152–1164.

[13] G. Fragnelli, G. R. Goldstein, J. A. Goldstein and S. Romanelli, Generators with interior degeneracy on spaces of L^2-type, Electron. J. Differential Equations, (2012), pp. 1–30.

[14] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, Australia, 1984.

[15] D. Henry, Geometric theory of semilinear parabolic equations, Springer Lecture Notes in Mathematics, 840, 1981.

[16] N. I. Karachalios and N. B. Zographopoulos, On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence, Calc. Var., 25 (2006), 3, pp. 361–393.

[17] R. V. Kohn and P. Sternberg, Local minimizers and singular perturbations, Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), pp. 69–84.

[18] K. Kurata and H. Matsuzawa, Multiple stable patterns in a balanced bistable equation with heterogeneous environments, Applicable Analysis, 7 (2010), 89, pp. 1023–1035.

[19] K. Nakashima, Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation, J. Differential Equations, 191 (2003), pp. 234–276.

[20] K. Nakashima and K. Tanaka, Clustering layers and boundary layers in spatially inhomogeneous phase transition problems, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20 (2003), pp. 107–143.

[21] J. Norbury and L. C. Yeh, The location and stability of interface solutions of an inhomogeneous parabolic problem SIAM J. Appl. Math., 61 (2000), pp. 1418–1430

[22] M. Sônego, On the weakly degenerate Allen-Cahn equation. Advances in Nonlinear Analysis, 9 (2020), 1, pp. 361–371.

[23] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems, Archive for Rational Mechanics and Analysis, 101 (3) (1988), pp. 209–260.

[24] E. Yanagida, Stability of stationary distributions in a space-dependent population growth process, J. Math. Biology, 15 (1982), pp. 37–50.

E-mail address: mcn.sonego@unifei.edu.br
E-mail address: arnaldon@dm.ufscar.br