Universidade Federal de São Carlos
Departamento de Matemática

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Falará sobre

Elliptic complexes: Hodge theory, Lefschetz formula

The aim of this short course is to give a sense of the main ideas underlying the topics in the title.
Throughout the three lectures, the underlying differential-topological objects will be a compact manifold M without boundary, vector bundles $E^k \to M$, and an elliptic complex

$$C^\infty(M;E^0) \xrightarrow{P_0} C^\infty(M;E^1) \xrightarrow{P_1} \dots \xrightarrow{P_{m-1}} C^\infty(M;E^m) \tag{$\dagger$}$$

of first order differential operators. Precise definitions will be given in time throughout the lectures as needed. This unfortunately somewhat extended abstract is intended to give some context.
The E^k are vector bundles over M, which means they are, locally, products $U \times \mathbb{C}^{d_k}$ with $U \subset M$ open. Also locally, $C^\infty(M;E^k)$ consist of functions $\phi : U \to \mathbb{C}^{d_k}$, and the P_k are matrices of first order partial differential operators. That ($\dagger$) is a complex means that $P_{k+1} \circ P_k = 0$ for all k. It therefore has associated cohomology spaces,

$$H^k_E(M) = \operatorname{rg} P_{k-1} / \ker P_k.$$

They represent the failure of being able to solve $P_k\psi = \phi$ when $P_k\phi = 0$.
Each space $C^\infty(M;E^k)$ will have an inner product with respect to which one can construct the adjoint $P_k^* : C^\infty(M;E^{k+1}) \to C^\infty(M;E^k)$ of P_k. Ellipticity of the complex means that the operators $\Box_k = P_{k-1}P_{k-1}^* + P_k^*P_k$ satisfy the inequality

$$\|\phi\|_2 \le C(\|\Box_k\phi\|_0 + \|\phi\|_0)$$

for some C. Here $\|\phi\|_2$ means the sum of the norms of all derivatives of ϕ up to second order. The norms on the right are with the inner product itself.
Examples of elliptic complexes are the de Rham complex (closely related to the topology of M) and the Dolbeault complex in complex geometry. I plan to give other examples subject to time constraints.
Assuming ellipticity of the complex, Hodge theory establishes that $H^k_E(M)$ is canonically isomorphic to $\ker \Box_k$. (By general elliptic theory the kernel is finite-dimensional.) Furthermore, one has the Hodge decomposition

$$C^\infty(M;E^k) = \ker \Box_k \oplus P_{k-1}C^\infty(M;E^{k-1}) \oplus P_k^*C^\infty(M;E^{k+1})$$

which is an orthogonal decomposition.

The classical Lefschetz formula involves a continuous map $f : M \to M$ with which one defines the Lefschetz number, L_f, of f. This number is an invariant of the homotopy class of f. For the purposes of this outline, L_f can be defined when f is smooth and the the complex (†) is the de Rham complex. In this case, f induces maps $f^* : C^\infty(M; E^k) \to C^\infty(M; E^k)$ such that $df^k = f^* d$, so maps

$$\mathfrak{f}_k : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}.$$

Then $L_f = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(\mathfrak{f}_k)$. The relevancy of the Lefschetz number lies in that $L_f \neq 0$ implies f has a fixed point.

The lectures will begin with recalling the de Rham and Dolbeault complexes, thus ensuring some concreteness to what will follow. Additional background will be provided throughout the lectures as needed. The plan for the main topics is to first discuss Hodge theory, then the Lefschetz number. In the last lecture we will prove a theorem of Atiyah and Bott that gives a formula for the Lefschetz number of f in the special case that $\Gamma(f) \subset M \times M$, the the graph of f, intersects the diagonal $\Delta \subset M \times M$ transversely. The formula involves restricting a distribution supported on $\Gamma(f)$ to Δ, which is a good opportunity to exhibit the usefulness of the notion of wave front set of a distribution. The proof of the Atiyah-Bott formula for the Lefschetz number uses Hodge theory.

Referências

[1] Atiyah, M. F., Bott, R., *A Lefschetz fixed point formula for elliptic complexes. I.*, Ann. of Math. **86** (1967), 374–407.

[2] Hodge, W. V. D. *The theory and applications of harmonic integrals*, 2d ed. Cambridge University Press, 1952.

[3] Hörmander, L., *Linear partial differential operators* Third revised printing. Die Grundlehren der mathematischen Wissenschaften, Band 116 Springer-Verlag New York Inc., New York 1969.

[4] Lefschetz, S., *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. 28 (1926), no. 1, 1–49.

Data: 11, 12 e 13 de janeiro de 2023
Horário: 14h15 - 16h
Local: Auditório do DM