Impenetrability of Aharonov-Bohm solenoids: proof of norm resolvent convergence

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Abstract

Consider bounded solenoids in the space or planar approximate models of infinitely long solenoids. It is proved that the solenoid impenetrability, by means of a sequence of Hamiltonians with diverging potentials, converges in the norm resolvent sense to the usual Aharonov-Bohm model with Dirichlet boundary condition. The framework is that of nonrelativistic quantum mechanics.

Keywords: Aharonov-Bohm Hamiltonian, norm resolvent convergence, bounded solenoids.

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1 Introduction

The Aharonov-Bohm (AB) effect is an authentic quantum phenomenon and was first considered in [19, 18, 1]. There are many discussions about the justification of the famous Aharonov-Bohm Hamiltonian and interpretations (see, for instance, [2, 7, 8, 9, 11, 15, 20, 22, 26, 28, 30] and references therein); the questions are particularly interesting for the more realistic case of solenoids $\mathcal{S}$ of radii greater than zero. Sometimes it involves the quantization in multiply connected regions, and the main points to be clarified

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are the presence of the vector potential $\mathbf{A}$ in the operator action (occasionally in regions with no magnetic field), and the (natural) choice of Dirichlet boundary conditions at the solenoid border.

Apparently, Kretzschmar [24] was the first one to propose a model of the solenoid impenetrability via increasing positive potentials V_n so that $V_n \to \infty$ precisely on the interior $\mathcal{S}^\circ$ of the solenoid (see also comments in [7]), but a mathematical rigorous approach was only published many years later by Magni and Valz-Gris [26], whose main result was a proof a strong resolvent convergence.

In [13], the present authors have considered an increasing sequence of cylindrical solenoids of finite length together with the same shielding process of [26], and it was proved that both limits commute and converge, in the strong resolvent sense, to the usual AB Hamiltonian $H_{\rm AB}$ given by

$$\begin{aligned} \operatorname{dom} H_{\rm AB} &= \mathcal{H}^2(\mathcal{S}') \cap \mathcal{H}^1_0(\mathcal{S}'), \\ H_{\rm AB} &= \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2, \quad \mathbf{p} = -i\nabla, \end{aligned}$$

where $\mathcal{S}'$ denotes the exterior region of the solenoid. Note that the Dirichlet boundary conditions are selected (due to the space $\mathcal{H}^1_0(\mathcal{S}')$ in the limit operator domain). c and q are the light speed and particle electric charge, respectively. Recently [15], other boundary conditions at the cylindrical solenoid that are compatible with quantum mechanics, in the sense that they give rise to self-adjoint Hamiltonians, were investigated, and the scattering of the AB Hamiltonian with Robin boundary conditions has been considered and compared with the traditional Dirichlet case.

The goal of this note is to prove that, in some cases of interest, one has the norm resolvent convergence instead of just the strong one. This is important since spectral and eigenfunctions convergences are usually guaranteed only if norm resolvent convergence holds. The important assumption, necessary to get such norm convergence, is that the region border $\mathcal{S} = \partial\mathcal{S}^\circ$ is a bounded set, since some arguments involving compactness will be evoked. Hence, the results obtained below apply, in particular, to any finite cylindrical solenoid as well as to toroidal solenoids in $\mathbb{R}^3$ [27], and, due to the symmetry along the vertical coordinate, to approximate models in $\mathbb{R}^2$ of infinitely long cylindrical solenoids.

The toroidal case is particularly important, since for the physics community this setting is considered to present the only trustful experimental evidence of the AB effect [27, 31, 32, 10]; in case of straight solenoids there is a leakage of magnetic flux and the experiments are not generally accepted. Note that the recent experiments reported in [10] show that the experimental results of [31, 32], with toroidal solenoids, can not be explained by the action of a force. It is also worth mention the rigorous proofs [4, 5, 6] that the original Ansatz of Aharonov and Bohm [1], adapted to the toroidal case, can be approximated by real solutions of the Schrödinger equation. Hence, it is the three-dimensional case with some bounded solenoids the physically important one, and this setting is included in Theorem 1 below.

In the next section the results of this work are precisely stated, whereas their proofs appear in Section 3. Concluding remarks are discussed in the last section of this note.

2 Statement of the Results

If $\lambda \in \mathbb{C}$ and T is a linear operator on a Hilbert space, $R_\lambda(T) = (T - \lambda \mathbf{1})^{-1}$ denotes the resolvent operator of T at λ. Let $\mathcal{S}^\circ$ denote a (nonempty) bounded open subset of $\mathbb{R}^d$, $d = 2, 3$, $\mathcal{S}$ its boundary, $\mathcal{S}'$ its exterior, that is, $\mathcal{S}' = \mathbb{R}^d \setminus (\mathcal{S} \cup \mathcal{S}^\circ)$, and its closure will be indicated by $\tilde{\mathcal{S}} = \mathcal{S}^\circ \cup \mathcal{S}$, whereas the closure of its exterior by $\hat{\mathcal{S}} = \mathcal{S}' \cup \mathcal{S}$.

Let $\mathbf{B}$ denote a (bounded) magnetic field on $\mathbb{R}^d$ and $\mathbf{A}$ a (bounded) magnetic potential so that $\mathbf{B} = \nabla \times \mathbf{A}$; furthermore, assume that $\mathrm{div}\mathbf{A} \in \mathrm{L}^2_{\mathrm{loc}}$, and so H_n below is essentially self-adjoint when defined on $\mathrm{C}_0^\infty(\mathbb{R}^d)$ [25]. In $\mathbb{R}^2$ it is supposed that the only possible nonzero component of $\mathbf{B}$ is the one perpendicular to the plane, so that, in fact, the magnetic field is reduced to a scalar function in this case.

As already indicated, think of $\mathcal{S}$ as a finite solenoid of general shape, and the most interesting situation here is when $\mathbf{B}$ vanishes on $\mathcal{S}'$ (but not necessarily on $\mathcal{S}^\circ$), but the results below are not restricted to this setting. Particularly relevant situations are: (1) $\mathcal{S}$ is a torus in $\mathbb{R}^3$, and (2) $\mathcal{S}$ is a circle in $\mathbb{R}^2$.

In order to prevent a quantum particle to penetrate the interior region $\mathcal{S}^\circ$, as in the Aharonov-Bohm effect, consider an increasing sequence of potential barriers $V_n(x) = n\chi(x)$, where $\chi = \chi_{\mathcal{S}^\circ}$ is the characteristic function of this set, that is, $\chi(x) = 1$ if $x \in \mathcal{S}^\circ$ and $\chi(x) = 0$ otherwise. The question is: What does happen with the particle Hamiltonian as $n \to \infty$? This is a way to model the impenetrability of the region $\mathcal{S}^\circ$ [24, 26], and so to find the quantum Hamiltonian describing the particle motion in a magnetic field and in $\mathbb{R}^d$ with a hole $\mathcal{S}^\circ$.

The Hamiltonian of a quantum particle in this situation is given by the self-adjoint operator, for $n \geq 0$ and $d = 2, 3$,

$$\mathrm{dom}\, H_n = \mathcal{H}^2(\mathbb{R}^d), \quad H_n = \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + V_n = H_0 + V_n,$$

where $\mathcal{H}^2$ denotes a usual Sobolev space in $\mathrm{L}^2(\mathbb{R}^d)$, i.e., the domain of the free Hamiltonian $-\Delta$ (the negative laplacian). Note that the operator H_0 is implicitly defined by this relation. Recall that, under the above hypotheses on $\mathbf{A}$, $\mathrm{C}_0^\infty(\mathbb{R}^d)$ is a core of H_n [25].

The impenetrable limit $n \to \infty$ was rigorously considered in [26] for a particular region, however the same arguments hold in a much more general situation. By using the Kato-Robinson theorem [12, 14], it was shown that H_n converges to H_{AB} in the strong resolvent sense as $n \to \infty$, and since elements of $\mathcal{H}_0^1(\mathcal{S}')$ vanish at the solenoid border (in the sense of Sobolev traces), Dirichlet boundary conditions have showed up in this limit. Note, however, that H_n and H_{AB} act in different Hilbert spaces; since $V_n \to \infty$ in $\mathcal{S}^\circ$, one restricts the resolvent convergence to the subspace of functions that vanish a.e. in $\mathcal{S}^\circ$, and if P' denotes the projection onto the subspace $\mathrm{L}^2(\mathcal{S}')$ of $\mathrm{L}^2(\mathbb{R}^d)$, the resolvent convergence will be understood in the sense $R_{-\lambda}(H_n)P' \to R_{-\lambda}(H_{\mathrm{AB}})P'$, for $\lambda > 0$ (see [12, 26, 29] for details); but in order to simplify the notation, here it will be simply written $R_{-\lambda}(H_n) \to R_{-\lambda}(H_{\mathrm{AB}})$. This construction works since H_n is a monotonically increasing sequence of self-adjoint operators, and so the strong resolvent convergent is directly related to the convergence of a monotonic family of quadratic forms [12, 29]. A discussion on the nontrivial relation between resolvents and quadratic forms for general bounded from below sequences of self-adjoint operators, with some applications to singular quantum limits, can be found in [16].

The main goal of this note is to show that, in case of bounded regions $\mathcal{S}$, the above strong convergence can be substantially improved, that is, it will be proven that the norm resolvent convergence is in effect.

Theorem 1. *Let $\mathcal{S}$ be a (nonempty) bounded open subset of $\mathbb{R}^d$, $d = 2, 3$, and H_n and H_{AB} as above. Then, H_n converges to H_{AB} in the norm resolvent sense as $n \to \infty$.*

3 Proofs

3.1 Preliminary Lemmas

In this subsection two technical results that will be used to prove Theorem 1 will be discussed. Such results are based on references [3, 25]. Without loss set $c = 1 = q$.

Let B and C be bounded operators on $\mathrm{L}^2(\mathbb{R}^d)$ and denote by $\mathrm{B}_\infty^\infty(\mathbb{R}^d)$ the set of bounded Borel functions that vanish at infinity (with the sup norm). Write $B \dot{\le} C$ to indicate that $|B\psi| \le C|\psi|$ for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, that is, $|B\psi|(x) \le (C|\psi|)(x)$. The following results will be used ahead, and references to their proofs are indicated:

(i) If $B \dot{\le} C$ and C is a compact operator, then B is also compact (Theorem 2.2 in [3]).

(ii) If $\lambda > 0$, $\mathbf{a} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d$ and the function $V \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is nonnegative, then
$$\left[(-i\nabla - \mathbf{a})^2 + V + \lambda\right]^{-1} \dot{\le} (-\Delta + \lambda)^{-1}$$
(Lemma 6 in [25]).

(iii) If $f, g \in \mathrm{B}_\infty^\infty(\mathbb{R}^d)$, the operator $f(x)g(p)$ on $\mathrm{L}^2(\mathbb{R}^d)$ is compact (Section 11.4.1 in [14]).

Lemma 1. *If $f \in \mathrm{B}_\infty^\infty(\mathbb{R}^d)$, $\mathbf{a} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d$ and $V \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is nonnegative, then $f(x)R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + V(x)\right)$ is compact, for all $\lambda > 0$.*

Proof. Let $\lambda > 0$ and $f \in \mathrm{B}_\infty^\infty(\mathbb{R}^d)$. By (ii) above,

$$R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + V\right) \dot{\leq} R_{-\lambda}(-\Delta),$$

that is,

$$\left|R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + V\right)\psi\right| \leq R_{-\lambda}(-\Delta)|\psi|, \qquad \forall\, \psi \in \mathrm{L}^2(\mathbb{R}^d).$$

Hence

$$\left|f(x)R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + V\right)\psi\right| \leq |f(x)|R_{-\lambda}(-\Delta)|\psi|, \qquad \forall\, \psi \in \mathrm{L}^2(\mathbb{R}^d),$$

that is,

$$f(x)R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + V\right) \dot{\leq} |f(x)|R_{-\lambda}(-\Delta).$$

Since the operator on the right hand side is compact by (iii), the lemma follows by (i). □

Lemma 2. *Let λ, f, $\mathbf{a}$ and V be as in Lemma 1. Let $H_\infty(\mathbf{a})$ be the strong resolvent limit of the sequence of operators $(-i\nabla - \mathbf{a})^2 + nV$ as $n \to \infty$. Then $fR_{-\lambda}(H_\infty(\mathbf{a}))$ is a compact operator.*

Proof. Note that such strong resolvent limit $H_\infty(\mathbf{a})$ is supposed to exist. By (ii), for all n one has

$$R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + nV\right) \dot{\leq} R_{-\lambda}(-\Delta),$$

that is,

$$\left|R_{-\lambda}\left((-i\nabla - \mathbf{a})^2 + nV\right)\psi\right| \leq R_{-\lambda}(-\Delta)|\psi|, \qquad \forall\, \psi \in \mathrm{L}^2(\mathbb{R}^d).$$

Take $n \to \infty$, and since the strong resolvent limit exists in $\mathrm{L}^2(\mathbb{R}^d)$, for each ψ there exists a subsequence that converges a.e., so that

$$|R_{-\lambda}(H_\infty(\mathbf{a}))\psi| \leq R_{-\lambda}(-\Delta)|\psi|, \qquad \forall \psi \in \mathrm{L}^2(\mathbb{R}^d).$$

Hence,

$$|f(x)R_{-\lambda}(H_\infty(\mathbf{a}))\psi| \leq |f(x)|\, R_{-\lambda}(-\Delta)|\psi|, \qquad \forall\, \psi \in \mathrm{L}^2(\mathbb{R}^d),$$

and so

$$f(x)R_{-\lambda}(H_\infty(\mathbf{a})) \dot{\leq} |f(x)|\, R_{-\lambda}(-\Delta).$$

The lemma follows by applying first (iii) and then (i). □

The next simple remarks will be used several times in the arguments of Subsection 3.2. If a bounded operator T on a Hilbert space is positive $T \geq 0$, then T is self-adjoint and

$$\begin{aligned} \|T\| &= \sup_{\|\xi\|=1} \langle T\xi, \xi \rangle = \sup_{\|\xi\|=1} \langle T^{1/2}\xi, T^{1/2}\xi \rangle \\ &= \sup_{\|\xi\|=1} \|T^{1/2}\xi\|^2 = \|T^{1/2}\|^2. \end{aligned}$$

And if S, P are bounded operators, then $\|SP\|^2 = \|P^* S^* SP\|$.

3.2 Proof of Theorem 1

Denote $R_n := R_{-1}(H_n)$, $n \geq 0$, and $R_{\mathrm{AB}} := R_{-1}(H_{\mathrm{AB}})$, and recall that R_n converges strongly to R_{AB} [26].

Pick an open subset $\mathcal{O} \subset \mathcal{S}^\circ$ so that $\mathrm{dist}\,(\mathcal{O}, \mathcal{S}) > 0$, and another open subset $\mathcal{U} \supset \tilde{\mathcal{S}} \backslash \mathcal{O} \supset \mathcal{S}$ whose closure $\overline{\mathcal{U}}$ is compact. Choose a nonnegative (real-valued) function $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ with $\varphi = 1$ on $\mathcal{U}$.

One may write

$$\begin{aligned} R_n - R_{\mathrm{AB}} &= \chi_{\mathcal{O}} \, (R_n - R_{\mathrm{AB}}) + \chi_{\mathcal{S}^\circ \backslash \mathcal{O}} \, \varphi \, (R_n - R_{\mathrm{AB}}) \\ &+ \chi_{\hat{\mathcal{S}}} \, \varphi \, (R_n - R_{\mathrm{AB}}) + \chi_{\hat{\mathcal{S}}} \, (1 - \varphi) \, (R_n - R_{\mathrm{AB}}) \,, \end{aligned}$$

and the proof that follows consists in showing that each term on the right hand side vanish in the norm of the space of bounded operators. This is the motivation for the calculations below.

Lemma 3. *Let φ also denote the multiplication operator by the function φ. Then*

$$0 \leq \varphi \, (R_n - R_{\mathrm{AB}}) \, \varphi \leq \varphi \, (R_0 - R_{\mathrm{AB}}) \, \varphi,$$

and the operators $\varphi \, (R_n - R_{\mathrm{AB}}) \, \varphi$ are compact, for all $n \geq 0$. Furthermore, $\varphi \, (R_n - R_{\mathrm{AB}}) \, \varphi$ converges strongly to zero as $n \to \infty$.

Proof. It is known that $R_n - R_{\mathrm{AB}} \geq 0$; see the proof of Theorem 10.4.2 in [14]. Thus, by Proposition 9.3.1 in [14], one has

$$\begin{aligned} \langle \psi, \varphi \, (R_n - R_{\mathrm{AB}}) \, \varphi\psi \rangle &= \langle \varphi\psi, (R_n - R_{\mathrm{AB}}) \, \varphi\psi \rangle \\ &= \left\langle (R_n - R_{\mathrm{AB}})^{1/2} \, \varphi\psi, (R_n - R_{\mathrm{AB}})^{1/2} \, \varphi\psi \right\rangle \\ &= \left\| (R_n - R_{\mathrm{AB}})^{1/2} \, \varphi\psi \right\| \geq 0, \qquad \forall \, \psi \in \mathrm{L}^2(\mathbb{R}^d). \end{aligned}$$

To conclude the second inequality in the statement of the lemma, consider the quadratic forms b^n associated with the operators H_n, $n \geq 0$, and note that $b^0 \leq b^n$, for all $n \geq 0$. Hence, by Lemma 10.4.4 in [14], one has $R_n \leq R_0$. Thus,

$$\begin{aligned} \langle \psi, [\varphi(R_0 &- R_{\mathrm{AB}})\varphi - \varphi(R_n - R_{\mathrm{AB}})\varphi]\psi \rangle \\ &= \langle \psi, \varphi \, (R_0 - R_n) \, \varphi\psi \rangle = \left\| (R_0 - R_n)^{1/2} \, \varphi\psi \right\| \geq 0, \end{aligned}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$.

Now apply Lemma 1 with $f = \varphi$ and $R_{-1}\left((-i\nabla - \vec{\mathbf{a}})^2 + V\right) = R_n$, to conclude that φR_n is compact and, so, that $\varphi R_n \varphi$ is compact since it is the composition of a compact operator with a bounded one. Similarly, apply Lemma 2 with $f = \varphi$ and $R_{-1}(H_\infty(\vec{\mathbf{a}})) = R_{\mathrm{AB}}$ to conclude that both φR_{AB} and $\varphi R_{\mathrm{AB}} \varphi$ are also compact. Since H_n converges to H_{AB} in the strong resolvent sense as $n \to \infty$, for $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ one has

$$\|\varphi\,(R_n - R_{\mathrm{AB}})\,\varphi\psi\| \le \|\varphi\|\,\|(R_n - R_{\mathrm{AB}})\,(\varphi\psi)\| \to 0,$$

as $n \to \infty$, since φ is a bounded function. Therefore, the sequence of operators $\varphi(R_n - R_{\mathrm{AB}})\varphi$ converges strongly to zero as $n \to \infty$. □

The above lemma, combined with Theorems VIII-3.3 and VIII-3.5 in [23], ensures the norm convergence

$$\|\varphi\,(R_n - R_{\mathrm{AB}})\,\varphi\| \to 0, \qquad n \to \infty,$$

and so

$$\left\|(R_n - R_{\mathrm{AB}})^{1/2}\,\varphi\right\|^2 = \|\varphi\,(R_n - R_{\mathrm{AB}})\,\varphi\| \to 0,$$

as $n \to \infty$. Hence, one also gets the following norm convergence

$$\begin{aligned} \|(R_n - R_{\mathrm{AB}})\,\varphi\| &= \left\|(R_n - R_{\mathrm{AB}})^{1/2}\,(R_n - R_{\mathrm{AB}})^{1/2}\,\varphi\right\| \\ &\le \left\|(R_n - R_{\mathrm{AB}})^{1/2}\right\|\,\left\|(R_n - R_{\mathrm{AB}})^{1/2}\,\varphi\right\| \\ &\le \left\|(R_0 - R_{\mathrm{AB}})^{1/2}\right\|\,\left\|(R_n - R_{\mathrm{AB}})^{1/2}\,\varphi\right\| \to 0, \end{aligned} \tag{1}$$

as $n \to \infty$.

By following the proof of Lemma 2.3 in [21], one finds that there exists a constant $\kappa \ge 1$ so that

$$\|R_n \chi_{\mathcal{O}}\| \le \kappa\, n^{-1/2}, \qquad \forall n \ge 1.$$

Indeed, it is possible to find nonnegative functions $\eta_1, \eta_2 \in \mathrm{C}^\infty(\mathbb{R}^d)$ so that $\eta_1^2 + \eta_2^2 = 1$, with $\eta_1 = 1$ on $\mathcal{O}$ and $\operatorname{supp}\eta_1 \subset \mathcal{S}^\circ$. Furthermore, it is possible to assume that $|\nabla\eta_1|$ and $|\nabla\eta_2|$ are bounded functions. Thus,

$$H_n = \eta_1 H_n \eta_1 + \eta_2 H_n \eta_2 - |\nabla\eta_1|^2 - |\nabla\eta_2|^2\,.$$

Since

$$\begin{aligned} \langle\psi, H_n\psi\rangle &= \langle\psi, H_0\psi\rangle + \langle\psi, n\chi_{\mathcal{S}^\circ}\psi\rangle \\ &= \langle H_0^{1/2}\psi, H_0^{1/2}\psi\rangle + n\int_{\mathcal{S}^\circ} |\psi|^2 \ge n\|\psi\|^2, \quad \forall\,\psi \in \mathrm{C}_0^\infty(\mathcal{S}^\circ), \end{aligned}$$

one has

$$\begin{aligned}\langle\psi,\eta_1 H_n\eta_1\psi\rangle &= \langle\eta_1\psi, H_n\eta_1\psi\rangle \geq n\langle\eta_1\psi,\eta_1\psi\rangle \\ &= n\langle\psi,\eta_1^2\psi\rangle \geq n\langle\psi,\chi_{\mathcal{O}}\psi\rangle, \quad \forall\,\psi\in \mathrm{C}_0^\infty(\mathbb{R}^d),\end{aligned}$$

that is,

$$\eta_1 H_n \eta_1 \geq n\eta_1^2 \geq n\chi_{\mathcal{O}},$$

and one may find a constant κ so that $H_n \geq n\chi_{\mathcal{O}} - \kappa\mathbf{1}$, that is, $H_n + \kappa\mathbf{1} \geq n\chi_{\mathcal{O}}$; without loss, one may assume that $\kappa \geq 1$.

Write $Q_n := R_{-\kappa}(H_n) = (H_n + \kappa\mathbf{1})^{-1}$ and $S = Q_n^{1/2}\chi_{\mathcal{O}}$. It then follows that

$$\begin{aligned}\langle\psi, Q_n^{1/2}\chi_{\mathcal{O}}Q_n^{1/2}\psi\rangle &= \left\langle Q_n^{1/2}\psi,\chi_{\mathcal{O}}Q_n^{1/2}\psi\right\rangle \\ &\leq n^{-1}\left\langle Q_n^{1/2}\psi,(H_n+\kappa\mathbf{1})Q_n^{1/2}\psi\right\rangle \\ &= n^{-1}\left\langle \psi, Q_n^{1/2}(H_n+\kappa\mathbf{1})Q_n^{1/2}\psi\right\rangle = n^{-1}\langle\psi,\psi\rangle,\end{aligned}$$

for all $\psi\in \mathrm{C}_0^\infty(\mathbb{R}^d)$, that is,

$$(H_n+\kappa\mathbf{1})^{-1/2}\,\chi_{\mathcal{O}}\,(H_n+\kappa\mathbf{1})^{-1/2} \leq n^{-1}.$$

Note that S is bounded, $S^* = \chi_{\mathcal{O}}Q_n^{1/2}$ and

$$SS^* = Q_n^{1/2}\chi_{\mathcal{O}}\chi_{\mathcal{O}}Q_n^{1/2} = Q_n^{1/2}\chi_{\mathcal{O}}Q_n^{1/2}$$

is self-adjoint and bounded. Hence,

$$\|Q_n^{1/2}\chi_{\mathcal{O}}\|^2 = \|S\|^2 = \|S^*S\| = \|Q_n^{1/2}\chi_{\mathcal{O}}Q_n^{1/2}\| \leq n^{-1},$$

and so $\|Q_n^{1/2}\chi_{\mathcal{O}}\| \leq n^{-1/2}$. Therefore,

$$\begin{aligned}\|Q_n\chi_{\mathcal{O}}\| &= \|Q_n^{1/2}Q_n^{1/2}\chi_{\mathcal{O}}\| \leq \|Q_n^{1/2}\|\|Q_n^{1/2}\chi_{\mathcal{O}}\| \\ &\leq \|Q_n^{1/2}\chi_{\mathcal{O}}\| \leq n^{-1/2},\end{aligned}$$

with $\|Q_n^{1/2}\| \leq 1$, since $\kappa \geq 1$. Thus,

$$\left\|(H_n+\kappa\mathbf{1})^{-1}\,\chi_{\mathcal{O}}\right\| \leq \left\|(H_n+\kappa\mathbf{1})^{-1/2}\,\chi_{\mathcal{O}}\right\| \leq n^{-1/2}.$$

By the first resolvent identity, one has $R_n = Q_n + (\kappa-1)R_nQ_n$, so that

$$R_n\chi_{\mathcal{O}} = Q_n\chi_{\mathcal{O}} + (\kappa-1)R_nQ_n\chi_{\mathcal{O}},$$

and since $\|R_n\| \leq 1$,

$$\begin{aligned}\|R_n\chi_{\mathcal{O}}\| &\leq \|Q_n\chi_{\mathcal{O}}\| + (\kappa-1)\|R_n\|\|Q_n\chi_{\mathcal{O}}\| \\ &\leq n^{-1/2} + (\kappa-1)n^{-1/2} = \kappa\, n^{-1/2},\end{aligned}$$

that is, $\|R_n \chi_{\mathcal{O}}\| \le \kappa\, n^{-1/2}$. This inequality implies that

$$\|(R_n - R_{\mathrm{AB}})\, \chi_{\mathcal{O}}\| = \|R_n \chi_{\mathcal{O}}\| \to 0, \tag{2}$$

as $n \to \infty$, since $R_{\mathrm{AB}}\chi_{\mathcal{O}} \equiv 0$.

Recall that $\hat{\mathcal{S}} = \mathcal{S} \cup \mathcal{S}'$ and write $\rho = (1-\varphi)\chi_{\hat{\mathcal{S}}}$; then one has $\rho \in \mathrm{C}^\infty$. Compute

$$\begin{aligned}(H_0 + \mathbf{1})\,[\rho\,(R_0 - R_{\mathrm{AB}})] = &(-\Delta\rho)\,(R_0 - R_{\mathrm{AB}}) - 2\,(\nabla\rho)\cdot\nabla\,(R_0 - R_{\mathrm{AB}})\\ &+ \rho\,(H_0 + \mathbf{1})\,R_0 - \rho\,(H_0 + \mathbf{1})\,R_{\mathrm{AB}}\\ &+ 2i\,(\mathbf{A}\cdot\nabla\rho)\,(R_0 - R_{\mathrm{AB}}),\end{aligned}$$

but since $(H_0 + \mathbf{1})\,R_0 = \mathbf{1}$ and, due to the presence of the projection operator P' (see just before Theorem 1), $(H_0 + \mathbf{1})\,R_{\mathrm{AB}} \equiv (H_{\mathrm{AB}} + \mathbf{1})\,R_{\mathrm{AB}} = \mathbf{1}$, it then follows that

$$\begin{aligned}(H_0 + \mathbf{1})\,[\rho\,(R_0 - R_{\mathrm{AB}})] = &(-\Delta\rho)\,(R_0 - R_{\mathrm{AB}}) - 2\,(\nabla\rho)\cdot\nabla\,(R_0 - R_{\mathrm{AB}})\\ &+ 2i\,(\mathbf{A}\cdot\nabla\rho)\,(R_0 - R_{\mathrm{AB}}),\end{aligned}$$

and, since $\rho = (1-\varphi)\chi_{\hat{\mathcal{S}}}$, one has

$$\begin{aligned}(H_0 + \mathbf{1})\,[\rho\,(R_0 - R_{\mathrm{AB}})] = &(\Delta\varphi)\,\chi_{\hat{\mathcal{S}}}\,(R_0 - R_{\mathrm{AB}})\\ &+ 2\,(\nabla\varphi)\,\chi_{\hat{\mathcal{S}}}\cdot\nabla\,(R_0 - R_{\mathrm{AB}})\\ &- 2i\,(\mathbf{A}\cdot\nabla\varphi)\,\chi_{\hat{\mathcal{S}}}(R_0 - R_{\mathrm{AB}}),\end{aligned}$$

so that, after applying R_0 on the left, and using Lemmas 1 and 2 again, it is found that the operator

$$\begin{aligned}\rho\,(R_0 - R_{\mathrm{AB}}) &= R_0\chi_{\hat{\mathcal{S}}}\,(\Delta\varphi)\,(R_0 - R_{\mathrm{AB}})\\ &+ 2R_0\chi_{\hat{\mathcal{S}}}\,(\nabla\varphi)\cdot\nabla\,(R_0 - R_{\mathrm{AB}})\\ &- 2iR_0\chi_{\hat{\mathcal{S}}}\,(\mathbf{A}\cdot\nabla\varphi)\,(R_0 - R_{\mathrm{AB}})\end{aligned}$$

is compact.

The same arguments above imply that $\rho\,(R_n - R_{\mathrm{AB}})$ is also compact, and the inequality $\rho\,(R_n - R_{\mathrm{AB}})\,\rho \le \rho\,(R_0 - R_{\mathrm{AB}})\,\rho$ implies that

$$\left\|(R_n - R_{\mathrm{AB}})\,(1-\varphi)\chi_{\hat{\mathcal{S}}}\right\| \to 0, \qquad n \to \infty. \tag{3}$$

Now, one can write

$$\begin{aligned}R_n - R_{\mathrm{AB}} &= \chi_{\mathcal{O}}\,(R_n - R_{\mathrm{AB}}) + \chi_{\mathcal{S}^\circ\setminus\mathcal{O}}\,\varphi\,(R_n - R_{\mathrm{AB}})\\ &+ \chi_{\hat{\mathcal{S}}}\,\varphi\,(R_n - R_{\mathrm{AB}}) + \chi_{\hat{\mathcal{S}}}\,(1-\varphi)\,(R_n - R_{\mathrm{AB}})\,,\end{aligned}$$

and by equations (1), (2) and (3), it follows that

$$\|R_n - R_{\mathrm{AB}}\| \to 0,$$

as $n \to \infty$, that is, H_n converges to H_{AB} in the norm resolvent sense. This finishes the proof of Theorem 1.

4 Final Remarks

An important point in the proof of the norm resolvent convergence here was the compactness of the closure of the region $\mathcal{S}^\circ$, since this allowed the choice of the nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi = 1$ on the compact set $\overline{\mathcal{U}} \supset \tilde{\mathcal{S}}$. In some sense, such function $\varphi \in C_0^\infty(\mathbb{R}^d)$ is responsible for the compactness of the auxiliary operators in the proof of Theorem 1.

The magnetic field may vanish outside $\tilde{\mathcal{S}}$ or not. Note that Theorem 1 also applies to regions like (assume that $a, L, \epsilon > 0$, and $\epsilon < a$)

$$\mathcal{S} = \left\{(x, y, z) \in \mathbb{R}^3 : -L < z < L,\ a - \epsilon < x^2 + y^2 < a\right\},$$

which is a "fat" cylinder of radius a and finite length L. This may be a way to take into account the difference between the particle running into the finite cylindrical solenoid at a point with $z = \pm L$, from entering through the lateral border of that cylinder.

Think of an infinitely long cylindrical solenoid in $\mathbb{R}^3$ (i.e., the standard Aharonov-Bohm setting). A common situation is to take into account the symmetry along the vertical coordinate by considering approximations by two-dimensional models [1, 28]. Note that the norm resolvent convergence holds for such approximate models in $\mathbb{R}^2$, but the proof presented here does not work for the original model in $\mathbb{R}^3$, since in this case the solenoid border is not included in a compact subset.

Similarly, for fixed intensity n, if one considers a sequence of solenoids whose lengths go to infinity in $\mathbb{R}^3$ (e.g., as discussed in [13]), the norm resolvent convergence is not expected to hold (furthermore, the convergence of the corresponding sequence of vector potentials is not uniform in general). In this way, the commutation diagram at the end of reference [13] should hardly be improved to accommodate norm resolvent for all convergences therein presented.

Finally, note that the result of Theorem 1 is easily adapted to subsets of $\mathbb{R}^d$, for any $d \geq 2$; the restriction to $d = 2, 3$, was only due to the physical interpretations related to the magnetic Aharonov-Bohm effect.

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