QUANTUM SINGULAR OPERATOR LIMITS OF THIN DIRICHLET TUBES VIA Γ-CONVERGENCE

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ABSTRACT. The Γ-convergence of lower bounded quadratic forms is used to study the singular operator limit of thin tubes (i.e., the vanishing of the cross section diameter) of the Laplace operator with Dirichlet boundary conditions; a procedure to obtain the effective Schrödinger operator (in different subspaces) is proposed, generalizing recent results in case of compact tubes. Finally, after scaling curvature and torsion the limit of a broken line is briefly investigated.

Keywords: quantum thin tubes, singular operators, Γ-convergence, broken line.

CONTENTS

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1. Introduction

Among the tools for studying limits of self-adjoint operators in Hilbert spaces are the resolvent convergence and the sesquilinear form convergence. In the context of quantum mechanics, it is well known that in case of monotone sequences of operators these approaches are strictly related, as discussed, for example, in Section 10.4 of [1] and Section VIII.7 (Supplementary Material) of [2]. These form convergences have the advantage of dealing with some singular limits in quantum mechanics, which is well exemplified by mathematical arguments supporting the Aharonov-Bohm Hamiltonian [3, 4].

Let $\mathbf{1}$ denote the identity operator and $0 \leq T_j$ be a sequence of positive (or uniformly lower bounded in general) self-adjoint operators acting in the Hilbert space $\mathcal{H}$. From the technical point of view what happens is that the monotone increasing of the sequence of resolvent operators $R_{-\lambda}(T_j) := (T_j + \lambda\mathbf{1})^{-1}$ ($\lambda > 0$) implies the monotone decreasing of the corresponding sequence of sesquilinear forms and vice versa. Due to monotonicity, in such cases one clearly understands the existence of limits and have some insight in limit form domains. However, in principle it is not at all clear what are the relations between strong resolvent convergence of operators and sesquilinear form convergence in more general cases, say, if one requires that the self-adjoint operators or closed forms are only uniformly bounded from below. It happens that such relations are well known among people interested in variational convergences and applied mathematics, and it is directly related to the concept of Γ-convergence; but when addressing sesquilinear forms these variational problems are usually formulated in real Hilbert spaces and the theory has been developed under this condition. However, in quantum mechanics the Hilbert spaces are usually complex, and an adaptation of the main results to complex Hilbert spaces will appear elsewhere [5]. Due to the particular class of applications we have in mind, here it will be enough to reduce some key arguments to real Hilbert spaces.

In Section 2 the very basics of Γ-convergence are recalled in a suitable way; for details the reader is referred to the important monographs in the field [6, 7].

As an application of Γ-convergence of forms we study the limit operator obtained from the Laplacian (with Dirichlet boundary conditions) restricted to a tube in $\mathbb{R}^3$ that shrinks to a smooth curve. This problem has been considered in the interesting work [8], where the strong Γ-convergence was employed to study the limit operator and convergence of eigenvalues and eigenvectors, but the curve was supposed to have finite length L and the convergence restricted to the subspace of vectors of the form $w(s)u_0(y)$ ("the first sector"), with $w \in \mathcal{H}_0^1[0, L]$ (i.e., the usual Sobolev space) and u_0 being the first eigenfunction of the restriction of the Laplacian to the tube cross section (s denotes the curve arc length and $y = (y_1, y_2)$ the cross section variables). Here we indicate how the proofs in [8] can be worked out to get an effective operator in case of curves of infinite length; in fact it will be necessary to go a step further and we prove both strong and weak Γ-convergences of forms which imply the strong resolvent convergence of operators to an effective Schrödinger operator on the curve. We also give an alternative proof of the spectral convergence discussed in [8] in case of finite length curves: compactness arguments due to the boundedness of the tube in this case will be essential; our proof also clarifies the mechanism behind the spectral convergence. These issues are discussed in Section 3.

The use of quadratic forms, and the possibility of considering Γ-convergence, in the application mentioned above is important because due to the intricate geometry of the allowed tubes the expressions of the actions of the involved operators are rather complicated, with the presence of mixed derivatives and nonlinear coefficients. In case the curve lives in $\mathbb{R}^2$ there are results about the limit operator in [9, 10] and the effective potential is written in terms of the curvature; the main novelty in case of $\mathbb{R}^3$ considered in [8] is the additional presence of "twisting" and torsion in the effective potential, since the case of untwisted tubes has also been previously studied in [11] (see also [12, 13]). Since in both $\mathbb{R}^2$ and untwisted tubes in $\mathbb{R}^3$ we have simpler expressions for the involved operators, it is possible to deal directly with the strong resolvent convergence and consider more general spaces than that related only to the first eigenfunction of the laplace operator in the cross section (i.e., the first sector). So, based on the Γ-convergence of sesquilinear forms, we propose here a rather natural procedure to handle the strong convergence in case of vectors of the form $w(s)u_n(y)$, i.e., the $(n+1)$th sector spanned by the general eigenvector u_n of the Laplacian restricted to the cross section, and we will impose that each eigenvalue of this operator is simple in order to simplify the implementation of the Γ-convergence of b_n^ε and the strong resolvent convergence of the associated self-adjoint operators H_n^ε, $n \geq 1$, which are introduced in Definition 2. It turned out that the resulting effective operator depends on the sector considered, an effect not present in $\mathbb{R}^2$ [9, 10]. This "new" effect is understood since in $\mathbb{R}^2$ there is a separation of variables s and y (see the top of page 14 of [10], but be aware of the different notations), which in general does not occur in $\mathbb{R}^3$ and so the necessity of introducing a specific procedure. These developments are also presented in Section 3. Roughly speaking, from the dynamical point of view the dependence of the limit operator on the sector of the Hilbert space would correspond to the dependence of the effective potential on the initial condition, a phenomenon already noticed in [14, 15].

It must be mentioned that the author has no intention to claim that Γ-convergence should replace the traditional operator techniques in such kind of problems with no separation of variables [16, 17, 18, 19], including some important spectral results for twisted tubes [22]. Further, we need a combination of strong and weak Γ-convergences of quadratic forms in order to get strong resolvent convergence of the related operators, and other assumptions and tools must be added to get norm resolvent convergence (see, for instance, Subsection 3.5). The Γ-convergence in this context must be seen as just another available tool for mathematical physicists.

With the effective limit operator at hand, we finally discuss the limit of a smooth curve approaching a broken line, similarly to [9, 10], but here we confine ourselves to take one limit at a time, that is, first we constrain the particle motion from the tube to the curve (as discussed above), then we take the limit when the curve approaches the broken line; we shall closely follow a discussion in [10]. Besides the negative term related to the curvature, in our case the effective potential $V^{\rm eff}(s)$ has the additional presence of a positive term related to "twist" and torsion so that the technical condition

$$\int_{\mathbb{R}} V^{\rm eff}(s)ds \neq 0$$

assumed in [10] might not hold in some cases. We have then checked that it is still possible to follow the same proofs, but with suitable adaptations, even if the above integral vanishes. The bottom line is that different self-adjoint realizations for the operator on the broken line are found. These broken-line limits are shortly discussed in Section 4.

Of course in this introduction we have skipped many technical details, some of them are fundamental to a correct understanding of the contents of this work. In the following sections I shall try to fill out those gaps. I expect this work will motivate researchers to seriously consider the Γ-convergence of forms as a useful way to study (singular) limits of observables in quantum mechanics.

2. Strong Resolvent and Γ Convergences

2.1. **Γ-Convergence.** Our sequences (more properly they should be called families) of self-adjoint operators T_ε, with domain dom T_ε in a separable Hilbert space $\mathcal{H}$, and the corresponding closed sesquilinear forms b_ε will be indexed by the parameter $\varepsilon > 0$ and, by definiteness, we think of the limit $\varepsilon \to 0$ and we want to study the limit T (resp. b) of T_ε (resp. b_ε). The domain of T will not be supposed to be dense in $\mathcal{H}$ and its closure will be denoted by $\mathcal{H}_0 = \overline{\mathrm{dom}\ T}$ (with rng $T \subset \mathcal{H}_0$); usually this is indicated by simply saying that "T is self-adjoint in $\mathcal{H}_0$."

We assume that a sesquilinear form $b(\zeta, \eta)$ is linear in the second entry and antilinear in the first one. As usual the real-valued function $\zeta \mapsto b(\zeta, \zeta)$ will be simply denoted by $b(\zeta)$ and called the associated quadratic form; we will use the terms sesquilinear and quadratic forms almost interchangeably, since usually the context makes it clear which one is being referred to. It will also be assumed that b is positive (or lower bounded in general) and $b(\zeta) = \infty$ if ζ does not belong to its domain dom b; this is important in order to guarantee that in some cases b is lower semicontinuous, which is equivalent to b be the sesquilinear form generated by a positive self-adjoint operator T, that is,

$$b(\zeta, \eta) = \langle T^{1/2}\zeta, T^{1/2}\eta\rangle, \quad \zeta, \eta \in \mathrm{dom}\ b = \mathrm{dom}\ T^{1/2};$$

see Theorem 9.3.11 in [1]. By allowing $b(\zeta) = \infty$ we have a handy way to work in the larger space $\mathcal{H}$ instead of only in $\mathcal{H}_0 = \overline{\text{dom } T}$. If $\lambda \in \mathbb{R}$, then $b + \lambda$ indicates the sesquilinear form $(b+\lambda)(\zeta, \eta) := b(\zeta, \eta) + \lambda\langle \zeta, \eta\rangle$, whose corresponding quadratic form is $b(\zeta) + \lambda\|\zeta\|^2$.

It is known that (Lemma 10.4.4 in [1]), for any $\lambda > 0$, one has $b_{\varepsilon_1} \le b_{\varepsilon_2}$ iff $R_{-\lambda}(T_{\varepsilon_2}) \le R_{-\lambda}(T_{\varepsilon_1})$, so that a sequence of quadratic forms is monotone iff the corresponding sequence of resolvent operators is monotone. This has been explored in the quantum mechanics literature in order to get strong resolvent limits of self-adjoint operators through the study of quadratic forms (see, for instance, Section 10.4 of [1] and Section VIII.7 of [2]). For more general sequences of operators, the form counterpart of the strong resolvent convergence is not so direct and it was found that the correct concept comes from the so-called Γ-convergence [6]. In what follows the concept of Γ-convergence will be recalled in a suitable way, and then applied to the study of singular limits of Dirichlet tubes in other sections.

The general concept of Γ-convergence is not restricted to quadratic forms and can be applied to quite general topological spaces, but in this section the ideas will mostly be suitably adapted to our framework; e.g., we try to restrict the discussion to Hilbert spaces and to lower semicontinuous functions, since the quadratic forms we are interested in have this property. The general theory is nicely presented in the book [6], to which we will often refer. $\mathcal{H}$ always denote a separable Hilbert space and $B(\zeta;\delta)$ the open ball cantered at $\zeta \in \mathcal{H}$ of radius $\delta > 0$; finally $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and the symbol l.sc. will be a shorthand to *lower semicontinuous*.

Definition 1. *The lower* Γ*-limit of a sequence of l.sc. functions* $f_\varepsilon : \mathcal{H} \to \overline{\mathbb{R}}$ *is the function* $f^- : \mathcal{H} \to \overline{\mathbb{R}}$ *given by*

$$f^-(\zeta) = \lim_{\delta\to 0}\ \liminf_{\varepsilon\to 0}\ \inf\{f_\varepsilon(\eta) : \eta \in B(\zeta;\delta)\}, \quad \zeta \in \mathcal{H}.$$

The upper Γ*-limit* $f^+(\zeta)$ *of* f_ε *is defined by replacing* $\liminf$ *by* $\limsup$ *in the above expression. If* $f^- = f^+ =: f$ *we say that such function is the* Γ*-limit of* f_ε *and it will be denoted by*

$$f = \Gamma\text{-}\lim_{\varepsilon\to 0} f_\varepsilon.$$

Remark 1. *It was assumed in Definition 1 that the topology of* $\mathcal{H}$ *is the usual norm topology, and in this case we speak of* strong Γ-convergence. *If the weak topology is considered, the balls* $B(\zeta;\delta)$ *must be replaced by the set of all open weak neighborhoods of* ζ [6]*, and in this case we speak of* weak Γ-convergence. *Both concepts will be important here, and in general they are not equivalent since the norm is not continuous in the weak topology (see Example 6.6 in* [6]*). When convenient, the symbols*

$$f_\varepsilon \xrightarrow{S\Gamma} f, \quad f_\varepsilon \xrightarrow{W\Gamma} f$$

will be used to indicate that f_ε Γ*-converges to* f *in the strong and weak sense in* $\mathcal{H}$*, respectively. See also Proposition 1 and Remark 4.*

Example 1. *The sequence* $f_\varepsilon : \mathbb{R} \to \mathbb{R}$, $f_\varepsilon(x) = \sin(x/\varepsilon)$, Γ*-converges to the constant function* $f(x) = -1$ *as* $\varepsilon \to 0$. *This simple example nicely illustrates the property of "convergence of minima" that motivated the introduction of the* Γ*-convergence. In quantum mechanics one is used to the convergence of averages, so that the natural guess (if any) for the limit in this example would be the null function.*

Remark 2. *The* Γ*-convergence is usually different from both the pointwise and weak limit of functions and roughly it can be illustrated as follows. Assume one is studying a heterogeneous material subject to strong tensions whose intensity in some regions is measured by the parameter* $1/\varepsilon$ *and it will undergo a kind of phase transition as* $\varepsilon \to 0$*; for each* $\varepsilon > 0$ *one computes the equilibrium configuration of this material via a minimum of certain energy functional; this transition would be computed as the limit of the equilibria (i.e., minima of such functionals), and this limit (i.e., minimum of an "effective limit functional") would be quite singular. In many instances* Γ*-convergence is the correct concept to describe such situations* [7, 6] *and its importance in the asymptotics of variational problems relies here.*

Remark 3. *From some points of view the notion of* Γ*-convergence is quite subtle, as exemplified by the following facts:*

a) *Due to the prominent role played by minima, in general*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} f_\varepsilon \neq -\left(\Gamma\text{-}\lim_{\varepsilon\to 0}(-f_\varepsilon)\right).$$

b) *Assume that* $f = \Gamma\text{-}\lim_{\varepsilon\to 0} f_\varepsilon$ *and* $g = \Gamma\text{-}\lim_{\varepsilon\to 0} g_\varepsilon$*; it may happen that* $(f_\varepsilon + g_\varepsilon)$ *is not* Γ*-convergent.*

c) *Clearly it is not necessary to restrict the definition of* Γ*-convergence to l.sc. functions. If* $f_\varepsilon = f$*, for all* ε*, and* f *is not l.sc., then* $\Gamma\text{-}\lim_{\varepsilon\to 0} f$ *is the greatest lower semicontinuous function majorized by* f *(the so-called l.sc. envelope of* f*), and so different from* f*!*

Since $\mathcal{H}$ satisfies the first axiom of countability, Proposition 8.1 of [6] implies the following handy characterization of strong and weak Γ-convergence:

Proposition 1. *The sequence* $f_\varepsilon : \mathcal{H} \to \overline{\mathbb{R}}$ *strongly* Γ*-converges to* f *(that is,* $f_\varepsilon \xrightarrow{S\Gamma} f$*) iff the following two conditions are satisfied:*

i) *For every* $\zeta \in \mathcal{H}$ *and every* $\zeta_\varepsilon \to \zeta$ *in* $\mathcal{H}$ *one has*

$$f(\zeta) \leq \liminf_{\varepsilon\to 0} f_\varepsilon(\zeta_\varepsilon).$$

ii) *For every* $\zeta \in \mathcal{H}$ *there exists a sequence* $\zeta_\varepsilon \to \zeta$ *in* $\mathcal{H}$ *such that*

$$f(\zeta) = \lim_{\varepsilon\to 0} f_\varepsilon(\zeta_\varepsilon).$$

Remark 4. *If instead of strong convergence* $\zeta_\varepsilon \to \zeta$ *one considers weak convergence* $\zeta_\varepsilon \rightharpoonup \zeta$ *in Proposition 1, then we have a characterization of*

$f_\varepsilon \xrightarrow{\text{W}\Gamma} f$. *Here this characterization will be used in practice, and so it justifies the lack of details with respect to weak* Γ*-convergence in Remark 1.*

Recall that a function $f : \mathcal{H} \to \overline{\mathbb{R}}$ is *coercive* if for every $x \in \mathbb{R}$ the set $f^{-1}(-\infty, x]$ is precompact in $\mathcal{H}$. Since $\mathcal{H}$ is reflexive, it turns out that a function f is coercive in the weak topology of $\mathcal{H}$ iff $\lim_{\|\zeta\|\to\infty} f(\zeta) = \infty$. A sequence of functions $f_\varepsilon : \mathcal{H} \to \overline{\mathbb{R}}$ is *equicoercive* if there exists a coercive $\varphi : \mathcal{H} \to \overline{\mathbb{R}}$ such that $f_\varepsilon \geq \varphi$, for all $\varepsilon > 0$. The following results, stated as Theorems 1 and 2, will be useful later on. The first one is about convergence of minimizers (for the proof see [6], Chapter 7), whereas the second one gives some conditions guaranteeing that the Γ-convergence is stable under continuous perturbations (see Proposition 6.21 in the book [6].).

Theorem 1. *Assume that $f_\varepsilon : \mathcal{H} \to \overline{\mathbb{R}}$ Γ-converges to f and let ζ_ε be a minimizer of f_ε, for all ε. Then any cluster point of (ζ_ε) is a minimizer of f. Further, if f_ε is equicoercive and f has a unique minimizer x_0, then ζ_ε converges to x_0.*

Theorem 2. *Let $b_\varepsilon, b \geq \beta > -\infty$ be closed and (uniformly) lower bounded sesquilinear forms in $\mathcal{H}$ and $f : \mathcal{H} \to \mathbb{R}$ a continuous function. Then b_ε Γ-converges to b iff $(b_\varepsilon + f)$ Γ-converges to $(b + f)$. In particular, in case of both weak and strong Γ-convergences in $\mathcal{H}$, it holds for the functional $f(\cdot) = \langle \eta, \cdot \rangle + \langle \cdot, \eta \rangle$, defined for each fixed $\eta \in \mathcal{H}$.*

2.2. **Γ and Resolvent Convergences.** Now we recall the main results with respect to the relation between the strong resolvent convergence of self-adjoint operators and Γ-convergence of the associated sesquilinear forms [6, 5].

Theorem 3. *Let b_ε, b be positive (or uniformly lower bounded) closed sesquilinear forms in the Hilbert space $\mathcal{H}$, and T_ε, T the corresponding associated positive self-adjoint operators. Then the following statements are equivalent:*

i) $b_\varepsilon \xrightarrow{\text{S}\Gamma} b$ *and, for each $\zeta \in \mathcal{H}$, $b(\zeta) \leq \liminf_{\varepsilon\to 0} b_\varepsilon(\zeta_\varepsilon)$, $\forall \zeta_\varepsilon \rightharpoonup \zeta$ in $\mathcal{H}$.*
ii) $b_\varepsilon \xrightarrow{\text{S}\Gamma} b$ *and* $b_\varepsilon \xrightarrow{\text{W}\Gamma} b$.
iii) $b_\varepsilon + \lambda \xrightarrow{\text{S}\Gamma} b + \lambda$ *and* $b_\varepsilon + \lambda \xrightarrow{\text{W}\Gamma} b + \lambda$, *for some $\lambda > 0$ (and so for all $\lambda \geq 0$).*
iv) *For all $\eta \in \mathcal{H}$ and $\lambda > 0$, the sequence*

$$\min_{\zeta\in\mathcal{H}} \left[b_\varepsilon(\zeta) + \lambda \|\zeta\|^2 + \frac{1}{2}(\langle \eta, \zeta \rangle + \langle \zeta, \eta \rangle) \right]$$

converges to

$$\min_{\zeta\in\mathcal{H}} \left[b(\zeta) + \lambda \|\zeta\|^2 + \frac{1}{2}(\langle \eta, \zeta \rangle + \langle \zeta, \eta \rangle) \right].$$

v) T_ε *converges to* T *in the strong resolvent sense in* $\mathcal{H}_0 = \overline{\text{dom } T} \subset \mathcal{H}$, *that is,*

$$\lim_{\varepsilon\to 0} R_{-\lambda}(T_\varepsilon)\zeta = R_{-\lambda}(T)P_0\zeta, \quad \forall\zeta \in \mathcal{H}, \forall\lambda > 0,$$

where P_0 *is the orthogonal projection onto* $\mathcal{H}_0$.

The next two results are included mainly because they help to elucidate the connection between forms, operator actions and domains on the one hand, and minimalization of suitable functionals on the other hand; this sheds some light on the role played by Γ-convergence in the convergence of self-adjoint operators.

Proposition 2. *Let* $b \geq 0$ *be a closed sesquilinear form on the Hilbert space* $\mathcal{H}$, $T \geq 0$ *the self-adjoint operator associated with* b *and* P_0 *be the orthogonal projection onto* $\mathcal{H}_0 = \overline{\text{dom } T} \subset \mathcal{H}$. *Then* $\zeta \in \text{dom } T$ *and* $T\zeta = P_0\eta$ *iff* ζ *is a minimum point (also called minimizer) of the functional*

$$g : \mathcal{H} \to \overline{\mathbb{R}}, \qquad g(\zeta) = b(\zeta) - \langle\eta, \zeta\rangle - \langle\zeta, \eta\rangle.$$

Proposition 3. *Let* $T : \text{dom } T \to \mathcal{H}$ *be a positive self-adjoint operator,* $\overline{\text{dom } T} = \mathcal{H}_0$, *and* $b^T : \mathcal{H} \to \overline{\mathbb{R}}$ *the quadratic form generated by* T. *Then*

$$\begin{aligned} b^T(\zeta) &= \sup_{\eta\in\text{dom } T} \left[\langle T\eta, \zeta\rangle + \langle\zeta, T\eta\rangle - \langle T\eta, \eta\rangle\right] \\ &= \sup_{\eta\in\text{dom } T} \left[b^T(\eta) + \langle T\eta, \zeta\rangle + \langle\zeta, T\eta\rangle - 2\langle T\eta, \eta\rangle\right], \end{aligned}$$

for all $\zeta \in \mathcal{H}_0$ *and* $b^T(\zeta) = \infty$ *if* $\zeta \in \mathcal{H} \setminus \mathcal{H}_0$.

Finally we recall Theorem 13.5 in [6]:

Theorem 4. *Let* $b_\varepsilon, b \geq \beta > 0$ *be sesquilinear forms on the Hilbert space* $\mathcal{H}$ *and* $T_\varepsilon, T \geq \beta\mathbf{1}$ *the corresponding associated self-adjoint operators, and let* $\overline{\text{dom } T} = \mathcal{H}_0 \subset \mathcal{H}$. *Then the following statements are equivalent:*

i) $b_\varepsilon \xrightarrow{\text{w}\Gamma} b$.
ii) $R_0(T_\varepsilon)$ *converges weakly to* $R_0(T)P_0$, *where* P_0 *is the orthogonal projection onto* $\mathcal{H}_0$.

2.3. **Norm Resolvent Convergence.** Before turning to an application of Theorem 3 in the next section, we introduce additional conditions in order to get norm resolvent convergence of operators from Γ-convergence. This condition will be used to recover a spectral convergence proved in [8] and, from the technical point of view, can be considered our first contribution.

Proposition 4. *Let* $b_\varepsilon, b \geq \beta > -\infty$ *be closed sesquilinear forms and* $T_\varepsilon, T \geq \beta\mathbf{1}$ *the corresponding associated self-adjoint operators, and let* $\overline{\text{dom } T} = \mathcal{H}_0 \subset \mathcal{H}$. *Assume that the following three conditions hold:*

a) $b_\varepsilon \xrightarrow{\text{s}\Gamma} b$ *and* $b_\varepsilon \xrightarrow{\text{w}\Gamma} b$.
b) *The resolvent operator* $R_{-\lambda}(T)$ *is compact in* $\mathcal{H}_0$ *for some real number* $\lambda > |\beta|$.

c) *There exists a Hilbert space $\mathcal{K}$, compactly embedded in $\mathcal{H}$, so that if the sequence (ψ_ε) is bounded in $\mathcal{H}$ and $(b_\varepsilon(\psi_\varepsilon))$ is also bounded, then (ψ_ε) is a bounded subset of $\mathcal{K}$.*

Then, T_ε converges in norm resolvent sense to T in $\mathcal{H}_0$ as $\varepsilon \to 0$.

Proof. We must show that $R_{-\lambda}(T_\varepsilon)$ converges in operator norm to $R_{-\lambda}(T)P_0$, where P_0 is the orthogonal projection onto $\mathcal{H}_0$; to simplify the notation the projection P_0 will be ignored.

If $R_{-\lambda}(T_\varepsilon)$ does not converge in norm to $R_{-\lambda}(T)$, there exist $\delta_0 > 0$ and vectors η_ε, $\|\eta_\varepsilon\| = 1$, for a subsequence (we tacitly keep the same notation after taking subsequences) of indices $\varepsilon \to 0$ so that

$$\|R_{-\lambda}(T_\varepsilon)\eta_\varepsilon - R_{-\lambda}(T)\eta_\varepsilon\| \geq \delta_0, \quad \forall \varepsilon > 0.$$

We will argue to get a contradiction with this inequality, so proving the proposition. Denote $\zeta_\varepsilon := R_{-\lambda}(T_\varepsilon)\eta_\varepsilon$. By the reflexivity of $\mathcal{H}$ one can suppose that $\eta_\varepsilon \rightharpoonup \eta$, for some $\eta \in \mathcal{H}$, and since $R_{-\lambda}(T)$ is compact we have $R_{-\lambda}(T)\eta_\varepsilon \to R_{-\lambda}(T)\eta$ in $\mathcal{H}$. The general inequalities

$$\|R_{-\lambda}(T_\varepsilon)\eta_\varepsilon\| \leq \frac{1}{|\beta - \lambda|} \quad \text{and} \quad \|T_\varepsilon R_{-\lambda}(T_\varepsilon)\eta_\varepsilon\| \leq \|\eta_\varepsilon\| = 1, \quad \forall \varepsilon,$$

imply that

$$|b_\varepsilon(\zeta_\varepsilon)| = |\langle T_\varepsilon\zeta_\varepsilon, \zeta_\varepsilon\rangle| \leq \|T_\varepsilon\zeta_\varepsilon\| \, \|\zeta_\varepsilon\| \leq \frac{1}{|\beta - \lambda|}, \quad \forall \varepsilon,$$

and so it follows by c) that $(R_{-\lambda}(T_\varepsilon)\eta_\varepsilon)$ is a bounded sequence in $\mathcal{K}$, and since this space is compactly embedded in $\mathcal{H}$ there exists a (strongly) convergent subsequence so that

$$R_{-\lambda}(T_\varepsilon)\eta_\varepsilon \to \zeta$$

for some $\zeta \in \mathcal{H}$. Next we will employ the Γ-convergence to show that $\zeta = R_{-\lambda}(T)\eta$.

By Proposition 2, for each ε fixed, ζ_ε is the minimizer in $\mathcal{H}$ of the functional

$$g_\varepsilon(\phi) = b_\varepsilon(\phi) + \lambda\|\phi\|^2 - \langle\eta_\varepsilon, \phi\rangle - \langle\phi, \eta_\varepsilon\rangle,$$

whereas $\zeta = R_{-\lambda}(T)\eta$ is the unique minimizer of

$$g(\phi) = b(\phi) + \lambda\|\phi\|^2 - \langle\eta, \phi\rangle - \langle\phi, \eta\rangle.$$

Since $\lambda > |\beta|$ it follows that g_ε is weakly equicoercive (see the discussion after Remark 4) and, by Theorem 1, $\zeta_\varepsilon \rightharpoonup \zeta$. By this weak convergence and Theorem 3 iv), for all $\mu \geq \lambda$, one has

$$b(\zeta) + \mu\|\zeta\|^2 - \langle\eta, \zeta\rangle - \langle\zeta, \eta\rangle = \lim_{\varepsilon\to 0}[b_\varepsilon(\zeta_\varepsilon) + \mu\|\zeta_\varepsilon\|^2 - \langle\eta, \zeta_\varepsilon\rangle - \langle\zeta_\varepsilon, \eta\rangle].$$

Theorem 3 and again $\zeta_\varepsilon \rightharpoonup \zeta$ imply

$$b(\zeta) + \lambda\|\zeta\|^2 \leq \liminf_{\varepsilon\to 0}[b_\varepsilon(\zeta_\varepsilon) + \lambda\|\zeta_\varepsilon\|^2],$$

whereas the lower semicontinuity of the norm with respect to weak convergence gives

$$(\mu-\lambda)\|\zeta\|^2 \le \liminf_{\varepsilon\to 0}(\mu-\lambda)\|\zeta_\varepsilon\|^2, \quad \mu>\lambda.$$

The last three relations imply $\|\zeta\| = \lim_{\varepsilon\to 0}\|\zeta_\varepsilon\|$, and together with $\zeta_\varepsilon \rightharpoonup \zeta$ one obtains the strong convergence $\zeta_\varepsilon \to \zeta$, that is,

$$R_{-\lambda}(T_\varepsilon)\eta_\varepsilon \to R_{-\lambda}(T)\eta,$$

which contradicts the existence of $\delta_0 > 0$ above. The proof of the proposition is complete. □

3. Singular Limit of Dirichlet Tubes

There are many occasions in which particles or waves are restricted to propagate in thin domains along one-dimensional structures, as a graph. Optical fibers, carbon nanotubes and the motion of valence electrons in aromatic molecules are good examples. One natural theoretical consideration is to neglect the small transversal sections and model these systems by the true one-dimensional versions by means of effective parameters and potentials. Besides the necessity of finding effective models, one has also to somehow decouple the transversal and the longitudinal variables. Such confinements can be realized by strong potentials (see [14, 23] and references therein) or boundary conditions, and the graph can be imbedded in spaces of different dimensions.

In this section we apply the Γ-convergence in Hilbert spaces to study the limit operator of the Dirichlet Laplacian in a sequence of tubes in $\mathbb{R}^3$ that is squeezed to a curve. This kind of problem (mainly in $\mathbb{R}^2$) has been considered in some papers (for instance, [10, 8, 9, 11, 13]), and here we show how part of the construction in [8] can be carried out to curves of infinite length. As already mentioned, since our main interest is in quantum mechanics, in principle one should use complex Hilbert spaces and adapt the results of Γ-convergence to this more general setting [5] before presenting applications, but since the Laplacian is a real operator, and its eigenfunctions can always be supposed to be real valued, one can reduce the arguments to the real setting.

In a second step we propose a procedure to deal with more general vectors than that generated by the first eigenvector of the restriction of the Laplacian to the tube cross section. In simpler situations, like tubes in $\mathbb{R}^3$ with vanishing "twisting" (see Definition 3) or in $\mathbb{R}^2$, it is possible to get a suitable separation of variables, it is not necessary to employ that procedure and the limit operator does not depend on the subspace considered. However, in the most general case the limit operator will depend on the subspace in the cross section due to an "additional memory of extra dimensions" that can be present in $\mathbb{R}^3$. This will appear explicitly in the expressions of effective potentials ahead.

3.1. **Tube and Hamiltonian.** Given a connected open set $\Omega \subset \mathbb{R}^3$, denote by $-\Delta_\Omega$ the usual negative Laplacian operator with Dirichlet boundary conditions, that is, the Friedrichs extensions of $-\Delta$ with domain $\mathrm{dom}\,(-\Delta) = C_0^\infty(\Omega)$. More precisely, $-\Delta_\Omega$ is the self-adjoint operator acting in $\mathrm{L}^2(\Omega)$ associated with the positive sesquilinear form

$$b_\Omega(\psi, \varphi) = \langle \nabla_x \psi, \nabla_x \varphi \rangle, \quad \mathrm{dom}\; b_\Omega = \mathcal{H}_0^1(\Omega);$$

the inner product is in the space $\mathrm{L}^2(\Omega)$ and ∇_x is the usual gradient in cartesian coordinates (x, y, z). The operator $-\Delta_\Omega$ describes the energy of a quantum free particle in Ω. We are interested in Ω representing the following kind of tubes, which will be described in some details. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a C^3 curve parametrized by its (signed) arc length s, and introduce

$$T(s) = \dot\gamma(s), \quad N(s) = \frac{1}{\kappa(s)} \dot T(s), \quad B(s) = T(s) \times N(s),$$

with $\kappa(s) = \|\ddot\gamma(s)\|$ being the curvature of γ; the dot over a function always indicates derivative with respect to s. These quantities are the well-known tangent, normal and binormal (orthonormal) vectors of γ, respectively, which constitute a distinguished Frenet frame for the curve [24]. If κ vanishes in suitable intervals one considers a constant Frenet frame and in many cases it is possible to join distinguished and constant Frenet frames, for instance if $\kappa > 0$ on a bounded interval I and vanishing in $\mathbb{R} \setminus I$ [22]; this will be implicitly used when we deal with the broken-line limit in Section 4. Here it is assumed that such a global Frenet frame exists, and so it changes along the curve γ according to the Serret-Frenet equations

$$\begin{pmatrix} \dot T \\ \dot N \\ \dot B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\tau(s)$ is the torsion of the curve $\gamma(s)$. Although in principle closed curves can be allowed we will exclude this case since our main interest is in curves of infinite length. Nevertheless, the case of closed curves would lead to a bounded tube and discrete spectrum of the associated Laplacian $-\Delta_\alpha^\varepsilon$ (see below) and would fit in a small variation of the discussion in Subsection 3.5.

Next an open, bounded and connected subset $\emptyset \neq S \subset \mathbb{R}^2$ will be transversally linked to the reference curve γ, so that S will be the cross section of the tube. However, S will also be rotated with respect to the Frenet frame as one moves along the reference curve γ, and with rotation angle given by a C^1 function $\alpha(s)$. Given $\varepsilon > 0$, the tube so obtained (i.e., by moving S along the curve γ together with the rotation $\alpha(s)$) is given by

$$\Omega_\alpha^\varepsilon := \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) = f_\alpha^\varepsilon(s, y_1, y_2), s \in \mathbb{R}, (y_1, y_2) \in S\},$$

with $f_\alpha^\varepsilon(s, y_1, y_2) = \gamma(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s)$, and

$$\begin{aligned} N_\alpha(s) &= \cos\alpha(s) N(s) - \sin\alpha(s) B(s) \\ B_\alpha(s) &= \sin\alpha(s) N(s) + \cos\alpha(s) B(s). \end{aligned}$$

The tube is then defined by the map $f_\alpha^\varepsilon : \mathbb{R} \times S \to \Omega_\alpha^\varepsilon$, and we will be interested in the singular limit case $\varepsilon \to 0$, that is, when the tube is squeezed to the curve γ and what happens to the Dirichlet Laplacian $-\Delta_{\Omega_\alpha^\varepsilon}$ in this process (see [13] for the corresponding construction in $\mathbb{R}^n$). This will result in the one-dimensional quantum energy operator that arises after the confinement onto γ; on basis of Proposition 8.1 in [14], it is expected that this will be the relevant operator also in the case of holonomic constraints (at least from the dynamical point of view for finite times); see also Theorem 3 in [9].

Note that the tube is completely determined by the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve $\gamma(s)$, together with the cross-section S and the rotation function $\alpha(s)$. Below some conditions will be imposed on f_α^ε so that it becomes a C^1-diffeomorphism. It will be assumed that γ has no self-intersection and that its curvature is a bounded function of s, that is, $\|\kappa\|_\infty < \infty$, and, for simplicity, unless explicitly specified that $\|\tau\|_\infty, \|\dot\alpha\|_\infty < \infty$.

As usual in this context we rescale and change variables in order to work with the fixed domain $\mathbb{R} \times S$; the price we have to pay is a nontrivial Riemannian metric $G = G_\alpha^\varepsilon$, which is induced by the embedding f_α^ε, that is, $G = (G_{ij})$, $G_{ij} = e_i \cdot e_j = G_{ji}$, $1 \le i, j \le 3$, with

$$e_1 = \frac{\partial f_\alpha^\varepsilon}{\partial s}, \quad e_2 = \frac{\partial f_\alpha^\varepsilon}{\partial y_1}, \quad e_3 = \frac{\partial f_\alpha^\varepsilon}{\partial y_2}.$$

Direct computations give

$$G_\alpha^\varepsilon = \begin{pmatrix} \beta_\varepsilon^2 + \frac{1}{\varepsilon^2}(\rho_\varepsilon^2 + \sigma_\varepsilon^2) & \rho_\varepsilon & \sigma_\varepsilon \\ \rho_\varepsilon & \varepsilon^2 & 0 \\ \sigma_\varepsilon & 0 & \varepsilon^2 \end{pmatrix},$$

with $\beta_\varepsilon(s, y_1, y_2) = 1 - \varepsilon\kappa(s)(y_1 \cos\alpha(s) + y_2 \sin\alpha(s))$, $\rho_\varepsilon(s, y_1, y_2) = -\varepsilon^2 y_2(\tau(s) - \dot\alpha(s))$ and $\sigma_\varepsilon(s, y_1, y_2) = \varepsilon^2 y_1(\tau(s) - \dot\alpha(s))$. Its determinant is

$$|\det G_\alpha^\varepsilon| = \varepsilon^4 \beta_\varepsilon^2,$$

so that f_α^ε is a local diffeomorphism provided β_ε does not vanish on $\mathbb{R} \times S$, which will occur if κ is bounded and ε small enough (recall that S is a bounded set), so that $\beta_\varepsilon > 0$. By requiring that f_α^ε is injective (that is, the tube is not self-intersecting) one gets a global diffeomorphism.

Coming back to the beginning of this subsection, we pass the sesquilinear form in usual coordinates (x, y, z),

$$b_{\Omega_\alpha^\varepsilon}(\psi, \varphi) = \langle \nabla_x \psi, \nabla_x \varphi \rangle, \quad \mathrm{dom}\, b_{\Omega_\alpha^\varepsilon} = \mathcal{H}_0^1(\Omega_\alpha^\varepsilon),$$

to coordinates (s, y_1, y_2) of $\mathbb{R} \times S$ and express the inner product and gradients appropriately. For $\psi \in \mathcal{H}_0^1(\Omega_\alpha^\varepsilon)$ set $\psi(s, y_1, y_2) := \psi(f_\alpha^\varepsilon(s, y_1, y_2))$. If ∇

denotes the gradient in the (s, y_1, y_2) coordinates, then by the chain rule $\nabla_x\psi = J^{-1}\nabla\psi$ where J is the 3×3 matrix, expressed in the Frenet frame (T, N, B),

$$\begin{aligned} J &= \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ &= \begin{pmatrix} \beta_\varepsilon & \varepsilon(\tau-\dot\alpha)(y_1\sin\alpha - y_2\cos\alpha) & \varepsilon(\tau-\dot\alpha)(y_2\sin\alpha + y_1\cos\alpha) \\ 0 & -\varepsilon\cos\alpha & \varepsilon\sin\alpha \\ 0 & \varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix}. \end{aligned}$$

Noting that $JJ^t = G$, $\det J = |\det G|^{1/2} = \varepsilon^2\beta_\varepsilon$, and introducing the notation

$$\langle\psi,\varphi\rangle_G = \int_{\mathbb{R}\times S} \overline{\psi(s,y_1,y_2)}\varphi(s,y_1,y_2)\ \varepsilon^2\beta_\varepsilon(s)\ dsdy_1dy_2,$$

it follows that

$$\langle\nabla_x\psi,\nabla_x\varphi\rangle = \langle J^{-1}\nabla\psi, J^{-1}\nabla\varphi\rangle_G = \langle\nabla\psi, G^{-1}\nabla\varphi\rangle_G$$

and the operator $-\Delta^\varepsilon_\alpha$ can be described as the operator associated with the positive sesquilinear form

$$\mathrm{dom}\ \tilde b^\varepsilon = \mathcal{H}^1_0(\mathbb{R}\times S, G), \quad \tilde b^\varepsilon(\psi,\phi) := \langle\nabla\psi, G^{-1}\nabla\varphi\rangle_G,$$

and the ε dependence is now in the Riemannian metric G. More precisely, the above change of variables is implemented by the unitary transformation

$$U : \mathrm{L}^2(\Omega^\varepsilon_\alpha) \to \mathrm{L}^2(\mathbb{R}\times S, G), \quad U\psi = \psi\circ f^\varepsilon_\alpha.$$

Note, however, that usually we will continue denoting $U\psi$ simply by ψ. Explicitly the quadratic form is given by (with $dy = dy_1dy_2$ and $\nabla_\perp\psi = (\partial_{y_1}\psi, \partial_{y_2}\psi)$, so that $\nabla = (\partial_s, \nabla_\perp)$)

$$\begin{aligned} \tilde b^\varepsilon(\psi) &= \left\|J^{-1}\nabla\psi\right\|^2_G \\ &= \varepsilon^2\int_{\mathbb{R}\times S} dsdy\left[\frac{1}{\beta_\varepsilon}\left|\nabla\psi\cdot(1, y_2(\tau-\dot\alpha), y_1(\tau-\dot\alpha))\right|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_\perp\psi|^2\right] \\ &= \varepsilon^2\int_{\mathbb{R}\times S} dsdy\left[\frac{1}{\beta_\varepsilon}\left|\nabla\psi\cdot(1, Ry(\tau-\dot\alpha))\right|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_\perp\psi|^2\right], \end{aligned}$$

where $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On functions $\psi\in C^\infty_0(\mathbb{R}\times S)$ a calculation shows that the corresponding operator has the following action

$$U(-\Delta^\varepsilon_\alpha)U^*\psi = -\varepsilon^2\frac{1}{\beta_\varepsilon}\mathrm{div}\beta_\varepsilon G^{-1}\nabla\psi,$$

and mixed derivatives will not be present iff $\tau(s) - \dot{\alpha}(s) = 0$, since this is the condition for G^{-1} be a diagonal matrix:

$$G^{-1} = \begin{pmatrix} \beta_\varepsilon^{-2} & 0 & 0 \\ 0 & \varepsilon^{-2} & 0 \\ 0 & 0 & \varepsilon^{-2} \end{pmatrix}.$$

This hypothesis simplifies the operator expression and it is the main reason this case attracted more attention [11, 12]; more recently the general case has also been considered and the spectral-geometric effects have been reviewed in [25]. With respect to the limit $\varepsilon \to 0$ for the more general tubes we consider here, the expression of the operator actions seem prohibitive to be manipulated in a useful way; so the main advantage in using sesquilinear forms and Γ-convergence, as studied in case of bounded tubes in [8].

Since we are interested in the limit of the tube approaching the reference curve γ, i.e., $\varepsilon \to 0$, it is still necessary to perform two kinds of "regularizations" in $\tilde{b}^\varepsilon$ in order to extract a meaningful limit; these are common approaches to balance singular problems, particularly due to the presence of regions that scale in different manners, and so to put them in a tractable form [7]. The first one is physically related to the uncertainty principle in quantum mechanics and is in fact a renormalization. Let $u_n \in \mathcal{H}_0^1(S)$ and $\lambda_n \in \mathbb{R}$, $n \geq 0$, be the normalized eigenfunctions and corresponding eigenvalues of the (negative) Laplacian restricted to the cross section S (since S is bounded the Dirichlet Laplacian on S has compact resolvent), and we suppose that $\lambda_0 < \lambda_1 < \lambda_2 \cdots$ and that all eigenvalues λ_n are simple; later on we will underline where this assumption is used (see, in particular, Subsection 3.3.1).

When the tube is squeezed there are divergent energies due to terms of the form λ_n/ε^2, and one needs a rule to get rid of these energies related to transverse oscillations in the tube. Since in quantum mechanics these quadratic forms $\tilde{b}^\varepsilon$ correspond to expectation values of total energy, one subtracts such diverging terms from $\tilde{b}^\varepsilon$. However, in principle it is not clear which expression one should use in the subtraction process and we will consider the following possibilities

$$\tilde{b}^\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2}\|\psi\|_G^2 = \tilde{b}^\varepsilon(\psi) - \lambda_n \int_{\mathbb{R}\times S} dsdy\; \beta_\varepsilon(s,y)|\psi(s,y)|^2.$$

The second regularization is simply a division by the global factor ε^2 so defining the family of quadratic forms we will work with:

$$\begin{aligned} b_n^\varepsilon(\psi) &:= \varepsilon^{-2}\left(\tilde{b}^\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2}\|\psi\|_G^2\right) \\ &= \int_{\mathbb{R}\times S} dsdy \left[\frac{1}{\beta_\varepsilon}\left|\nabla\psi\cdot(1, Ry(\tau-\dot{\alpha}))\right|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_\perp\psi|^2 - \lambda_n|\psi|^2\right)\right], \end{aligned}$$

dom $b_n^\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S)$. In [8] only b_0^ε was considered, and ahead we propose a procedure to deal with b_n^ε, for all $n \geq 0$. After such regularizations we finally obtain the operators H_n^ε, associated with these forms, for which we will investigate the limit $\varepsilon \to 0$. Let $\mathcal{H}^\varepsilon$ denote the Hilbert space $\mathrm{L}^2(\mathbb{R} \times S, \beta_\varepsilon)$, that is, the inner product is given by

$$\langle \psi, \varphi \rangle_\varepsilon := \int_{\mathbb{R}\times S} \overline{\psi(s,y)}\varphi(x,y)\, \beta_\varepsilon(s,y)\, dsdy.$$

Definition 2. *The operator H_n^ε is the self-adjoint operator associated with the sesquilinear form b_n^ε (see* [1]*, page 101), whose domain* dom H_n^ε *is dense in* dom b_n^ε *and*

$$b_n^\varepsilon(\psi, \varphi) = \langle \psi, H_n^\varepsilon \varphi \rangle_\varepsilon, \quad \forall \psi \in \mathrm{dom}\ b_n^\varepsilon, \forall \varphi \in \mathrm{dom}\ H_n^\varepsilon.$$

In case the functions $\kappa, \tau, \dot\alpha$ are bounded and S has a smooth boundary then, by elliptic regularity [26]*,* dom $H_n^\varepsilon = \mathcal{H}^2(\mathbb{R} \times S) \cap \mathcal{H}_0^1(\mathbb{R} \times S)$*.*

Remark 5. *In the above construction of $\mathcal{H}^\varepsilon$ and H_n^ε it was very important the uniform bound of β_ε, which implies that for all $\varepsilon > 0$ small enough there exist $0 < a_\varepsilon \le a^\varepsilon < \infty$ with*

$$a_\varepsilon \| \cdot \| \le \| \cdot \|_\varepsilon \le a^\varepsilon \| \cdot \|,$$

(similar inequalities also hold for the associated quadratic forms b_n^ε) so that all Hilbert spaces $\mathcal{H}^\varepsilon$ coincide algebraically with $\mathrm{L}^2(\mathbb{R} \times S)$ *and also have equivalent norms. Furthermore, it is possible to assume that both $a_\varepsilon \to 1$ and $a^\varepsilon \to 1$ hold as $\varepsilon \to 0$, since the sequence of functions $\beta_\varepsilon \to 1$ uniformly. These properties imply the equality of the quadratic form domains for all $\varepsilon > 0$ and permit us to speak of* Γ*-convergence of b_n^ε and resolvent convergence of H_n^ε in* $\mathrm{L}^2(\mathbb{R}\times S)$ *even though they act in, strictly speaking, different Hilbert spaces. Such facts will be freely used ahead.*

Definition 3. *Given $\Omega_\alpha^\varepsilon$, let $C_n(S) := \int_S dy\, |\nabla_\perp u_n(y) \cdot Ry|^2$, where, as before, u_n is the $(n+1)$th normalized eigenfunction of the negative Laplacian on S. The tube $\Omega_\alpha^\varepsilon$ is said to be* quantum twisted *if for some n the function*

$$\mathcal{A}_n(s) := (\tau(s) - \dot\alpha(s))^2 C_n(S) \neq 0.$$

Note that the quantities $C_n(S)$ are parameters that depend only on the cross section S, and that $\mathcal{A}_n$ is zero if either the cross section rotation compensates the torsion of the curve (i.e., $\tau - \dot\alpha = 0$) or if the eigenfunction u_n is radial (i.e., $C_n(S) = 0$); $\mathcal{A}_n$ has a geometrical nature and $\mathcal{A}_0$ was first considered [8]. In [25] there is an interesting list of equivalent formulations of the condition $\tau - \dot\alpha = 0$.

It should be mentioned that the case of twisting has been addressed also in the case of infinite curve γ in [22], and that there are some studies realized in $\mathbb{R}^n$ for $n \ge 3$ [13].

3.2. **Confinement: Statements.** In this subsection we discuss results about the limit $\varepsilon \to 0$ of H_n^ε. We begin with H_0^ε and will follow closely [8], where Γ-convergence was used to study the limit operator upon confinement and convergence of eigenvalues in case of bounded curves γ (so bounded tubes). We will point out the necessary modifications in their approach in order to prove Theorem 5 below, which is in the setting of $\mathrm{L}^2(\mathbb{R}\times S)$ and no restriction on the curve length, although our results on spectral convergence are limited to the standard consequences of strong operator resolvent convergence (not stated here; see, for instance, Corollary 10.2.2 in [1]); in any event, ahead we shall recover their spectral convergence by means of our Proposition 4.

For each $n \geq 0$, denote by L_n (resp. h_n) the Hilbert subspace of $\mathrm{L}^2(\mathbb{R}\times S)$ (resp. $\mathcal{H}_0^1(\mathbb{R}\times S)$) of vectors of the form $\psi(s,y) = w(s)u_n(y)$, $w \in \mathrm{L}^2(\mathbb{R})$ (resp. $w \in \mathcal{H}^1(\mathbb{R}) = \mathcal{H}_0^1(\mathbb{R})$), so that

$$\begin{aligned} \mathrm{L}^2(\mathbb{R}\times S) &= \mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \mathrm{L}_2 \oplus \cdots, \\ \mathcal{H}_0^1(\mathbb{R}\times S) &= h_0 \oplus h_1 \oplus h_2 \oplus \cdots, \end{aligned}$$

and each h_n is a dense subspace of L_n. Note that h_n is related to b_n^ε. In Theorem 5 we deal with the limit of b_0^ε and generalize some results of [8] for elements of h_0; then we propose a procedure to deal with the relation between the limit of b_n^ε and h_n, for all n.

Now define the real potentials

$$\begin{aligned} V_n^{\mathrm{eff}}(s) &:= \mathcal{A}_n(s) - \frac{1}{4}\kappa(s)^2 \\ &= (\tau(s) - \dot\alpha(s))^2\, C_n - \frac{1}{4}\kappa(s)^2, \quad n \geq 0, \end{aligned}$$

and the corresponding Schrödinger operators

$$(H_n^0\psi)(s) = -\frac{d^2}{ds^2}\psi(s) + V_n^{\mathrm{eff}}(s)\psi(s),$$

whose domains are $\mathrm{dom}\, H_n^0 = \mathcal{H}^2(\mathbb{R})$, for all $n \geq 0$, in case of bounded functions $\kappa, \tau, \dot\alpha$. Here the curvature κ is always assumed to be bounded, but if either the torsion τ or the $\dot\alpha$ is not bounded, the domain must be discussed on an almost case-by-case basis. The subspace L_n can be identified with $\overline{\mathrm{dom}\, H_n^0} = \mathrm{L}^2(\mathbb{R})$ via $wu_n \mapsto w$. Let b_n^0 be the sesquilinear form generated by H_n^0, that is,

$$b_n^0(\psi) = \int_{\mathbb{R}} ds\ \left(|\dot\psi(s)|^2 + V_n^{\mathrm{eff}}(s)|\psi(s)|^2\right)$$

whose $\mathrm{dom}\, b_n^0 = \mathcal{H}^1(\mathbb{R})$ (for bounded $\kappa, \tau, \dot\alpha$) can be identified with h_n; hence $b_n^0(\psi) = \infty$ if $\psi \in \mathrm{L}_n \setminus h_n$.

Theorem 5. *The sequence of self-adjoint operators H_0^ε converges in the strong resolvent sense to H_0^0 in L_0 as $\varepsilon \to 0$.*

Remark 6. *We have then obtained an explicit form of the effective operator that describes the confinement of a free quantum particle (in the subspace* L_0*) from $\Omega_\alpha^\varepsilon$ to the curve γ. The action of the operator H_0^0 is the same as that in* [8] *and, as already anticipated, the arguments below for its proof are based on Γ-convergence of quadratic forms and mainly consist of indications* *of the necessary modifications to take into account unbounded curves, that is, the results of Section 2 and the important additional verification of weak* Γ*-convergence of the involved quadratic forms.*

It is possible to get some intuition about what should be expected for the convergence of H_n^ε, $n \geq 1$, by considering the form b_n^ε evaluated at vectors of h_n, i.e., wu_n, $w \in \mathcal{H}^1(\mathbb{R})$, and some formal arguments. A direct substitution gives an integral with four terms

$$\begin{aligned} b_n^\varepsilon(wu_n) &= \int_{\mathbb{R}\times S} dsdy \Big[\frac{1}{\beta_\varepsilon}|\dot{w}|^2|u_n|^2 + |w|^2\Big(\frac{1}{\beta_\varepsilon}|\nabla_\perp u_n \cdot Ry(\tau-\dot{\alpha})|^2 \\ &+ \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_\perp u|^2 - \lambda_0|u|^2\right)\Big) + \frac{1}{\beta_\varepsilon} 2\mathrm{Re}\,\left(\overline{\dot{w}u_n}\, w\nabla_\perp u_n \cdot Ry\right)\Big], \end{aligned}$$

and if we approximate $\beta_\varepsilon \approx 1$, except in the third term, we find that the last term vanishes due to the Dirichlet boundary condition and thus (recall that u_n is normalized in $\mathrm{L}^2(S)$)

$$\begin{aligned} b_n^\varepsilon(wu_n) &\approx \int_{\mathbb{R}} ds\left[|\dot{w}(s)|^2 + C_n(S)(\tau(s)-\dot{\alpha}(s))^2|w(s)|^2\right] \\ &+ \int_{\mathbb{R}} ds\,|w(s)|^2\int_S dy\,\frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_\perp u_n(y)|^2 - \lambda_n|u_n(y)|^2\right)\Big), \end{aligned}$$

so that the first term $\mathcal{A}_n(s)$ in the effective potential $V_n^{\mathrm{eff}}(s)$ is clearly visible. However, the remaining term

$$K_n^\varepsilon(u,s) := \int_S dy\,\frac{\beta_\varepsilon(s,y)}{\varepsilon^2}\left(|\nabla_\perp u(y)|^2 - \lambda_n|u(y)|^2\right)$$

is related to the curvature and requires much more work; this was a major contribution of [8] in case $n = 0$ through the study of minima and strong Γ-convergence. It is shown in [8] that K_0^ε attains a minimum $-\kappa^2/4 > -\infty$ since λ_0 is the bottom of the spectrum of the Laplacian restricted to the cross section S; however, a minimum is not expected to occur in case of K_n^ε, $n \geq 1$, since, if we take again the approximation $\beta_\varepsilon \approx 1$, there will be vectors $u \in h_j$, $j < n$, such that

$$K_n^\varepsilon(u,s) \approx \frac{1}{\varepsilon^2}(\lambda_j - \lambda_n) \to -\infty, \quad \varepsilon \to 0,$$

and this unboundedness from below pushes u away of the natural range of applicability of quadratic forms and Γ-convergence. We then impose that if $b_n^\varepsilon(\psi) \to \pm\infty$, then the vector ψ is simply excluded from the domain of the limit sesquilinear form, and so also from the domain of the limit operator. Since Theorem 5 tells us how to deal with vectors in L_0, by taking the above discussion into account we propose to study the convergence of b_1^ε restricted to the orthogonal complement of L_0 in $\mathrm{L}^2(\mathbb{R}\times S)$ and, more generally, to study the Γ-convergence of b_n^ε restricted to

$$\mathcal{E}_n := \left(\mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_{n-1}\right)^\perp, \quad n \geq 1.$$

In what follows these domain restrictions will also apply to the resolvent convergence of the restriction to $\mathcal{E}_n$ of the associated self-adjoint operators and will be used without additional warnings. Under such procedure we shall obtain the following result (recall that it is assumed that all eigenvalues λ_n of the Laplacian restricted to the cross section S are simple).

Theorem 6. *The sequence of self-adjoint operators obtained by the restriction of H_n^ε to* dom $H_n^\varepsilon \cap \mathcal{E}_n$ *converges, in the strong resolvent sense in L_n, to H_n^0 as $\varepsilon \to 0$.*

Remark 7. *Since quadratic forms can conveniently take the value $+\infty$, here we have rather different nomenclatures for singular convergences, that is, the sequence of forms b_n^ε converges in $\mathcal{E}_n$ whereas the matching sequence of operators H_n^ε converges in* L_n.

Remark 8. *Note that this procedure leads to a kind of decoupling among the subspaces h_n, which is supported by some cases of plane waveguides treated in* [10] *where the decoupling is found through an explicitly separation of variables; see the presence of Kronecker deltas in the expressions for resolvents in Theorem 1 and Lemma 3 in* [10]. *In 3D such separation occurs if there is no twisting* [11].

Remark 9. *In the first approximation, in the vicinity of the overall spectral threshold (i.e., by considering λ_0), the effective Hamiltonian describes bound states. However, at higher thresholds (i.e., $\lambda_n, n \geq 1$), the effective Hamiltonians are usually related to resonances in thin tubes (see* [21, 20]*), and so this physical content is another motivation for the consideration of $b_n, n \geq 1$.*

Remark 10. *A novelty with respect to similar situations in $\mathbb{R}^2$* [10, 9] *is that the effective operator H_n^0 describing the particle confined to the curve $\gamma(s)$ depends on n, that is, we have different quantum dynamics for different subspaces L_n. As mentioned in the Introduction, this is a counterpart of some situations found in* [15, 14], *since there the effective dynamics may depend on the initial condition.*

Since the above procedure was strongly based on the mentioned estimate

$$K_n^\varepsilon(u,s) \approx \frac{1}{\varepsilon^2}(\lambda_j - \lambda_n) \to -\infty, \quad j < n,$$

as well as the rather natural expectation that it is "spontaneously" possible an energy transfer only from higher to lower energy states, we present a rigorous version of such estimate.

Proposition 5. *For normalized $u(s,y) = w(s)u_j(y)$, $w \in \mathcal{H}^1(\mathbb{R})$, one has*

$$\lim_{\varepsilon \to 0} \varepsilon^2 b_n^\varepsilon(wu_j) = (\lambda_j - \lambda_n).$$

In particular, for small ε and a.e. s, $K_n^\varepsilon(u,s) \approx \frac{1}{\varepsilon^2}(\lambda_j - \lambda_n) \to -\infty$ if $j < n$ as $\varepsilon \to 0$.

Proof. Again a direct substitution and taking into account the Dirichlet boundary condition yield

$$\begin{aligned} b_n^\varepsilon(wu_j) &= \int_{\mathbb{R}\times S} dsdy \Big[\frac{1}{\beta_\varepsilon} |\dot w|^2 |u_j|^2 \\ &+ |w|^2 \Big(\frac{1}{\beta_\varepsilon} |\nabla_\perp u_j \cdot Ry(\tau - \dot\alpha)|^2 + \frac{\beta_\varepsilon}{\varepsilon^2} \left(|\nabla_\perp u_j|^2 - \lambda_0 |u_j|^2 \right) \Big) \Big], \end{aligned}$$

and the only term that may not vanish as $\varepsilon \to 0$ in $\varepsilon^2 b_n^\varepsilon(wu_j)$ is the last one, so it is enough to analyze

$$K_n^\varepsilon(u_j, s) := \int_S dy \, \frac{\beta_\varepsilon(s,y)}{\varepsilon^2} \left(|\nabla_\perp u_j(y)|^2 - \lambda_n |u_j(y)|^2 \right).$$

Write $\beta_\varepsilon = 1 - \xi \cdot y$, with $\xi = \varepsilon\kappa(s) z_\alpha$ and $z_\alpha = (\cos\alpha, \sin\alpha)$. Since $u_j \in \mathcal{H}^2(S)$ and u_j satisfies the Dirichlet boundary condition, upon integrating by parts one gets, for a.e. s,

$$\begin{aligned} K_n^\varepsilon(u_j, s) &= -\frac{1}{\varepsilon^2} \int_S dy \, \overline{u_j} \left(\nabla_\perp \cdot (\beta_\varepsilon \nabla_\perp u_j) + \lambda_n \beta_\varepsilon u_j \right) \\ &= -\frac{1}{\varepsilon^2} \int_S dy \, \overline{u_j} \left(\nabla_\perp \beta_\varepsilon \cdot \nabla_\perp u_j + \beta_\varepsilon (\Delta_\perp u_j + \lambda_n u_j) \right) \\ &= -\frac{1}{\varepsilon^2} \int_S dy \, \overline{u_j} \left(-\xi \cdot \nabla_\perp u_j + \beta_\varepsilon (-\lambda_j + \lambda_n) u_j \right) \\ &= \frac{1}{\varepsilon} \kappa(s) z_\alpha \cdot \int_S dy \, \overline{u_j} \nabla_\perp u_j + \frac{(\lambda_j - \lambda_n)}{\varepsilon^2} \int_S dy \, \beta_\varepsilon |u_j|^2. \end{aligned}$$

Exchanging the roles of u_j and $\overline{u_j}$ yields

$$K_n^\varepsilon(u_j, s) = \frac{1}{\varepsilon} \kappa(s) z_\alpha \cdot \int_S dy \, u_j \nabla_\perp \overline{u_j} + \frac{(\lambda_j - \lambda_n)}{\varepsilon^2} \int_S dy \, \beta_\varepsilon |u_j|^2,$$

and by adding such expressions

$$K_n^\varepsilon(u_j, s) = \frac{1}{2\varepsilon} \kappa(s) z_\alpha \cdot \int_S dy \, \nabla_\perp |u_j|^2 + \frac{(\lambda_j - \lambda_n)}{\varepsilon^2} \int_S dy \, \beta_\varepsilon |u_j|^2.$$

The Dirichlet boundary condition implies $\int_S dy \, \nabla_\perp |u_j|^2 = 0$. Since by dominated convergence $\int_S dy \, \beta_\varepsilon |u_j|^2 \to \|u_j\|^2 = 1$ as $\varepsilon \to 0$, it follows that $\varepsilon^2 K_{n,j}(s) \to (\lambda_j - \lambda_n)$ for a.e. $s \in \mathbb{R}$. Again by dominated convergence

$$\lim_{\varepsilon \to 0} \varepsilon^2 b_n^\varepsilon(wu_j) = (\lambda_j - \lambda_n).$$

The proof is complete. □

The reader may protest at this point that the above procedure of considering restriction to $\mathcal{E}_n$ when dealing with b_n, $n > 0$, should in fact be deduced instead of assumed. However, we think that such deduction is certainly beyond the range of Γ-convergence, since by beginning with the form b_n^ε the vectors $\psi \in h_j$, $j > n$, will not belong to the domain of the limit form b_n^0 since $\lim_{\varepsilon\to 0} b_n^\varepsilon(\psi) = +\infty$, while for vectors $\psi \in h_j$, $j < n$, the $\lim_{\varepsilon\to 0} b_n^\varepsilon(\psi) = -\infty$ indicates that they would be outside the usual scope of limit of forms and Γ-convergence. Further, we have got reasonable expectations that the domain of the limit form b_n^0 should consists of only vectors in $h_n \subset \mathrm{L}_n$. By taking into account the proposed procedure of suitable domain restrictions, in the next subsection we will support such expectations by proving the above theorems; we stress that there is no mathematical issue in the formulation of the above procedure (e.g., if $\psi \in \mathrm{dom}\ H_n^\varepsilon \cap \mathcal{E}_n$, then $H_n^\varepsilon \psi \in \mathcal{E}_n$) and that it rests only on physical interpretations.

3.3. **Confinement: Convergence.** In this subsection the proofs of Theorems 5 and 6 are presented. There are two main steps; the first one is to check the strong Γ-convergence (recall the restriction to the subspace $\mathcal{E}_n$)

$$b_n^\varepsilon \xrightarrow{\mathrm{S}\Gamma} b_n^0,$$

and the second one is a verification that if $\psi_\varepsilon \rightharpoonup \psi$ in $\mathcal{E}_n \subset \mathrm{L}^2(\mathbb{R} \times S)$, then

$$b_n^0(\psi) \le \liminf_{\varepsilon \to 0} b_n^\varepsilon(\psi_\varepsilon).$$

The theorems will then follow by Theorem 3, including the additional information $b_n^\varepsilon \xrightarrow{\mathrm{W}\Gamma} b_n^0$ in $\mathcal{E}_n$. The proof of the first step is just a verification that the proof of the corresponding result in [8] can be explicitly carried out for unbounded tubes and for b_n^ε with $n \ge 0$. Then we shall address the proof of the second step and at the end of this subsection we briefly discuss how the spectral convergences in [8] can be recovered from our Proposition 4.

3.3.1. *First Step.* I will be very objective and just indicate the adaptations to the proofs in [8]; I will also use a notation similar to the one used in that work. The first point to be considered is a variation of Proposition 4.1 in [8], that is, the study of the quantities

$$\lambda_n(\xi) := \inf_{\substack{0 \ne v \in \mathcal{H}_0^1(S) \\ v \in [u_0, \cdots, u_{n-1}]^\perp}} \frac{\int_S (1 - \xi \cdot y) |\nabla_\perp v(y)|^2}{\int_S (1 - \xi \cdot y) |v(y)|^2}, \quad n > 0,$$

and

$$\gamma_{\varepsilon,n}(s) := \frac{1}{\varepsilon^2} \left(\lambda_n(\xi) - \lambda_n \right), \quad \xi = \varepsilon \kappa(s) z_\alpha(s);$$

recall that $z_\alpha = (\cos\alpha, \sin\alpha)$. In case $n = 0$ the above infimum is taken over $0 \ne v \in \mathcal{H}_0^1(S)$; $[u_0, \cdots, u_n]$ denotes the subspace in $\mathcal{H}_0^1(S)$ spanned by the first eigenvectors $u_0, \cdots, u_n$ of the (negative) Laplacian restricted to the cross section S and recall that here its eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 \cdots$ are supposed to be simple.

The prime goal is to show that

$$\gamma_{\varepsilon,n}(s) \to -\frac{1}{4} \kappa^2(s)$$

uniformly on $\mathbb{R}$ (Proposition 4.1 in [8]; don't confuse $\gamma_{\varepsilon,n}(s)$ with the reference curve $\gamma(s)$). In order to prove this we consider the unique solution

$u_{\xi,n} \in \mathcal{H}_0^1(S)$ of the problem

$$-\Delta_\perp u_{\xi,n} - \lambda_n u_{\xi,n} = -\xi \cdot \nabla_\perp u_n, \quad u_{\xi,n} \in [u_0, \cdots, u_n]^\perp,$$

which exists by Fredholm alternative since $[u_0, \cdots, u_n]^\perp$ is invariant under the operator $(-\Delta_\perp)$, the real number λ_n belongs to the resolvent set of the restriction $(-\Delta_\perp)|_{[u_0,\cdots,u_n]^\perp}$ (again because the eigenvalues are simple) and $\xi \cdot \nabla_\perp u_n$ is orthogonal to u_n. Note that the simplicity of eigenvalues was again invoked to guarantee the unicity of solution and that $u_{\xi,n}$ is real since its complex conjugate is also a solution of the same problem.

Since we can take real-valued eigenfunctions u_n, the Lemma 4.3 in [8] will read

$$\inf_{\substack{v \in \mathcal{H}_0^1(S) \\ v \in [u_0,\cdots,u_{n-1}]^\perp}} \int_S dy \ \left[|\nabla_\perp v|^2 - \lambda_n |v|^2 + (\xi \cdot \nabla_\perp u_n)(v + \bar{v}) \right] = -\frac{1}{4}\xi^2,$$

with the infimum reached precisely for $u_{\xi,n}$, also a real-valued function. With such remarks the proof of Lemma 4.3 in [8] also works for all $n \geq 0$. The same words can be repeated for the proof of the above mentioned Proposition 4.1, the only necessary additional remark is that the function φ in their equation (4.12) can also be taken real, since in our context it always appears in the form $(\varphi + \bar{\varphi})$. With such modifications we obtain suitable versions of that proposition, that is, $\gamma_{\varepsilon,n}(s) \to -\frac{1}{4}\kappa^2(s)$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$ for all $n \geq 0$. With such tools at hand the proof of $b_n^\varepsilon \xrightarrow{\text{S}\Gamma} b_n^0$ in $\mathcal{E}_n$, for all $n \geq 0$, is exactly that done in Section 4.3 of [8]. This completes our first main step.

3.3.2. *Second Step.* Now we are going to complete the proofs of Theorems 5 and 6. However we consider a simple modification of the forms b_n^ε, that is, we shall consider

$$b_{n,c}^\varepsilon(\psi) := b_n^\varepsilon(\psi) + c\|\psi\|_\varepsilon^2, \quad \text{dom } b_{n,c}^\varepsilon = \text{dom } b_n^\varepsilon \subset \mathcal{E}_n,$$

for some $c \geq \|\kappa\|_\infty^2$ so that we work with the more natural set of positive forms. This implies that if H_n^ε is the self-adjoint operator associated with b_n^ε, then $H_{n,c}^\varepsilon := H_n^\varepsilon + c\mathbf{1}$ is the operator associated with $b_{n,c}^\varepsilon$ and vice versa; so strong resolvent convergence $H_n^\varepsilon \to H_n^0$ in L_n is equivalent to strong resolvent convergence $H_{n,c}^\varepsilon \to H_{n,c}^0 := H_n^0 + c\mathbf{1}$ in L_n as $\varepsilon \to 0$.

For each $\varepsilon > 0$ and $\psi \in \text{dom } b_{n,c}^\varepsilon$ we have the important lower bound (by the definitions of the forms and $\gamma_{\varepsilon,n}$; see also equation 4.19 in [8])

$$b_{n,c}^\varepsilon(\psi) \geq \int_{\mathbb{R}\times S} dsdy \left(\frac{1}{\beta_\varepsilon} \left| \dot{\psi} + \nabla_\perp \psi \cdot Ry(\tau - \dot{\alpha}) \right|^2 + \beta_\varepsilon (c - \gamma_{\varepsilon,n}) |\psi|^2 \right).$$

Our interest in considering the modified forms $b_{n,c}^\varepsilon$ is also that, for ε small enough,

$$b_{n,c}^\varepsilon(\psi) \geq \left(c - \frac{1}{2}\|\kappa\|_\infty^2 \right) \|\psi\|^2 \geq \frac{1}{2}\|\kappa\|_\infty^2 \, \|\psi\|^2$$

and so both $b^\varepsilon_{n,c}$ and $H^\varepsilon_{n,c}$ are positive; this will be important when dealing with weak convergence ahead. Furthermore, the arguments in First Step above and Theorem 2, with f playing the role of the norm, which is strongly continuous, also show that

$$b^\varepsilon_{n,c} \stackrel{\mathrm{S}\Gamma}{\longrightarrow} b^0_{n,c} \quad \text{in} \quad \mathcal{E}_n,$$

where

$$b^0_{n,c}(\psi) := \int_{\mathbb{R}} ds \, \left(|\dot\psi(s)|^2 + \left[(\tau(s) - \dot\alpha(s))^2 \, C_n + c - \frac{1}{4}\kappa(s)^2 \right] |\psi(s)|^2 \right)$$

is the form generated by $H^0_{n,c}$, with dom $b^0_{n,c} = \mathcal{H}^1_0(\mathbb{R})$ identified with h_n.

Now we have the task of checking that if $\psi_\varepsilon \rightharpoonup \psi$ in $\mathcal{E}_n \subset \mathrm{L}^2(\mathbb{R} \times S)$, then

$$\liminf_{\varepsilon \to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) \geq b^0_{n,c}(\psi),$$

in other words, the second main step mentioned above. By Theorem 3 i) we will then conclude the strong resolvent convergence $H^\varepsilon_{n,c} \to H^0_{n,c}$ in L_n, which will complete the desired proofs. So assume the weak convergence $\psi_\varepsilon \rightharpoonup \psi$. If ψ_ε does not belong to $\mathcal{H}^1_0(\mathbb{R} \times S) \cap \mathcal{E}_n$, then $b^\varepsilon_{n,c}(\psi_\varepsilon) = \infty$, for all $\varepsilon > 0$; so we can assume that $(\psi_\varepsilon) \subset \mathcal{H}^1_0(\mathbb{R} \times S) \cap \mathcal{E}_n$ and, up to subsequences, that

$$\liminf_{\varepsilon \to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) = \lim_{\varepsilon \to 0} b^\varepsilon_{n,c}(\psi_\varepsilon).$$

Since (ψ_ε) is a weakly convergent sequence, it is bounded in $\mathrm{L}^2(\mathbb{R} \times S)$ and we can also suppose that $\sup_\varepsilon b^\varepsilon_{n,c}(\psi_\varepsilon) < \infty$; hence, as on page 804 of [8], it follows that (ψ_ε) is a bounded sequence in $\mathcal{H}^1_0(\mathbb{R} \times S)$. Since Hilbert spaces are reflexive, (ψ_ε) has a subsequence, again denoted by (ψ_ε), so that $\psi_\varepsilon \rightharpoonup \phi$ in $\mathcal{H}^1_0(\mathbb{R} \times S)$. Since also $\psi_\varepsilon \rightharpoonup \psi$ in $\mathrm{L}^2(\mathbb{R} \times S)$ it follows that $\psi = \phi$. Therefore

$$\dot\psi_\varepsilon + \nabla_\perp \psi_\varepsilon \cdot Ry(\tau - \dot\alpha) \rightharpoonup \dot\psi + \nabla_\perp \psi \cdot Ry(\tau - \dot\alpha)$$

and the weak lower semicontinuity of the L^2-norm (again the importance of introducing the parameter c above), the above lower bound $b^\varepsilon_{n,c}(\psi) \geq 1/2\|\kappa\|^2_\infty \|\psi\|^2$, together with the uniform convergences

$$\beta_\varepsilon \to 1, \quad \gamma_{\varepsilon,n} \to -\frac{1}{4}\kappa^2,$$

imply that

$$\begin{aligned} \lim_{\varepsilon \to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) \quad &\geq \quad \mathcal{G}(\psi) \\ &:= \quad \int_{\mathbb{R} \times S} dsdy \left(\left| \dot\psi + \nabla_\perp \psi \cdot Ry(\tau - \dot\alpha) \right|^2 + \left(c - \frac{1}{4}\kappa^2 \right) |\psi|^2 \right). \end{aligned}$$

If $\psi \in$ dom $b^0_{n,c}$, that is, $\psi = w(s)u_n(y)$, $w \in \mathcal{H}^1(\mathbb{R})$, then a direct substitution infers that $\mathcal{G}(wu_n) = b^0_{n,c}(wu_n)$ and so

$$\lim_{\varepsilon \to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) \geq b^0_{n,c}(\psi)$$

in this case. Now we will show that, for ψ with a nonzero component in the complement of dom $b^0_{n,c}$, necessarily $\lim_{\varepsilon\to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) = \infty$. In fact, if ψ does not belong to dom $b^0_{n,c}$ then $\|P_{n+1}\psi\| > 0$ where P_{n+1} is the orthogonal projection onto $\mathcal{E}_{n+1}$. Because $\psi_\varepsilon \rightharpoonup \psi$ in $\mathcal{E}_n \cap \mathcal{H}^1_0(\mathbb{R}\times S)$ it follows that $P_{n+1}\psi_\varepsilon \rightharpoonup P_{n+1}\psi$, and since the L^2-norm is weakly l.sc. we find

$$\liminf_{\varepsilon\to 0} \|P_{n+1}\psi_\varepsilon\| \geq \|P_{n+1}\psi\| > 0.$$

Hence for ε small enough the function ψ_ε has a nonzero component $P_{n+1}\psi_\varepsilon$ in $\mathcal{E}_{n+1}$ and the $\mathrm{L}^2(\mathbb{R}\times S)$-norm of such components are uniformly bounded from zero by $\|P_{n+1}\psi\|$. Now, $\sup_\varepsilon \|\psi_\varepsilon\|_\varepsilon < \infty$ and recalling that

$$\beta_\varepsilon(s,y) = 1 - \xi\cdot y, \quad \xi = \varepsilon\kappa(s) z_\alpha,$$

one has

$$\begin{aligned}
b^\varepsilon_{n,c}(\psi_\varepsilon) &= \int_{\mathbb{R}\times S} dsdy \Big[\frac{1}{\beta_\varepsilon} |\nabla\psi_\varepsilon \cdot (1, Ry(\tau - \dot\alpha))|^2 \\
&+ \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_\perp \psi_\varepsilon|^2 - \lambda_n |\psi_\varepsilon|^2\right)\Big] + c\|\psi_\varepsilon\|^2_\varepsilon \\
&\geq \int_{\mathbb{R}\times S} dsdy\, \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_\perp \psi_\varepsilon|^2 - \lambda_n |\psi_\varepsilon|^2\right) \\
&= \frac{1}{\varepsilon^2}\int_{\mathbb{R}\times S} dsdy\, \left(|\nabla_\perp \psi_\varepsilon|^2 - \lambda_n |\psi_\varepsilon|^2\right) \\
&- \frac{1}{\varepsilon^2}\int_{\mathbb{R}\times S} dsdy\, (\xi\cdot y)\left(|\nabla_\perp \psi_\varepsilon|^2 - \lambda_n |\psi_\varepsilon|^2\right).
\end{aligned}$$

Let us estimate the remanning two integrals above; for

$$\phi \in \mathcal{H}^1_0(S) \cap [u_0, \cdots, u_{n-1}]^\perp$$

denote by $\phi^{(n)}$ the component of ϕ in $[u_n]$ and by Q_{n+1} the orthogonal projection onto $[u_0, \cdots, u_n]^\perp$ in $\mathcal{H}^1_0(S)$. The first integral is positive and divergent as $1/\varepsilon^2$ to $+\infty$ since for ε small enough

$$
\begin{aligned}
\int_{\mathbb{R}\times S} & dsdy \quad \frac{1}{\varepsilon^2}\left(|\nabla_\perp \psi_\varepsilon|^2 - \lambda_n |\psi_\varepsilon|^2\right) \\
= & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\left(\|\nabla_\perp \psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} - \lambda_n \|\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)}\right) \\
= & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\left(\|\psi_\varepsilon(s)\|^2_{\mathcal{H}^1_0(S)} - (\lambda_n+1)\|\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)}\right) \\
= & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\Big(\|Q_{n+1}\psi_\varepsilon(s)\|^2_{\mathcal{H}^1_0(S)} + \|\psi^{(n)}_\varepsilon(s)\|^2_{\mathcal{H}^1_0(S)} \\
- & \quad (\lambda_n+1)\|\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)}\Big) \\
= & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\Big(\|\nabla_\perp Q_{n+1}\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} + \|Q_{n+1}\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} \\
+ & \quad \|\nabla_\perp \psi^{(n)}_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} + \|\psi^{(n)}_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} \\
- & \quad (\lambda_n+1)\|\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)}\Big) \\
\geq & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\Big(\lambda_{n+1}\|Q_{n+1}\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} + \lambda_n\|\psi^{(n)}_\varepsilon(s)\|^2_{\mathrm{L}^2(S)} \\
- & \quad \lambda_n\|\psi_\varepsilon(s)\|^2_{\mathrm{L}^2(S)}\Big) \\
= & \quad \frac{1}{\varepsilon^2}\int_{\mathbb{R}\times S} ds\ \left(\lambda_{n+1}-\lambda_n\right)\|Q_{n+1}\psi_\varepsilon\|^2_{\mathrm{L}^2(S)} \\
= & \quad \frac{(\lambda_{n+1}-\lambda_n)}{\varepsilon^2}\|P_{n+1}\psi_\varepsilon\|^2 \geq \frac{(\lambda_{n+1}-\lambda_n)}{\varepsilon^2}\|P_{n+1}\psi\|,
\end{aligned}
$$

and, by hypothesis, $\lambda_{n+1} > \lambda_n$. The absolute value of the second integral diverges at most as $1/\varepsilon$; indeed, since (ψ_ε) is a bounded sequence in $\mathcal{H}^1_0(\mathbb{R}\times S)$ and we have

$$
\begin{aligned}
& \frac{1}{\varepsilon^2}\left|\int_{\mathbb{R}\times S} dsdy\ (\xi\cdot y)\left(|\nabla_\perp\psi_\varepsilon|^2 - \lambda_n|\psi_\varepsilon|^2\right)\right| \\
\leq & \quad \frac{\|\kappa\|_\infty}{\varepsilon}\int_{\mathbb{R}\times S} dsdy\ |z_\alpha\cdot y|\left(|\nabla_\perp\psi_\varepsilon|^2 + \lambda_n|\psi_\varepsilon|^2\right) \\
\leq & \quad \frac{\|\kappa\|_\infty}{\varepsilon}\sup|z_\alpha\cdot y| \times \max\{\lambda_n, 1/\lambda_n\}\|\psi_\varepsilon\|^2_{\mathcal{H}^1_0}.
\end{aligned}
$$

Therefore, if ψ does not belong to dom $b^0_{n,c}$, then

$$\liminf_{\varepsilon\to 0} b^\varepsilon_{n,c}(\psi_\varepsilon) = +\infty.$$

We have then verified the statement i) of Theorem 3, and so Theorem 6 follows by Theorem 3v); Theorem 5 is just a particular case with $n = 0$.

3.4. **Spectral Possibilities.** There is a competition between the curvature and the twisting terms in the effective potential

$$V_n^{\rm eff}(s) = (\tau(s) - \dot\alpha(s))^2\, C_n(S) - \frac{1}{4}\kappa(s)^2, \quad n \geq 0,$$

since the curvature gives an attractive term and the twisting a repulsive one.

This effective potential is the net result of a memory of higher dimensions that takes into account the geometry of the confining region. In planar (2D) cases only the curvature term is present and (if its not zero) a bound state does always exist. In the spatial (3D) case, by tuning up the tubes, and so the functions that define effective potentials, one finds a huge amount of spectral possibilities for the effective Schrödinger operator in the reference curve $\gamma(s)$. Below some of them are selected; it will be assumed that $C_n(S) \neq 0$ and that $\tau, \dot\alpha$ are not necessarily bounded; this is only related to the domain of the forms and involved operators, differently from the essential technical condition of bounded curvature. It is worth mentioning that for lower bounded potentials V (in particular for $V_n^{\rm eff}$ above) that belong to $\mathrm{L}^2_{\rm loc}(\mathbb{R}^k)$ the operators $-\Delta + V$ are essentially self-adjoint when defined on $C_0^\infty(\mathbb{R}^k)$; see, for instance, Section 6.3 in [1].

(1) **No twisting.** In this case $\mathcal{A}_n(s) = 0$ and the 3D situation is quite similar to the 2D one; the effective potential $V_n^{\rm eff}(s) = -1/4\,\kappa(s)^2$ is purely attractive and the spectrum of H_n^0 has at least one negative eigenvalue. See Subsection 11.4.4 in [1].
(2) **Periodic.** By choosing bounded $\kappa, \tau, \dot\alpha$ so that $V_n^{\rm eff}(s)$ becomes periodic the resulting effective operators H_n^0 have purely absolutely continuous spectra and with a band-gap structure [27]. Such periodicity may come from different combinations; for instance, the tube curvature and torsion could be periodic (with the same period) and $\alpha(s)$ a constant function, or the tube could be straight so that $\kappa(s) = 0 = \tau(s)$ but the cross section S rotates at a periodic speed $\dot\alpha(s)$. See also [28].
(3) **Purely discrete.** The operators H_n^0 will have this kind of spectrum if $\lim_{|s|\to\infty} V_n^{\rm eff}(s) = \infty$ (see Section 11.5 in [1]); this happens iff the torsion τ or $\dot\alpha$, as well as their difference, diverge at both $\pm\infty$. In particular H_n^0 will have discrete spectrum in case this limit operator is obtained from a straight tube with growing rotation speed of the cross section such that $\lim_{|s|\to\infty} \dot\alpha(s) = \infty$.
(4) **Quasiperiodic.** For one of the simplest situations select $\dot\alpha$ and κ periodic functions with (minimum) periods $t_\alpha > 0$ and $t_\kappa > 0$, respectively; if t_α/t_κ is an irrational number we are in the case of quasiperiodic potentials. In this case there are many spectral possibilities that usually are very sensitive to details of the potential. Of course one may also take $V_n^{\rm eff}$ in the more general class of almost periodic functions; see, for instance, [29].

(5) **Singular continuous.** An appealing possibility is the choice of decaying potentials $V_n^{\rm eff}$ in the class studied by Pearson [30], which leads to singular continuous spectrum for H_n^0. See also explicit examples in [31].

The reader can play with his/her imagination in order to consider tubes that give rise to previously selected spectral types.

3.5. **Bounded Tubes.** Now we say something about the particular case of bounded tubes; the goal is to recover the spectral results of [8]. Since the cross section S is a bounded set, the boundedness of the tube

$$\Omega^\varepsilon_{\alpha,L} := \left\{(x,y,z)\in\mathbb{R}^3 : (x,y,z) = f^\varepsilon_\alpha(s,y_1,y_2), s\in[0,L], (y_1,y_2)\in S\right\},$$

is a consequence of a bounded generating curve $\gamma(s)$ defined, say, on a compact set $s\in[0,L]$ instead of on the whole line $\mathbb{R}$ as before. In this case the negative Laplacian operator $-\Delta_{\Omega^\varepsilon_{\alpha,L}}$ has compact resolvent and its spectrum is composed only of eigenvalues λ_j^ε, $j\in\mathbb{N}$; denote by ψ_j^ε the normalized eigenfunction associated with λ_j^ε. Let b_L^ε and $H_0^\varepsilon(L)$ be the the corresponding sesquilinear form and self-adjoint operator, after the "regularizations" and acting in subspaces of $\mathrm{L}^2([0,L]\times S)$, suitably adapted from Subsection 3.1 (the same quantities as in [8]).

Theorem 7. *For each $j\in\mathbb{N}$ one has*

$$\lim_{\varepsilon\to0}\left(\lambda_j^\varepsilon - \frac{\lambda_0}{\varepsilon^2} - \mu_j\right) = 0,$$

where μ_j are the eigenvalues of the of the Schrödinger operator

$$\begin{aligned}\mathrm{dom}\ H_0^0(L) &= \mathcal{H}^2(0,L)\cap\mathcal{H}_0^1(0,L),\\ (H_0^0(L)\psi)(s) &= -\ddot\psi(s) + \left((\tau(s)-\dot\alpha(s))^2C_0(S) - \frac14\alpha^2(s)\right)\psi(s).\end{aligned}$$

Furthermore, there are subsequences of $f_\alpha^\varepsilon(\psi_j^\varepsilon)$ that converge to $w_j(s)u_0(y)$ in $\mathrm{L}^2([0,L]\times S)$ as $\varepsilon\to0$, where w_j are the normalized eigenfunctions corresponding to μ_j.

Proof. The proof will be an application of Proposition 4, with $T = H_0^0(L)$, $T_\varepsilon = H_0^\varepsilon(L)$, $\mathcal{H}_0 = \{w(s)u_0(y) : w\in\mathcal{H}_0^1(0,L)\}$. Let b_L^0 be the form generated by $H_0^0(L)$. Previously discussed results in the case of unbounded tubes apply also here and they show that

$$b_L^\varepsilon \xrightarrow{\rm S\Gamma} b_L^0, \qquad b_L^\varepsilon \xrightarrow{\rm W\Gamma} b_L^0,$$

that is, item a) of Proposition 4 holds in this setting. Since $H_0^0(L)$ has compact resolvent, item b) in that proposition follows at once.

Finally, for each $\varepsilon>0$ the two hypotheses, (ψ_ε) is bounded in $\mathrm{L}^2(\mathbb{R}\times S)$ and $b_L^\varepsilon(\psi_\varepsilon)$ is bounded, imply that (see page 804 of [8]) (ψ_ε) is a bounded sequence in $\mathcal{K} = \mathcal{H}_0^1([0,L]\times S)$. By Rellich-Kondrachov Theorem the space $\mathcal{K}$ is compactly embedded in $\mathrm{L}^2([0,L]\times S)$ (due to the boundedness of

$[0, L] \times S$), and so item c) of Proposition 4 holds. By that proposition $H_0^\varepsilon(L)$ converges in the norm resolvent sense to $H_0^0(L)$ in $\mathcal{H}_0$, and it is well known that the spectral assertions in Theorem 7 follow by this kind of convergence, that is, the convergence of eigenvalues of $H_0^\varepsilon(L)$ to eigenvalues of $H_0^0(L)$ as well as the assertion about convergence of eigenfunctions. Taking into account that in the construction of $H_0^\varepsilon(L)$ there was the "regularization" subtraction of $(\lambda_0/\varepsilon^2)\|\psi\|_\varepsilon^2$ from the original form of the Laplacian $-\Delta_{\Omega^\varepsilon_{\alpha,L}}$, the conclusions of Theorem 7 follow. □

Remark 11. *Theorem 7 makes clear the mechanism behind the spectral approximations in case of bounded tubes, that is, the powerful* norm resolvent convergence *is in action!*

Remark 12. *Although we expect that for Theorem 6 the norm resolvent convergence takes place, we were not able to prove it; at the moment, to get norm convergence we need a combination of* Γ*-convergence and compactness of the tube (as in Theorem 7). A very simple example indicates how subtle those properties can be combined and that our expectations might be wrong.*

Consider the sequence of multiplication operators $T_n\psi(x) = x\psi(x)/n$ and $T = 0$. In the space $\mathrm{L}^2(\mathbb{R})$*, dominated convergence implies that T_n converges to T in the strong resolvent sense. Now, $\sigma(T_n) = \mathbb{R}$, for all n, while $\sigma(T) = \{0\}$; thus, T_n does not converge in the norm resolvent sense to T. However, for the same operator actions in* $\mathrm{L}^2[0, 1]$ *one gets that T_n converges to T in the norm resolvent sense (due to the compactness of* $[0, 1]$*).*

Remark 13. *It is also possible to consider semi-infinite tubes, that is, $s \in [0, \infty]$. In this case all previous constructions apply and the limit operator has the expected action but with Dirichlet boundary condition at zero. The details are similar to the arguments previously discussed here and in* [8] *and will be omitted.*

4. Broken-Line Limit

In this section we discuss the operators H_n^0, defined on a spacial curve $\gamma(s)$ with compactly supported curvature, that approximate another singular limit, now given by two infinite straight edges with one vertex at the origin. The angle between the straight edges is θ and is kept fixed during the approximation process. In case of planar curves this problem has been considered in [9, 10] and a variation of it in [32], and those authors had at hand explicitly expressions for the resolvents of the Hamiltonians as integral kernels.

This geometrical broken line is a simple instance of a *quantum graph* and the main question is about the boundary conditions that is selected at the vertex in the convergence process; that is expected to be the physical boundary conditions. We refer to the above cited references for more physical and mathematical details. Note that in this work we have restricted ourselves to first confine the quantum system from the tube to the curve, and then take the broken-line limit. In the planar curve cases both limits are taken together (as in [9, 10, 32]) and with no reference to Γ-convergence; but since it may involve sequences of operators that are not uniformly bounded from below [9], it is not clear that in our 3D setting we could address both limits together by using Γ-convergence; this seems to be an interesting open problem.

Of course a novelty here is the possibility of quantum twisting in the effective potentials

$$V_n^{\text{eff}}(s) = (\tau(s) - \dot\alpha(s))^2\, C_n - \frac{1}{4}\kappa(s)^2, \quad n \geq 0,$$

since in the plane cases [9, 10] only the curvature term $-\kappa(s)^2/4$ is present. Another interesting point is the dependence of the effective potential on the $(n+1)$th sector spanned by the eigenvector u_n of the Laplacian restricted to the cross section S. Thus the limit operator depends on n and since this additional term $\mathcal{A}_n(s)$ is positive we have a wide range of possibilities in the 3D case, that is, not just an attractive potential as in 2D. From the technical point of view we will follow closely the proof of Lemma 1 in [10], which uses results of [33]. However, differently than the planar situation, the condition

$$\langle V_n^{\text{eff}} \rangle := \int_{\mathbb{R}} ds\, V_n^{\text{eff}}(s) \neq 0$$

may not hold in 3D, but we will see that the same proof can be adapted to the case $\langle V_n^{\text{eff}} \rangle = 0$ by using results of [34]; the boundary conditions at the vertex depend explicitly on the curvature and twisting.

We will be rather economical in the proofs below, since we do not intend to just repeat whole parts of published works; we are sure that from the statements below and references to papers and specific equations, the interested reader will have no special difficulties in filling out the missing details.

Assume that the curvature, torsion and the speed of the rotation angle $\dot\alpha$ are compactly supported in $(-1, 1)$ and scale them as

$$\kappa_\delta(s) := \frac{1}{\delta}\kappa\left(\frac{s}{\delta}\right), \quad \tau_\delta(s) := \frac{1}{\delta}\tau\left(\frac{s}{\delta}\right), \quad \dot\alpha_\delta(s) := \frac{1}{\delta}\dot\alpha\left(\frac{s}{\delta}\right),$$

and the continuations of the half-lines to the left and to the right of that support joint at the origin with an angle θ; this angle is exactly the integral

$$\theta = \int_{\mathbb{R}} ds\, \kappa_\delta(s) = \int_{\mathbb{R}} ds\, \kappa(s), \quad \forall \delta > 0.$$

Of course, as above, the curve γ is supposed to be smooth and without self-intersection. Here we consider only the above scales. Our concern now is to study the limit $\delta \to 0$ of the families of operators

$$(H_n^0(\delta)\psi)(s) = -\ddot\psi(s) + V_{n,\delta}^{\text{eff}}(s)\psi(s), \quad \text{dom } H_n^0(\delta) = \mathcal{H}^2(\mathbb{R}), \quad n \geq 0,$$

where

$$V_{n,\delta}^{\text{eff}}(s) := (\tau_\delta(s) - \dot\alpha_\delta(s))^2\, C_n - \frac{1}{4}\kappa_\delta(s)^2.$$

It turns out that this limit $\delta \to 0$ is related to the low energy expansion of the resolvent $R_{k^2}(H_n^0)$, Im $k > 0$, as explained on page 8 of [10]. The operator H_n^0 is said to have a *resonance at zero* if there exists $\psi_r \in \mathrm{L}^\infty(\mathbb{R})$, $\psi \notin \mathrm{L}^2(\mathbb{R})$, such that $H_n^0\psi_r = 0$ in the sense of distributions; in this case ψ_r can be chosen real and is unique (as a subspace). Since all $V_n^{\rm eff}$ have compact support, one has $\int_{\mathbb{R}} ds\, e^{as}|V_n^{\rm eff}(s)| < \infty$ for some $a > 0$, which is a technical condition necessary for what follows [10, 34, 33]. Now we consider two complementary cases: $\langle V_n^{\rm eff}\rangle \neq 0$ and $\langle V_n^{\rm eff}\rangle = 0$.

4.1. $\langle V_n^{\rm eff}\rangle \neq 0$**.** Assume that this condition holds. In this case we may directly apply Lemma 1 in [10], which employs results of [33], to obtain:

Proposition 6. (a) *If H_n^0 has no resonance at zero, then $H_{n,\delta}^0$ converges in the norm resolvent sense, as $\delta \to 0$, to the one-dimensional Laplacian $-\Delta^D$ with Dirichlet boundary condition at the origin, that is,*

$$\begin{aligned} \text{dom } (-\Delta^D) &= \left\{\psi \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R}\setminus\{0\}) : \psi(0) = 0\right\}, \\ (-\Delta^D\psi)(s) &= -\ddot{\psi}(s). \end{aligned}$$

(b) *If H_n^0 has a resonance at zero, then $H_{n,\delta}^0$ converges in the norm resolvent sense, as $\delta \to 0$, to the one-dimensional Laplacian $-\Delta^r$ given by*

$$\begin{aligned} \text{dom } (-\Delta^r) &= \big\{\psi \in \mathcal{H}^2(\mathbb{R}\setminus\{0\}) : (c_1^n + c_2^n)\psi(0^+) = (c_1^n - c_2^n)\psi(0^-), \\ &\qquad (c_1^n - c_2^n)\dot{\psi}(0^+) = (c_1^n + c_2^n)\dot{\psi}(0^-)\big\}, \\ (-\Delta^r\psi)(s) &= -\ddot{\psi}(s), \end{aligned}$$

where

$$\begin{aligned} c_1^n &= \frac{1}{2\langle V_n^{\rm eff}\rangle}\int_{\mathbb{R}\times\mathbb{R}} dsdy\; V_n^{\rm eff}(s)\,|s-y|\,V_n^{\rm eff}(y)\psi_r(y), \\ c_2^n &= -\frac{1}{2}\int_{\mathbb{R}} ds\; sV_n^{\rm eff}(s)\psi_r(s). \end{aligned}$$

Moreover, c_1^n and c_2^n do not vanish simultaneously.

4.2. $\langle V_n^{\rm eff}\rangle = 0$**.** Assume that this condition holds. Now we cannot apply directly Lemma 1 in [10], but by invoking results of [34] we can check that the proof of such Lemma 1 may be replicated to conclude:

Proposition 7. (a) *If H_n^0 has no resonance at zero, then $H_{n,\delta}^0$ converges in the norm resolvent sense, as $\delta \to 0$, to the one-dimensional Laplacian $-\Delta^D$ with Dirichlet boundary condition at the origin.*

(b) *If H_n^0 has a resonance at zero, then $H_{n,\delta}^0$ converges in the norm resolvent sense, as $\delta \to 0$, to the one-dimensional Laplacian $-\Delta^r$, as in Proposition 6*(b)*, but now*

$$\begin{aligned} c_1^n &= \frac{1}{2W}\int_{\mathbb{R}^3} dsdxdy\; V_n^{\rm eff}(s)\,|s-x|\,V_n^{\rm eff}(x)\,|x-y|\,V_n^{\rm eff}(y)\psi_r(y),\\ c_2^n &= -\frac{1}{2}\int_{\mathbb{R}} ds\; sV_n^{\rm eff}(s)\psi_r(s),\\ W &= \int_{\mathbb{R}^2} dsdy\; V_n^{\rm eff}(s)\,|s-y|\,V_n^{\rm eff}(y)\; > 0. \end{aligned}$$

Moreover, c_1^n and c_2^n do not vanish simultaneously.

Note the different expressions for the parameter c_1^n from the case $\langle V_n^{\rm eff}\rangle \neq 0$; in both Propositions 6 and 7, the expressions for c_1^n, c_2^n were obtained by working with relations in references [33] and [34], respectively. In order to replicate the proof of the above mentioned Lemma 1, it is enough to check some key properties that can be found spread along reference [34]; there is a complete parallelism between both cases, although the expressions defining the involved quantities are different (that was a chief contribution of [34]). In what follows we indicate what are such properties, where their versions in case $\langle V_n^{\rm eff}\rangle = 0$ can be found in [34] and we use the notation of [10, 34] without explaining the meaning of some of the symbols employed (e.g., $t_j, M_j, \phi_0, \cdots$). Unfortunately a short explanation of the involved symbols will not be very helpful to the understanding of the large amount of involved technicalities; at any rate, they are not necessary to state the above results, they can be easily found in the references and the equations in [10, 34] we shall use in the proof below will be explicitly indicated.

Proof. Introduce the functions

$$v = |V_n^{\rm eff}|^{1/2}, \quad u = |V_n^{\rm eff}|^{1/2}({\rm sgn} V_n^{\rm eff}),$$

so that $V_n^{\rm eff} = vu$ and $\langle V_n^{\rm eff}\rangle = (v,u)$ (inner product in $\mathrm{L}^2(\mathbb{R})$). The properties needed for the proof of Proposition 7(a) appear in equation (25) of [10], that is,

$$(v, t_0 u) = 0, \quad ((\cdot)v, t_0 u) = (v, t_0 u(\cdot)) = 0, \quad (v, t_1 u) = -2.$$

The first and the third ones can be found in equation (3.83) of [34], while the second one is obtained by combining equations (2.8) and (3.98) of that work.

For the proof of Proposition 7(b) one need to check equations (17) and (18) of [10]; equation (18) reads

$$\begin{aligned} ((\cdot)v, t_{-1}u(\cdot)) &= \frac{2(c_2^n)^2}{(c_1^n)^2+(c_2^n)^2}, \\ ((\cdot)v, t_0 u) &= \frac{2c_1^n c_2^n}{(c_1^n)^2+(c_2^n)^2}, \\ (v, t_1 u) &= -\frac{2(c_2^n)^2}{(c_1^n)^2+(c_2^n)^2}; \end{aligned}$$

these relations are found in equations (4.16), (4.15) and (3.91) of [34], respectively. Now equation (17) of [10] reads

$$t_{-1}u = 0, \quad t_{-1}^* v = 0, \quad (v, t_0 u) = 0.$$

The third relation follows from equation (3.90) of [34], and their equation (3.93) implies (recall we are using their notation)

$$t_{-1}^* v = (\mathrm{sgn} V_n^{\mathrm{eff}}) t_{-1} (\mathrm{sgn} V_n^{\mathrm{eff}}) v = (\mathrm{sgn} V_n^{\mathrm{eff}}) t_{-1} u = 0,$$

that is, we have got the second relation by accepting that the first one holds. Now we show how to derive the first one from [34]. By equations (3.45) and (3.5) in [34] it is found that

$$\begin{aligned} t_{-1} &= -c_0 P_0 \hat{Q} = -c_0 P_0 (\mathbf{1} - \hat{P}) \\ &= -c_0 P_0 \left(\mathbf{1} - \frac{1}{c} M_0 P \right) \\ &= -c_0 P_0 + \frac{c_0}{c} M_0 P, \end{aligned}$$

with $P(\cdot) = (v, \cdot)u$ and (see also equations (3.2) and (3.3) in [34])

$$P_0(\cdot) = (\hat{\phi_0}, \cdot)\phi_0, \quad \hat{\phi_0} = (\mathrm{sgn} V_n^{\mathrm{eff}}) M_0 \phi_0.$$

The proof finishes as soon as we check that $P_0 u = 0$ and $Pu = 0$. By the hypothesis on the potential we have

$$Pu = (v, u)u = \langle V_n^{\mathrm{eff}} \rangle u = 0,$$

and

$$\begin{aligned} P_0 u &= (\hat{\phi_0}, u)\phi_0 = ((\mathrm{sgn} V_n^{\mathrm{eff}}) M_0 \phi_0, u)\phi_0 \\ &= (M_0 \phi_0, (\mathrm{sgn} V_n^{\mathrm{eff}}) u)\phi_0 = (M_0 \phi_0, v)\phi_0 \end{aligned}$$

which vanishes by equation (3.10) in [34]. □

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