Elliptic complexes:

Hodge theory, Lefschetz formula

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Preface

The aim of this short course is to give a sense of the main ideas underlying Hodge theory and the Atiyah-Bott formula for the Lefschetz number.

Throughout the three lectures, the underlying differential-topological objects will be a compact manifold $\mathcal{M}$ without boundary, vector bundles $E^k \to \mathcal{M}$, and an elliptic complex

$$(\dagger) \qquad C^\infty(\mathcal{M};E^0) \xrightarrow{P_0} C^\infty(\mathcal{M};E^1) \xrightarrow{P_1} \cdots \xrightarrow{P_{m-1}} C^\infty(\mathcal{M};E^m)$$

of first order differential operators. Precise definitions will be given in time throughout the lectures as needed. We give here a brief outline of the topics, and later, details about each.

The E^k are vector bundles over $\mathcal{M}$, which means they are, locally, products $U \times \mathbb{C}^{d_k}$ with $U \subset \mathcal{M}$ open. Also locally, $C^\infty(\mathcal{M};E^k)$ consist of functions $\phi : U \to \mathbb{C}^{d_k}$, and the P_k are matrices of first order partial differential operators. That (†) is a complex means that $P_{k+1} \circ P_k = 0$ for all k. It therefore has associated cohomology spaces,

$$H^k_E(\mathcal{M}) = \operatorname{rg} P_{k-1} / \ker P_k.$$

They represent the failure of being able to solve $P_k\psi = \phi$ when $P_k\phi = 0$.

Each space $C^\infty(\mathcal{M};E^k)$ will have an inner product with respect to which one can construct the adjoint $P_k^* : C^\infty(\mathcal{M};E^{k+1}) \to C^\infty(\mathcal{M};E^k)$ of P_k. Ellipticity of the complex means that the operators $\Box_k = P_{k-1}P_{k-1}^* + P_k^*P_k$ satisfy the inequality

$$\|\phi\|_2 \le C(\|\Box_k\phi\|_0 + \|\phi\|_0)$$

for some C. Here $\|\phi\|_2$ means the sum of the norms of all derivatives of ϕ up to second order. The norms on the right are with the inner product itself.

Examples of elliptic complexes are the de Rham complex (closely related to the topology of $\mathcal{M}$) and the Dolbeault complex in complex geometry. I plan to give other examples subject to time constraints.

Assuming ellipticity of the complex, Hodge theory establishes that $H^k_E(\mathcal{M})$ is canonically isomorphic to $\ker \Box_k$. (By general elliptic theory the kernel is finite-dimensional.) Furthermore, one has the Hodge decomposition

$$C^\infty(\mathcal{M};E^k) = \ker\Box_k \oplus P_{k-1}C^\infty(\mathcal{M};E^{k-1}) \oplus P_k^*C^\infty(\mathcal{M};E^{k+1})$$

which is an orthogonal decomposition.

The classical Lefschetz formula involves a continuous map $f : \mathcal{M} \to \mathcal{M}$ with which one defines the Lefschetz number, L_f, of f. This number is an invariant of the homotopy class of f. For the purposes of this outline, L_f can be defined when f is smooth and the the complex (†) is the de Rham complex. In this case, f induces maps $f^* : C^\infty(\mathcal{M}; E^k) \to C^\infty(\mathcal{M}; E^k)$ such that $df^k = f^* d$, so maps

$$\mathfrak{f}_k : H^k_{\mathrm{dR}}(\mathcal{M}) \to H^k_{\mathrm{dR}}.$$

Then $L_f = \sum_{k=0}^n (-1)^k \operatorname{tr}(\mathfrak{f}_k)$. The relevancy of the Lefschetz number lies in that $L_f \neq 0$ implies f has a fixed point.

The lectures will begin with recalling the de Rham and Dolbeault complexes, thus ensuring some concreteness to what will follow. Additional background will be provided throughout the lectures as needed. The plan for the main topics is to first discuss Hodge theory, then the Lefschetz number. In the last lecture we will prove a theorem of Atiyah and Bott that gives a formula for the Lefschetz number of f in the special case that $\Gamma(f) \subset \mathcal{M} \times \mathcal{M}$, the the graph of f, intersects the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$ transversely. The formula involves restricting a distribution supported on $\Gamma(f)$ to Δ, which is a good opportunity to exhibit the usefulness of the notion of wave front set of a distribution. The proof of the Atiyah-Bott formula for the Lefschetz number uses Hodge theory.

Contents

CHAPTER I

Vector bundles, differential operators

1. Vector bundles

In the following we assume all manifolds are paracompact and they and all maps are of class C^∞. Let $\mathcal{M}$ be a manifold, let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. A vector bundle over $\mathcal{M}$ of rank r is the datum of

(a) a manifold E and a map $\rho : E \to \mathcal{M}$;

(b) an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathcal{M}$ and, for each $\alpha \in A$, bijective maps

$$\Phi_\alpha : \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^r$$

such that

$$\begin{array}{ccc} \rho^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times \mathbb{F}^r \\ & {\scriptstyle \rho|_{\rho^{-1}(U_\alpha)}} \searrow \quad \swarrow {\scriptstyle \rho_\alpha} & \\ & U_\alpha & \end{array}$$

commutes, where $\rho_\alpha : U_\alpha \times \mathbb{F}^r \to U_\alpha$ is the canonical projection;

(c) for each α, $\beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\Phi_{\alpha\beta}(p) : \mathbb{F}^r \to \mathbb{F}^r.$$

obtained from

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, z) : (U_\alpha \cap U_\beta) \times \mathbb{F}^r \to (U_\alpha \cap U_\beta) \times \mathbb{F}^r,$$
$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, z) = (p, \Phi_{\alpha\beta}(p)(z)),$$

is linear.

The functions Φ_α are called local trivializations of E, and E is said to be trivial over U_α. The maps

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \operatorname{Aut}(\mathbb{F}^r) \tag{1.1}$$

are called transition functions. They satisfy

$$\begin{gathered} \text{for all } \alpha, \beta, \gamma \in A, \text{ for all } p \in U_\alpha \cap U_\beta \cap U_\gamma : \\ \Phi_{\alpha\alpha}(p) = I, \ \Phi_{\alpha\beta}(p) \circ \Phi_{\beta\gamma}(p) \circ \Phi_{\gamma\alpha}(p) = I, \ \Phi_{\alpha\beta}(p) = \Phi_{\beta\alpha}^{-1}(p) \end{gathered} \tag{1.2}$$

Inverse and composition refer to the operations in $\operatorname{Aut}(\mathbb{F}^r)$, and I refers to the constant map $p \mapsto I \in \operatorname{Aut}(\mathbb{F}^r)$. One often says that E is a vector bundle over $\mathcal{M}$ and leaves ρ and the trivializations implicit.

If $\mathbb{F} = \mathbb{R}$, the vector bundle is a real vector bundle, if $\mathbb{F} = \mathbb{C}$ then it is a complex vector bundle. If $r = 1$, the bundle is said to be a (real or complex) line bundle. The manifold E is the total space of the vector bundle.

The trivial vector bundle of rank r is $\mathcal{M} \times \mathbb{F}^r$ with the canonical projection. Any vector bundle $E \to \mathcal{M}$ which is isomorphic to the trivial bundle (trivialized by a single trivialization $E \to \mathcal{M} \times \mathbb{F}^r$) is also called trivial.

The set $E_p = \rho^{-1}(p)$ is the fiber of E over $p \in \mathcal{M}$. The trivializations determine vector space structures on each fiber, as follows. The trivializations are of the form

$$\Phi_\alpha = (\rho, \phi_\alpha), \quad \phi_\alpha : \rho^{-1}(U_\alpha) \to \mathbb{F}^r. \tag{1.3}$$

The maps

$$\phi_\alpha|_{E_p} : E_p \to \mathbb{F}^r \tag{1.4}$$

are bijective. Let $p \in \mathcal{M}$, let Φ_α be a trivialization of E near p (i.e., $p \in U_\alpha$). If η_1, $\eta_2 \in E_p$ and $a \in \mathbb{F}$, then $a\eta_1 + \eta_2$ is defined to be the element $\eta \in E_p$ such that if $\Phi_\alpha(\eta_i) = (p, z_i)$, $i = 1, 2$, then

$$\Phi_\alpha(\eta) = (p, az_1 + z_2).$$

This definition of a vector space structure on E_p is independent of the trivialization chosen: if $\Phi_\beta(\eta_i) = (p, z_i')$, $i = 1, 2$, and $\Phi_\beta(\eta') = (p, az_1' + z_2')$, then $\eta' = \eta$ since $\Phi_{\alpha\beta}(p)$ is linear:

$$\Phi_{\alpha\beta}(p)(az_1' + z_2') = a\Phi_{\alpha\beta}(p)(z_1') + \Phi_{\alpha\beta}(p)(z_2') = a\Phi_\alpha(\eta_1) + \Phi_\alpha(z_2) = \Phi_\alpha(\eta)$$

The maps (2.4) are vector space isomorphisms.

Suppose E and F are vector bundles over $\mathcal{M}$. A map $a : E \to F$ is a vector bundle homomorphism (covering the identity) if for each p, $a(E_p) \subset F_p$ and $a|_{E_p} : E_p \to F_p$ is linear. The map a is an isomorphism if it is bijective (the inverse is automatically smooth). The vector bundles E and F are called isomorphic if there is an isomorphism between them. Usually one does not distinguish between isomorphic vector bundles.

More generally, if $E \to \mathcal{M}$ and $F \to \mathcal{N}$ are vector bundles, a vector bundle morphism from $\rho_E : E \to \mathcal{M}$ to $\rho_F : F \to \mathcal{N}$ is a pair of maps $(a : E \to F, f : \mathcal{M} \to \mathcal{N})$ such that $\rho_F \circ a = f \circ \rho_E$ with linear $a(p) : E_p \to F_{f(p)}$. The map a is said to cover f. With this definition of morphism one has a category whose objects are vector bundles and whose morphisms are vector bundle morphisms.

$$\begin{array}{ccc} E & \xrightarrow{\ a\ } & F \\ \downarrow{\scriptstyle \rho_E} & & \downarrow{\scriptstyle \rho_F} \\ \mathcal{M} & \xrightarrow{\ f\ } & \mathcal{N} \end{array}$$

If $E, F \to \mathcal{M}$ are vector bundles overt he same manifold, then a homomorphism $\psi : E \to F$ is a morphism covering the identity $\mathcal{M} \to \mathcal{M}$. The space of homomorphism $E \to F$, denoted $\mathrm{Hom}(E, F)$, is a vector space over $\mathbb{F}$ (and a ring over $C^\infty(\mathcal{M})$). If $F = E$ we just write $\mathrm{End}(E)$ and call its elements endomorphisms. The subset of bijective maps is $\mathrm{Iso}(E)$.

2. Spaces of sections

Let $\rho : E \to \mathcal{M}$ be a vector bundle of rank r. A section of E over $U \subset \mathcal{M}$ is a map $\sigma : U \to E$ such that $\rho \circ \sigma = I_U$. Since E is a manifold, it makes sense to talk about continuous sections, and if U is an open subset of $\mathcal{M}$ or more generally a submanifold, it also makes sense to say that σ is of class C^m on U. Explicitly, a section $\sigma : U \to E$ is C^m if for every trivialization $\Phi_\alpha = (\rho, \phi_\alpha) : \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}$ as in (2.3), the map

$$U \cap U_\alpha \ni p \mapsto \phi_\alpha(\sigma(p)) \in \mathbb{F}^r \tag{2.1}$$

is C^m.

If U is open, the set of C^m sections of E over U is denoted by $C^m(U; E|_U)$ and the set of sections of E over $\mathcal{M}$ by $C^m(\mathcal{M}; E)$. We will usually consider C^∞ (i.e., smooth) sections. Observe that a section σ over U is smooth if and only if for any trivialization Φ_α, $\Phi_\alpha \circ \sigma$ is smooth where defined.

Suppose U is open. If σ_1, $\sigma_2 \in C^m(U; E|_U)$ and $f \in C^m(U, \mathbb{F})$, then $p \mapsto f(p)\sigma_1(p) + \sigma_2(p)$ is defines a section $f\sigma_1 + \sigma_2 \in C^m(U; E|_U)$. If σ is a section of E and f is a C^m function on U with values in $\mathbb{F}$, then $f\sigma$ is the section defined by $(f\sigma)(p) = f(p)\sigma(p)$. Thus $C^m(U; E|_U)$ is a vector space over $\mathbb{F}$ as well as a module over the ring $C^m(U, F)$.

Let $U \subset \mathcal{M}$ be an open set. An element $\sigma \in C^m(\mathcal{M}; E)$ vanishes on U if $\sigma(p) = 0$ (the 0 element of E_p) for every$p \in U$. The support of σ is the complement largest open set on which σ vanishes. Equivalently, $\operatorname{supp}\sigma$ is the closure of $\{p : \sigma(p) \neq 0\}$. The space of compactly supported C^m sections is $C^\infty_c(\mathcal{M}; E)$.

A frame for E over an open set $U \subset \mathcal{M}$ is the datum of r sections $\eta_1, \dots, \eta_r \in C^\infty(U; E|_U)$ such that at every $p \in U$, the vectors $\eta_1(p), \dots, \eta_r(p)$ form a basis of E_p. If E admits a frame over the set U, then E is said to be trivial over U. If U is one of the open sets U_α, then one gets a C^∞ frame $\tilde\eta^\alpha_\mu$ by setting $\tilde\eta^\alpha_\mu(p) = \Phi^{-1}_\alpha(p, e_\mu)$ where the $e_\mu \in \mathbb{F}^r$ form the canonical basis. If $\eta_1, \dots, \eta_r$ is a C^∞ frame over the open set U, then for each $p \in U$ every element $\eta \in E_p$ can be written uniquely as $\eta = \sum_\mu a^\mu \eta_\mu(p)$. The map $\Phi : \rho^{-1}(U) \to U \times \mathbb{F}^r$ defined by $\Phi(\eta) = (p, (a^1, \dots, a^r))$ is C^∞ and is also called a trivialization of E over U. That such Φ is C^∞ is proved by showing that $\Phi \circ \Phi^{-1}_\alpha$ is C^∞ for any α, as follows. Since each η_μ is C^∞, the map

$$p \mapsto \Phi_\alpha(\eta_\mu(p)) = (p, b_\mu(p))$$

is C^∞, that is, $b_\mu : U \to \mathbb{F}^r$ is C^∞. Writing $b_\mu = \sum_\nu b^\nu_\mu e_\nu$ we see that $\eta_\mu(p) = \Phi^{-1}_\alpha(p, b_\mu(p)) = \sum_\nu b^\nu_\mu(p)\tilde\eta^\alpha_\nu(p)$. This gives a formula for the η_μ in terms of the $\tilde\eta^\alpha_\nu$. Since both the η_μ and the $\tilde\eta^\alpha_\nu$ give bases at every point of $U \cap U_\alpha$, the matrix $[b^\mu_\nu]$ is invertible, and the inverse is necessarily of class C^∞.

Let $c = [c^\mu_\nu]$ be this inverse, so $\tilde\eta^\alpha_\mu = \sum_\nu c^\nu_\mu \eta_\nu$. Then

$$\Phi \circ \Phi_\alpha^{-1}(p, z) = \Phi(\sum_\mu z^\mu \tilde\eta^\alpha_\mu(p)) = \Phi(\sum_\nu(\sum_\mu c^\nu_\mu(p) z^\mu)\eta_\nu(p)) = (p, c(p)z)$$

which is C^∞.

If $\eta_1 \ldots, \eta_r$ is a frame of E in a neighborhood U of p_0 and $\chi \in C^\infty_c(U)$ is such that $\chi(p) = 1$ for p near p_0, then the sections $\chi\eta_\mu$, $\mu = 1, \ldots, r$ form a frame near p_0. These sections, extended as 0 to $\mathcal{M} \setminus U$, are smooth, globally defined, and a frame near p_0.

The space $C^m(\mathcal{M}; E)$ is given a topology by declaring that a sequence $\{\sigma_k\}_{k=1}^\infty$ converges to σ if, with the notation of (3.1), the maps $p \mapsto \phi_\alpha(\sigma_k(p))$ converge in $U \cap U_\alpha$ in the the C^m topology.

For Hodge theory we will need L^2 spaces of sections of vector bundles. To define it for a given vector bundle $\rho : E \to \mathcal{M}$, the needed ingredients are an inner product on E (Hermitian in the case of a complex vector bundle) and a smooth density on $\mathcal{M}$. We begin with the latter.

A smooth density on $\mathcal{M}$ is a measure $\mathfrak{m}$ on the Borel subsets of $\mathcal{M}$ with the property that for every choice of local coordinates $x^1, \ldots, x^n$ on $\mathcal{M}$ there is a smooth positive function g defined on the domain of the chart such that $\mathfrak{m} = g\, dx^1\, dx^2 \cdots dx^n$.

Assuming E is a complex vector bundle, a Hermitian metric on E is the specification of a Hermitian inner product h_p on each fiber E_p with the condition that for every smooth local frame $\eta_1, \ldots, \eta_r$, the functions $p \mapsto h_p\big(\eta_\mu(p), \eta_\nu(p)\big)$ are, for all μ, ν, smooth on the domain of the frame. Thus, if $\eta, \varphi \in C^m(\mathcal{M}; E)$, then the map $p \mapsto h_p(\eta(p), \varphi(p))$ is C^m.

Fix the smooth density and inner product on E. Then $C^\infty_c(\mathcal{M}; E)$ is a pre-Hilbert space with inner product

$$(\eta, \varphi) = \int h_p(\eta(p), \varphi(p)\, d\mathfrak{m}(p)).$$

Its completion is $L^2(\mathcal{M}; E)$.

If $\mathcal{M}$ is not compact, then $L^2(\mathcal{M}; E)$ depends very strongly on the density and metric. But if $\mathcal{M}$ Is compact, then $L^2(\mathcal{M}; E)$ is well defined as a topological vector space. All inner products as above become equivalent.

3. Pull back

$$\begin{array}{ccc} f^*E & & E \\ \downarrow{\scriptstyle \rho_{f^*E}} & & \downarrow{\scriptstyle \rho_E} \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M} \end{array}$$

Suppose given a map $f : \mathcal{N} \to \mathcal{M}$ and a vector bundle $\rho_E : E \to \mathcal{M}$. Then one can form a new vector bundle, denoted $f^*E \to \mathcal{N}$ as follows. The total space is

$$\{(q, \eta) \in \mathcal{N} \times E : f(q) = \rho(\eta)\},$$

the projection map $\rho_{f^*E} : f^*E \to \mathcal{N}$ is

$$\rho_{f^*E}(q, \eta) = q$$

and if $\Phi_\alpha = (\rho, \phi_\alpha) : \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^r$ is a trivialization of $E \to \mathcal{M}$, then the map

$$f^*\Phi_\alpha : \rho^{-1}_{f^*E}(f^{-1}(U_\alpha)) \to f^{-1}(U_\alpha) \times \mathbb{F}^r$$

given by

$$f^*\Phi_\alpha(q, \eta) = (\rho_{f^*E}(q, \eta), \phi_\alpha(\eta))$$

is a trivialization of f^*E.

This construction will be used in various places, especially when defining the principal symbol of a differential operator, which will be a homomorphism

$$\dot{\pi}^*E \to \dot{\pi}^*F$$

where $\dot{\pi} : \dot{T}^*\mathcal{M} \to \mathcal{M}$ is the canonical projection, $\dot{T}^*\mathcal{M}$ the cotangent bundle minus the zero section, ad later in the discussion of the Lefschetz number.

Lemma 3.1. *Suppose given a vector bundle $\rho_E : E \to \mathcal{M}$ and a map $f : \mathcal{N} \to \mathcal{M}$. Then there is an induced map*

$$f^* : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; f^*E)$$

given by

$$(f^*u)(q, \eta) = (q, u(q)),\ q \in \mathcal{N},\ u \in C^\infty(\mathcal{M}; E).$$

4. Differential operators

Let $E, F \to \mathcal{M}$ be vector bundles. We will define linear partial differential operators $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ inductively on the order.

Definition 4.1. P is of order $k < 0$ if it is the 0 operator. Once the class $\text{Diff}^k(\mathcal{M}; E, F)$ of differential operators of order $k \in \mathbb{N}$ has been defined, we let $\text{Diff}^{k+1}(\mathcal{M}; E, F)$ be the space of linear maps $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ such that

$$\text{for all } f \in C^\infty(\mathcal{M}) : \ C^\infty(\mathcal{M}; E) \ni u \mapsto P(fu) - fP(u) \in C^\infty(\mathcal{M}; F)$$

is of order k. We will write $[P, f](u) = P(fu) - fP(u)$.

For example, of P is of order 0, then for any $f \in C^\infty(\mathcal{M})$,

$$u \mapsto P(fu) - fP(u)$$

is of order -1, so it vanishes: $P(fu) = fP(u)$ for any f and u.

Example 4.2. The de Rham differential

$$d : C^\infty(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M}) \to C^\infty(\mathcal{M}; \textstyle\bigwedge^{q+1} \mathcal{M})$$

is a first order linear differential operator in the sense of the definition. Indeed, linearity is clear. And if u is a q-from and f a smooth function, then $d(fu) =$

$f du + df \wedge u$ gives that $u \mapsto [d,f](u) = d(fu) - f du = df \wedge u$. The latter is of order 0: if g is a function, then

$$[d,f](gu) - g[d,f](u) = df \wedge (gu) - g df \wedge u = 0.$$

EXAMPLE 4.3. A connection on the vector bundle $E \to \mathcal{M}$ is a linear operator

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$$

such that

$$\nabla(fu) = f\nabla u + u \otimes df, \quad f \in C^\infty(\mathcal{M}),\ u \in C^\infty(\mathcal{M}; E)$$

Since $u \mapsto \nabla(fu) - f\nabla u = u \otimes df$ is of order 0, ∇ is of order 1.

EXAMPLE 4.4. Let $E \to \mathcal{M}$ be a complex vector bundle of complex rank r. Let

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$$

be a connection. Define

$$\nabla^q : C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^q \mathcal{M}) \to C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^{q+1} \mathcal{M})$$

by $\nabla^0 = \nabla$ and

$$\text{if } u \in C^\infty(\mathcal{M}; E) \text{ and } \alpha \in C^\infty(\mathcal{M}; {\textstyle\bigwedge}^q \mathcal{M}), \text{ then}$$
$$\nabla^q(u \otimes \alpha) = (\nabla^0 u) \wedge \alpha + u \otimes d\alpha$$

That this works follow from the fact that if f isa nonvanishing function, then the computation of $\nabla(\eta' \otimes \alpha')$ with $\eta' = f^{-1}\eta$ and $\alpha' = f\eta$ via the formula yields the same section as computing with η and α;

$$\nabla(\eta' \otimes \alpha') = \nabla(\eta \otimes \alpha)$$

Then ∇^q is a differential operator of order 1. To see this, let f be a function. Keeping the above notation,

$$\begin{aligned}\nabla^q(fu \otimes \alpha) = f(\nabla^0 u) + u \otimes d(f\alpha) = f((\nabla^0 u) \wedge \alpha + u \otimes d\alpha) + u \otimes df \wedge \alpha \\ = f\nabla^q(u \otimes \alpha) + u \otimes df \wedge \alpha.\end{aligned}$$

Since $u \otimes \alpha \mapsto u \otimes df \wedge \alpha$ is of order 0, ∇^q is of order 1. The above formula also gives: if w is a section of $E \otimes \bigwedge^q \mathcal{M}$ and f is a function, then

$$\nabla^q(fw) = f\nabla^q + (-1)^q w \wedge df.$$

With this formula we have:

$$\begin{aligned}\nabla^{q+1}\nabla^q(fw) = f\nabla^{q+1}\nabla^q w + (-1)^{q+1}\nabla^q \wedge df \\ + (-1)^q(\nabla_q w) \wedge df + (-1)^{2q+1} w \wedge ddf \\ = f\nabla^{q+1}\nabla^q w\end{aligned}$$

so $\nabla^{q+1}\nabla^q$ is of order 0. In particular, $\nabla^1 \circ \nabla^0$ is defined by a homomorphism

$$\Omega : E \to E \otimes \bigwedge^2 \mathcal{M}.$$

This is the curvature of the connection. In general, $\nabla^{q+1} \circ \nabla^q$ is the homomorphism $\Omega : E \otimes \bigwedge^q \to E \otimes \bigwedge^{q+2} \mathcal{M}$ defined by $u \otimes \alpha \mapsto \Omega(u) \wedge \alpha$. Thus if $\Omega = 0$, then $\nabla^{q+1} \circ \nabla^q = 0$ for all q.

Lemma 4.5. *A linear operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ is a differential operator of order 0 iff there is a homomorphism $\psi : E \to F$ such that $P(u)(p) = \psi(u(p))$ for every $u \in C^\infty(\mathcal{M}; E)$ and $p \in \mathcal{M}$.*

Proof. Suppose $\psi \in \mathrm{Hom}(E, F)$ and let P be defined by $P(u)(p) = \psi(u(p))$. If f is a function, then

$$P(fu)(p) = \psi(f(p)u(p)) = f(p)\psi(u(p)) = f(p)P(u(p))$$

so $P(fu) = fP(u)$, and thus P os of order 0.

To prove the converse we first show that $\operatorname{supp} P(\varphi) \subset \operatorname{supp}(\varphi)$ for any $\varphi \in C^\infty(\mathcal{M}; E)$. Indeed, suppose $\varphi \in C^\infty(\mathcal{M}; E)$ vanishes near some p_0 and let $\chi \in C^\infty(\mathcal{M})$ be such that $\chi = 1$ on $\operatorname{supp}(\varphi)$ and $\chi(p) = 0$ for p near p_0. Then $\varphi = \chi\varphi$ and so

$$P(\varphi) = P(\chi\varphi) = \chi P(\varphi)$$

so $P(\varphi)$ vanishes on the support of χ.

Next, let $\eta_1, \dots, \eta_r \in C^\infty(\mathcal{M}; E)$ form a frame in a neighborhood of p_0. For an arbitrary $u \in C^\infty(\mathcal{M}; E)$ we have

$$u = \sum_\mu \tilde{u}^\mu \eta_\mu$$

in a neighborhood U of p_0. Suppose $\chi \in C^\infty(\mathcal{M})$ is supported in U and equal to 1 near p_0. Then

$$u = \chi u + (1 - \chi)u = \sum_\mu \chi \tilde{u}^\mu \eta_\mu + (1 - \chi)u,$$

that is,

$$u = \sum_\mu u^\mu \eta_\mu + \varphi \tag{4.6}$$

with $u^\mu = \chi\tilde{u}^\mu \in C^\infty(\mathcal{M})$ and $\varphi = (1 - \chi)\eta \in C^\infty(\mathcal{M}; E)$ vanishing near p_0. From

$$P(u) = \sum_\mu u^\mu P(\eta_\mu) + P(\varphi).$$

we get $P(u)(p) = \sum_\mu a^\mu(p)P(\eta_\mu)(p)$ when p is near p_0. Define $\psi_p : E_p \to E_p$ for p near p_0 by

$$\psi_p(\sum_\mu a^\mu \eta_\mu(p)) = \sum_\mu a^\mu P(\eta_\mu)(p).$$

This defines ψ near p_0. The construction at other points gives local definitions that will agree on overlapping domains. □

LEMMA 4.7. *Suppose $P \in \mathrm{Diff}^k(\mathcal{M}; E, F)$. If $f_1, \dots, f_{k+1} \in C^\infty(\mathcal{M})$ all vanish at p_0, then $P(f_1 \cdots f_{k+1} u)$ vanishes at p_0 for every $u \in C^\infty(\mathcal{M}; E)$.*

PROOF. Proof by induction. If $k = 0$ and f_1 vanishes at p_0, then

$$P(f_1 u)(p_0) = f_1(p_0) P(u) = 0.$$

Suppose the statement has been proved for $k-1$, let P be of order k and let the f_j, $j = 1, \dots, k+1$ be as in the statement of the lemma. Since

$$[P, f_{k+1}](u) = P(f_{k+1} u) - f_{k+1} P(u)$$

is of order $k-1$, the induction hypothesis gives that

$$[P, f_{k+1}](f_1 \cdots f_k u)(p_0) = 0.$$

Therefore

$$P(f_1 \cdots f_{k+1} u)(p_0) = [P, f_{k+1}](f_1 \cdots f_k u)(p_0) + f_{k+1}(p_0) P(u)(p_0) = 0.$$

□

COROLLARY 4.8. *Let $P \in \mathrm{Diff}^k(\mathcal{M}; E, F)$. If $\varphi \in C^\infty(\mathcal{M}; E)$ vanishes on an open set $U \subset \mathcal{M}$, then $P(\varphi)$ vanishes on U. Thus, if $\psi, \varphi \in C^\infty(\mathcal{M}; E)$ and $\psi = \varphi$ on an open set U, then $P(\psi) = P(\varphi)$ on U.*

The following proposition combines the lemma and the corollary to give the link to a more traditional version of differential operator.

PROPOSITION 4.9. *Let $x = (x^1, \dots, x^n)$ be coordinates in a neighborhood U assumed to be a coordinate ball. Let $\eta_1, \dots, \eta_r$ be smooth global sections which form a frame of E on U. Let $E, F \to \mathcal{M}$ be vector bundles, $P \in \mathrm{Diff}^k(\mathcal{M}; E, F)$. If $u \in C^\infty(\mathcal{M}; E)$ is given by*

$$u = \sum_\mu u^\mu \eta_\mu + \varphi \tag{4.10}$$

with smooth u^μ and φ, the latter vanishing near p_0, then

$$P(u)(p) = \sum_\mu \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}(p) P(\eta_{\mu,\alpha})(p) \quad \text{when } p \text{ is near } p_0,$$

where the $\eta_{\mu,\alpha}$ are certain global sections independent of u.

PROOF. Any u can be expressed as in (5.10) using the argument leading to (5.6). The Taylor expansion of u^μ centered at p to order k with remainder of order $k+1$ is

$$u^\mu(p') = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}(p)(x(p') - x(p))^\alpha + \sum_{|\alpha| = k+1} \tilde{u}^\mu_\alpha(p', p)(x(p') - x(p))^\alpha.$$

Let $\chi \in C_c^\infty(U)$ be equal to 1 near p_0. Using $u^\mu = \chi u^\mu + (1-\chi)u^\mu$ get

$$u^\mu(p') = \sum_\mu \sum_{|\alpha|\le k} \frac{1}{\alpha!}\frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}(p)\chi(p')(x(p')-x(p))^\alpha \\ + \sum_{|\alpha|=k+1} \tilde{u}^\mu_\alpha(p',p)\chi(p')(x(p')-x(p))^\alpha + (1-\chi(p'))u^\mu(p')$$

in which each term is viewed as globally defined ($p' \in \mathcal{M}$); the factors $\partial_x^\alpha u^\mu(p)$ are constants depending on p. Using this expression for u^μ in (5.10) and applying P we get

$$P(u) = \sum_\mu \sum_{|\alpha|\le k} \frac{1}{\alpha!}\frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}(p)P(\chi(p')(x(p')-x(p))^\alpha \eta_\mu(p')) \tag{4.11}$$

$$+ \sum_{|\alpha|=k+1} P(\tilde{u}^\mu_\alpha(p',p)\chi(p')(x(p')-x(p))^\alpha\eta_\mu) + P((1-\chi(p'))u^\mu(p')\eta_\mu(p')) + P(\varphi).$$

All terms in the second line vanish when evaluated at p where $\chi(p)=1$, the remainder because of Lemma 5.7 and the last two because of Corollary 5.8. Thus

$$P(u)(p) = \sum_\mu \sum_{|\alpha|\le k} \frac{1}{\alpha!}\frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}(p)P(\eta_{\mu,\alpha})(p)$$

with $\eta_{\mu,\alpha}(p') = \chi(p')(x(p')-x(p))^\alpha \eta_\mu(p)$. □

Continuing with the notation of the proposition and its proof, suppose $\theta_1, \ldots, \theta_s \in C^\infty(\mathcal{M}; F)$ give a frame of F near p_0. Then

$$\frac{1}{\alpha!}P(\eta_{\mu,\alpha})(p) = \sum_\nu a^\nu_{\mu,\alpha}(p)\theta_\nu(p)$$

for p near p_0 with some functions $a^\nu_{\mu,\alpha}$. With this,

$$P(u) = \sum_{\mu,\nu} \sum_{|\alpha|\le k} a^\nu_{\mu,\alpha}\frac{\partial^\alpha u^\mu}{\partial x^{|\alpha|}}\theta_\nu.$$

Letting v be smooth, defined near p_0 and defining

$$P^\nu_\mu(v) = \sum_{|\alpha|\le k} a^\nu_{\mu,\alpha}\frac{\partial^\alpha v}{\partial x^{|\alpha|}}$$

as a scalar differential operator, we have

$$P(\sum u^\mu \eta_\mu) = \sum_{\nu,\mu} P^\nu_\mu(u^\nu)\theta_\nu$$

So using frames we see that P is a matrix of scalar differential operators.

Corollary 4.12. *Let* $P \in \operatorname{Diff}^k(\mathcal{M}; E, F)$. *Then* $\operatorname{supp} P(\varphi) \subset \operatorname{supp}\varphi$ *for every* $\varphi \in C^\infty(\mathcal{M}; E)$.

Proposition 4.13. *Let $E, F, G \to \mathcal{M}$ be vector bundles. If $P \in \operatorname{Diff}^k(\mathcal{M}; E, F)$ and $Q \in \operatorname{Diff}^\ell(\mathcal{M}; F, G)$, then $QP \in \operatorname{Diff}^{k+\ell}(\mathcal{M}; E, G)$.*

Proof. Let $f \in C^\infty(\mathcal{M})$. If $u \in c^\infty(\mathcal{M}; E)$, then

$$\begin{aligned}[QP, f](u) &= QP(fu) - fQP(u) = Q(fPu + [P, f](u)) - fQP(u) \\ &= fQPu + [Q, f](Pu) + [P, f](u) - fQP(u) = [Q, f](Pu) + [P, f](u)\end{aligned}$$

Here $[Q, f]$ is of order $\ell - 1$ and $[P, f]$ is of order $k - 1$. Using induction, the above is of order $k + \ell - 1$, so QP is of order $k + \ell$. □

Suppose now that E and F are complex vector bundles with Hermitian metrics and $\mathfrak{m}$ is a smooth positive density on $\mathcal{M}$. If $P \in \operatorname{Diff}(\mathcal{M}; E, F)$ and $u \in C_c^\infty(\mathcal{M}; E)$, then $u \in C_c^\infty(\mathcal{M}; F)$. So we can use the pre-Hilbert structures on these spaces of sections to define the formal adjoint $P^\star$ of P:

$$(Pu, v) = (u, P^\star v), \quad u \in C_c^\infty(\mathcal{M}; E),\ v \in C_c^\infty(\mathcal{M}; F)$$

The operator $P^\star$ is uniquely determined by this condition.

Lemma 4.14. *If $P \in \operatorname{Diff}^k(\mathcal{M}; E, F)$, then $P^\star \in \operatorname{Diff}^k(\mathcal{M}; F, E)$.*

Proof. Let f be some function. The formal adjoint of

$$u \mapsto -[P, \overline{f}](u) = \overline{f}Pu - P(\overline{f}u)$$

is

$$v \mapsto [P^\star, \overline{f}](v) = P^\star(fv) - fP^\star v.$$

If P is of order 0, then $[P, \overline{f}] = 0$ so $[P^\star, f] = -[P, \overline{f}]^\star = 0$ and thus $P^\star$ is of order 0. Once it has been proved that the adjoint of an operator of order $k-1$ is of order $k-1$, the above gives that $P^\star$ is of order k if P is of order k. □

5. Principal symbol, ellipticity

Again let $E, F \to \mathcal{M}$ be vector bundles and $P \in \operatorname{Diff}(\mathcal{M}; E, F)$. Using frames we see that P is a matrix of scalar differential operators of order at most k, and using this we see that if f is a function then for any $u \in C^\infty(\mathcal{M}; E)$

$$\mathbb{R} \ni \tau \mapsto e^{-i\tau f(p)} P(e^{i\tau f} u)(p) \in F_p$$

is a polynomial of degree k in τ with values in F_p. The leading coefficient depends on f only through the value of df at p, and on u only through its value at p and furthermore is linear in $u(p)$. This is the principal symbol of P at $df(p)$ evaluated at $\eta(p)$:

Definition 5.1. Let $\boldsymbol{\xi} \in T_p^*\mathcal{M}$, let $\eta \in E_p$, let f be real valued with $df(p) = \boldsymbol{\xi}$ and let u be section of E with $u(p) = \eta$. The principal symbol of P at $\boldsymbol{\xi}$ is the linear function

$$\eta \mapsto \boldsymbol{\sigma}_k(P)(\boldsymbol{\xi})(\eta) = \lim \frac{1}{\tau^k} e^{-i\tau f(p)} P(e^{i\tau f} u)(p)$$

REMARK 5.2. $v_\tau = 1/\tau^k e^{-i\tau f} P(e^{i\tau f} u)$ is a smooth section of F and $v_\tau \to \boldsymbol{\sigma}_k(P)(df)(u)$ in C^∞.

If $Q \in \mathrm{Diff}^{k-1}(\mathcal{M}; E, F)$ then $\boldsymbol{\sigma}_k(Q)$ makes sense, since $\mathrm{Diff}^{k-1}(\mathcal{M}; E, F) \subset \mathrm{Diff}^{k-1}(\mathcal{M}; E, F)$. However $\boldsymbol{\sigma}_k(Q) = 0$.

EXAMPLE 5.3. We compute the principal symbol of the de Rham differential. Let u be a q-form, let f be a real-valued function. Then

$$\frac{1}{\tau} e^{-i\tau f} d(e^{-i\tau f} u) = i df \wedge u + \frac{1}{\tau} d\eta$$

gives

$$\boldsymbol{\sigma}(d)(\boldsymbol{\xi})(\eta) = i\boldsymbol{\xi} \wedge \eta, \quad \boldsymbol{\xi} \in T^*_p\mathcal{M},\ \eta \in {\textstyle\bigwedge}^q_p \mathcal{M}.$$

EXAMPLE 5.4. In Example 5.4,

$$\boldsymbol{\sigma}(\nabla^q)(\boldsymbol{\xi})(\eta) = i\eta \otimes \boldsymbol{\xi}, \quad \eta \in E \otimes T^*_p\mathcal{M},\ \boldsymbol{\xi} \in T^*_p\mathcal{M}$$

PROPOSITION 5.5. *If $Q \in \mathrm{Diff}^k(\mathcal{M}; E, F)$ and $\boldsymbol{\sigma}_k(P) = 0$, then $Q \in \mathrm{Diff}^{k-1}(\mathcal{M}; E, F)$.*

For a differential operator, the principal symbol can also be defined for complex covectors, and, as we did implicitly, for $\boldsymbol{\xi} = 0$; both features are useful in some way, even though one always has $\boldsymbol{\sigma}_k(P)(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} = 0$. Neither feature carries over to pseudodifferential operators.

Let then $\dot{T}^*\mathcal{M}$ denote the cotangent bundle minus its zero section, let $\dot{\pi} : \dot{T}^*\mathcal{M} \to \mathcal{M}$ denote the restriction of the canonical projection. The usual point of view is that the principal symbol of P is a homomorphism

$$\boldsymbol{\sigma}_k(P) : \dot{\pi}^* E \to \dot{\pi}^* F$$

From the definitions of $\dot{\pi}^* E$ we get for each $\tau > 0$ a map $\varrho_\tau : \dot{\pi}^* E \to \dot{\pi}^* E$. Namely if $(\boldsymbol{\xi}, \eta) \in \dot{\pi}^* E$ (so $\eta \in E_{\dot{\pi}(\boldsymbol{\xi})}$), then $(\tau\boldsymbol{\xi}, \eta) \in \dot{\pi}^* E$. Using this (with $\dot{\pi} F$) we have that

$$\boldsymbol{\sigma}_k(P)(\tau\boldsymbol{\xi}) = \tau^k \varrho_\tau \circ \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) \circ \varrho_{1/\tau}.$$

PROPOSITION 5.6. *Let $E, F, G \to \mathcal{M}$ be vector bundles. If $P \in \mathrm{Diff}^k(\mathcal{M}; E, F)$ and $Q \in \mathrm{Diff}^\ell(\mathcal{M}; F, G)$, then $\boldsymbol{\sigma}_{k+\ell}(QP) = \boldsymbol{\sigma}_\ell(Q) \circ \boldsymbol{\sigma}_k(P)$.*

PROOF. Let f be a smooth function, $u \in C^\infty(\mathcal{M}; E)$, $p \in \mathcal{M}$, $\tau \in \mathbb{R}$. Then

$$\frac{1}{\tau^{\ell+k}} e^{-i\tau f} QP(e^{i\tau f} u) = (\frac{1}{\tau^\ell} e^{-i\tau f} Q e^{i\tau f})(\frac{1}{\tau^k} e^{-i\tau f} P e^{i\tau f})(u)$$

Taking advantage of the remark above one gets

$$\lim_{\tau\to\infty} \frac{1}{\tau^{\ell+k}} e^{-i\tau f} QP(e^{i\tau f} u) = \boldsymbol{\sigma}_\ell(Q)(df) \circ \boldsymbol{\sigma}_k(P)(df)(u)$$

from which the conclusion follows. ∎

PROPOSITION 5.7. *Let $E, F \to \mathcal{M}$ be Hermitian vector bundles, $\mathfrak{m}$ a smooth positive density on $\mathcal{M}$. and $P \in \operatorname{Diff}^k(\mathcal{M}; E, F)$. For any $\boldsymbol{\xi} \in T^*\mathcal{M}$, $\boldsymbol{\sigma}(P^\star)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P^\star)(\boldsymbol{\xi})^*$.*

PROOF. Let f be a real valued function, $u \in C_c^\infty(\mathcal{M}; E)$, $v \in C_c^\infty(\mathcal{M}; F)$. Then

$$\begin{aligned}(\boldsymbol{\sigma}(P)(df)(u), v) &= \lim_{\tau\to\infty} (\frac{1}{\tau^k} e^{-i\tau f} P(e^{i\tau f} u), v) \\ &= \lim_{\tau\to\infty} (u, \frac{1}{\tau^k} e^{-i\tau f} P^\star(e^{i\tau f} v)) \\ &= (u, \boldsymbol{\sigma}(P^\star)(df)(\boldsymbol{\xi})(v)).\end{aligned}$$

Thus $\boldsymbol{\sigma}(P^\star)(df)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P)(df)^*$. □

DEFINITION 5.8. We say that $P \in \operatorname{Diff}^k(\mathcal{M}; E, F)$ is elliptic if $\boldsymbol{\sigma}_k(P)$ is an isomorphism.

We often omit the subindex indicating the order of the operator. The context makes it clear what is meant. Note that in the definition of ellipticity, $\boldsymbol{\xi}$ is a real (nonzero) covector.

CHAPTER II

Complexes of differential operators

1. First order complexes

Let $E^0, \dots, E^m \to \mathcal{M}$ be vector bundles. A first order differential complex on the E^q is a cochain complex

$$C^\infty(\mathcal{M}; E^0) \xrightarrow{P_0} C^\infty(\mathcal{M}; E^1) \xrightarrow{P_1} \cdots \xrightarrow{P_{m-1}} C^\infty(\mathcal{M}; E^m) \tag{1.1}$$

in which $P_q \in \mathrm{Diff}^1(\mathcal{M}; E^q, E^{q+1})$ and $P_q \circ P_{q-1} = 0$.

The condition that $P_{q+1} \circ P_q = 0$ is equivalent to $\operatorname{rg} P_{q-1} \subset \ker P_q$ (as operators on smooth sections, at least), so one can form

$$H^q_E(\mathcal{M}) = \ker P_q / \operatorname{rg} P_{q-1};$$

This is the q-th cohomology space of the complex. For complexes of a topological nature, the spaces $H^q_E(\mathcal{M})$ have topological information (of course). In other contexts, one can view the cohomology spaces as the obstructions to solving

$$P_{q-1} u = f \quad \text{given } f \in \ker P_q.$$

If $\mathcal{M}$ is not compact and the complex is not elliptic (see the definition below), the determination of properties of the cohomology spaces is a subtle problem. Instead of specifying smooth sections as domain, one can specify other domains, leading to other cohomology spaces.

The condition $P_{q+1} \circ P_q = 0$ implies $\boldsymbol{\sigma}(P_{q+1})(\boldsymbol{\xi}) \circ \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}) = 0$, so the symbol sequence at $\boldsymbol{\xi}$,

$$0 \to \dot{\pi}^* E^0 \xrightarrow{\boldsymbol{\sigma}(P_0)(\boldsymbol{\xi})} \dot{\pi}^* E^1 \xrightarrow{\boldsymbol{\sigma}(P_1)(\boldsymbol{\xi})} \cdots \xrightarrow{\boldsymbol{\sigma}(P_{m-1})(\boldsymbol{\xi})} \dot{\pi}^* E^m \to 0, \tag{1.2}$$

is a cochain complex.

Definition 1.3. The complex (1.1) is called elliptic if for each $\boldsymbol{\xi} \in \dot{T}^*\mathcal{M}$, the sequence (1.2) is exact.

Note again that in the definition of ellipticity, $\boldsymbol{\xi}$ is a real (nonzero) covector.

Example 1.4. The best known example is the de Rham complex, in which P_q is the de Rham differential $d_q : C^\infty(\mathcal{M}; \bigwedge^q \mathcal{M}) \to C^\infty(\mathcal{M}; \bigwedge^{q+1} \mathcal{M})$. Once has $d_{q+1} \circ d_q = 0$, thus $\operatorname{rg} \boldsymbol{\sigma}(d_{q-1})(\boldsymbol{\xi}) \subset \ker \boldsymbol{\sigma}(d_q)(\boldsymbol{\xi})$ as pointed out earlier. The de Rham complex is elliptic. To see this, let $\boldsymbol{\xi} \in \dot{T}^*_p\mathcal{M}$, suppose $\eta \in \bigwedge^q_p \mathcal{M}$ is such that $\boldsymbol{\sigma}(\boldsymbol{\xi})(\eta) = 0$. Thus $\boldsymbol{\xi} \wedge \eta = 0$. Since $\boldsymbol{\xi} \neq 0$ we can use it as part of a basis $\omega^1, \dots, \omega^n$ of $T_p^*\mathcal{M}$, say with $\omega^1 = \boldsymbol{\xi}$. Then[1]

$$\eta = \sum_{|I|=q}{}' a_I\, \omega^I = \sum_{\substack{|J|=q-1 \\ 1 \notin J}}{}' a_{\langle 1,J\rangle} \omega^1 \wedge \omega^J + \sum_{\substack{|I|=q \\ 1 \notin I}}{}' a_I\, \omega^I$$

Thus

$$0 = \boldsymbol{\xi} \wedge \eta = \sum_{\substack{|I|=q \\ 1 \notin I}}{}' a_I\, \omega^1 \wedge \omega^I$$

gives that $a_I = 0$ if $1 \in I$. Thus

$$\eta = \sum_{\substack{|J|=q-1 \\ 1 \notin J}}{}' a_{\langle 1,J\rangle} \omega^1 \wedge \omega^J$$

and we have $\eta = \boldsymbol{\sigma}(d)(\boldsymbol{\xi})(\theta)$ with

$$\theta = -i \sum_{\substack{|J|=q-1 \\ 1 \notin J}}{}' a_{\langle 1,J\rangle} \omega^J.$$

Thus $\ker \boldsymbol{\sigma}(d_q)(\boldsymbol{\xi}) \subset \operatorname{rg} \boldsymbol{\sigma}(d_{q-1})(\boldsymbol{\xi})$. We already know the opposite inclusion, so the symbol sequence is exact at $\boldsymbol{\xi}$.

EXAMPLE 1.5. Let $E \to \mathcal{M}$ be a vector bundle and $\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$ a connection. Let

$$\nabla^q : C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^q \mathcal{M}) \to C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^{a+1} \mathcal{M})$$

be the operators defined in Example 5.4. If the curvature of ∇ vanishes (∇ is a flat connection), then we have a complex

$$\cdots \to C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^q \mathcal{M}) \to C^\infty(\mathcal{M}; E \otimes {\textstyle\bigwedge}^{q+1} \mathcal{M}) \to \cdots$$

This is again an elliptic complex, sometimes called a twisted de Rham complex. In this context one write d instead of ∇ for the operators of the complex (also omitting the subscript).

Another important example of an elliptic sequence is the Dolbeault complex. We first discuss a more general example of first order complexes.

EXAMPLE 1.6. An involutive structure on $\mathcal{M}$ is a subbundle $\mathcal{V} \subset \mathbb{C}TM$ of the complexification of the tangent bundle. This is a subbundle with the property that

(1.7) If X and Y are vector fields with values in $\mathcal{V}$, then $[X, Y]$ also has values in $\mathcal{V}$.

[1]The notation means $I = (i_1, \dots, i_q)$ with $1 \leq q \leq \operatorname{rk} \mathcal{V}$. The prime by the summation symbol means to add only through indices I with strictly increasing components.

Let $\overline{\mathcal{V}} = \{v \in \mathbb{C}T\mathcal{M} : \overline{v} \in \mathcal{V}\}$ be the conjugate of $\mathcal{V}$, let $\overline{\mathcal{V}}^*$ be the dual bundle, $\bigwedge^q \overline{\mathcal{V}}^*$ the exterior powers. Define

$$\mathbb{D}_q : C^\infty(\mathcal{M}; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; \bigwedge^{q+1} \overline{\mathcal{V}}^*) \tag{1.8}$$

by: $\phi \in C^\infty(\mathcal{M}; \bigwedge^q \overline{\mathcal{V}}^*)$, if $v_0, \dots, v_q \in \overline{\mathcal{V}}_p$ and $X_0, \dots, X_q$ are sections of $\overline{\mathcal{V}}$ extending the v_j, then

$$\begin{aligned}(q+1)\mathbb{D}_q\phi(X_0, \dots, X_q) &= \sum_{j=0}^{q} (-1)^{j+1} X_j \phi(X_0, \dots, \hat{X}_j, \dots, X_q) \\ &\quad + \sum_{0 \le j < k \le q} (-1)^{j+k} \phi([X_j, X_k])\phi(X_0, \dots, \hat{X}_j, \dots \hat{X}_k, \dots, X_q).\end{aligned}$$

If $\mathcal{V} = \mathbb{C}T\mathcal{M}$ then this is the formula for the de Rham differential **[6]**. One verifies that

$$\mathbb{D}_q\phi(aX_0, \dots, X_q) = a\mathbb{D}_q\phi(X_0, \dots, X_q)$$

for any function a, so if ϕ and $X_1, \dots, X_q$ are fixed, the operator $X_1 \mapsto \mathbb{D}_q\phi(X_0, \dots, X_q)$ is of order 0: it is a bundle homomorphism. Thus $\mathbb{D}_q\phi(X_0, \dots, X_q)(p)$ depends only n the initially fixed $v_0, \dots, v_q$, and not on the extensions of these to vector fields, and $\mathbb{D}_q$ is well defined by the formula as a map (1.8).

On functions f, $(q = 0)$, the formula reduces to $\mathbb{D}_0 f(v_0) = v_0 f$. Since $v_0 f = \langle df, v_0 \rangle$, the formula for $\mathbb{D}_0$ is

$$\mathbb{D}_0 f = \iota^* df$$

where $\iota^* : \mathbb{C}T^*\mathcal{M} \to \overline{\mathcal{V}}^*$ is the dual ap of the inclusion $\iota : \overline{\mathcal{V}} \to \mathbb{C}T\mathcal{M}$. Using the Leibniz rule $\mathbb{D}_q(f\phi) = \mathbb{D}_0 f \wedge \phi + f\mathbb{D}_q\phi$ one finds that the principal symbol of $\mathbb{D}_q$ is

$$\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})(\eta) = i\iota^*(\boldsymbol{\xi}) \wedge \eta, \quad \boldsymbol{\xi} \in T^*_p\mathcal{M},\ \eta \in \textstyle\bigwedge^q_p \overline{\mathcal{V}}^*$$

One verifies that $\mathbb{D}_{q+1}\mathbb{D}_q = 0$, so one has a first order complex. We analyze ellipticity: The condition on a real covector $\boldsymbol{\xi}$ that

$$\text{for any } \eta,\ \iota^*(\boldsymbol{\xi}) \wedge \eta = 0 \text{ implies } \eta = \iota^*(\boldsymbol{\xi}) \wedge \theta$$

is equivalent to

$$\iota^*(\boldsymbol{\xi}) \neq 0.$$

So ellipticity is equivalent to:

$$\text{if } \boldsymbol{\xi} \text{ is a real covector, } \iota^*(\boldsymbol{\xi}) = 0 \text{ implies } \boldsymbol{\xi} = 0.$$

The kernel of ι^* is the annihilator of $\overline{\mathcal{V}}$,

$$\overline{\mathcal{V}}^\perp = \{\boldsymbol{\zeta} \in \mathbb{C}T^*\mathcal{M} : \langle \boldsymbol{\zeta}, v \rangle = 0 \text{ for all } v \in \overline{\mathcal{V}}\}$$

So ellipticity is the condition that $\overline{\mathcal{V}}^\perp \cap T^*\mathcal{M} = 0$ (the zero section). This is equivalent to the condition that

$$\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}T\mathcal{M}.$$

Such subbundles are called elliptic structures.

EXAMPLE 1.9. Let $\mathcal{M}$ be a complex manifold. This is a manifold together with an involutive subbundle $T^{1,0}\mathcal{M}$ such that with $T^{0,1}\mathcal{M} = \overline{T^{1,0}\mathcal{M}}$,

(a) $T^{1,0}\mathcal{M} + T^{1,0}\mathcal{M} = \mathbb{C}T\mathcal{M}$
(b) $T^{1,0}\mathcal{M} \cap T^{1,0}\mathcal{M} = 0$.

Thus $T^{1,0}\mathcal{M}$ is an elliptic structure. The second condition implies that the dual bundles can be identified with subbundles of $\mathbb{C}T^*\mathcal{M}$: The dual bundle of $T^{1,0}\mathcal{M}$ is $\bigwedge^{1,0}\mathcal{M}$, that of $T^{0,1}\mathcal{M}$ is $\bigwedge^{0,1}\mathcal{M}$. The dual of the inclusion $\iota : T^{0,1}\mathcal{M} \to \mathbb{C}T\mathcal{M}$ becomes the projection $\mathbb{C}T^*\mathcal{M} \to \bigwedge^{0,1}\mathcal{M}$ according to the decomposition

$$\mathbb{C}T^*\mathcal{M} = \bigwedge\nolimits^{1,0}\mathcal{M} \oplus \bigwedge\nolimits^{1,0}\mathcal{M}.$$

The q-th exterior power of $\bigwedge^{0,1}\mathcal{M}$ is $\bigwedge^{0,q}\mathcal{M}$. The associated complex is

$$\cdots \to C^\infty(\mathcal{M}; \bigwedge\nolimits^{0,q}\mathcal{M}) \xrightarrow{\overline{\partial}} C^\infty(\mathcal{M}; \bigwedge\nolimits^{0,q+1}\mathcal{M}) \to \cdots .$$

This is elliptic because $T^{1,0}\mathcal{M}$ is an elliptic structure.

EXAMPLE 1.10. Suppose $\mathcal{M}$ is a complex manifold, $E \to \mathcal{M}$ a vector bundle and $\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$ a connection. Using the decomposition

$$\mathbb{C}T^*\mathcal{M} = \bigoplus_{p+q=r} \bigwedge\nolimits^{p,q}\mathcal{M}$$

where $\bigwedge^{p,q}\mathcal{M} = \bigwedge^{p,0}\mathcal{M} \wedge \bigwedge^{0,q}\mathcal{M}$, let $\pi^{0,q}$ be the projection on $\bigwedge^{0,q}\mathcal{M}$. Define

$$\nabla^{0,q} : C^\infty(\mathcal{M}; E \otimes \bigwedge\nolimits^{0,q}\mathcal{M}) \to C^\infty(\mathcal{M}; E \otimes \bigwedge\nolimits^{0,q+1}\mathcal{M})$$

by

$$\nabla^{0,q} = (I \otimes \pi^{0,q+1}) \circ \nabla^q.$$

Then $\nabla^{0,1} \circ \nabla^{0,0}$ is the $(0,2)$ component of the curvature of ∇, a homomorphism

$$\nabla^{0,1} \circ \nabla^{0,0} = \Omega^{0,2} : E \to E \otimes \bigwedge\nolimits^{0,2}\mathcal{M}$$

If $\Omega^{0,2} = 0$, the connection is called holomorphic. If this is the case, then one again has an elliptic complex

$$\cdots \to C^\infty(\mathcal{M}; E \otimes \bigwedge\nolimits^{0,q}\mathcal{M}) \xrightarrow{\overline{\partial}} C^\infty(\mathcal{M}; E \otimes \bigwedge\nolimits^{0,q+1}\mathcal{M}) \to \cdots ,$$

This is a Dolbeault complex with coefficients in E. It again is elliptic.

EXAMPLE 1.11. Given a Hermitian metric to each of the vector bundles in the the first order complex (1.1), fix a smooth positive density $\mathfrak{m}$ on $\mathcal{M}$. Then we have an adjoint complex

$$C^\infty(\mathcal{M}; E^0) \xleftarrow{P_0^\star} C^\infty(\mathcal{M}; E^1) \xleftarrow{P_1^*} \cdots \xleftarrow{P_{m-1}^\star} C^\infty(\mathcal{M}; E^m). \tag{1.12}$$

Indeed, $P_{q-1}^\star \circ P_q^\star = (P_q \circ P_{q-1})^\star = 0$. Moreover, the adjoint complex is elliptic iff the original complex is. This complex is the formal adjoint complex.

2. The formal Hodge Laplacians

Let $E^0, \dots, E^m \to \mathcal{M}$ be vector bundles and

$$C^\infty(\mathcal{M}; E^0) \xrightarrow{P_0} C^\infty(\mathcal{M}; E^1) \xrightarrow{P_1} \cdots \xrightarrow{P_{m-1}} C^\infty(\mathcal{M}; E^m) \tag{2.1}$$

a first order differential complex. Give each vector bundle a Hermitian metric, h_q, and fix a smooth positive density $\mathfrak{m}$ on $\mathcal{M}$. Recall form Proposition 6.7 that then $\boldsymbol{\sigma}(P_q^\star)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi})^*$. Fix $p \in \mathcal{M}$ and $\boldsymbol{\xi} \in \dot{T}_p^*\mathcal{M}$. From linear algebra we have that the range of

$$\boldsymbol{\sigma}(P_{q-1})(\boldsymbol{\xi}) : E_p^{q-1} \to E_p^q$$

is orthogonal to the kernel of the adjoint,

$$\boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}) : E_p^q \to E_p^{q-1}$$

Also, since $\operatorname{rg} \boldsymbol{\sigma}(P_{q-1})(\boldsymbol{\xi}) \subset \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi})$,

$$(\ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}))^\perp \subset \ker \boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}).$$

So

$$\ker \boldsymbol{\sigma}(P_{q-1})(\boldsymbol{\xi})^* + \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}) = E_q$$

in any case, and the sum is direct iff $\operatorname{rg} \boldsymbol{\sigma}(P_{q-1})(\boldsymbol{\xi}) = \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi})$, that is, iff the symbol sequence at $\boldsymbol{\xi}$ is exact in degree q. This proves:

LEMMA 2.2. *The complex* (2.1) *is elliptic iff*

$$\ker \boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}) \cap \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}) = \{0\} \quad \textit{for all } \boldsymbol{\xi} \in \dot{T}^*\mathcal{M}.$$

Continuing with the complex (2.1), observe that $P_{q-1} \circ P_{q-1}^\star$ and $P_q^\star \circ P_q$ both map $C^\infty(\mathcal{M}, E^q)$ to itself, so one can form

$$\Box_q = P_{q-1} \circ P_{q-1}^\star + P_q^\star \circ P_q.$$

This is the formal[2] Hodge Laplacian of the complex. By Propositions 6.6 and 6.7, the principal symbol of $\Box_q$ at $\boldsymbol{\xi} \in T_p^*\mathcal{M}$ is

$$\boldsymbol{\sigma}(\Box_q)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P_{q-1})(\xi) \circ \boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}) + \boldsymbol{\sigma}(P_q^\star)(\boldsymbol{\xi}) \circ \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}). \tag{2.3}$$

LEMMA 2.4. *We have*

$$\ker \boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}) \cap \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi}) = \ker(\Box_q)(\boldsymbol{\xi}). \tag{2.5}$$

for all $\xi \in \dot{T}^*\mathcal{M}$.

PROOF. The formula (2.3) immediately implies that the set on the left in (2.5) is contained in the set on the right. For the opposite inclusion we note that

$$h_q\big(\boldsymbol{\sigma}(\Box_q)(\boldsymbol{\xi})(\eta), \eta\big) = \|\boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi})(\eta)\|^2 + \|\boldsymbol{\sigma}(P_q)(\boldsymbol{\xi})(\eta)\|^2$$

so if $\eta \in \ker \boldsymbol{\sigma}(\Box_q)(\boldsymbol{\xi})$, then $\eta \in \ker \boldsymbol{\sigma}(P_{q-1}^\star)(\boldsymbol{\xi}) \cap \ker \boldsymbol{\sigma}(P_q)(\boldsymbol{\xi})$. □

[2]It is formal because we are not specifying any domains.

Since $\boldsymbol{\varpi}(P_q^\star)(\boldsymbol{\xi}) : E_p^q \to E_p^q$, invertibility is equivalent to injectivity. This and the previous two lemmas immediately give

THEOREM 2.6. *The operator* $\Box_q$ *is elliptic iff the symbol sequence of the complex is exact at each* $\boldsymbol{\xi} \in \dot{T}^*\mathcal{M}$ *in degree* q.

Incidentally, the formal Hodge Laplacians for the adjoint complex (1.12) are the same as those of the original complex.

CHAPTER III

Hodge theory

1. Set-up

Throughout this chapter we will be work with a a first order differential complex

$$C^\infty(\mathcal{M}; E^0) \xrightarrow{P_0} C^\infty(\mathcal{M}; E^1) \xrightarrow{P_1} \cdots \xrightarrow{P_{m-1}} C^\infty(\mathcal{M}; E^m) \tag{1.1}$$

Each vector bundle will have a Hermitian metric, h_q, and $\mathfrak{m}$ on $\mathcal{M}$ will be a smooth positive density.

Recall that $L^2(\mathcal{M}; E^q)$ is the completion of the $C_c^\infty(\mathcal{M}; E^q)$ with respect to the inner product given by

$$(u, v) = \int_{\mathcal{M}} h_p(u(p), v(p))\, d\mathfrak{m}(p), \quad u, v \in C_c^\infty(\mathcal{M}; E^q).$$

For the moment we do not assume that $\mathcal{M}$ is closed (compact without boundary) but write $\mathring{\mathcal{M}}$ for the interior of $\mathcal{M}$ in case there is a boundary. We also don't assume that the complex is elliptic until explicitly stated so.

2. L^2 Cohomology

Accommodating L^2, there are two canonical domains one can give to the operators of the complex, beyond smooth or smooth compactly supported sections. The biggest one is the maximal domain,

$$\mathcal{D}^q_{\max} = \{u \in L^2(\mathring{\mathcal{M}}; E^q) : P_q u \in L^2(\mathring{\mathcal{M}}; E^{q+1})\}.$$

From $P_{q+1}u \in L^2(\mathring{\mathcal{M}}; E^{q+1})$ and $P_{q+2}(P_{q+1}u) = 0 \in L^2(\mathring{\mathcal{M}}; E^{q+3})$ we conclude $P_{q+1}u \in \mathcal{D}_{\max}(P_{q+2})$, so there is a complex

$$0 \to \mathcal{D}^0_{\max} \xrightarrow{P_0} \mathcal{D}^1_{\max} \to \cdots \to \mathcal{D}^{m-1}_{\max} \xrightarrow{P_{m-1}} \mathcal{D}^m_{\max} \to 0,$$

the maximal or absolute complex. Each space $\mathcal{D}^q_{\max}$ is a Hilbert space with the inner product

$$(u, v)_{P_q} = (u, v)_q + (P_q u, P_q v)_{q+1}, \quad u, v \in \mathcal{D}^q_{\max} \tag{2.1}$$

where $(u, v)_q$ is the inner product in $L^2(\mathcal{M}; E^q)$. We write $\|u\|_{P_q}$ to indicate the norm with respect to this inner product. Complexes of this kind, where the operators

$$P_q : \mathcal{D}^q_{\max} \subset L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^{q+1})$$

are closed densely defined are called Hilbert complexes [**2**]. Since $C^\infty_c(\mathring{\mathcal{M}}; E^q) \subset \mathcal{D}^q_{\max}$, we can also define

$$\mathcal{D}^q_{\min} = \overline{C^\infty_c(\mathring{\mathcal{M}}; E^q)},$$

the closure in $\mathcal{D}^q_{\max}$. With these spaces we obtain the minimal or relative complex

$$0 \to \mathcal{D}^0_{\min} \xrightarrow{P_0} \mathcal{D}^1_{\min} \to \cdots \to \mathcal{D}^{m-1}_{\min} \xrightarrow{P_{m-1}} \mathcal{D}^m_{\min} \to 0$$

which is also a Hilbert complex. To see that this is indeed a complex, let $u \in \mathcal{D}^q_{\min}$, let $\{u_k\}_{k=1}^\infty \subset C^\infty_c(\mathring{\mathcal{M}}; E^q)$ be a sequence converging to u in the norm of $\mathcal{D}^q_{\max}$. To see that $v = P_q u \in \mathcal{D}^{q+1}_{\min}$, we note that $v_k = P_q u_k \in C^\infty_c(\mathring{\mathcal{M}}; E^{q+1})$, that $v_k \to v$ due to the convergence of u_k to u in $\mathcal{D}^q_{\max}$, and that $P_{q+1} v_k \to P_{q+1} v$, simply because $P_{q+1} v_k = 0 = P_{q+1} v$.

If the complex is elliptic and $\mathcal{M}$ is closed (compact without boundary), then the minima and maximal complexes are the same. In all other cases these complexes may be different. If the minimal and maximal domains are different we may take closed subspaces $\mathcal{D}^q \subset \mathcal{D}^q_{\max}$ with $\mathcal{D}^q_{\min} \subset \mathcal{D}^q$ y $P_q(\mathcal{D}^q) \subset \mathcal{D}^{q+1}$ thus getting a general Hilbert complex

$$0 \to \mathcal{D}^0 \xrightarrow{P_0} \mathcal{D}^1 \to \cdots \to \mathcal{D}^{m-1} \xrightarrow{P_{m-1}} \mathcal{D}^m \to 0 \tag{2.2}$$

where $\mathcal{D}^m = L^2(\mathring{\mathcal{M}}; E^m)$. Using $\mathcal{D}$ to refer to this complex, let $Z^q_{\mathcal{D}}$ be the kernel of P_q in $\mathcal{D}^q$ and $B^q_{\mathcal{D}}$ the image of $\mathcal{D}^{q-1}$ by P_{q_1}. The quotient

$$H^q_{\mathcal{D}} = Z^q_{\mathcal{D}} / B^q_{\mathcal{D}}, \tag{2.3}$$

is the cohomología space of the complex (2.2) in degree q. The space $Z^q_{\mathcal{D}}$ is closed in $\mathcal{D}_{\max}$, and the norm on $Z^q_{\mathcal{D}}$ (inherited from $\mathcal{D}^q_{\max}$) determines a seminorm on $H^q_{\mathcal{D}}$: if $\mathbf{u} \in H^q_{\mathcal{D}}$ is the equivalence class of u, then

$$\|\mathbf{u}\| = \inf\{\|u + P_{q-1} v\| : v \in \mathcal{D}^{q-1}\}.$$

This seminorm is a norm iff $B^q_{\mathcal{D}}$ is a closed subspace of $\mathcal{D}^q_{\max}$. This need not be true in general. If $B^q_{\mathcal{D}}$ is a closed subspace, then $H^q_{\mathcal{D}}$ is again a Hilbert space, isomorphic to the subspace of $Z^q_{\mathcal{D}}$ orthogonal to $B^q_{\mathcal{D}}$. In the particular case that all the cohomology spaces $H^q_{\mathcal{D}}$ are finite dimensional we say that the Hilbert complex (2.2) is a Fredholm complex.

The possible difference between the minimal and maximal domains disappears when $\mathcal{M}$ is closed:

THEOREM 2.4. *If $\mathcal{M}$ is closed, then $\mathcal{D}^q_{\min} = \mathcal{D}^q_{\max}$.*

PROOF. The proof consists of observing that $C^\infty(\mathcal{M}; E^q) = C^\infty_c(\mathcal{M}; E^q)$ which is trivial because $\mathcal{M}$ is compact, and showing that $C^\infty(\mathcal{M}; E^q)$ is dense in $\mathcal{D}^q_{\max}$. This last part requires pseudodifferential operator theory but the basic idea is to use convolution mollifiers, that is, convolution with a sequence of functions $\ell^n \chi(\ell x)$ where χ is smooth, compactly supported in a coordinate patch, with integral 1. There is a sequence of pseudodifferential operators A_ℓ of order $-\infty$ converging to I weakly in L^2 as $\ell \to \infty$, such that also $[A_\ell, P_q]u$ converges in L^2 when $\ell \to \infty$ if $u \in \mathcal{D}^q_{\max}$. Then for such u, $A_\ell u = u$ and $P_q A_\ell u = [P_q, A_\ell]u + A_\ell P_q u$ both converge. So $u \in \mathcal{D}^q_{\min}$. $\square$

If $\mathcal{M}$ is compact with boundary, or non-compact, then the issue of domains is delicate.

3. The adjoint complex

Under the right conditions, Hodge theory relates the cohomology spaces (2.3) with the kernel of a Hodge Laplacian. To get there we first need a brief discussion if the adjoint complex.

As we saw already, since $0 = (P_q \circ P_{q-1})^\star = P^\star_{q-1} \circ P^\star_q$, we have a formal adjoint complex

$$(3.1) \quad 0 \leftarrow C^\infty(\mathring{\mathcal{M}}; E^0) \xleftarrow{P_0^\star} C^\infty(\mathring{\mathcal{M}}; E^1) \leftarrow \cdots$$
$$\cdots \leftarrow C^\infty(\mathring{\mathcal{M}}; E^{m-1}) \xleftarrow{P^\star_{m-1}} C^\infty(\mathring{\mathcal{M}}; E^m) \leftarrow 0$$

We let $\mathfrak{D}^{q+1}_{\max}$ and $\mathfrak{D}^{q+1}_{\min}$ be, respectively, the maximal and minimal domains of the operators $P^\star_q$. With the inner product of $\mathfrak{D}^{q+1}_{\max}$ defined using $P^\star_q$, this is a Hilbert space and $\mathfrak{D}^{q+1}_{\min}$ is a closed subspace.

In the case of the Hilbert complex (2.2), the adjoint complex is the Hilbert space adjoint. Each formal adjoint $P^\star_q$ acquires as domain the domain f the adjoint of

$$(3.2) \qquad P_q : \mathcal{D}_q \subset L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^{q+1}).$$

This domain is by definition

$$\mathfrak{D}^{q+1} = \{v \in L^2(\mathcal{M}; E^{q+1}) : \mathcal{D}^q \ni u \mapsto (P_q u, v) \in \mathbb{C} \text{ is } L^2\text{-continuous}\}$$

If $v \in \mathfrak{D}^{q+1}$, the continuity of

$$\mathcal{D}^q \ni u \mapsto (P_q u, v)_{q+1} \in \mathbb{C}$$

and the fact that $\mathcal{D}^q$ is dense in $L^2(\mathcal{M}; E^q)$ imply, by the Riesz Representation Theorem (see Rudin [**11**]), that there is a unique $f \in L^2(\mathcal{M}; E^q)$ such that

$$(P_q u, v)_{q+1} = (u, f)_q \text{ for all } u \in \mathcal{D}^q.$$

This element f is by definition $P^*_q v$ (with $*$ in place of $\star$). This equality also holds if $u \in C^\infty_c(\mathring{\mathcal{M}}; E^q)$, and interpreted in the distributional sense, says that $P^*_q v = P^\star_q v$. For this reason we are concerned only about the domain:, the adjoint operator is the formal adjoint acting distributionally on the elements of the adjoint domain.

Using adjoints, the adjoint of the complex 2.2 is

$$0 \leftarrow \mathfrak{D}^0 \xleftarrow{P_0^\star} \mathfrak{D}^1 \leftarrow \cdots \leftarrow \mathfrak{D}^{m-1} \xleftarrow{P_{m-1}^\star} \mathfrak{D}^m \leftarrow 0 \tag{3.3}$$

where $\mathfrak{D}^0 = L^2(\mathcal{M}; E^0)$. In particular, the adjoint of P_q with domain $\mathcal{D}^q_{\min}$ is $P_q^\star$ with domain $\mathfrak{D}^{q+1}_{\max}$. To see thus, note that by definition $\mathfrak{D}^{q+1}_{\max}$ consists of those elements $v \in L^2(\mathcal{M}; E^{q+1})$ such that $P_q^\star v = f \in L^2(\mathcal{M}; E^q)$, where $P_q^\star v$ is interpreted in the distributional sense. that $P_q^\star v = f$ in this sense means exactly that

$$(f, u)_q = (v, P_q u)_{q+1} \text{ for all } u \in C_c^\infty(\mathring{\mathcal{M}}; E^q). \tag{3.4}$$

This implies the continuity of

$$C_c^\infty(\mathring{\mathcal{M}}; E^q) \ni u \mapsto (v, P_q u)_{q+1} \in \mathbb{C} \tag{3.5}$$

on L^2, because the right hand side is (f, u), therefore $|(v, P_q u)| \leq \|f\| \|u\|$. Since $C_c^\infty(\mathring{\mathcal{M}}; E^q)$ is dense in $\mathcal{D}^q_{\min}$, the domain of the adjoint of P_q with minimal domain contains $\mathfrak{D}^{q+1}_{\max}$. On the other hand, if v belongs to the domain of the adjoint of P_q with minimal domain, then (3.5) is continuous in $L^2(\mathcal{M}; E^{q+1})$. Using that $C_c^\infty(\mathring{\mathcal{M}}; E^q)$ is dense in $L^2(\mathcal{M}; E^q)$, the Riesz Representation Theorem implies the existence of $f \in L^2(\mathcal{M}; E^q)$ such that (3.4) holds, therefore $P_q^\star v = f$ in the distributional sense. That is, $P_q v \in L^2(\mathcal{M}; E^q)$, and then by definition, $v \in \mathfrak{D}^{q+1}_{\max}$.

Because of the symmetry of the situation and the fact that the adjoint of the adjoint of a closed densely defined operator is the original operator, we have:

LEMMA 3.6. *The adjoint of P_q with domain $\mathcal{D}^q_{\min}$ is $P_q^\star$ with domain $\mathfrak{D}^{q+1}_{\max}$. The adjoint of P_q with domain $\mathcal{D}^q_{\max}$ is $P_q^\star$ with domain $\mathfrak{D}^{q+1}_{\min}$.*

As in Theorem 2.4,

THEOREM 3.7. *If $\mathcal{M}$ is closed, then $\mathfrak{D}^q_{\min} = \mathfrak{D}^q_{\max}$.*

4. Hodge theory

The Laplacians of the complex (2.2) are the formal Laplacians of the complex 1.1, with the domain of $\square_q$ given by

$$\mathcal{D}(\square_q) = \{u \in \mathcal{D}^q \cap \mathfrak{D}^q : P_q u \in \mathfrak{D}^{q+1} \text{ and } P_{q-1}^\star u \in \mathcal{D}^{q-1}\}$$

Thus if $u \in \mathcal{D}(\square_q)$, then both $P_{q-1} P_{q-1}^\star u$ and $P_q^\star P_q u$ are legal, so $\square_q u$ is defined.

LEMMA 4.1. *If $u, v \in \mathcal{D}(\square_q)$, then*

$$(\square_q u, v) = (P_{q-1}^\star u, P_{q-1}^\star v) + (P_q u, P_q v) \tag{4.2}$$

In particular

$$\square_q : \mathcal{D}(\square_q) \subset L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q) \tag{4.3}$$

is nonnegative and satisfies

$$\|u\| \le \|(\Box_q + I)u\| \tag{4.4}$$

PROOF. If $u \in \mathcal{D}(\Box_q)$ then

$$(\Box_q u, u) = (P_{q-1}P^\star_{q-1}u, u) + (P^\star_q P_q u, u)$$

Since $P^\star_{q-1}u \in \mathcal{D}^{q-1}$ and $v \in \mathfrak{D}_q$,

$$(P_{q-1}P^\star_{q-1}u, v) = (P^\star_{q-1}u, P^\star_{q-1}v)$$

Likewise

$$(P^\star_q P_q u, v) = (P_q u, P_q v).$$

This proves (4.2) and that

$$(\Box_q u, u) = \|P^\star_{q-1}u\|^2 + \|P_q u\|^2$$

so (4.3) is positive. Finally

$$\|u\|^2 \le \|u\|^2 + (\Box_q u, u) = ((I + \Box_q)u, u) \le \|(I + \Box_q)u\|\|u\|, \quad u \in \mathcal{D}(\Box_q)$$

so (4.4) holds. $\Box$

Let

$$\mathcal{H}^q = \{u \in \mathcal{D}(\Box_q) : \Box_q u = 0\}.$$

This is a closed subspace of $L^2(\mathcal{M}; E^q)$. We will later make use of the orthogonal projection

$$\Pi_q : L^2(\mathcal{M}; E^q) \to \mathcal{H}^q.$$

LEMMA 4.5. *We have* $\mathcal{H}^q = \{u \in \mathcal{D}^q \cap \mathfrak{D}^q : P_q u = 0,\ P^\star_{q-1}u = 0\}$. *In particular there is a well defined map*

$$\mathcal{H}^q \ni u \mapsto [u] \in H^q_{\mathcal{D}}.$$

Furthermore, $\mathcal{H}^q$ *is a closed subspace of* $L^2(\mathcal{M}; E^q)$.

PROOF. If $u \in \mathcal{H}^q$ then by definition $u \in \mathcal{D}^q \cap \mathfrak{D}^q$, and we have

$$0 = (\Box_q u, u) = \|P^\star_{q-1}u\|^2 + \|P_q u\|^2.$$

Thus $P_q u = 0$ and $P^\star_{q-1}u = 0$. Conversely, if $u \in \mathcal{D}^q \cap \mathfrak{D}^q$ satisfies $P_q u = 0$ and $P^\star_{q-1}u = 0$, then $P_q u \in \mathfrak{D}^{q+1}$ because in fact $P_q u = 0$, likewise $P^\star_{q-1}u \in \mathcal{D}^{q-1}$. Thus $u \in \mathcal{D}(\Box_q)$. Furthermore, clearly $\Box_q u = 0$. So $u \in \mathcal{H}^q$.

Of course if $u \in \mathcal{H}^q$, then $u \in \mathcal{D}^q$ and since $P_q u = 0$, u defines an element in the q-th cohomology space of (2.2).

Finally, suppose $\{u_\ell\}_{\ell=1}^\infty \subset \mathcal{H}^q$ converges to u in $L^2(\mathcal{M}; E^q)$. Since P_q with domain $\mathcal{D}^q$ is closed and $P_q u_\ell = 0$, $u \in \mathcal{D}^q$ and $P_q u = 0$. Likewise $u \in \mathfrak{D}^q$ and $P^\star_{q-1}u = 0$. Thus $u \in \mathcal{H}^q$. $\Box$

Let

$$\mathrm{rg}(P_{q-1}) = P_{q-1}(\mathcal{D}^q), \quad \mathrm{rg}(P^\star_q) = P^\star_{q-1}(\mathfrak{D}^{q+1}).$$

Proposition 4.6. *The spaces* $\mathcal{H}^q$, $\operatorname{rg}(P_{q-1})$, *and* $\operatorname{rg}(P_q^\star)$ *are orthogonal to each other.*

Proof. Suppose $u \in \mathcal{H}^q$ and $v \in \mathcal{D}^{q-1}$. Since $u \in \mathfrak{D}^q$,

$$(u, P_{q-1}v) = (P_{q-1}^\star u, v) = 0$$

Thus $\mathcal{H}^q \perp \operatorname{rg}(P_{q-1})$. Likewise $\mathcal{H}^q \perp \operatorname{rg}(P_q^\star)$. To show that $\operatorname{rg}(P_{q-1}) \perp \operatorname{rg}(P_q^\star)$, let $v \in \mathcal{D}^{q-1}$, $w \in \mathfrak{D}^{q+1}$. Since (2.2) is a complex, $u = P_{q-1}v \in \mathcal{D}^q$. Thus $(u, P_q^\star w) = (P_q u, w)$. But $P_q u = P_q P_{q-1} v = 0$. So $(P_{q-1}v, P_q^\star w) = 0$. Thus $\operatorname{rg}(P_{q-1}) \perp \operatorname{rg}(P_q^\star)$. □

Theorem 4.7. *The operator* (4.3) *is selfadjoint.*

Proof. We first show that (4.3) is closed. Let $\{u_\ell\}_{\ell=1}^\infty \subset \mathcal{D}(\Box_q)$ be such that u_ℓ and $\Box_q u_\ell = f_\ell$ converge in $L^2(\mathcal{M}; E^q)$. Then

$$(f_\ell - f_{\ell'}, u_\ell - u_{\ell'}) = (\Box_q(u_\ell - u_{\ell'}), u_\ell - u_{\ell'}) \to 0 \quad \text{as } \ell, \ell' \to \infty,$$

thus

$$\|P_{q-1}^\star(u_\ell - u_{\ell'})\|^2 + \|P_q(u_\ell - u_{\ell'})\|^2 \to 0 \quad \text{as } \ell, \ell' \to \infty.$$

Hence $\{P_{q-1}^\star u_\ell\}_{\ell=1}^\infty$ and $\{P_q u_\ell\}_{\ell=1}^\infty$ are Cauchy sequences in $L^2(\mathcal{M}; E^q)$. Thus they converge. Consequently both $u \in \mathfrak{D}^q$ and $u \in \mathcal{D}^q$, since $P_{q-1}^\star$ and P_q with these domains are closed. Next, since $\{\Box_q u_\ell\}_{\ell=1}^\infty$ converges and the ranges of $P_{q-1}P_{q-1}^\star$ and $P_q^\star P_q$ are orthogonal to each other, both $\{P_{q-1}P_{q-1}^\star u_\ell\}_{\ell=1}^\infty$ and $\{P_q^\star P_q u_\ell\}_{\ell=1}^\infty$ converge. Since $P_{q-1}^\star u_\ell \in \mathcal{D}^q$, this implies $P_{q-1}^\star u \in \mathcal{D}_{q-1}$ and since $P_q u_\ell \in \mathfrak{D}^q$, $P_q u \in \mathfrak{D}^q$, since these operators with their given domains are closed.

Note that since (4.3) is positive, it is symmetric. Having established that it is closed, we observe that because of (4.4) its deficiency indices both vanish. Consequently (see [**10**, pg. 141]), (4.3) is selfadjoint. □

Theorem 4.8. *Suppose that for each q there is a continuous operator*

$$G_q : L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q), \quad \operatorname{rg} G_q \subset \mathcal{D}(\Box_q)$$

such that

$$G_q \Box_q = I - \Pi_q \ \text{ on } \mathcal{D}(\Box_q) \text{ and } \Box_q G_q = I - \Pi_q \ \text{ on } L^2(\mathcal{M}; E^q). \tag{4.9}$$

Then

$$L^2(\mathcal{M}; E^q) = \mathcal{H}^q \oplus \operatorname{rg}(P_{q-1}) \oplus \operatorname{rg}(P_q^\star) \tag{4.10}$$

as an orthogonal decomposition. Furthermore, the canonical map

$$\mathcal{H}^q \to H^q_{\mathcal{D}} \tag{4.11}$$

is an isomorphism.

Equation (4.10) is the Hodge decomposition. Equation (4.11) is the Hodge isomorphism.

PROOF. The sum of the spaces in the right hand side of (4.10) is direct because of the pairwise orthogonality of the spaces. Using $I = \Pi_q + \Box_q G_q =$, we have

$$u = \Pi_q u + P_{q-1} P_{q-1}^\star G_q u + P_q^\star P_q G_q u$$

for any $u \in L^2(\mathcal{M}; E^q)$. Since $G_q u \in \mathcal{D}(\Box_q)$, $P_{q-1}^\star G_q u \in \mathcal{D}^{q-1}$, so

$$P_{q-1} P_{q-1}^\star G_q u \in \mathrm{rg}(P_{q-1}).$$

Likewise $P_q^\star P_q G_q u \in \mathrm{rg}(P_q^\star)$ and by defintion $\Pi_q u \in \mathcal{H}^q$. Thus we have equality in (4.10).

To prove that the natural map (4.11) is an isomorphism, we will first show that

$$P_q G_q = G_{q+1} P_q, \quad P_{q-1}^\star G_q = G_{q-1} P_{q-1}^\star. \tag{4.12}$$

We have

$$P_q \Box_q = \Box_{q+1} P_q, \quad P_{q-1}^\star \Box_q = \Box_{q-1} P_{q-1}^\star.$$

Thus $G_{q+1} P_q \Box_q G_q = G_{q+1} \Box_{q+1} P_q G_q$. Using (4.9) this becomes

$$G_{q+1} P_q (I - \Pi_q) = (I - \Pi_q) P_q G_q$$

The left hand side is $G_{q+1} P_q$. Since the range of $P_q G_q$ is orthogonal to $\mathcal{H}^q$, the right hand side is $P_q G_q$. This and the analogous argument in the case of $P_{q-1}^\star \Box_q = \Box_{q-1} P_{q-1}^\star$ give the formulas in (4.12). Using (4.9), these formulas give in particular

$$u = \Pi_q u + P_{q-1} G_{q-1} P_{q-1}^\star u + P_q^\star G_{q+1} P_q u. \tag{4.13}$$

The natural map $\theta : \mathcal{H}^q \to H^q_{\mathcal{D}}$ is $\theta(u) = \mathbf{u}$ where $\mathbf{u} = u \mod \mathrm{rg}(P_{q-1})$. This map is injective: If $\theta(u) = 0$ then $u = P_{q-1} v$ for some $v \in \mathcal{D}_{q-1}$. Since $u \in \mathcal{H}^q$ is orthogonal to $P_{q-1} v$, $u = 0$. The map is also surjective. Suppose now $\mathbf{u} \in H^q_{\mathcal{D}}$ is represented by $u \in \ker P_q$. Then (4.13) gives

$$u = \Pi_q u + P_{q-1} G_{q-1} P_{q-1}^\star u,$$

therefore $\mathbf{u} = \theta(\Pi_q u)$. Thus θ is also surjective. □

Note that the proof only involves the degrees $q - q$, q, and $q + 1$.

The Hodge decomposition also implies that $\mathrm{rg}(\Box_q)$ is closed. Indeed, on the one hand, the argument in the proof of (4.10) shows that $\mathrm{rg}(P_{q-1}) \oplus \mathrm{rg}(P_q^\star) \subset \mathrm{rg}(\Box_q)$, and on the other, $\mathrm{rg}(\Box)$ is a subset of the space orthogonal to the kernel of the adjoint of (4.3), which is $\mathcal{H}^q$ because (4.3) is selfadjoint.

A typical situation in which the above theorem holds is as follows:

THEOREM 4.14. *Suppose there are compact operators*

$$Q_q, R_{\mathrm{L},q}, R_{\mathrm{R},q} : L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q)$$

where Q_q has image in $\mathcal{D}(\Box_q)$, such that

$$\Box_q Q_q = I - R_{\mathrm{R},q}, \quad Q_q \Box_q = I - R_{\mathrm{L},q}.$$

Then Theorem 4.8 holds. Furthermore $\mathcal{H}^q$, hence $H^q_{\mathcal{D}}$, is finite dimensional, $G_q : L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q)$ is compact and the spectrum of $\Box_q$ consists of a sequence of eigenvalues of finite multiplicity without points of accumulation.

Proof. We first show that $\mathcal{H}^q$ is finite-dimensional. For this we use that if bounded sets in a Banach space are precompact, then the space is finite dimensional. Recall from Lemma 4.5 that $\mathcal{H}^q$ is closed. Suppose then that $\{u_\ell\}_{\ell=1}^\infty$ is a bounded sequence in $\mathcal{H}^q$. From $Q_q\Box_q u_\ell = 0$ we get $u_\ell - R_{\mathrm{L},q}u_\ell = 0$, that is, $u_\ell = R_{\mathrm{L},q}u_\ell$. Since $R_{\mathrm{L},q}$ is compact, there is a subsequence $\{u_{\ell_k}\}$ such that $\{R_{\mathrm{L},q}u_{\ell_k}\}$ converges. Then $\{u_{\ell_k}\}$ converges. Thus $\mathcal{H}^q$ is finite-dimensional.

Next we show that $\Box_q$ has closed range. Let $\{f_\ell\}_{\ell=1}^\infty$ be a sequence in $\operatorname{rg}(\Box_q)$ converging in $L^2(\mathcal{M}; E^q)$ to some f. Let $\{u_\ell\}_{\ell=1}^\infty \subset \mathcal{D}(\Box_q)$ be such that $\Box_q u_\ell = f_\ell$. We may assume $u_\ell \perp \mathcal{H}^q$. If $\{u_\ell\}$ is bounded, pass to a subsequence $\{u_{\ell_k}\}$ such that $R_{\mathrm{L},q}u_{\ell_k}$ converges. Then

$$Q_q f_{\ell_k} = Q_q\Box_q u_{\ell_k} = u_{\ell_k} - R_{\mathrm{L},q}u_{\ell_k}$$

gives that $\{u_{\ell_k}\}_{k=1}^\infty$ converges in $L^2(\mathcal{M}; E^q)$ to some u. Since (4.3) is a closed operator, $u \in \mathcal{D}(\Box_q)$ and $\Box_q u = f$. Thus $f \in \operatorname{rg}(\Box_q)$. If $\{u_\ell\}$ is unbounded, replacing it with a subsequence we may assume $\|u_\ell\| \to \infty$. The sequence with terms $v_\ell = \|u_\ell\|^{-1}u_\ell$ is bounded, the v_ℓ are orthogonal to $\mathcal{H}^q$, and $\Box_q v_\ell = \|u_\ell\|^{-1}f_\ell \to 0$. By the previous argument there is a subsequence $\{v_{\ell_k}\}$ that converges to some $v \in \mathcal{D}(\Box_q)$. But then $\Box_q v = 0$ and $v \perp \mathcal{H}^q$, since $\mathcal{H}^q$ is closed. This is a contradiction. So on fact the original sequence was bounded, and so $f \in \operatorname{rg}(\Box_q)$.

Now, since $\Box_q$ with domain $\mathcal{D}(\Box_q)$ is selfadjoint and has closed range, $\operatorname{rg}(\Box_q) = (\mathcal{H}^q)^\perp$. Let $\iota : (\mathcal{H}^q)^\perp \to L^2(\mathcal{M}; E^q)$ be the inclusion map. The restriction of $\Box_q$ to $(\mathcal{H}^q)^\perp$ is bijective, hence invertible. Let $\tilde{G}^q$ be the inverse of this operator. Define

$$G_q : L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q), \quad G_q = \iota \circ \tilde{G}_q \circ (I - \Pi_q). \tag{4.15}$$

Then (4.9) holds. The space $\mathcal{D}(\Box_q)$ with the norm defined by

$$\|u\|_\Box]^2 = \|u\|^2 + \|\Box_q u\|^2$$

is complete (a Hilbert space) and (4.3) is continuous in this norm. So G_q is continuous, Since the inclusion $\mathcal{D}(\Box_q) \to L^2(\mathcal{M}; E^q)$ is continuous, (4.15) is continuous. Thus the hypotheses of Theorem 4.8 are satisfied.

Finally, applying Q_q to both sides of $\Box_q G_q = I - \Pi_q$ gives

$$Q_q\Box_q G_q = Q_q(I - \Pi_q) \quad \text{and} \quad Q_q\Box_q G_q = (I - R_{\mathrm{L},q})G_q$$

so

$$G_q = Q_q(I - \Pi_q) + R_{\mathrm{L},q}G_q$$

so G_q is compact.

The compactness of G_q and the finite-dimensionality of $\mathcal{H}^q$ implies the statement about the eigenvalues. $\square$

When $\mathcal{M}$ is closed the only closed domains for the complex and its adjoint are the maximal domains. If the complex is elliptic, then the Hodge Laplacians are elliptic and the hypotheses of Theorem 4.15 are satisfied: In this case the Q_q are pseudodifferential operators of order -2 and the $R_{\mathrm{L},q}, R_{\mathrm{R},q}$ are smoothing operators.

CHAPTER IV

The Atiyah-Bott-Lefschetz formula

1. The Lefschetz number

Suppose $\mathcal{M}$ is closed, orientable, n-dimensional. Let $f : \mathcal{M} \to \mathcal{M}$ is continuous. It induces linear maps

$$\mathfrak{f}_q^* : H^q(\mathcal{M}, \mathbb{R}) \to H^q(\mathcal{M}, \mathbb{R})$$

Since the spaces $H^q(\mathcal{M}, \mathbb{R})$ are finite-dimensional, $\mathfrak{f}_q^*$ has a well-defined trace, and we can form

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{tr} \mathfrak{f}_q^*$$

This is the Lefschetz number of f.

Theorem 1.1 (Lefschetz [**9**]). *If $L_f \neq 0$, then f has a fixed point.*

To understand the theorem we turn to homology. Note first that L_f depends only on the homotopy class of f. So we may replace f by a new function homotopic to f but with only discrete fixed points. Then we may take advantage of the fact that $\mathcal{M}$ admits a triangulation (see [**3**, Cairns]) and further refine the triangulation and deform f so that it maps simplices to simplices, and no simplex contains more that one fixed point. Let S_q be the space (over $\mathbb{R}$) spanned by the simplices of the triangulation in dimension q, let $H_q(\mathcal{M};\mathbb{R})$ be the homology space in degree $\mathbb{R}$. Then f induces maps $f_*^q : S_q \to S_q$, whose trace is again well defined. These maps in turn induce maps $\mathfrak{f}_*^q : H_q(\mathcal{M};\mathbb{R}) \to H_q(\mathcal{M};,\mathbb{R})$. The actual Lefschetz number is in homology,

$$L_f' = \sum_{q=0}^{n} (-1)^q \operatorname{tr} \mathfrak{f}_*^q.$$

This is just $(-1)^n L_f$ because of Poincare duality. In any case,

$$\sum_{q=0}^{n} (-1)^q \operatorname{tr} \mathfrak{f}_*^q = \sum_{q=0}^{n} (-1)^q \operatorname{tr} f_*^q$$

Now, if f has no fixed point, then f_*^q maps a simplex of dimension q to a different simplex of dimension q, so the matrix of f_*^q with respect to the bases formed by these simplices has zeros on the diagonal. Thus the trace is 0 for all q.

2. Lefschetz number in finite dimensional complexes

The general scheme for proving the Atiyah-Bott formula for the Lefschetz number has some similarity with that in the finite-dimensional case. Suppose

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & S^{q-1} & \xrightarrow{\mathfrak{a}_{q-1}} & S^q & \xrightarrow{\mathfrak{a}_q} & S^{q+1} & \longrightarrow & \cdots \\ & & \downarrow \mathfrak{f}^{q-1} & & \downarrow \mathfrak{f}^q & & \downarrow \mathfrak{f}^{q+1} & & \\ \cdots & \longrightarrow & S^{q-1} & \xrightarrow{\mathfrak{a}_{q-1}} & S^q & \xrightarrow{\mathfrak{a}_q} & S^{q+1} & \longrightarrow & \cdots \end{array}$$

is a cochain map of a complex of finite dimensional vector spaces, with the degrees q ranging from from 0 to m. Thus $\mathfrak{a}_q\mathfrak{a}_{q-1} = 0$ and each square commutes, so we have finite dimensional cohomology spaces H^q and each $\mathfrak{f}_q$ induces a map $\boldsymbol{\mathfrak{f}}_q : H^q \to H^q$. So we have on the one hand the trace of this map and on the other that of $\mathfrak{f}_q : S^q \to S^{q+1}$. Assuming the complex is finite, we have

Proposition 2.1.

$$\sum_{q=0}^{r}(-1)^q \operatorname{tr} \boldsymbol{\mathfrak{f}}_q = \sum_{q=0}^{r}(-1)^q \operatorname{tr} \mathfrak{f}_q.$$

The proof will occupy the rest of this section. Fix an inner product (Hermitian metric if the field is $\mathbb{C}$) on each S^q and do Hodge theory, which works equally well in this context. Let then $\mathfrak{h}^q$ be the kernel of $\mathfrak{d}_q = \mathfrak{a}_{q-1}\mathfrak{a}^*_{q-1} + \mathfrak{a}^*_q\mathfrak{a}_q$, let π^q_0 be the orthogonal projection on $\mathfrak{h}^q$, let $\mathfrak{g}_q$ be the Green's operator,

$$\mathfrak{d}_q\mathfrak{g}_q = g_q\mathfrak{d}_q = I - \pi^q_0.$$

From this one gets the Hodge decomposition which is based on

$$I = \pi^0_q + \pi^\uparrow_q + \pi^\downarrow_q. \tag{2.2}$$

with the projections

$$\pi^\uparrow_q, \pi^\downarrow_q : S^q \to S^q$$

defined by

$$\pi^\uparrow_q = \mathfrak{a}_{q-1}\mathfrak{g}_{q-1}\mathfrak{a}^*_{q-1}, \quad \pi^\downarrow_q = \mathfrak{a}^*_q\mathfrak{g}_{q+1}\mathfrak{a}_q.$$

Thus

$$S^q = \mathfrak{h}^q \oplus \operatorname{rg} \mathfrak{a}_{q-1} \oplus \operatorname{rg} \mathfrak{a}^*_q. \tag{2.3}$$

Note that $\mathfrak{a}_q\pi^\downarrow_q = \pi^\uparrow_{q+1}\mathfrak{a}_q$ and $\mathfrak{a}^*_{q-1}\pi^\uparrow_q = \pi^\downarrow_{q-1}a_{q-1}$. The proof that H^q and $\mathfrak{h}^q$ are isomorphic is the same as in the case of first order differential complexes we saw earlier.

We will first show that the maps

$$\boldsymbol{\mathfrak{f}}_q : H^q \to H^q, \quad \pi^0_q\mathfrak{f}^q|_{\mathfrak{h}^q} : \mathfrak{h}^q \to \mathfrak{h}^q$$

have the same trace. If $u \in \ker \mathfrak{a}_q$ represents an element $\mathbf{u} \in H^q$, then

$$\mathfrak{f}^q u = (\pi_q^0 + \pi_q^{\uparrow} + \pi_q^{\downarrow})\mathfrak{f}^q u = \pi_q^0 \mathfrak{f}^q u + \pi_q^{\uparrow}\mathfrak{f}^q u,$$

the last equality because

$$\pi_q^{\downarrow}\mathfrak{f}^q u = \mathfrak{a}_q^* g_{q+1} \mathfrak{a}_q \mathfrak{f}^q u = \mathfrak{a}_q^* g_{q+1} \mathfrak{f}^{q+1} \mathfrak{a}_q u.$$

Since $\pi_q^{\uparrow}\mathfrak{f}^q u$ is exact, the class $\boldsymbol{\mathfrak{f}}^q \mathbf{u}$ of $\mathfrak{f}^q u$ in H^q is represented by the element $\pi_q^0 \mathfrak{f}^q u \in \mathfrak{h}^q$. We see that $\boldsymbol{\mathfrak{f}}^q$, as an operator $\mathfrak{h}^q \to \mathfrak{h}^q$, is $\pi_q^0 \mathfrak{f}^q|_{\mathfrak{h}^q}$. This shows that with the canonical isomorphism $\theta_q : \mathfrak{h}^q \to H^q$,

$$\theta_q \pi_q^0 \mathfrak{f}_q u = \boldsymbol{\mathfrak{f}}_q \theta_q u.$$

Since θ_q^0 is an isomorphism that intertwines $\boldsymbol{\mathfrak{f}}^q$ and $\pi_q \mathfrak{f}^q|_{\mathfrak{h}^q}$,

$$\operatorname{tr} \boldsymbol{\mathfrak{f}}^q = \operatorname{tr} \pi_q^0 \mathfrak{f}^q|_{\mathfrak{h}^q}$$

as we wanted to show.

The trace of $\mathfrak{f}_q$ is the same as that of its diagonal part according to the decomposition (2.3):

$$\operatorname{tr} \mathfrak{f}^q = \operatorname{tr} \pi_q^0 \mathfrak{f}^q \pi_q^0 + \operatorname{tr} \pi_q^{\uparrow} \mathfrak{f}^q \pi_q^{\uparrow} + \operatorname{tr} \pi_q^{\downarrow} \mathfrak{f}^q \pi_q^{\downarrow}$$

Defining

$$\mathfrak{f}_{\square}^q = \pi_q^0 \mathfrak{f}^q \pi_q^0 + \pi_q^{\uparrow} \mathfrak{f}^q \pi_q^{\uparrow} + \pi_q^{\downarrow} \mathfrak{f}^q \pi_q^{\downarrow}$$

we have

$$\mathfrak{a}_q \mathfrak{f}_{\square}^q = \mathfrak{a}_q \pi_q^{\downarrow} \mathfrak{f}^q \pi_q^{\downarrow} = \pi_{q+1}^{\uparrow} \mathfrak{a}_q \mathfrak{f}^q \pi_q^{\downarrow} = \pi_{q+1}^{\uparrow} \mathfrak{f}^{q+1} \mathfrak{a}_q \pi_q^{\downarrow} = \pi_{q+1}^{\uparrow} \mathfrak{f}^{q+1} \pi_{q+1}^{\uparrow} \mathfrak{a}_q = \mathfrak{f}_{\square}^{q+1} \mathfrak{a}_q.$$

Thus the maps $\mathfrak{f}_{\square}^q$ also define maps $\boldsymbol{\mathfrak{f}}_{\square}^q : H^q \to H^q$. If $u \in \ker \mathfrak{a}_q$, then

$$\mathfrak{f}_{\square}^q u = \pi_q^0 \mathfrak{f}^q \pi_q^0 u + \pi_q^{\uparrow} \mathfrak{f}^q \pi_q^{\uparrow} u,$$

so $\boldsymbol{\mathfrak{f}}_{\square}^q$, as a map $\mathfrak{h}^q \to h^q$, is equal to $\boldsymbol{\mathfrak{f}}^q$. Thus

$$\operatorname{tr} \boldsymbol{\mathfrak{f}}^q = \operatorname{tr} \boldsymbol{\mathfrak{f}}_{\square}{}^q.$$

We will now relate $\pi_q^{\uparrow} \mathfrak{f}^q \pi_q^{\uparrow}$ and $\pi_{q-1}^{\downarrow} \mathfrak{f}^{q-1} \pi_{q-1}^{\downarrow}$. Using (2.2) and

$$\mathfrak{f}^q \mathfrak{a}_{q-1} = \mathfrak{a}_{q-1} f^{q-1},\ \mathfrak{g}_q \mathfrak{a}_{q-1} = \mathfrak{a}_{q-1} \mathfrak{g}_{q-1},\ \text{y } \mathfrak{a}_{q-1}^* = \pi_{q-1}^{\downarrow} \mathfrak{a}_{q-1}^*$$

we get

$$\begin{aligned}
\pi_q^{\uparrow} \mathfrak{f}^q \pi_q^{\uparrow} &= \mathfrak{a}_{q-1} \mathfrak{a}_{q-1}^* \mathfrak{g}_q \mathfrak{f}^q \mathfrak{a}_{q-1} \mathfrak{a}_{q-1}^* \mathfrak{g}_q \\
&= \mathfrak{a}_{q-1} \mathfrak{a}_{q-1}^* \mathfrak{g}_q \mathfrak{a}_{q-1} \mathfrak{f}^{q-1} \mathfrak{a}_{q-1}^* \mathfrak{g}_q \\
&= \mathfrak{a}_{q-1} \mathfrak{a}_{q-1}^* \mathfrak{a}_{q-1} \mathfrak{g}_{q-1} \mathfrak{f}^{q-1} \mathfrak{a}_{q-1}^* \mathfrak{g}_q \\
&= \mathfrak{a}_{q-1} \pi_{q-1}^{\downarrow} \mathfrak{f}^{q-1} \pi_{q-1}^{\downarrow} \mathfrak{a}_{q-1}^* \mathfrak{g}_q.
\end{aligned}$$

The operator

$$\mathfrak{a}_{q-1}^* \mathfrak{g}_q|_{\operatorname{rg} \mathfrak{a}_{q-1}} : \operatorname{rg} \mathfrak{a}_{q-1} \to \operatorname{rg} \mathfrak{a}_{q-1}^*$$

is invertible, with inverse

$$\mathfrak{a}_{q-1}|_{\operatorname{rg}\mathfrak{a}^*_{q-1}} : \operatorname{rg}\mathfrak{a}^*_{q-1} \to \operatorname{rg}\mathfrak{a}_{q-1}.$$

Indeed $\mathfrak{a}_{q-1}\mathfrak{a}^*_{q-1}\mathfrak{g}_q = \pi^\uparrow_q$, which as an operator $\operatorname{rg}\mathfrak{a}_{q-1} \to \operatorname{rg}\mathfrak{a}_{q-1}$ is the identity. We conclude that

$$\operatorname{tr}\pi^\uparrow_q \mathfrak{f}^q \pi^\uparrow_q = \operatorname{tr}\pi^\downarrow_{q-1}\mathfrak{f}^{q-1}\pi^\downarrow_{q-1}.$$

Hence

$$\begin{aligned}\sum_{q=0}^r(-1)^q\operatorname{tr}\mathfrak{f}^q &= \sum_{q=0}^r(-1)^q\operatorname{tr}\mathfrak{f}^q_\square\\ &= \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^0_q\mathfrak{f}^q\pi^0_q + \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^\uparrow_q\mathfrak{f}^q\pi^\uparrow_q + \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^\downarrow_q\mathfrak{f}^q\pi^\downarrow_q\\ &= \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^0_q\mathfrak{f}^q\pi^0_q + \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^\downarrow_{q-1}\mathfrak{f}^{q-1}\pi^\downarrow_{q-1} + \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^\downarrow_q\mathfrak{f}^q\pi^\downarrow_q\\ &= \sum_{q=0}^r(-1)^q\operatorname{tr}\pi^0_q\mathfrak{f}^q\pi^0_q\\ &= \sum_{q=0}^r(-1)^q\operatorname{tr}\boldsymbol{\mathfrak{f}}^q.\end{aligned}$$

This completes the proof of the proposition.

In particular, for the complex associated to a triangulation of a manifold, the Lefschetz number computed using homology. and that using the maps on the spaces generated by the simplices is the same.

3. The Lefschetz theorem of Atiyah-Bott

Let $\mathcal{M}$ be a closed orientable manifold of dimension n. We will discuss the Atiyah-Bott theorem only in the case of the de Rham complex,

$$0 \to C^\infty(\mathcal{M}) \xrightarrow{d} C^\infty(\mathcal{M};\textstyle\bigwedge^1\mathcal{M}) \to \cdots \\ \cdots \to C^\infty(\mathcal{M};\textstyle\bigwedge^{n-1}\mathcal{M}) \xrightarrow{d} C^\infty(\mathcal{M};\textstyle\bigwedge^n\mathcal{M}) \to 0. \tag{3.1}$$

Fix a Riemannian metric, let $\mathfrak{m}$ be the Riemannian density on $\mathcal{M}$ view the vector bundles $\bigwedge^q\mathcal{M}$ as Hermitian vector bundles using the metric. The use of the metric is auxiliary, the result is independent of the metric.

The maps

$$f^*_q : C^\infty(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}) \to C^\infty(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}) \tag{3.2}$$

satisfy

$$d\circ f^*_q = f^*_q\circ d$$

so they induce mappings in cohomology:

$$\mathfrak{f}_q : H^q_{dR}(\mathcal{M}) \to H^q_{dR}(\mathcal{M}).$$

Since the cohomology spaces are finite-dimensional, the traces $\operatorname{tr} \mathfrak{f}_q$ are well defined, and we set

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{tr} \mathfrak{f}_q. \tag{3.3}$$

Suppose $p_0 \in \mathcal{M}$ is a fixed point of f. Then $f^*_{p_0} : T^*_{p_0}\mathcal{M} \to T^*_{p_0}\mathcal{M}$, so again $\operatorname{tr} d_{p_0}$ is well defined. We say that p_0 is a simple fixed point if 1 is not an eigenvalue of df_{p_0}, so that $\det(I - f^*_{p_0}) \neq 0$. The geometric meaning of this condition will be clarified later. Let Fix denote the set of fixed points of f.

THEOREM 3.4 (Atiyah-Bott). *Suppose all fixed points of f are simple. Then*

$$L_f = \sum_{p \in \mathrm{Fix}} \sum_{q=0}^{n} \frac{(-1)^q \operatorname{tr} f^*_p}{|\det(I - f^*_p)|}.$$

To prove this one attempts to show that

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{Tr} f^*_q$$

using the traces of the maps

$$f^*_q : L^2(\mathcal{M}; {\textstyle\bigwedge}^q \mathcal{M}) \to L^2(\mathcal{M}; {\textstyle\bigwedge}^q \mathcal{M}) \tag{3.5}$$

obtained by extending the maps (3.2) to the L^2 spaces. Unfortunately these maps are not trace class, so this does not quite work.

The strategy will be to approximate the f^*_q by a family $\{f^*_{q,\lambda} : \lambda < 0\}$ of operators

$$f^*_{q,\lambda} : L^2(\mathcal{M}; {\textstyle\bigwedge}^q \mathcal{M}) \to L^2(\mathcal{M}; {\textstyle\bigwedge}^q \mathcal{M})$$

that are trace class, then show on the one hand that

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{Tr} f^*_{q,\lambda} \tag{3.6}$$

and on the other, that with $K_{q,\lambda}$ denoting the Schwartz kernel of $f^*_{q,\lambda}$,

$$\operatorname{Tr} f^*_{q,\lambda} = \int_{\mathcal{M}} \operatorname{tr} K_{q,\lambda}(p,p)\, d\mathfrak{m}(p) \to \sum_{p \in \mathrm{Fix}} \frac{\operatorname{tr} f^*_q}{|\det(I - df_p)|} \tag{3.7}$$

as $\lambda \to -\infty$. Of course we can take the parameter λ to be discrete, $\lambda = -k$, $k \in \mathbb{N}$.

We will define the operators $f^*_{q,\lambda}$ in the next section and prove (3.6). In the section after that we will prove (3.7). In the remainder of this section we discuss the relevance of the condition that the fixed points are simple.

Let $\Gamma \subset \mathcal{M} \times M$ be the graph of f,

$$\Gamma = \{(p, f(p)) \in \mathcal{M} \times \mathcal{M} : p \in \mathcal{M}\}$$

and let $\operatorname{diag}(M) = \{(p,p) : p \in \mathcal{M}\}$ be the diagonal. These are submanifolds of $\mathcal{M} \times \mathcal{M}$ of dimension n. The fixed points of f correspond to the points where these two submanifolds intersect. Let (p_0, p_0) be one such point. The condition

$$T_{p_0,p_0}\Gamma + T_{p_0,p_0} \operatorname{diag}(\mathcal{M}) = T_{p_0,p_0}(\mathcal{M} \times \mathcal{M})$$

says that the two manifolds intersect transversely at (p_0, p_0). This condition is equivalent to the condition that p_0 is a simple fixed point of f. To see this, let $x_1, \dots, x_n$ be coordinates in a neighborhood U of p_0 in $\mathcal{M}$, vanishing at p_0, let $y_1, \dots, y_n$ be the same functions thought of as in the second copy of $\mathcal{M}$ so $x_1, \dots, x_n, y_1, \dots, y_n$ are coordinates on $\mathcal{M} \times \mathcal{M}$ centered at (p_0, p_0). Let $f_j = y_j \circ f$, so that in coordinates, the graph of f near (p_0, p_0) is

$$\{(x_1, \dots, x_n, f_1(x), \dots, f_n(x)) : x \in U\}$$

A basis of $T_{p_0,p_0}\Gamma$ is

$$\frac{\partial}{\partial x_j} + \sum_\ell \frac{\partial f_\ell}{\partial x_j} \frac{\partial}{\partial y_\ell}$$

all evaluated at $p_0 = 0$, and a basis of the diagonal is

$$\frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j}$$

The matrix of this basis with respect to the basis $\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}$ is

$$\begin{bmatrix} I & I \\ \dfrac{\partial f}{\partial x} & I \end{bmatrix}$$

where $\partial f/\partial x$ is the Jacobian matrix of f. Transverse intersection means that the determinant of this matrix is nonzero. But the determinant is $\det(\partial f/\partial x - I)$.

The next aspect to understand is how this comes into the calculation of the trace. With some more generality, suppose $E, F \to \mathcal{M}$ are Hermitian vector bundles and $f : \mathcal{M} \to \mathcal{M}$ is smooth. Recall from Section I.4 that there is a vector bundle $f^*E \to \mathcal{M}$ and from Lemma I.4.1 an induced map

$$f^* : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; f^*E).$$

Suppose $\Psi : f^*E \to F$ is a vector bundle morphism over the identity, so it maps the fiber f^*E_p of f^*E to the fiber F_p of F. Then we can form a composition

$$\Psi \circ f^* : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E), \tag{3.8}$$

explicitly if $u \in C^\infty(\mathcal{M}; E)$, then $(\Psi \circ f^*)(u)$ is the section

$$(\Psi \circ f^*)(u)(p) = \Psi(p, u(f(p)))$$

With $F = E$ this is analogous to pull-back of q-forms.

The transformation (3.8) just defined has Schwartz kernel K supported on Γ, the graph of f. To explain this, define the projections

$$\begin{array}{ccc} & \mathcal{M}\times\mathcal{M} & \\ {}^{\pi_{\mathrm{L}}}\swarrow & & \searrow^{\pi_{\mathrm{R}}} \\ \mathcal{M} & & \mathcal{M} \end{array} \tag{3.9}$$

let $E^* \to \mathcal{M}$ be the dual bundle to $E \to \mathcal{M}$, let $E \boxtimes E^* \to \mathcal{M} \times \mathcal{M}$ be the Whitney product of $\pi_{\mathrm{L}}^* E$ and $\pi_{\mathrm{R}}^* E$. If $v \in C^\infty(\mathcal{M}; E^*)$ and $u \in C^\infty(\mathcal{M}; E)$ then $v \boxtimes u$ is the element $\pi_{\mathrm{L}}^* v \otimes \pi_{\mathrm{R}}^* u$ of $C^\infty(\mathcal{M} \times \mathcal{M}; E^* \boxtimes E)$. Then K is the element of $C^{-\infty}(\mathcal{M} \times \mathcal{M}; E \boxtimes E^*)$ such that

$$\langle K, v \boxtimes u\rangle = \int \langle v(p), (\Psi \circ f^*)(u)(p))\rangle \, d\mathfrak{m}(p).$$

for every $v \in C^\infty(\mathcal{M}; E^*)$ and $u \in C^\infty(\mathcal{M}; E)$. The statement about the support goes like this. Suppose V and U are open in $\mathcal{M}$ and $V \times U$ is disjoint from Γ. Suppose u has support in U and v has support in V. The support of $(\Psi \circ f^*)u$ is contained in

$$\{p' \in \mathcal{M} : f(p') \in \operatorname{supp}(u)\}$$

If $\langle v(p), (\Psi \circ f^*)(u)(p)\rangle \neq 0$, then $p \in \operatorname{supp} v \subset V$ and $f(p) \in \operatorname{supp}(u) \subset U$, so $(p, f(p)) \in V \times U$, contradicting $(V \times U) \cap \Gamma = \emptyset$.

We will now use local computations to get a formula for K. Let $p_0 \in \mathcal{M}$, let η_μ, $\mu = 1, \dots, r$, be a frame of E in a neighborhood U of $q_0 = f(p_0)$. Then

$$p \mapsto (f^*\eta_\mu)(p) = \big(p, \eta_\mu(f(p))\big), \quad \mu = 1, \dots, r,$$

is a frame of f^*E in the neighborhood $V = f^{-1}(U)$ of p_0. So if u is a section of E supported in U, then $u = \sum u^\mu \eta_\mu$ and

$$(f^*u)(p) = (p, \sum_\mu u_\mu(f(p))\eta_\mu(f(p))) = \sum_\mu u^j(f(p)) f^*\eta_\mu(p)$$

on V. Let now θ_ν, $\nu = 1, \dots, r$, be a frame of E over V. The homomorphism Ψ is given by

$$(f^*E)_p \ni \sum c^\mu f^*\eta_\mu(p) \mapsto \sum_\nu \big[\sum_\mu \psi^\nu_\mu(p) c^\mu\big]\theta_\nu(p) \in E_p.$$

over V, and thus

$$(\Psi \circ f^*)(u)(p) = \sum_\nu \big[\sum_\mu \psi^\nu_\mu(p) u^\mu(f(p))\big]\theta_\nu(p) \tag{3.10}$$

Let θ^ν denote the frame of E^* dual to the frame θ_ν, let v be a section of E^* supported in V, $v = \sum v_\nu \theta^\nu$ on V with smooth functions v^ν supported on V.

Then

$$\langle K, v \boxtimes u\rangle = \sum_{\nu,\mu} \int_V v_\nu(p)\psi^\nu_\mu(p)u^\mu(f(p))\, d\mathfrak{m}(p) \tag{3.11}$$

Let η^μ be the frame dual to the frame η_μ, define δ_Γ globally on $\mathcal{M}\times\mathcal{M}$ as a distribution on functions by

$$\langle \delta_\Gamma, w\rangle = \int_{\mathcal{M}} w(p, f(p))\, d\mathfrak{m}(p), \tag{3.12}$$

symbolically $\delta_\Gamma(p,p') = \delta_0(p' - f(p))$ with δ_0 the Dirac delta at 0. Then (3.11) gives

$$K = \sum \psi^\nu_\mu \pi^*_{\mathrm{L}}\theta_\nu \otimes \pi^*_{\mathrm{R}}\eta^\mu \otimes \delta_\Gamma. \tag{3.13}$$

on $V\times U$ with smooth coefficients ψ^ν_μ there.

Returning to the Atiyah-Bott theorem, the problem with the traces of the maps (3.5) becomes clear: to compute $\operatorname{Tr} f^*_q$ we need to somehow make sense of the restriction of δ_Γ to the diagonal in $\mathcal{M}\times\mathcal{M}$. Recall that the fixed points of f correspond to the points of intersection of Γ and $\operatorname{diag}(\mathcal{M})$.

4. The Lefschetz number, rewritten

We return to the Lefschetz number of a smooth $f:\mathcal{M}\to\mathcal{M}$ defined in (3.3). We aim at showing that

$$L_f = \sum_{q=0}^n (-1)^q \operatorname{Tr} f^*_q$$

with some convenient interpretation of the trace of

$$f^*_q : L^2(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}) \to L^2(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}).$$

The auxiliary datum is a Riemannian metric on $\mathcal{M}$ which we use to establish the Hodge theory of the de Rham complex, which is elliptic. With the aid of the Laplacian we will find continuous operators

$$f^*_{q,\lambda} : L^2(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}) \to L^2(\mathcal{M};\textstyle\bigwedge^q\mathcal{M}),\quad \lambda<0$$

which are genuinely of trace class and such that

$$L_f = \sum_{q=0}^n (-1)^q \operatorname{Tr} f^*_{q,\lambda}.$$

Let Δ_q be the Hodge Laplacian on forms of degree q, let G_q be the Green's operator, so

$$\Delta_q G_q = I - \Pi_q,\quad G_q\Delta_q = I - \Pi_q$$

with G_q mapping $L^2(\mathcal{M};\bigwedge^q\mathcal{M})$ into $\mathcal{D}(\Delta_q)$ as in the previous chapter and Π_q the orthogonal projection on $\ker\Delta_q$. Green's operator exists since $\mathcal{M}$ is closed (and the Laplacian is elliptic). Let $\mathrm{sp}(\Delta_q)$ denote the spectrum of Δ_q (a discrete infinite subset of $[0,\infty)$ without points of accumulation), let

$$\mathcal{E}^q_\zeta = \ker(\Delta_q - \zeta I).$$

Because of the ellipticity, this is a finite-dimensional subspace of $C^\infty(\mathcal{M}; E^q)$. In particular, $\mathcal{E}^q_0 = \mathcal{H}^q$. In general, if ζ is not an eigenvalue, then $\mathcal{E}^q_\zeta = \{0\}$. Define

$$\Pi^\zeta_q : L^2(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M}) \to L^2(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M})$$

to be the orthogonal projection on $\mathcal{E}^q_\zeta$. As a special case, $\Pi^0_q = \Pi_q$. The resolvent of Δ_q is

$$(\Delta_q - \lambda I)^{-1} = \sum_{\zeta \in \mathrm{sp}(\Delta_q)} \frac{1}{\zeta - \lambda} \Pi^\zeta_q. \tag{4.1}$$

The right hand side of this formula converges pointwise (but not in norm). If $u \in L^2(\mathcal{M}; E^q)$, then

$$\|u\|^2 = \sum_{\zeta \in \mathrm{sp}(\Delta_q)} \|\Pi^\zeta_q u\|^2$$

as a consequence of Parseval's identity. Thus

$$\sum_{\substack{\lambda \notin \mathrm{sp}(\Delta_q) \\ \zeta' \le \zeta < \zeta''}} \frac{1}{|\zeta - \lambda|^2} \|\Pi^\zeta_q u\|^2 \to 0 \text{ when } \zeta', \zeta'' \to \infty \text{ with } \zeta' < \zeta''$$

so the sequence of partial sums is Cauchy. The operator

$$(\Delta_q - \lambda I)^{-1} : L^2(\mathcal{M}; E^q) \to L^2(\mathcal{M}; E^q)$$

is continuous, a pseudodifferential operator of order -2 depending analytically on $\lambda \in \mathrm{res}(\Delta_q)$. Its Schwartz kernel is not a function (unless $n = 1$). However, the Schwartz kernel of

$$(\Delta_q - \lambda I)^{-\ell} = \sum_{\zeta \in \mathrm{sp}(\Delta_q)} \frac{1}{(\zeta - \lambda)^\ell} \Pi^\zeta_q. \tag{4.2}$$

is continuous if $\ell > n/2$, and in fact of class C^m if $\ell > n/2 + m$. Thus if ℓ is large enough, (4.2) is trace class (later we'll assume $\ell > n$). We will write K^λ_q for the Schwartz kernel, omitting the reference to ℓ, which will be assumed fixed, but large enough. With such large ℓ, the operator

$$f^*_{q,\lambda} : C^\infty(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M}) \to C^\infty(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M}), \quad f^*_{q,\lambda} = (-\lambda)^\ell f^*_q \circ (\Delta_q - \lambda I)^{-\ell}$$

has an extension $L^2(\mathcal{M}; \bigwedge^q \mathcal{M}) \to L^2(\mathcal{M}; \bigwedge^q \mathcal{M})$ which is trace class. The factor $(-1)^\ell$ will eventually allow us to argue that

$$(-1)^\ell (\Delta_q - \lambda I)^{-1} \to I$$

as $\lambda \to -\infty$ in the next section to finish the proof of the Atiyah-Bott theorem.

THEOREM 4.3. *Fix $\lambda < 0$ and ℓ large enough so that (4.2) is trace class. Then*

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{Tr} f^*_{q,\lambda}.$$

PROOF. The proof is very similar to that of Proposition 2.1. The trace of $f^*_{q,\lambda}$ is equal to that of

$$f^*_{q,\lambda,\Box} = \sum_{\zeta \in \operatorname{sp}(\Delta_q)} \Pi_q^\zeta \circ f^*_{q,\lambda} \circ \Pi_q^\zeta.$$

Here, for the purposes of the trace, we may view $\Pi_q^\zeta \circ f^*_{q,\lambda} \circ \Pi_q^\zeta$ indistinctly as an operator on $L^2(\mathcal{M}; \bigwedge^q \mathcal{M})$ or on $\mathcal{E}^q_\zeta$. We will first argue that

$$\operatorname{Tr}(\Pi_q^\zeta \circ f^*_{q,\lambda} \circ \Pi_q^\zeta) = \operatorname{tr} \mathfrak{f}_q \quad \text{when } \zeta = 0 \tag{4.4}$$

(where $\mathfrak{f}_q : H^q_{\mathrm{dR}}(\mathcal{M}) \to H^q_{\mathrm{dR}}(\mathcal{M})$ as in Theorem 3.4), then show that

$$\sum_{q=0}^{n} (-1)^q \operatorname{Tr}(\Pi_q^\zeta \circ f^*_{q,\lambda} \circ \Pi_q^\zeta) = 0 \quad \text{if } \zeta > 0. \tag{4.5}$$

With fixed $\lambda < 0$ and $\zeta \in \bigcup_q \operatorname{sp}(\Delta_q)$, let

$$\mathfrak{f}^*_{q,\zeta} = \Pi_q^\zeta \circ f^*_{q,\lambda} \circ \Pi_q^\zeta$$

(where on the left hand side we omitted the reference to λ). Directly from the definitions we have

$$\mathfrak{f}^*_{q,\zeta} = \frac{(-\lambda)^\ell}{(\zeta - \lambda)^\ell} \Pi_q^\zeta \circ f^*_q \circ \Pi_q^\zeta$$

Thus for (4.4) we just observe that the diagram

$$\begin{array}{ccc} \mathcal{H}^q & \xrightarrow{\mathfrak{f}^*_{q,0}} & \mathcal{H}^q \\ \theta \downarrow & & \downarrow \theta \\ H^q_{dR}(\mathcal{M}) & \xrightarrow{\mathfrak{f}_q} & H^q_{dR}(\mathcal{M}) \end{array}$$

commutes and θ is an isomorphism.

The proof of (4.5) is longer. We begin by noting that d maps $\mathcal{E}^q_\zeta$ into $\mathcal{E}^{q+1}_\zeta$, indeed

$$d \circ \Pi_q^\zeta = \Pi_{q+1}^\zeta \circ d \tag{4.6}$$

To see this, let $u \in \mathcal{E}^q_\zeta$, so that $\Delta_q u = \zeta u$. Since $\Delta_{q+1} du = d\Delta_q u = d(\zeta)u = \zeta du$, we have $du \in \mathcal{E}^{q+1}_\zeta$. In particular there is a complex

$$\cdots \to \mathcal{E}^{q-1}_\zeta \xrightarrow{d} \mathcal{E}^q_\zeta \xrightarrow{d} \mathcal{E}^{q+1}_\zeta \to \cdots \tag{4.7}$$

defined by the operators d. Next we observe that as a consequence of (4.6) and of the fact that $d \circ f_q^* = f_{q+1}^* \circ d$,

$$d \circ \mathfrak{f}_{q,\zeta}^* = \mathfrak{f}_{q+1,\zeta}^* \circ d,$$

as follows:

$$\begin{aligned} d \circ (\Pi_q^\zeta \circ f_q^* \circ \Pi_q^\zeta) &= \Pi_{q+1}^\zeta \circ d \circ f_q^* \circ \Pi_q^\zeta \\ &= \Pi_{q+1}^\zeta \circ f_{q+1}^* \circ d \circ \Pi_q^\zeta \\ &= \Pi_{q+1}^\zeta \circ f_{q+1}^* \circ \Pi_{q+1}^\zeta \circ d. \end{aligned}$$

Thus the maps $\{\mathfrak{f}_{q,\zeta}^*\}_q$ define a cochain map

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{E}_\zeta^{q-1} & \xrightarrow{d} & \mathcal{E}_\zeta^q & \xrightarrow{d} & \mathcal{E}_\zeta^{q+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mathfrak{f}_{q-1,\zeta}^*} & & \downarrow{\scriptstyle \mathfrak{f}_{q,\zeta}^*} & & \downarrow{\scriptstyle \mathfrak{f}_{q+1,\zeta}^*} & & \\ \cdots & \longrightarrow & \mathcal{E}_\zeta^{q-1} & \xrightarrow{d} & \mathcal{E}_\zeta^q & \xrightarrow{d} & \mathcal{E}_\zeta^{q+1} & \longrightarrow & \cdots \end{array} \tag{4.8}$$

Thus there is a Lefschetz number for the complex (4.7) defined by the cochain map (4.8). By Proposition 2.1,

$$\sum_{q=0}^{n} (-1)^q \operatorname{Tr} \mathfrak{f}_{q,\zeta}^* \tag{4.9}$$

is equal to the alternating sum of the traces of the induced operators on the cohomology groups of the complex (4.7). However, this complex is acyclic, as we proceed to show.

Let G_q denote, as before, the Green's operator for Δ_q. The formula

$$u = \Pi_q^0 + d^\star G_{q+1} du + d G_{q-1} d^\star u$$

applied to $u \in \mathcal{E}_\zeta^q$ with $du = 0$ gives

$$u = d G_{q-1} d^\star u. \tag{4.10}$$

because $\Pi_q^0 u = 0$ if $\zeta \neq 0$. Since $d^\star \circ \Delta_q = \Delta_{q-1} \circ d^\star$, $d^\star u \in \mathcal{E}_\zeta^{q-1}$. The operator G_{q-1} is given by (4.1) with $\lambda = 0$ (and $q-1$ in place of q), so G_{q-1} maps $\mathcal{E}_\zeta^{q-1}$ to itself. Thus $G_{q-1} d^\star u \in \mathcal{E}_\zeta^{q-1}$ if $u \in \mathcal{E}_\zeta^q$. Hence u is exact in the complex (4.7). Consequently, (4.9) vanishes: (4.5) holds.

This completes the proof of Theorem (4.3). □

5. Taking the limit

The next step is to determine analytically

$$\lim_{\lambda \to -\infty} \operatorname{Tr} f_{q,\lambda}^* \tag{5.1}$$

for any q. This will then give an analytic formula for L_f via Theorem 4.3.

We will work in the more general context described at the end of Section 3, see (3.8) in particular. We begin by establishing a convenient formula for the trace of an operator

$$A : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

with sufficiently smooth Schwartz kernel, at least continuous. As in Section 3, $E \to \mathcal{M}$ is a vector bundle with Hermitian metric h and $\mathfrak{m}$ is a smooth positive density on $\mathcal{M}$. The inner product on $L^2(\mathcal{M}; E)$ is given by

$$(u, v) = \int_{\mathcal{M}} h_p\big(u(p), v(p)\big)\, d\mathfrak{m}(p).$$

Let K be the Schwartz kernel of A. Thus

$$K \in C^\ell(\mathcal{M} \times \mathcal{M}; E \boxtimes E^*)$$

(see (3.9) for the notation) and

$$Au(p) = \int K(p, p')u(p')\, d\mathfrak{m}(p'),\ u \in C^\infty(\mathcal{M}; E),$$

where (for fixed p), $k(p, p')u(p') \in E_p$ means the pairing of the right factor of $K(p, p') \in E_p \otimes E^*_{p'}$ with $u(p') \in E_{p'}$. In particular $K(p, p) \in E_p \otimes E^*_p$. But the latter space is isomorphic to $\mathrm{Hom}(E_p, E_p)$, so we can define $\operatorname{tr} K(p, p)$. Since we are assuming K is smooth enough, this gives a continuous function $p \to \operatorname{tr} K(p, p)$. Thus one has the following well known formula which we will prove for the sake of completeness:

Lemma 5.2. *Let $\iota : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the diagonal map, $\iota(p) = (p, p)$, let $\operatorname{tr} : \mathrm{Hom}(E, E) \to E$ be the fiberwise trace map. The trace of A is*

$$\mathrm{Tr}(A) = \int_{\mathcal{M}} \operatorname{tr} \iota^* K(p)\, d\mathfrak{m}(p).$$

We fix some notation before proving the lemma. Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ be a differential operator of order 2, elliptic and positive: $(Pu, u) \geq 0$ for all $u \in C^\infty(\mathcal{M}; E)$. The spectrum of P is discrete since $\mathcal{M}$ is compact and P is elliptic, and nonnegative, since P is positive. In the proof of the lemma, also later, we will use an orthonormal basis $\{\phi_k\}_{k=1}^\infty$ of eigenvectors of P. In the lemma this is just for specificity; the trace is independent of the orthonormal basis.

Proof. The trace of A is

$$\sum_k (A\phi_k, \phi_k) = \sum_k \int h_p\Big(\int K(p, p')\phi_k(p')\, d\mathfrak{m}(z), \phi_k(p)\Big)\, d\mathfrak{m}(x). \tag{5.3}$$

We have

$$h_p\Big(\int K(p,p')\phi_k(p')\,d\mathfrak{m}(p'),\phi_k(p)\Big) = \int h_p\big(K(p,p')\phi_k(p'),\phi_k(p)\big)\,d\mathfrak{m}(p')$$
$$= \Big[\int h_p\big(K(p,p'),\phi_k(p)\big)\,d\mathfrak{m}(p)\Big]\phi_k(p')$$

where the first equality uses the continuity of the inner product and the convergence of the Riemann sums of the integral. In the second equality, the integral of $h_x\big(K(p,p'),\phi_k(p)\big)$ yields an element of $E^*_{p'}$ which is then paired with $\phi_k(p') \in E_{p'}$; the result is a scalar (depending on p') which we can integrate as we do next. Replacing in (5.3), changing the order of integration and commuting the sum and an integral we get

$$\sum_k (A\phi_k,\phi_k) = \int \sum_k \Big[\int h_p\big(K(p,p'),\phi_k(p)\big)\,d\mathfrak{m}(p)\Big]\phi_k(p')\,d\mathfrak{m}(p') \tag{5.4}$$

The integral in brackets is the k-th Fourier coefficient of $K(\cdot,p')$ with respect to the basis ϕ_k. Thus

$$K(p,p') = \sum_k \phi_k(p)\otimes\int h_p\big(K(p'',p'),\phi_k(p'')\big)\,d\mathfrak{m}(p'')$$

for any $(p,p') \in \mathcal{M}\times\mathcal{M}$, and in particular

$$K(p,p) = \sum_k \phi_k(p)\otimes\int h_x\big(K(p'',p),\phi_k(p'')\big)\,d\mathfrak{m}(p''). \tag{5.5}$$

The integrand in (5.4) is obtained from this by pairing the left factor of $K(p,p) \in E_p \otimes E^*_p$ with its right factor. Let $\eta_1,\dots,\eta_r$ be a frame of E over some open set, let $\eta^1,\dots,\eta^r$ be the dual frame. Then

$$K(p,p) = \sum_{\mu,\nu} \kappa^\nu_\mu(p)\eta_\nu(p)\otimes\eta^\mu(p)$$

and the pairing is $\sum_{\mu,\nu}\kappa^\nu_\mu(p)\eta^\mu(p)\big(\eta_\nu(p)\big) = \sum_\nu \kappa^\nu_\nu(p) = \operatorname{tr} K(p,p)$. Using this in (5.5) yields

$$\operatorname{Tr} A = \int_{\mathcal{M}} \operatorname{tr} K(p,p)\,d\mathfrak{m}(p).$$

□

Before continuing we will discuss the continuity of the Schwartz kernel of $A_\lambda = (P-\lambda)^{-\ell}$ when ℓ is large enough; here and below we assume $\lambda < 0$. Thus

$$A_\lambda = \sum_{\zeta\in\operatorname{sp}(P)} \frac{(-\lambda)^\ell}{(\zeta-\lambda)^\ell}\Pi^\zeta, \quad \lambda < 0. \tag{5.6}$$

The notation follows the one used in (4.2). With the orthonormal basis introduce above and letting ζ_k be the corresponding eigenvalues we have

$$A_\lambda v = \sum_k \frac{(-\lambda)^\ell}{(\zeta_k - \lambda)^\ell}(v, \phi_k)\phi_k \tag{5.7}$$

for $v \in C^\infty(\mathcal{M}; E)$. Recall that $\lambda < 0$. We do not include ℓ in the notation for this operator because it is some fixed sufficiently large integer. Let $\jmath_p : E_p \to E_p^*$ denote the operator such that

$$h_p(a, b) = \langle \jmath_p(b), a\rangle.$$

Clearly, $\jmath_p$ is conjugate-linear and by the Riesz representation theorem, it is surjective. We use this to define a Hermitian metric on E^* so that $h_p^*(j_p u, j_p v) = h(v, u)$ if $u, v \in E_p$. If v is a section of E, then $p \mapsto \jmath_p(v(p))$ is a section of E^* which we denote $\mathrm{J}(v)$. With this notation we have

$$(u, \phi_k) = \int \langle \mathrm{J}(\phi_k), u\rangle \, d\mathfrak{m}(p)$$

and so the the Schwartz kernel of A_λ in (5.7) is

$$K_\lambda = \sum_k \frac{(-\lambda)^\ell}{(\zeta_k - \lambda)^\ell} \pi_{\mathrm{L}}^* \phi_k \otimes_{\mathrm{R}}^* \mathrm{J}(\phi_k) \tag{5.8}$$

with convergence in $C^{-\infty}(\mathcal{M} \times \mathcal{M}; E \boxtimes E^*)$.

We will now determine the regularity of K_λ. From Weyl's estimate we have

$$\zeta_k \sim c_0 k^{2/n} \quad \text{as } k \to \infty$$

for some c_0, see Seeley [**12**, p. 291]. Since P is elliptic positive of order 2, we can define the L^2-based Sobolev spaces of sections of E, $\mathrm{H}^s(\mathcal{M}; E)$, using the norm

$$\|u\|_s = \|(A + I)^{s/2} u\|$$

where the norm on the right is that of $L^2(\mathcal{M}; E)$. From the Sobolev embedding lemma we have that $\mathrm{H}^s(\mathcal{M}; E) \subset C^m(\mathcal{M}; E)$ if $s > n/2 + m$ with

$$\|u\|_{C^m} \le c_{s,m} \|u\|_s$$

for some c_s. Here

$$\|u\|_{C^0} = \sup_{p \in \mathcal{M}} \sqrt{h_p(u(p), u(p))}$$

while for $m > 0$ we use some connection to differentiate u and the compute the maximum of the pointwise norms. The Sobolev estimate gives in particular

$$\|\phi_k\|_{C^0} \le c_{s,m}(1 + \zeta_k)^{s/2}$$

$$\|\pi_{\mathrm{L}}^* \phi_k \otimes \pi_{\mathrm{R}}^* \mathrm{J}(\phi_k)\|_{C^0} \le C(1 + \zeta_k)^s$$

with some constant $C > 0$. Thus

$$\Big\| \frac{(-\lambda)^\ell}{(\zeta_k - \lambda)^\ell} \pi_{\mathrm{L}}^* \phi_k \otimes_{\mathrm{R}}^* \mathrm{J}(\phi_k) \Big\|_{C^0} \leq C(1 + \zeta_k)^{s-\ell} \sim C k^{2(s-\ell)/n}$$

for $\lambda < -1$. If $s = n/2 + \varepsilon$, $\varepsilon > 0$, then the uniform convergence of the series (5.8) will be guaranteed if $\ell > n + \varepsilon$. We will assume ℓ satisfies this condition. Note in passing that the pseudodifferential operator $(A+I)^{-\ell'}$ is of order $-2\ell'$, so the condition $\ell' > n/2$ is sufficient to get the continuity of K_λ; nevertheless we will assume $\ell > n$ to ensure convergence of the series.

As should be expected, K_λ converges in $C^{-\infty}(\mathcal{M} \times \mathcal{M}; E \boxtimes E^*)$ to the Schwartz kernel of the identity map, which is supported on the diagonal. We shall omit the details of the proof of this fact.

Let now $f : \mathcal{M} \to \mathcal{M}$ be smooth, let $\Psi : f^*E \to E$ be a homomorphism. Then as discussed in Section 3, we have an induce map $\Psi \circ f^* : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$, see (3.8), whose Schwartz kernel is supported on the graph $\Gamma = \{(p, f(p)) : p \in \mathcal{M}\}$ of f.

Using the orthonormal basis $\{\phi_k\}$. Let $v \in C^\infty(\mathcal{M}; E)$, $v = \sum_k (v, \phi_k)\phi_k$. Then

$$(\Psi \circ f^* v)(p) = \sum_k (v, \phi_k)\big((\Psi \circ f^*)(\phi_k)\big)(p) = \langle \sum_k (\Psi \circ f^*)(\phi_k)(p) \otimes \mathrm{J}\,\phi_k, v\rangle,$$

thus the Schwartz kernel of $\Psi \circ f^*$ is

$$K_{\Psi\circ f^*} = \sum \pi_{\mathrm{L}}^* (\Psi \circ f^*)(\phi_k) \otimes \pi_{\mathrm{R}}^* \mathrm{J}\,\phi_k.$$

Using the last version of $K_{\Psi\circ \mathfrak{f}^*}$, the Schwartz kernel of $A_\lambda \circ [\Psi \circ f^*]$ is thus

$$K_{\Psi\circ f^*,\lambda} = \sum_{\zeta \in \operatorname{sp} P} \sum_k \frac{(-\lambda)^\ell}{(\zeta - \lambda)^\ell} \pi_{\mathrm{L}}^* \Pi^\zeta\big((\Psi \circ f^*)(\phi_k)\big) \otimes \pi_{\mathrm{R}}^* \mathrm{J}\,\phi_k.$$

The supremum of the pointwise norms $\|\Pi^\zeta\big((\Psi \circ f^*)(\phi_k)\big)(p)\|$ can be estimated as we did above just for the ϕ_k. This will give the uniform convergence of the series for any $\lambda < 0$, so $K_{\Psi\circ f^*,\lambda}$ is continuous. Thus we can take the trace of $A_\lambda \circ [\Psi \circ f^*]$:

$$\operatorname{Tr}(A_\lambda \circ [\Psi \circ f^*]) = \int_{\mathcal{M}} \operatorname{tr} \iota^* K_{\Psi\circ f^*,\lambda}(p)\, d\mathfrak{m}(p).$$

Furthermore $K_{\Psi\circ f^*,\lambda}$ converges in $C^{-\infty}(\mathcal{M} \times \mathcal{M}; E \boxtimes E^*)$ to the Schwartz kernel of $\Psi \circ f^*$, which is supported on Γ. In particular the restriction of $\iota^* K_{\Psi\circ f^*,\lambda} \to 0$ to the complement of Fix converges to 0.

Proposition 5.9. *Assume that the fixed points of f are simple. Then the sections $\iota^* K_{\Psi\circ f^*,\lambda}$ of $E \otimes E^*$ converge in $C^{-\infty}(\mathcal{M}; E \times E^*)$ as $\lambda \to -\infty$ to a distribution supported on* Fix.

The important statement is the convergence of $\iota^* K_{\Psi\circ f^*,\lambda}$.

PROOF. The operator $\Psi \circ f^*$ is a Fourier integral operator associated with Λ, the complement of the zero section of the conormal bundle to Γ. Since A_λ is pseudodifferential operator, also $A_\lambda \circ [\Psi \circ f^*]$ is a Fourier integral operator associated with Λ. As $\lambda \to -\infty$, $A_\lambda \to I$ so $K_{\psi\circ f^*,\lambda} \to K_{\psi\circ f^*}$ in the space $C^{-\infty}_\Lambda(\mathcal{M}\times\mathcal{M}; E\boxtimes E^*)$, the space whose elements are generalized sections of with wavefront set contained in Λ (see [**5**, Guillemin-Sternberg, p. 333] for the definition). The fact that the fixed points of f are simple is equivalent to the condition that Λ is disjoint from $N^*(\mathrm{diag}) \setminus 0$. It follows from Proposition 3.7 in [**5**] that $\iota^* K_{\Psi\circ f^*}$ converges in $C^{-\infty}_{\iota^*\Lambda}(\mathcal{M}; E\otimes E^*)$. (The statements in [**5**] use projective wavefront set, but our use here is valid.) □

We now identify $\iota^* K_{\Psi\circ f^*}$. To this end we will rewrite $K_{\Psi\circ f^*}$ along the lines of (3.13).

Giving the homomorphism $\Psi : f^*E \to E$ is the same as giving a homomorphism from the part of $\pi_{\mathrm{R}}^* E$ over Γ to the part of $\pi_{\mathrm{L}}^* E$ also over Γ:

$$\hat{\Psi} : \pi_{\mathrm{R}}^* E_\Gamma \to \pi_{\mathrm{L}}^* E_\Gamma.$$

Namely giving the linear map $\Psi(p, f(p)) : (f^*E)_p \to E_p$ is the same as giving the map $\hat{\Psi}(p, f(p)) : E_{f(p)} \to E_p$. Extend $\hat{\Psi}$ to a smooth homomorphism

$$\hat{\Psi} : \pi_{\mathrm{R}}^* E \to \pi_{\mathrm{L}}^* E.$$

and let $\hat{\Psi}$ also denote the section of $E\boxtimes E^*$ this extension induces. Then the Schwartz kernel of $\Psi\circ f^*$ is

$$K_{\Psi\circ f^*} = \hat{\Psi}\otimes\delta_\Gamma \tag{5.10}$$

where δ_Γ is given by (3.12).

It follows that $\iota *^{K}_{\Psi\circ f^*} = \iota^*\hat{\Psi}\otimes \iota^{\delta}_{\Gamma}$. But we have already determined that

$$\iota^*\delta_\gamma = \sum_{p\in\mathrm{Fix}} \frac{1}{|\det(df_p - I)|}\delta_p.$$

Thus

THEOREM 5.11. *If f has only simple fixed points, then*

$$\lim_{\lambda\to-\infty} \iota^*\mathcal{K}_{\Psi\circ f^*,\lambda} = \sum_{p\in\mathrm{Fix}} \frac{1}{|\det(df_p - I)|}\iota^*\hat{\Psi}\otimes\delta_p$$

in $C^{-\infty}(\mathcal{M}; E\otimes E^*)$. Consequently

$$\lim_{\lambda\to-\infty} \operatorname{tr}\iota^*\mathcal{K}_{\Psi\circ f^*,\lambda} = \sum_{p\in\mathrm{Fix}} \frac{1}{|\det(df_p - I)|}\iota^*\operatorname{tr}(\hat{\Psi})\otimes\delta_p$$

and so

$$\lim_{\lambda\to-\infty} A_\lambda\circ[\Psi\circ f^*] = \sum_{p\in\mathrm{Fix}} \frac{1}{|\det(df_p - I)|}\iota^*\operatorname{tr}(\hat{\Psi}(p)).$$

Applied to specific case of (5.1) we conclude

$$\lim_{\lambda \to -\infty} \operatorname{Tr} f^*_{q,\lambda} = \sum_{p \in \mathrm{Fix}} \frac{1}{|\det(df_p - I)|} \iota^* \operatorname{tr} df(p).$$

Since

$$L_f = \sum_{q=0}^{n} (-1)^q \operatorname{Tr} f^*_{q,\lambda}$$

is independent of λ,

$$L_f = \sum_{p \in \mathrm{Fix}} \sum_{q=0}^{n} (-1)^q \frac{1}{|\det(df_p - I)|} \iota^* \operatorname{tr} df(p).$$

With this we have completed the proof of Theorem 3.4

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