Relative differential cohomology

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Abstract

We briefly review the classical construction of the Cheeger-Simons characters, the Deligne cohomology groups and the differential K-theory groups, which are representatives of the absolute differential refinement of the corresponding cohomology theories. We present the axiomatic framework for the differential refinement of a generic cohomology theory in the absolute case, together with the important results of existence and uniqueness developed by Bunke and Schick. Motivated by the introduction of the relative Cheeger-Simons characters, we propose a suitable set of axioms for the *relative* differential extension of a cohomology theory, we construct a family of long exact sequences involving the differential groups and we extend to the relative case the results of existence and uniqueness. Furthermore, we generalize the notion of Cheeger-Simons character to any cohomology theory and we extend to the relative case the construction of the integration map.

Keywords: Differential cohomology, Cheeger-Simons characters, Deligne cohomology, differential K-theory, Gysin map.

Resumo

Lembramos brevemente a construção clássica dos carácteres de Cheeger-Simons, dos grupos de cohomologia de Deligne e dos grupos de K-teoria diferencial, os quais são representantes do refinamento diferencial absoluto das teorias cohomológicas correspondentes. Apresentamos a estrutura axiomática do refinamento diferencial de uma teoria da cohomologia genérica no caso absoluto, juntamente com os importantes resultados de existência e unicidade desenvolvidos por Bunke e Schick. Motivados pela introdução dos carácteres de Cheeger-Simons relativos, propomos uma estrutura axiomática adequada para a extensão diferencial relativa de uma teoria cohomológica, construímos uma família de sequências exatas longas que envolvem os grupos diferenciais e estendemos ao caso relativo os resultados de existência e unicidade. Além disso, generalizamos a noção de carácter de Cheeger-Simons a qualquer teoria cohomológica e estendemos ao caso relativo a construção do mapa de integração.

Palavras chave: Cohomologia diferencial, carácteres de Cheeger-Simons, cohomologia de Deligne, K-teoria diferencial, mapa de Gysin.

Contents

Notational Conventions

Common Symbols

Symbol	Term
$\Omega^n(M)$	Group of differential n-forms over M
$\Omega^n_{cl}(M)$	Group of closed differential n-forms over M
$\Omega^n_0(M)$	Group of closed differential n-forms over M with integral periods
$\Omega^n(M;\mathfrak{g})$	Group of differential n-forms over M with values in $\mathfrak{g}$
$C_n(M)$	Group of smooth n-chains over M
$Z_n(M)$	Group of smooth n-cycles over M
$B_n(M)$	Group of smooth n-boundaries over M
$H^n(M;\mathbb{Z})$	Integral singular cohomology group of degree n over M
$\hat{H}^n(M)$	Cheeger-Simons characters of degree n over M
$H^n_{dR}(M)$	de Rham cohomology group of degree n over M
ch	Generalized Chern character
$\underline{H}^n(\mathfrak{F}^\bullet)$	Cohomology sheaf associated to the sheaf $\mathfrak{F}^\bullet$
$H^n(M,\mathfrak{F}^\bullet)$	Hypercohomology group of M associated to the sheaf $\mathfrak{F}^\bullet$
$C^n(\mathcal{U},\mathfrak{F})$	Group of Čech n-cochains of M with respect to the sheaf $\mathfrak{F}$ and the covering $\mathcal{U}$
$\check{H}^n(\mathcal{U},\mathfrak{F})$	Čech cohomology group of M of degree n with respect to the sheaf $\mathfrak{F}$ and the covering $\mathcal{U}$
$\check{H}^n(\mathcal{U},\mathfrak{F}^\bullet)$	Čech hypercohomology group of M of degree n with respect to the sheaf complex $\mathfrak{F}^\bullet$ and the covering $\mathcal{U}$
$\underline{\Omega}^\bullet$	Sheaf of complex differential forms
$\underline{\mathbb{C}}^*$	Sheaf of non-vanishing complex smooth functions
$\Omega^\bullet(k)$	Sheaf complex truncated at k (smooth Deligne complex)
$H^n_D(M,\Omega^\bullet)$	Deligne cohomology groups of degree n over M
$ch(\nabla)$	Chern character associated to the connection ∇
$CS(\nabla,\nabla')$	Chern-Simons class associated to ∇ and ∇'
$\hat{K}^n(M)$	Differential K-theory group of degree n over M
$\mathfrak{h}^\bullet$	Cohomology groups of the point in the theory h, i.e. $\bigoplus_n h^n(\{*\})$
$\mathfrak{h}^\bullet_{\mathbb{R}}$	Cohomology groups of the point tensor $\mathbb{R}$, i.e. $\mathfrak{h}^\bullet \otimes \mathbb{R}$
$\hat{E}^\bullet$	Differential extension of a generic cohomology theory $E^\bullet$

$H^n_{dR}(M;V^\bullet)$	de Rham cohomology group of degree n with values in $V^\bullet$ over M
$\sum X$	Suspension of the space X
ΩX	Loop space of the space X
$[X,Y]$	Family of homotopy classes of maps from X to Y
$\pi_n(X)$	Homotopy group of degree n over X, i.e $[S^n,X]$
$C^n_{sm}(M;V^\bullet)$	Group of smooth n-cochains over M with values in $V^\bullet$
$C^n_{cl}(M,V^\bullet)$	Group of smooth closed n-cochains over M with values in $V^\bullet$
$C_n(\varphi)$	Group of smooth n-chains over the smooth map φ
$\Omega^n(\varphi)$	Group of differential n-forms over the smooth map φ
$H^n(\varphi;\mathbb{Z})$	Group of relative singular cohomology of degree n over the map φ
$H^n_{dR}(\varphi;V^\bullet)$	de Rham cohomology group of degree n with values in $V^\bullet$ over the smooth map φ
$\hat{H}^n(\varphi)$	Relative Cheeger-Simons characters of degree n over the smooth map φ

Categories

Symbol	**Category**
$\mathcal{M}$	Category of smooth manifolds
Top_2	Category with objects continuous maps between topological spaces
$\mathcal{M}_2$	Category with objects smooth maps between manifolds
$\mathcal{A}_{\mathbb{Z}}$	Category of $\mathbb{Z}$-graded abelian groups
$\mathcal{R}_{\mathbb{Z}}$	Category of $\mathbb{Z}$-graded rings

1 Introduction

Secondary invariants are the main topic of this thesis.[1] Intuitively, they represent a refinement of homotopical (topological) invariants, usually expressed through (co)homology classes, enriched with differential information that encodes the geometry of the objects being considered. The secondary invariants that we will treat in this work are characterized by the use of connections and their generalizations, in order to produce such enrichments. Hence, the new invariants will be defined only on smooth categories. The usual name given to these new objects is "differential cohomology classes".

One of the first examples of secondary invariants is provided by the Cheeger-Simons characters, introduced in [12]. The intention behind the definition was to interpret the Chern-Simons form [13], defined on the total space of a principal bundle with connection, in terms of a differential form on the base space. In view of this aim the theory was developed, generalizing to higher dimensions the well known fact that S^1-bundles with connection can be represented via their holonomy. The main applications, at least in this initial stage, were the following. In [12] the authors applied the new invariants to refine the characteristic classes of Euler, Chern and Pontryagin, obtaining obstructions to conformal immersions of Riemannian manifolds into the Euclidean space or, more generally, into non-negative space forms. The authors also used the new invariants to recast the geometric Atiyah-Singer index theorem in this new context and draw some conclusions in the case of flat line bundles.

The second grand example of secondary invariants comes in the form of Deligne cohomology, introduced around 1972. Historically, Deligne cohomology appeared as the natural generalization of the Kostant [25] and Weil [40] theory, that identifies $H^2(M;\mathbb{Z})$ with the group of line bundles over M. This identification is constructed via a local representation of a line bundle through transition maps satisfying the cocycle condition, i.e. as cocycles in the Čech complex. In higher dimensions a similar construction describes abelian p-gerbes, that have direct interpretations in physics. In Deligne cohomology, the differential refinement is implemented through the inclusion of the connection, through a modification of the local data used to represent the bundle. Concretely, the sheaf of local transitions is extended to an appropriated complex of sheaves, including the local potentials that describe the connection.

Both examples above (Cheeger-Simons characters and Deligne cohomology) give secondary invariants refining the integral singular cohomology and they turn out to be isomor-

[1] The expression "secondary invariant" is not used uniformly in the literature. Here a secondary invariant is a differential cohomology class; sometimes it is just a flat class or even a flat and topologically trivial one.

phic. A construction of an explicit isomorphism can be found in [15]. In the same vein, a more systematic approach towards a theory of differential refinements of *singular cohomology* was developed by Simons and Sullivan in [35], where they propose an axiomatic framework and settle the problem of uniqueness of the extension within this broader context.

Early refinements of a cohomology theory are not exclusive to singular cohomology. Already in 1987 Karoubi [24], trying to give a model of K-theory with coefficients, introduced the equivalent of the flat part of the modern version of differential K-theory. Further developments towards a complete differential refinement of K-theory were achieved by Freed and Lott in [18], inspired by the work of Hopkins and Singer in [21] and by previous work of Lott [26, 27] and Freed [16, 17] independently. With the model defined in [18], Freed and Lott obtain a meaningful generalization of the geometric Atiyah-Singer index theorem [3, 2]. Moreover, in [18], the authors argue that the index theorem in the setting of differential K-theory has direct applications to string theory, in particular, to prove the Green-Schwarz mechanism, an anomaly cancellation property. Independently from Freed and Lott, in [9] Bunke and Schick developed an alternative analytic model for smooth K-theory, extending the integration map and the Chern character to the differential setting. As in the case of Freed and Lott, in [9] the authors obtain a differential version of the Atiyah-Singer theorem. Apart from the works cited above, there are alternative models for differential K-theory, most notably the one in [36], which gives a simple description in terms of vector bundles with connection. For a more complete survey of the matter see [11].

In the more general setting of differential refinements of arbitrary cohomology theories, foundational work was laid down in [21] by Hopkins and Singer. There, the authors proposed a model that produces a differential extension out of a cohomology theory represented by a spectrum; in particular, they were a able to present models for differential bordism and differential K-theory. Their motivation was based on the extension to higher dimensions of the construction of quadratic functions à la Riemann, which in turn have applications in M-theory [41, 42]. Much of the recent development of the theory is based on the model of Hopkins-Singer.

The existence of various models of differential extensions for singular cohomology and K-theory, the possibility to produce at will differential extensions for arbitrary cohomology theories via the Hopkins-Singer model, the various applications to modern physics and the refinement of classic results, make natural to establish the theory of differential extensions as an area of study of its own. One of the principal results in this direction was introduced by Bunke and Schick in [10]. There the authors propose a set of axioms for the differential extension of a generalized cohomology theory and prove, under rather mild hypothesis, that the extension of a fixed theory is unique. Later, under this new axiomatic setting, Upmeier studied in [39] the introduction of a multiplicative structure into the differential cohomology groups; the construction uses extensively the properties of the Hopkins-Singer model; in particular, it extends to the differential case the product in terms of spectra. More recently, in [34], Ruffino used the axiomatic framework (in particular the differential extension of the

Gysin map) to define a pairing between the flat theory associated to a differential refinement and the topological theory. Then, the author generalizes the notion of Cheeger-Simons character to any cohomology theory.

Up to this point we have given a panoramic view of the development of differential cohomology in the *absolute* case, that is, as a contravariant functor from the category of smooth manifolds to graded abelian groups. It is natural to construct the relative version of these secondary invariants and of their applications to index theory. Moreover, as we said before, differential cohomology has useful interpretations in mathematical physics, in particular in quantum theory. One important instance is that of string and D-brane theory [43], where the necessity of an extension to the relative case became apparent. Important results in the relative setting of singular cohomology are present in the paper [6] by Brightwell and Turner, where they introduce two different models for relative Cheeger-Simons characters. Such a construction and its main properties are described in detail in the work of Bär and Becker [4]. In the case of Deligne cohomology, Ruffino proposes in [32] four candidates for the relative version and he extends to the relative case the formulas for the transgression map, the integration and the holonomy. The four candidates are generalized to any cohomology theory in [33], using the Hopkins-Singer model.

In this thesis we propose an axiomatic framework for the relative differential cohomology groups, generalizing the one developed for the absolute case. One of the salient features of the proposed framework is the deduction, from the axioms alone, of the existence of a family of long exact sequences that combine the differential and topological groups. Using an adaptation of the Hopkins-Singer model, we show the existence of the relative differential refinement for any cohomology theory. Then we show its uniqueness, using the techniques of Bunke and Schick plus a homology argument. After that, we develop the integration theory for the relative differential case, extending the definitions of orientation and Gysin map. Finally, we generalize the notion of relative Cheeger-Simons character to any cohomology theory.

The thesis is organized as follows. Chapter 2 provides a brief review of the theory of differential refinements of a cohomology theory in the absolute case. In section 1 we present a classical example that motivates the definition of the Cheeger-Simons characters. Then we proceed to define formally the characters and we prove their principal properties. In section 2 we revisit the example motivating the definition of the characters from a local point of view. After that, we introduce the language of sheaves, sheaf cohomology and hypercomology. Using such a language, we recast the local description of our motivating example and we obtain a suitable generalization to higher degrees, defining Deligne cohomology. Then we prove its principal properties and we state the natural identification between the Deligne cohomology and the Cheeger-Simons characters. In section 3 we construct a model for differential K-theory. As in the preceding sections, we exhibit the principal features of the model, in particular its periodicity. In section 4 we recall the axiomatic framework for differential refinements in the absolute case and in section 5 we prove their uniqueness.

Chapter 2 ends with the introduction of the Hopkins-Singer model, which solves the problem of the existence of a differential refinement of a generic cohomology theory.

Chapter 4 contains the original part of the work. Section 1 begins with a brief review of (topological) cohomology theory in the category of maps between spaces and some generalities about fiber integration over smooth maps. Next, we set forth the axioms that define a differential refinement of a relative cohomology theory and we show the existence of a family of long exact sequences involving the differential groups. As a preliminary step, we give an explicit construction of the Bockstein morphism, that extends to the differential case the topological one. Then we prove the exactness of the sequences. Next, we extend to the relative case the theorems of existence and uniqueness of the differential refinement. Section 2 deals with the problem of orientation. To motivate the theory in the differential case, we begin by introducing its topological counterpart. With the topological case settled, the generalization to the differential case becomes in the most part very natural. However, in order to the define the orientation in the relative differential case we have to make some additional adjustments in order to take into account the geometry of the situation. In section 3 we apply the results of section 2 to develop further the theory of differential integration, in particular, the case of maps between manifolds with boundary and integration to the point from manifolds with or without boundary, the last one being a key ingredient in the construction of the generalized relative Cheeger-Simons characters, the subject of the next section. In section 4 we extend to the relative case the geometrical description of the dual homology of a fixed cohomology theory $h^\bullet$. Then, after refining to the differential case the data of the description of the dual homology, we define the generalized relative Cheeger-Simons characters. We close the section with some properties of the generalized characters, in particular, their relation with a generic differential refinement of $h^\bullet$. In section 5 we define differential integration in the case of maps from manifolds with boundary to manifolds without boundary. Finally, in section 6 we show some properties of (topological) relative cohomology theory that do not have a direct contribution on the development of the differential case, so were better placed at the end of the chapter where they do not disrupt the flow of the arguments.

2 Absolute Differential Cohomology

2.1 Cheeger-Simons Differential Characters

2.1.1 An important example

Let $E \to M$ be a smooth principal S^1-bundle with connection $\theta \in \Omega(E; i\mathbb{R})$. Denote by $\Theta \in \Omega^2(M)$ its associated curvature form. By standard results, $\Theta/2i\pi$ represents the characteristic class of the bundle and has integral periods. Also the connection yields parallel transport, thus for a given smooth closed path γ we may define its holonomy as a number $h(\gamma) \in S^1$ or $\chi(\gamma) \in \mathbb{R}/\mathbb{Z}$, such that $h(\gamma) = e^{2\pi i \chi(\gamma)}$.

Observe that any given cycle $x \in Z_1(M)$ is given as $x = \gamma + \partial y$, where $y \in C_2(M)$ and γ is smooth closed path. So the function χ defined above can be extended to the group of all cycles by setting

$$\chi(x) = \chi(\gamma) + \frac{\tilde{\Theta}(y)}{2i\pi}.$$

Clearly $\chi \circ \partial = \tilde{\Theta}/2i\pi$ (here $\tilde{\Theta}$ represents the morphism obtained by integration of the form Θ over smooth 2-chains mod $\mathbb{Z}$). If one defines $\hat{H}^2(M)$ as the set all S^1-bundles with connection up to isomorphism, the construction above defines a pair of maps $I : \hat{H}^2(M) \to H^2(M; \mathbb{Z})$ and $R : \hat{H}^2(M) \to \Omega^2(M)$, such that for a S^1-bundle $E \to M$ with connection we have $R(\chi) = \Theta/2\pi$ and $I(\chi) = c(E)$, where $c(E)$ is the first Chern class of E. Observe that, while $R(\chi)$ and $I(\chi)$ may vanish for a particular χ, the map χ itself may not be trivial, that is, χ holds more information than $R(\chi)$ and $I(\chi)$ separately. This situation is prototypical for differential cohomology theories.

2.1.2 Generalization

Definition and morphisms

First we fix some notation. Let M be a smooth manifold, we will denote by $C_n(M)$ the group of *smooth chains* over M, that is, the group generated by the set of continuous maps $\sigma\colon \Delta^n \to M$ that can be extended to smooth ones over a neighborhood of Δ. Using the restriction of the boundary operator to smooth chains, we denote by $Z_n(M)$ and $B_n(M)$ subgroups of smooth cycles and smooth boundaries, respectively. We also define $\Omega^k_0(M)$ as the subgroup of closed forms with integral periods, that is, forms whose integral over smooth cycles takes integral values.

Definition 2.1.1. Let M be a smooth manifold. We define the degree-k differential characters of M as the family

$$\hat{H}^k(M) = \{f \in Hom(Z_{k-1}(M) \to \mathbb{R}/\mathbb{Z}) \quad | \quad f \circ \partial \in \Omega_0^k(M)\}.$$

The notation $f \circ \partial \in \Omega^k(M)$ means that there is a form $\omega \in \Omega^k(M)$ such that for any $\sigma \in C_k(M)$ it holds

$$f \circ \partial(\sigma) = \int_\sigma \omega \mod \mathbb{Z},$$

in particular, it follows that ω has integral periods.

The form ω is unique. Indeed, if ω and ω' are different k forms there is a x where $\omega(x) - \omega'(x) \neq 0$, so we may choose an appropriated open set U around x such that

$$\int_\sigma \omega - \omega' \neq 0$$

for a $k-$cycle $\sigma \subseteq U$. A similar reasoning shows that ω is a closed form, for $\int_\sigma d\omega = \int_{\partial\sigma} \omega = f\partial^2\sigma = 0$, for all $\sigma \in C_{k+1}(M)$.

An element $f \in \hat{H}^k(M)$ also determines a class $g \in H^k(M;\mathbb{Z})$. Indeed, since $Z_{k-1}(M)$ is a projective module, the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and the morphism $f : Z_{k-1}(M) \to \mathbb{R}/\mathbb{Z}$, induce a lifting $\tilde{f} : Z_{k-1}(M) \to \mathbb{R}$. Then we define

$$\begin{aligned} g : C_k(M) &\to \mathbb{Z} \\ \sigma &\mapsto \int_\sigma \omega - \tilde{f} \circ \partial(\sigma). \end{aligned} \tag{2-1}$$

Note that g in fact takes values in $\mathbb{Z}$, since by definition $q(\tilde{f} \circ \partial(\sigma)) = f \circ \partial(\sigma) = \int_\sigma \omega$, hence $\tilde{f} \circ \partial(\sigma)$ and $\int_\sigma \omega$ differ by an integer. The morphism g defines a cohomology class, for

$$\delta g(\sigma) = \delta \int_\sigma \omega - \delta(\tilde{f} \circ \partial(\sigma)) = \int_{\partial\sigma} \omega - \tilde{f} \circ \partial^2\sigma = \int_\sigma d\omega = 0$$

for all $\sigma \in C_{k+1}(M)$. The class is well defined, for even if the lifting of f is not unique the difference between any two liftings takes integral values, hence if $\tilde{f}$, $\tilde{f}'$ are two liftings of f and g, g' are the associated morphisms, we have that $g - g' = \delta(\tilde{f}' - \tilde{f})$, so they define the same cohomology class.

The assignation $f \mapsto \omega$ defines a function $R : \hat{H}^k(M) \to \Omega_0^k(M)$, called the *curvature* morphism. The assignation $f \mapsto \int \omega - \tilde{f} \circ \partial$ defines a function $I : \hat{H}^k(M) \to H^k(M; Z)$ called *characteristic class*. There is a natural additive operation defined on the set $Hom(Z_k(M), \mathbb{R}/\mathbb{Z})$ making it into a group. With this structure, the functions R and I defined above become morphisms of groups. Note also that the image in $H^k(M;\mathbb{R})$ of the class $f \mapsto \int_\sigma \omega - \tilde{f} \circ \partial(\sigma)$ is just $f \mapsto \int_\sigma \omega$, for $\tilde{f} \circ \partial$ is a real coboundary. Hence for each k we have the commutative diagram of groups

$$\begin{array}{ccc} \hat{H}^k(M) & \xrightarrow{I} & H^k(M;\mathbb{Z}) \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle ch} \\ \Omega_0^k(M) & \xrightarrow[dR]{} & H_{dR}^k(M) \end{array}$$

We also have morphisms

$$a : \frac{\Omega^{k-1}(M)}{\Omega_0^{k-1}(M)} \to \hat{H}^k(M)$$
$$[\omega] \mapsto f_\omega$$

where $f_\omega(\sigma) = \int_\sigma \omega \mod \mathbb{Z}$, for all $\sigma \in Z_{k-1}(M)$ and

$$b : H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \to \hat{H}^k(M)$$
$$[c] \mapsto f_c$$

where $f_c(\sigma) = c(\sigma)$.

Both morphisms are well defined. Indeed, take $\omega = \omega' + \eta$ with $\eta \in \Omega_0^{k-1}(M)$ and $\sigma \in Z_{k-1}(M)$ then we have

$$\begin{aligned} f_\omega(\sigma) &= \int_\sigma \omega \mod \mathbb{Z} = \int_\sigma (\omega' + \eta) \mod \mathbb{Z} = \int_\sigma \omega' + \int_\sigma \eta \mod \mathbb{Z} \\ &= \int_\sigma \omega' \mod \mathbb{Z} = f_{\omega'}(\sigma) \end{aligned}$$

A similar argument shows that in the definition of b, the element f_c is well defined.

Short exact sequences

More generally, the morphisms defined above fit in the following exact sequences:

$$0 \longrightarrow \frac{\Omega^{k-1}(M)}{\Omega_{cl}^{k-1}(M)} \xrightarrow{a} \hat{H}^k(M) \xrightarrow{I} H^k(M;\mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{b} \hat{H}^k(M) \xrightarrow{R} \Omega_0^k(M) \longrightarrow 0$$

In fact,

- for the first sequence, by definition of a it is clear that the form associated to $a(\omega)$ is given by $d\omega$, that is
$$a(\omega) \circ \partial(\sigma) = \int_{\partial\sigma} \omega = \int_\sigma d\omega.$$
Hence the composition $I \circ a$ is given by
$$I \circ a(\omega)(\sigma) = \int_\sigma d\omega - \widetilde{a(\omega)} \circ \partial\sigma = \int_\sigma d\omega - \int_\sigma d\omega = 0.$$
Conversely, let $f \in \hat{H}^k(M)$ be a differential character such that $I(f) = 0$, that is, $\int_\sigma \omega = \tilde{f} \circ \partial(\sigma)$, for all $\sigma \in C_k(M)$. It follows that $\omega = \delta\tilde{f}$ (up to an integer constant) and by the de Rham isomorphism ω is exact and $\tilde{f} = \eta$, where $d\eta = \omega$. Hence $f = \int \eta$ mod $\mathbb{Z}$.
The morphism $I : \hat{H}^k(M) \to H^k(M;\mathbb{Z})$ is surjective. In fact, let $[g] \in H^k(M;\mathbb{Z})$ then via the inclusion of coefficients $i : \mathbb{Z} \to \mathbb{R}$, we obtain a morphism $i : H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{R})$; denote $[g'] = i([g])$. By the de Rham isomorphism there is a closed form $\omega \in \Omega^k(M)$ such that $\int \omega = g' + \delta\tilde{f}$, where $\tilde{f} : C_{k-1}(M) \to \mathbb{R}$. We claim that
$$f := \tilde{f}|_{Z_{k-1}} \mod \mathbb{Z}$$
holds $f \in \hat{H}^k(M)$ and $I(f) = g$. First, note that g' only takes values in $\mathbb{Z}$, thus
$$f \circ \partial = (\int \omega - g') \mod \mathbb{Z} = \int \omega.$$
Also, using $\tilde{f}$ as the lifting of f, it is clear that $\int \omega - \tilde{f} \circ \delta = g'$, but $g' = i \circ g$, so it represents an integral cohomology class. The fact that a is injective follows directly by the definition.

- About the exactness of the second sequence, note that if $f \in H^{k-1}(M)$ then $f \circ \partial = 0$, hence the curvature of $b(f)$ is null, for there are not non-vanishing differential forms that only take integral values. Conversely, let $f \in \hat{H}^k(M)$ be a differential character with vanishing curvature. We show that f can be extended to a morphism $f' : C_{k-1}(M) \to \mathbb{R}/\mathbb{Z}$ such that $\delta f' = 0$. Indeed, observe that $C_{k-1}(M)/Z_{k-1}(M) \cong \partial(C_{k-1}(M)/Z_{k-1}(M))$, since the right side is a submodule of a free module, it's free and hence projective. Therefore, being $C_{k-1}(M) \to C_{k-1}(M)/Z_{k-1}(M)$ an epimorphism, we have a splitting $C_{k-1}(M) = Z_{k-1}(M) \oplus Q$, where Q is the image of $C_{k-1}(M)/Z_{k-1}(M)$ in $C_{k-1}(M)$ under the splitting map. Then we can define $f' : C_{k-1}(M) \to \mathbb{R}/\mathbb{Z}$ as $f'(\sigma) = f(\sigma)$, if $\sigma \in Z_{k-1}(M)$, and $f'(\sigma) = 0$, if $\sigma \in Q$, and extend linearly. It is clear that $\delta f' = 0$.
The morphism b is injective. This follows by the universal coefficient theorem and the fact that $Ext^1_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R}/\mathbb{Z}) = 0$.

The morphism R is surjective. In fact, let $\omega \in \Omega_0^k(M)$, then via the de Rham isomorphism there is a morphism $g : C_k(M) \to \mathbb{R}$, not unique in general, such that $g|_{B_k} \equiv 0$, that is, g represents a cohomology class in $H^k(M;\mathbb{R})$. By hypothesis the morphism g takes values in the integers when restricted to cycles defining a morphism $\gamma : Z_k(M) \to \mathbb{Z}$; so by the splitting $C_k(M) = Z_k(M) \oplus Q$ used above, we may extend γ to a morphism $\gamma : C_k(M) \to \mathbb{Z}$ and via the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ we identify it, abusing notation, with a morphism $\gamma : C_k(M) \to \mathbb{R}$. Note that $(g-\gamma)|_{Z_k} \equiv 0$, then by the universal coefficient theorem we have that $g - \gamma = \delta f'$ for some $[f'] \in H^{k-1}(M)$ i.e. $g-\gamma$ represents a coboundary. Consider now $f\colon = f'|_{Z_{k-1}} \mod \mathbb{Z}$, it is clear that $R(f) = \omega$, for

$$f \circ \partial = (g-\gamma) \mod \mathbb{Z} = g \mod \mathbb{Z} = \int \omega \mod \mathbb{Z}.$$

Ring structure

In this section we define the product of differential characters following closely the original source [12]. First we recall a result from Kervaire, relating the wedge product of differential forms and the cup product of forms thought as cochains via integration; this relation expresses the infinitesimal behavior of forms via subdivision. Let $\Delta : C_k(M) \to C_k(M)$ denote the subdivision morphism in the cubical theory then

$$\lim_{n\to\infty} \Delta^n(\omega_1 \cup \omega_2) = \omega_1 \wedge \omega_2$$

It is a classical result that subdivision is chain homotopic to the identity, the next lemma says that this chain homotopy extends inductively.

Lemma 2.1.2. Let M be a smooth manifold, $C_k(M)$ its group of cubical chains of dimension k and $\Delta : C_k(M) \to C_k(M)$ the subdivision morphism, then for $\psi : C_k(M) \to C_{k+1}(M)$ the canonical chain homotopy between Δ and the identity, the morphism $\sum_{i=0}^n \psi\Delta^i$ defines a chain homotopy between Δ^{n+1} and the identity.

Proof. Indeed, note that $\Delta \circ \partial = \partial \circ \Delta$, so by direct calculation one has

$$\begin{aligned}\partial \circ (\sum_{i=0}^n \psi\Delta^i) + (\sum_{i=0}^n \psi\Delta^i) \circ \partial &= \sum_{i=0}^n \partial\psi\Delta^i + \sum_{i=0}^n \psi\partial\Delta^i \\ &= \sum_{i=0}^n (\partial\psi + \psi\partial)\Delta^i = \sum_{i=0}^n (1-\Delta)\Delta^i = 1 - \Delta^{n+1}.\end{aligned}$$

□

Remark 2.1.3. Let $\omega \in \Omega^{k+1}(M)$, then the image of the chain homotopy lives in the kernel of ω when interpreted through evaluation of chains via integration, for $\psi(\sigma) \in C_{k+1}(M)$ has $(k+1)$-dimensional content zero for all $\sigma \in C_k(M)$; however, it is not true in general that $\omega_1 \cup \omega_2 \circ \psi = 0$, for two forms $\omega_1, \omega_2 \in \Omega^*(M)$.

Remark 2.1.4. If one supposes that $\omega_1 \cup \omega_2$ vanish on boundaries, then the previous lemma says that for all $z \in C_k(M)$ it holds that

$$\omega_1 \cup \omega_2(z) - \omega_1 \cup \omega_2(\Delta^{n+1} z) = \omega_1 \cup \omega_2((\sum_{i=0}^{n} \psi \Delta^i \partial) z)$$

The result of Kervaire and the last remark make it natural to associate to the difference $\omega_1 \cup \omega_2 - \omega_1 \wedge \omega_2$ the element

$$E(\omega_1, \omega_2) = \omega_1 \cup \omega_2(\sum_{i=0}^{\infty} \psi \Delta^i)$$

Lemma 2.1.5. Let M be a manifold and $\omega_1 \in \Omega^{k_1}(M)$, $\omega_2 \in \Omega^{k_2}(M)$ be closed forms then for any $\sigma \in C_{k_1+k_2}(M)$ it holds

$$\delta E(\omega_1, \omega_2)(\sigma) = \omega_1 \cup \omega_2(\sigma) - \omega_1 \wedge \omega_2(\sigma).$$

Proof. By direct computation we have

$$\begin{aligned}
\delta E(\omega_1, \omega_2)(\sigma) &= \omega_1 \cup \omega_2(\sum_{i=0}^{\infty} \psi \Delta^i)(\partial \sigma) \\
&= \omega_1 \cup \omega_2(\sum_{i=0}^{\infty} \psi \partial \Delta^i \sigma) \\
&= \omega_1 \cup \omega_2(\sum_{i=0}^{\infty} (1 - \Delta - \partial \psi) \Delta^i \sigma) \\
&= \lim_{n \to \infty} \omega_1 \cup \omega_2(\sum_{i=0}^{n} (1 - \Delta) \Delta^i \sigma) - \lim_{n \to \infty} \omega_1 \cup \omega_2(\sum_{i=0}^{n} \partial(\psi \Delta^i \sigma)) \\
&= \lim_{n \to \infty} \omega_1 \cup \omega_2((1 - \Delta^{n+1}) \sigma) \\
&= \omega_1 \cup \omega_2(\sigma) - \omega_1 \wedge \omega_2(\sigma).
\end{aligned}$$

□

We are ready to define the product of differential characters. Given a differential character $f \in \hat{H}^n(M)$, we denote by T_f the lifting of f to a morphism $C_{n-1}(M) \to \mathbb{R}$ and the induced morphism of the lifting with values in $\mathbb{R}/\mathbb{Z}$, $T_f \mod \mathbb{Z}$, is denoted by $\widetilde{T_f}$.

Definition 2.1.6. Let $f \in \hat{H}^{k_1}(M)$ and $g \in \hat{H}^{k_2}(M)$. The product $f * g \in \hat{H}^{k_1+k_2}(M)$ is defined as

$$f * g = \widetilde{T_f \cup \omega_g} + (-1)^{k_1} \widetilde{\omega_f \cup T_g} - \widetilde{T_f \cup \delta T_g} - \widetilde{E(\omega_f, \omega_g)}$$

where $\omega_f = R(f)$ and $\omega_g = R(g)$.

Proposition 2.1.7. The product of differential characters has the following properties:

1). *Compatibility with curvature.* For $f \in \hat{H}^{k_1}(M)$ and $g \in \hat{H}^{k_2}(M)$ it holds $R(f * g) = R(f) \wedge R(g)$.

2). *Compatibility with characteristic classes.* For $f \in \hat{H}^{k_1}(M)$ and $g \in \hat{H}^{k_2}(M)$ it holds $I(f * g) = I(f) \cup I(g)$.

3). *Associativity.* For $f \in \hat{H}^{k_1}(M)$, $g \in \hat{H}^{k_2}(M)$ and $h \in \hat{H}^{k_3}(M)$ it holds $(f * g) * h = f * (g * h)$.

4). *Naturality.* For a map of manifolds $\varphi : N \to M$ and $f \in \hat{H}^{k_1}(M)$, $g \in \hat{H}^{k_2}(M)$ it holds $\varphi^*(f * g) = \varphi^*(f) * \varphi^*(g)$.

5). *Graded commutativity.* For $f \in \hat{H}^{k_1}(M)$ and $g \in \hat{H}^{k_2}(M)$ it holds $f * g = (-1)^{k_1 k_2} g * f$.

Proof. By relation (2-1), we have that $\delta T_f = \omega_f - c_f$ and $\delta T_g = \omega_g - c_g$, where $[c_f] = I(f)$ and $[c_g] = I(g)$. Then we get

$$\begin{aligned}\delta(f * g) =& ((\omega_f - c_f) \cup \omega_g)^{\sim} + (\omega_f \cup (\omega_g - c_g))^{\sim} \\ & - ((\omega_f - c_f) \cup (\omega_g - c_g))^{\sim} - (\omega_f \cup \omega_g - \omega_f \wedge \omega_g)^{\sim} \\ =& - \widetilde{c_f \cup c_g} + \widetilde{\omega_f \wedge \omega_g} \\ =& \widetilde{\omega_f \wedge \omega_g}.\end{aligned}$$

Hence $R(f * g) = \omega_f \wedge \omega_g$. To prove property 2)., note that the formula of the product gives immediately an expression for the lifting in the definition of I, which we still denote by $f * g$, so the calculation above shows that $\int \omega_f \wedge \omega_g - \delta(f * g) = c_f \cup c_g$.

About property 3). a long calculation shows that

$$\begin{aligned}(f * g) * h - f * (g * h) = -(-1)^{k_1+k_2} \delta E(\omega_f,\omega_g) \cup T_h - E(\omega_f, \omega_g) \cup \omega_h + E(\omega_f, \omega_g) \cup \delta T_h + \\ E(\omega_f, \omega_g \wedge \omega_h) + (-1)^{k_1} \omega_f \cup E(\omega_g, \omega_h) - E(\omega_f \wedge \omega_g, \omega_h)\end{aligned}$$

And after applying the coboundary operator one gets

$$\begin{aligned}\delta((f * g) * h - f * (g * h)) &= -\delta E(\omega_f, \omega_g) \cup \delta T_h - \delta E(\omega_f, \omega_g) \cup \omega_h + \delta E(\omega_f, \omega_g) \cup \delta T_h + \\ &\quad \delta E(\omega_f, \omega_g \wedge \omega_h) + \omega_f \cup \delta E(\omega_g, \omega_h) - \delta E(\omega_f \wedge \omega_g, \omega_h) \\ &= -(\omega_f \cup \omega_g - \omega_f \wedge \omega_g) \cup \omega_h + \omega_f \cup (\omega_g \wedge \omega_h) - \omega_f \wedge \omega_g \wedge \omega_h + \\ &\quad \omega_f \cup (\omega_g \cup \omega_h - \omega_g \wedge \omega_h) - ((\omega_f \wedge \omega_g) \cup \omega_h - \omega_f \wedge \omega_g \wedge \omega_h) \\ &= 0\end{aligned}$$

So $(f*g)*h - f*(g*h)$ is in fact a cocycle. Note that by definition of $E(\omega_1, \omega_2)$, one necessarily has that $\lim_{n\to\infty} E(\omega_1, \omega_2)(\Delta^n \sigma) = 0$ and similarly for terms of the form $\omega_1 \cup E(\omega_2, \omega_3)$. It follows that $\lim_{n\to\infty} (f*g)*h - f*(g*h)(\Delta^n \sigma) = 0$, therefore it has integral periods and by the previous remarks it defines the zero character.

About the naturality of the product, observe that by the naturality of the cup product and the definition of $E(\omega_f, \omega_g)$ we have

$$
\begin{aligned}
\varphi^*(f * g) &= \varphi^*(\widetilde{T_f \cup \omega_g} + (-1)^{k_1}\widetilde{\omega_f \cup T_g} - \widetilde{T_f \cup \delta T_g} - E(\widetilde{\omega_f, \omega_g})) \\
&= \varphi^*(\widetilde{T_f) \cup \varphi^*(\omega_g}) + (-1)^{k_1}\varphi^*(\widetilde{\omega_f) \cup \varphi^*(T_g}) - \varphi^*(\widetilde{T_f) \cup \varphi^*(\delta T_g}) - \varphi^*(E(\widetilde{\omega_f, \omega_g})) \\
&= \widetilde{T_{\varphi^* f} \cup \omega_{\varphi^* g}} + (-1)^{k_1}\widetilde{\omega_{\varphi^* f} \cup T_{\varphi^* g}} - \widetilde{T_{\varphi^* f} \cup \delta T_{\varphi^* g}} - E(\widetilde{\omega_{\varphi^* f}, \omega_{\varphi^* g}}) \\
&= (\varphi^* f) * (\varphi^* g)
\end{aligned}
$$

The last property follows directly from the properties of the cup product i.e. $\alpha^i \cup \beta^k = (-1)^{ik}\beta^k \cup \alpha^i$. $\square$

Integration

For differential characters there is also a well defined integration theory for bundles, more concretely, given an oriented bundle $\pi : E \to M$ with fiber of dimension n there is a morphism

$$\pi_! : \hat{H}^{k+n}(E) \to \hat{H}^k(M)$$

for every k such that:

1). $\pi_!$ is compatible with the curvature, characteristic class and topological trivialization morphisms. Equivalently, the diagram

$$
\begin{tikzcd}
\dfrac{\Omega^{k+n-1}(E)}{im(d)} \arrow[r, "a"] \arrow[d, "\int"] & \hat{H}^{k+n}(E) \arrow[r, "R"] \arrow[d, "\pi_!"'] \arrow[rr, bend left, "I"] & \Omega^{k+n}(E) \arrow[d, "\int"] & H^{k+n}(E;\mathbb{Z}) \arrow[d, "\int"] \\
\dfrac{\Omega^{k-1}(M)}{im(d)} \arrow[r, "a"'] & \hat{H}^{k+n}(M) \arrow[r, "R"'] \arrow[rr, bend right, "I"'] & \Omega^{k}(M) & H^{k}(M;\mathbb{Z})
\end{tikzcd}
$$

commutes.

2). $\pi_!$ is natural, that is, for any smooth map $\varphi : N \to M$ and any n-bundle $\pi : E \to M$ the diagram

$$
\begin{tikzcd}
\hat{H}^{k+n}(E) \arrow[r, "\Phi^*"] \arrow[d, "\pi_!"'] & \hat{H}^{k+n}(\varphi^* E) \arrow[d, "\pi_!"'] \\
\hat{H}^{k}(M) \arrow[r, "\varphi^*"'] & \hat{H}^{k}(N)
\end{tikzcd}
$$

commutes.

The explicit construction of the integration morphism can be found in [4]. The construction requires the use of stratifolds, so to give a detailed description here would take us to far afield from our objectives, the interested reader can consult the cited reference.

Instead, we will give the construction of the simpler case of a trivial S^1-bundle, that is, $\int_{S^1} : \hat{H}^{k+1}(S^1 \times M) \to \hat{H}^k(M)$; this case is enough to establish important results such as the uniqueness of differential extensions. The basic idea is to use the prism construction to interpret a cycle $\sigma \in Z_{k-1}(M)$ as a cycle $\tilde{\sigma} \in Z_k(S^1 \times M)$. First we recall briefly how the prism operator is defined. Given the standard n-simplex Δ^n, take the space $\Delta^n \times I$ and consider the functions $\varphi_i : \Delta^n \to I$, defined as $\varphi_i(t_0, t_1, ..., t_n) = t_{i+1} + t_{i+2} + ... + t_n$. The graph of each function φ_i defines a $(n+1)$-simplex inside $\Delta^n \times I$, such that together they exhaust all of $\Delta^n \times I$ and the graphs of φ_i and φ_{i+1} intersect just in one face. The $(n+1)$-simplex generated by φ_i will be denoted by $[v_0, v_1, ..., v_i, w_i, ..., w_n]$, where v_j represents a vertex on the base of $\Delta^n \times I$ and w_j represents a vertex on the top of $\Delta^n \times I$. Now, if f, $g : M \to N$ are two homotopic functions, say through $\Psi \colon M \times I \to N$, we define a morphism of chains $P : C_n(M) \to C_{n+1}(N)$ by

$$P(\sigma) = \sum_i (-1)^i \Psi \circ (\sigma \times \mathrm{id})|_{[v_0, v_1, ..., v_i, w_i, ..., w_n]}.$$

It can be shown (see [20]) that the relation

$$\partial P = f_\# - g_\# - P\partial$$

holds.

In our case, we will use $Y = S^1 \times M$ and $X = M$, whereas the functions will be the inclusion $M \hookrightarrow S^1 \times M$, defined by $x \mapsto (1, x)$ and the homotopy will be $(x, t) \mapsto (e^{2\pi i t}, x)$; note that the relation above specializes to $\partial P = -P\partial$. With this setup we define the S^1-integration as follows: for a differential character $F : Z_k(S^1 \times M) \to \mathbb{R}/\mathbb{Z}$ we set

$$\left(\int_{S^1} F\right)(\sigma) = F \circ P(\sigma).$$

Observe that

$$\begin{aligned} \left(\int_{S^1} F\right)(\partial\sigma) &= F \circ P(\partial\sigma) = -F \circ \partial(P(\sigma)) = -\int_{P(\sigma)} \omega_F \\ &= \sum (-1)^{i+1} \int_{\Psi \circ (\sigma \times \mathrm{id})|_{[v_0, v_1, ..., v_i, w_i, ..., w_n]}} \omega_F \qquad (2\text{-}2) \\ &= \int_{S^1 \times \sigma} \omega_F = \int_\sigma \left(\int_{S^1} \omega_F\right) \end{aligned}$$

The fifth equality follows from the observation that the orientation induced by the ordering of the vertices makes it so that adjacent simplices have the same orientations, so after multiplying each simplex by $(-1)^{i+1}$ we get compatible orientations. From this it follows that $R \circ \int_{S^1} = \int_{S^1} \circ R$.

The S^1-integration is also compatible with the characteristic class. Indeed, if $\tilde{F}$ is a real lifting of F then the composition $\tilde{F} \circ P$ defines a lifting of $\int_{S^1} F$. Observe that

$$\left(I \circ \int_{S^1}\right)(F)(\sigma) = I(F \circ P)(\sigma) = \int_\sigma \left(\int_{S^1} \omega_F\right) - \tilde{F} \circ P \circ \partial(\sigma)$$

while

$$\begin{aligned}
\left(\int_{S^1} \circ I\right)(F)(\sigma) &= \int_{S^1} \left(\int \omega_F - \tilde{F} \circ \partial\right)(\sigma) \\
&= \left(\int \omega_F - \tilde{F} \circ \partial\right) \circ P(\sigma) \\
&= \int_{P(\sigma)} \omega_F + \tilde{F} \circ P \circ \partial(\sigma) \\
&= -\int_\sigma \left(\int_{S^1} \omega_F\right) + \tilde{F} \circ P \circ \partial(\sigma)
\end{aligned}$$

The last equality follows by 2-2. Hence $\int_{S^1} \circ I = -\int_{S^1} \circ I$.

2.2 Deligne Cohomology

2.2.1 The important example again

We revisit the example considered at the beginning from a different point of view. Consider a complex line bundle $\pi' \colon F \to M$. Alternatively, instead of describing the map π' and the total space F explicitly, one may give a local description of the bundle and a way to glue coherently all the local information. In fact, it is usual to define a vector bundle with fiber V through an open covering $\{U_\alpha\}$ of M and set of transition functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to Gl(V)$, such that the transition functions have the cocycle property

- $g_{\alpha\alpha} = \mathrm{id}$,
- for a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the relation $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ holds.

Intuitively, the cocycle condition amounts to say that on the intersections the local data patch nicely. To recover the picture of the total space F and the map $\pi' \colon F \to M$, one defines $F = \coprod U_\alpha \times V / \sim$, where the relation $\sim$ is defined by: given $(x, v) \in U_\alpha \times V$ and $(y, u) \in U_\beta \times V$ we have $(x, v) \sim (y, u)$ if $x = y$ and $g_{\alpha\beta}(x) v = u$. The map $\pi' : F \to M$ is defined as $\pi([x, v]) = x$.

For the case of complex line bundles we have $V = \mathbb{C}$ and $Gl(\mathbb{C}) = \mathbb{C}^*$. So the transition functions are just maps $g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathbb{C}^*$.

Now suppose that the line bundle has a connection. We want to use a local description for the connection as above. First, we recall that a connection ∇ on a vector bundle $\pi' \colon F \to M$ is a $\mathbb{K}$-linear[1] map $\nabla \colon \Gamma(F) \to \Gamma(T^*M \otimes F)$ such that for any map $f \colon M \to \mathbb{K}$ and any section $X \in \Gamma(F)$ it holds

$$\nabla(fF) = df \otimes F + f\nabla(F)$$

Now we give a local description of the connection. For this consider an open set U_α on M where the bundle is trivial. The trivialization induces a set of linearly independent local sections $\{e_i\}$ representing the standard basis on V, called a moving frame or local gauge. Now if X is an arbitrary section, when we consider its restriction to U_α, we have that $X = a^i e_i$, where the coefficients are maps $a^i : U_\alpha \to \mathbb{K}$. By the properties of the connection it follows

$$\nabla(X) = da^i \otimes e_i + a^i \nabla e_i$$

Let us be more specific. The term ∇e_i is by definition an element in $\Gamma(T^* \otimes F)$, so can be represented locally as

$$\nabla e_i = \Gamma^j_{ik} dx^k \otimes e_j$$

Hence, the local expression for $\nabla(X)$ becomes

$$\nabla(X) = da^i \otimes e_i + a^i \Gamma^j_{ik} dx^k \otimes e_j$$

Interpreting $\Gamma = (\Gamma^j_{ik})$ as a matrix valued 1-form and writing $(\Gamma^j_{ik}) = dx^i \otimes \Gamma_i$ the relation above transforms into

$$\nabla(X) = dX + \Gamma X$$

Now we show what happens when another trivialization is used. Suppose that U_β is another open set where the bundle is trivial such that $U_\alpha \cap U_\beta \neq \emptyset$, with transition function $g_{\alpha\beta}$. Take $\{f_i\}$ the local gauge associated to the new trivialization and represent the connection with respect to this local gauge as

$$\nabla f_i = \tilde{\Gamma}^j_{ik} dx^k \otimes f_j.$$

For every $x \in U_\alpha \cap U_\beta$, we have $e_i(x) = (g_{\alpha\beta}(x))^j_i \, f_j(x)$. From this it follows that

$$\nabla(e_i) = \Gamma^j_{ik} dx^k \otimes e_j = \Gamma^j_{ik} dx^k \otimes (g_{\alpha\beta})^l_j \, f_l.$$

On the other hand

$$\begin{aligned}\nabla(e_i) &= \nabla((g_{\alpha\beta})^j_i \, f_j) \\ &= d\,(g_{\alpha\beta})^j_i \otimes f_j + (g_{\alpha\beta})^j_i \, \nabla(f_j) \\ &= d\,(g_{\alpha\beta})^l_i \otimes f_l + (g_{\alpha\beta})^j_i \, \tilde{\Gamma}^l_{jk} dx^k \otimes f_l.\end{aligned}$$

[1] $\mathbb{K}$ being the field of definition of the vector space representing the fibers.

This produces the relation

$$dg_{\alpha\beta} + \tilde{\Gamma} g_{\alpha\beta} = g_{\alpha\beta}\Gamma$$

or equivalently

$$\Gamma = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1}\tilde{\Gamma} g_{\alpha\beta}.$$

Interpreting all the preceding discussion in the case of a complex line bundle, we have that if s_α is a non-vanishing local section (local gauge) then $\nabla s_\alpha = \Gamma s_\alpha$, where Γ is simply a 1-form with complex values which we denote by A_α. Then $A_\alpha = \left(\frac{\nabla s_\alpha}{s_\alpha}\right)$ and the expression for the transformation of the connection under two trivializations reduces to

$$A_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + A_\beta.$$

For this local representation of the connection there is an associated local description of the curvature. Consider a non-vanishing local section s', one may ask: how can s' be modified to become horizontal, that is, can we find $f\colon U \to \mathbb{C}^*$ such that $\nabla(fs') = 0$?. This is equivalent to ask for a f such that

$$\frac{df}{f} = -\frac{\nabla s'}{s'}.$$

A necessary condition for f to exists is that $d\left(\frac{\nabla s'}{s'}\right) = 0$. The 2-form $d\left(\frac{\nabla s'}{s'}\right)$ gives a local representation of the curvature of the connection. In general, the curvature form of a complex line bundle with connection ∇ is defined as the unique 2-form K such that for any non-vanishing local section s it holds that $K = d\left(\frac{\nabla s}{s}\right)$. From the preceding discussion, it is obvious that K does not depend on the choice of the local section s.

Summing up, given a complex line bundle $\pi\colon E \to M$ with connection ∇, we can give a local description taking an open cover $\{U_\alpha\}$ of M, a family of transition functions $\{g_{\alpha\beta}\}$ and a family of complex valued 1-forms A_α such that

- the family $\{g_{\alpha\beta}\}$ has the cocycle condition,
- the family $\{A_\alpha\}$ holds the relation

$$A_\alpha - A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

Also the curvature K of the connection holds $K = dA_\alpha$, over each U_α.

2.2.2 Deligne cohomology

The local data describing a line bundle with connection can be best represented in the language of sheaves and Čech cohomology. First , we recall the basic elements of sheaf theory.

Sheaf Cohomology and Hypercohomology

In order to give a more transparent treatment of sheaves we introduce the following notation. Let M be a topological space, the topology of M can be represented through a category which we denote by τ_M. Its objects are the open sets of M and the morphisms represent the inclusion relation, that is, there is a morphism $V \to U$ if $V \subseteq U$, so for each pair of objects there is at most one morphism.

Definition 2.2.1. Let M be a topological space. A presheaf of groups over M is a contravariant functor $\mathfrak{F}\colon \tau_M \to Grp$.

That is, to every open set $U \subseteq M$ we associate a group $\mathfrak{F}(U)$, the elements of $\mathfrak{F}(U)$ are called sections over U. Given a morphism $V \to U$ in τ_M, its image through $\mathfrak{F}$ is denoted $P_V^U : \mathfrak{F}(U) \to \mathfrak{F}(V)$, intuitively it represents the restriction to V of the sections in U. Finally, the morphisms between presheaves are given by natural transformations between contravariant functors.

Definition 2.2.2. Let $\mathfrak{F}\colon \tau_M \to Grp$ be a presheaf. We say that $\mathfrak{F}$ is a sheaf, if for all $U \in \tau_M$, all coverings $\{U_\alpha\}$ of U and all families $\{s_\alpha\}$, such that

- $s_\alpha \in \mathfrak{F}(U_\alpha)$,
- $P_{U_{\alpha\beta}}^{U_\alpha}(s_\alpha) = P_{U_{\alpha\beta}}^{U_\beta}(s_\beta)$ there exists a unique $s \in \mathfrak{F}(U)$ such that $P_{U_\alpha}^U(s) = s_\alpha$ for all $\{s_\alpha\}$.

The morphisms of sheaves are the same as those of presheaves. We observe that a morphism of sheaves $\varphi\colon \mathfrak{F} \to \mathfrak{E}$ determines two *presheaves* $Im(U) = Im\{\varphi : \mathfrak{F}(U) \to \mathfrak{E}(U)\}$ and $Ker(U) = Ker\{\varphi : \mathfrak{F}(U) \to \mathfrak{E}(U)\}$, which in principle may not be sheaves. In fact, $Ker(U)$ always is sheaf, which we denote by $Ker(\varphi)$, whereas $Im(U)$ usually is not. To recover a sheaf one must apply the sheafification process to the presheaf $Im(U)$ (see [19]). The sheaf thus obtained is denoted by $Im(\varphi)$. We note that a similar situation occurs when consider the quotient of sheaves,that is, the quotient of sheaves is not in general a sheaf but only a presheaf.

Definition 2.2.3. Let $\{\mathfrak{F}^i\}_{i\in\mathbb{Z}}$ be a set of sheaves and $d^i\colon \mathfrak{F}^i \to \mathfrak{F}^{i+1}$ a set of morphism of sheaves. We say that

$$\dots \longrightarrow \mathfrak{F}^i \xrightarrow{d^i} \mathfrak{F}^{i+1} \xrightarrow{d^{i+1}} \mathfrak{F}^{i+2} \xrightarrow{d^{i+2}} \dots$$

is a *complex of sheaves* if $d^{i+1} \circ d^i = 0$, for all $i \in \mathbb{Z}$.

To ease the notation, a complex of sheaves as defined above will be denoted simply as $(\mathfrak{F}^\bullet, d)$ or just by $\mathfrak{F}^\bullet$. Also we say that a complex of sheaves $\mathfrak{F}^\bullet$ is *bounded below* if there is a $k \in \mathbb{Z}$ such that $\mathfrak{F}^i = 0$ for $i < k$.

Definition 2.2.4. Given a complex of sheaves $(\mathfrak{F}^\bullet, d)$ we define the cohomology sheaf $\underline{H}^j(\mathfrak{F}^\bullet)$ as the sheaf associated to the presheaf $Ker(d^j)/Im(d^{j-1})$.

Definition 2.2.5. A sheaf $\mathfrak{F}$ of abelian groups over M is said *injective* if for any pair of morphisms of sheaves $i\colon \mathfrak{F} \to \mathfrak{E}$ and $f\colon \mathfrak{F} \to \mathfrak{L}$ with $Ker(i) = 0$, there exists a morphism of sheaves $g\colon \mathfrak{E} \to \mathfrak{L}$ such that $g \circ i = f$.

Definition 2.2.6. Let $\mathfrak{F}$ be a sheaf of groups. A *resolution* of $\mathfrak{F}$ is a complex of sheaves $(R^\bullet, d)$ together with a morphism $i\colon \mathfrak{F} \to R^0$ such that

- i is a monomorphism with image equal to $Ker(d^0)$,
- for $n \geq 1$, the $Ker(d^n) = Im(d^{n-1})$.

A resolution of $\mathfrak{F}$ for which each R^i is an injective sheaf is called an *injective resolution.*

Definition 2.2.7. A *double complex* $J^{\bullet\bullet}$ is a collection of groups $J^{p,q}$ with $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and a collection of morphisms

- vertical differentials $d\colon J^{p,q} \to J^{p,q+1}$,
- horizontal differentials $\delta\colon J^{p,q} \to J^{p+1,q}$,

such that $d \circ \delta = \delta \circ d$ and $\delta \circ \delta = d \circ d = 0$.

This definition can be represented by the following commutative diagram, where each column and each row defines a complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "\delta"] & J^{p-1,q+1} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p,q+1} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p+1,q+1} \arrow[r, "\delta"] \arrow[u, "d"'] & \cdots \\
\cdots \arrow[r, "\delta"] & J^{p-1,q} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p,q} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p+1,q} \arrow[r, "\delta"] \arrow[u, "d"'] & \cdots \\
\cdots \arrow[r, "\delta"] & J^{p-1,q-1} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p,q-1} \arrow[r, "\delta"] \arrow[u, "d"'] & J^{p+1,q-1} \arrow[r, "\delta"] \arrow[u, "d"'] & \cdots \\
 & \vdots \arrow[u, "d"'] & \vdots \arrow[u, "d"'] & \vdots \arrow[u, "d"'] &
\end{tikzcd}$$

In order to associate a cohomology group to this structure, we introduce a complex which gathers all the information of the double complex into a single structure:

Definition 2.2.8. Let $J^{\bullet\bullet}$ be a double complex with horizontal differentials δ and vertical differentials d. We define the *total complex*, $J^\bullet$ associated to $J^{\bullet\bullet}$ as the complex with n-component given by

$$J^n = \bigoplus_{p+q=n} J^{p,q}$$

and differentials

$$D\colon J^n \to J^{n+1}$$

where the differential takes $\{a^{p,q}\} \in J^n$, with $a^{p,q} \in J^{p,q}$, into $D(\{a^{p,q}\})$ and the $(i, n+1-i)$-term is given by $\delta(a^{i-1,n+1-i}) + (-1)^i d(a^{i,n-i})$.

The sign in the definition of the differential is necessary to obtain an actual differential, i.e $D \circ D = 0$. Now it is a straight matter to define

Definition 2.2.9. Given a double complex $J^{\bullet\bullet}$, we define the n-th cohomology group, $H^n(J^{\bullet\bullet})$, as the n-th cohomology group of the total complex $J^\bullet$ the associated to $J^{\bullet\bullet}$.

In general to compute cohomology groups of a double complex requires the use of spectral sequences.

Definition 2.2.10. Let $\mathfrak{F}^\bullet$ be a bounded below complex of sheaves over a space M. A double complex $(I^{\bullet\bullet}, \delta, d)$ with $I^{p,q} = 0$ for $p < 0$, is called an *injective resolution* of $\mathfrak{F}^\bullet$ if there is a morphism of complexes $u\colon \mathfrak{F}^\bullet \to (I^{0,\bullet}, d)$ such that

- for each $q \in \mathbb{Z}$, the complex of sheaves $(I^{\bullet,q}, \delta)$ is an injective resolution of $\mathfrak{F}^q$;
- for each $q \in \mathbb{Z}$, the complex of sheaves $d(I^{\bullet,q-1}) \subseteq I^{\bullet,q}$ is an injective resolution of $d(\mathfrak{F}^{q-1})$;
- for each $q \in \mathbb{Z}$, the complex of sheaves $Ker(d) \subseteq I^{\bullet,q}$ is an injective resolution of $Ker(d\colon \mathfrak{F}^q \to \mathfrak{F}^{q+1})$;
- for each $q \in \mathbb{Z}$, the complex of sheaves $\underline{H}^{\bullet,q}(I^{\bullet\bullet})$ is an injective resolution of $\underline{H}^q(\mathfrak{F}^\bullet)$.

Definition 2.2.11. Let $\mathfrak{F}^\bullet$ be a complex of sheaves bounded below over a space M. We define the *hypercohomology group* $H^n(M, \mathfrak{F}^\bullet)$ as the n-th cohomology group of the double complex $\Gamma(M, I^{p,q})$, where $I^{\bullet\bullet}$ is an injective resolution of the complex $\mathfrak{F}^\bullet$ and $\Gamma(M, I^{p,q}) = I^{p,q}(M)$, is the group of global sections on the sheaf $I^{p,q}$.

Čech Cohomology

We change our point of view and now we define a cohomology group for a sheaf over M by fixing an open cover of M and considering a combinatorial structure over its sections.

Definition 2.2.12. Given an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M and a sheaf $\mathfrak{F}$ of abelian groups over M, we define the Čech complex of M associated to the cover $\mathcal{U}$ as:

- set $C^p(\mathcal{U},\mathfrak{F}) = \prod_{i_0,i_1,i_2\dots i_p} \mathfrak{F}(U_{i_0,i_1,i_2\dots i_p})$ as the group of cochains of degree p. For $\sigma \in C^p(\mathcal{U},\mathfrak{F})$, we denote its component in $U_{i_0,i_1,i_2\dots i_p}$ by $(\sigma)_{i_0,i_1,i_2\dots i_p}$,

- set the boundary morphism

$$\delta\colon C^p(\mathcal{U},\mathfrak{F}) \to C^{p+1}(\mathcal{U},\mathfrak{F})$$

defined as

$$(\delta\sigma)_{i_0,i_1,i_2\dots i_{p+1}} = \sum_k (-1)^k P^{U_{i_0,i_1,i_2,i_{k-1},i_{k+1},\dots i_{p+1}}}_{U_{i_0,i_1,i_2\cdots i_{p+1}}}(\sigma_{i_0,i_1,i_2,i_{k-1},i_{k+1},\dots i_{p+1}}).$$

The morphism δ has the standard property of a boundary operators, that is, $\delta \circ \delta = 0$, so we obtain the complex

$$\dots \xrightarrow{\delta} C^p(\mathcal{U},\mathfrak{F}) \xrightarrow{\delta} C^{p+1}(\mathcal{U},\mathfrak{F}) \xrightarrow{\delta} \dots$$

The cohomology groups obtained from this complex are called the Čech cohomology groups of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathfrak{F}$ and denoted by $\check{H}^p(\mathcal{U},\mathfrak{F})$. This definition obviously depends on the cover, however using standard arguments one can come up with a definition of Čech cohomology independent of the cover taking the colimit

$$\check{H}^q(M,\mathfrak{F}) := \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathfrak{F})$$

over the poset of all covers of M ordered by the refinement relation.

We note that for manifolds, the cohomology groups $\check{H}^q(M,\mathfrak{F})$ are isomorphic to $\check{H}^q(\mathcal{U},\mathfrak{F})$ when $\mathcal{U}$ is a *good cover*.[2]

Fixing an open cover of M, one can also consider the information of a complex of sheaves through the apparatus of Čech cohomology. That is, for complex of sheaves

$$\dots \xrightarrow{d} \mathfrak{F}^q \xrightarrow{\delta} \mathfrak{F}^{q+1} \xrightarrow{\delta} \dots$$

and an open cover $\mathcal{U}$ of M one obtains the *Čech double complex*

[2] $\mathcal{U}$ is a good cover if it is an open cover such that every non-empty finite intersection $U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_p}$, with $U_{\alpha_i} \in \mathcal{U}$, is contractible.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\big\uparrow d & & \big\uparrow d & & \big\uparrow d & & \\
C^0(\mathcal{U},\mathfrak{F}^{q+2}) & \xrightarrow[\delta]{} & C^1(\mathcal{U},\mathfrak{F}^{q+2}) & \xrightarrow[\delta]{} & C^2(\mathcal{U},\mathfrak{F}^{q+2}) & \xrightarrow[\delta]{} & \cdots \\
\big\uparrow d & & \big\uparrow d & & \big\uparrow d & & \\
C^0(\mathcal{U},\mathfrak{F}^{q+1}) & \xrightarrow[\delta]{} & C^1(\mathcal{U},\mathfrak{F}^{q+1}) & \xrightarrow[\delta]{} & C^2(\mathcal{U},\mathfrak{F}^{q+1}) & \xrightarrow[\delta]{} & \cdots \\
\big\uparrow d & & \big\uparrow d & & \big\uparrow d & & \\
C^0(\mathcal{U},\mathfrak{F}^{q}) & \xrightarrow[\delta]{} & C^1(\mathcal{U},\mathfrak{F}^{q}) & \xrightarrow[\delta]{} & C^2(\mathcal{U},\mathfrak{F}^{q}) & \xrightarrow[\delta]{} & \cdots \\
\big\uparrow d & & \big\uparrow d & & \big\uparrow d & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

where the horizontal differentials are the Čech differential and the vertical differentials are the ones induced from the complex of sheaves.

Definition 2.2.13. Let $\mathfrak{F}^\bullet$ be a bounded below complex of sheaves over M and $\mathcal{U}$ an open cover of M. We define the Čech hypercohomology group of the complex $\mathfrak{F}^\bullet$ with respect to $\mathcal{U}$, denoted $\check{H}^n(\mathcal{U},\mathfrak{F}^\bullet)$, as the n-th cohomology group of the total complex associated to the Čech double complex.

From now on most of the sheaves we will be using are of the form $\underline{\Omega}^\bullet$. For a fixed k, the sheaf $\underline{\Omega}^k$ associates to an open set U the group $\Omega^k(U)$ of all complex valued k-forms over U and the induced morphisms $P^U_V : \Omega^k(U) \to \Omega^k(V)$ are defined as the restriction of the forms. The resulting Čech complex is denoted as

$$\Omega^k(U_\alpha) \xrightarrow{\delta} \Omega^k(U_{\alpha\beta}) \xrightarrow{\delta} \ldots$$

It is clear that Čech cohomology is better suited for actual computations when compared to sheaf cohomology, specially if one has a good cover. The following proposition shows that in most interesting cases Čech cohomology is all one needs (See [19])

Proposition 2.2.14. Čech cohomology is isomorphic to sheaf cohomology for any sheaf on a paracompact Hausdorff space.

The important example in the language of sheaves

There is an obvious structure that makes $\Omega^\bullet$ into a complex of sheaves. This observation leads to the definition of the double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^0(U_{\alpha\beta\gamma}) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^1(U_{\alpha\beta\gamma}) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^2(U_{\alpha\beta\gamma}) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots \\
\Omega^0(U_{\alpha\beta}) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^1(U_{\alpha\beta}) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^2(U_{\alpha\beta}) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots \\
\Omega^0(U_\alpha) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^1(U_\alpha) \arrow[r, "d"'] \arrow[u, "\delta"] & \Omega^2(U_\alpha) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots
\end{tikzcd}$$

where d is the usual differential of forms. It can be shown that $d \circ \delta = \delta \circ d$, so d actually defines a morphism of sheaves.

In order to apply this construction to our situation we make two adjustments. First, we modify the beginning of the complex and replace the sheaf Ω^0 by the sheaf $\underline{\mathbb{C}}^*$ of complex non-vanishing smooth functions and replace the first differential for $dLog$. Secondly, our k-cohomology groups will be taken from the truncated double complex

$$\begin{tikzcd}
\vdots & \vdots & \dots & \vdots \\
\underline{\mathbb{C}}^*(U_{\alpha\beta\gamma}) \arrow[r, "dLog"'] \arrow[u, "\delta"] & \Omega^1(U_{\alpha\beta\gamma}) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots \arrow[r, "d"'] & \Omega^{k-1}(U_{\alpha\beta\gamma}) \arrow[u, "\delta"] \\
\underline{\mathbb{C}}^*(U_{\alpha\beta}) \arrow[r, "dLog"'] \arrow[u, "\delta"] & \Omega^1(U_{\alpha\beta}) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots \arrow[r, "d"'] & \Omega^{k-1}(U_{\alpha\beta}) \arrow[u, "\delta"] \\
\underline{\mathbb{C}}^*(U_\alpha) \arrow[r, "dLog"'] \arrow[u, "\delta"] & \Omega^1(U_\alpha) \arrow[r, "d"'] \arrow[u, "\delta"] & \dots \arrow[r, "d"'] & \Omega^{k-1}(U_\alpha) \arrow[u, "\delta"]
\end{tikzcd}$$

So for $k-1 \leq l$, the groups of degree l on the total complex take the form

$$E^l(\mathcal{U}) = \bigoplus_{p+q=l, p\leq k-1} \Omega^p(U_{\alpha_0\alpha_1\dots\alpha_q}).$$

Within this new framework we may now describe more compactly the set of complex line bundles with connection. Previously we noted that a complex line bundle ∇ can be described by the local data of an open cover $\{U_\alpha\}$ of M, a family of transition functions $\{g_{\alpha\beta}\}$ and a family of complex valued 1-forms A_α such that

- the family $\{g_{\alpha\beta}\}$ has the cocycle condition,
- the family $\{A_\alpha\}$ holds the relation

$$A_\beta - A_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

This is equivalent to demand that the cochain $(g_{\alpha\beta}, A_\alpha) \in E^1(\mathcal{U})$ holds the condition $D(g_{\alpha\beta}, A_\alpha) = 0$ on the complex truncated at 1, that is, $(g_{\alpha\beta}, A_\alpha)$ is a representative of a class in

$$\check{H}^2_D(\mathcal{U}) = \frac{Ker(D\colon E^1(\mathcal{U}) \to E^2(\mathcal{U}))}{Im(D\colon E^0(\mathcal{U}) \to E^1(\mathcal{U}))}.$$

In fact, it can be shown that two representatives of the same class determine isomorphic complex line bundles with connection.

2.2.3 The generalization

As in the case of Cheeger-Simons characters, the preceding situation induces a natural generalization when we consider higher cohomology groups defined over appropriated truncated complexes. We denote the complex of sheaves

$$\underline{\mathbb{C}}^* \xrightarrow[dLog]{} \underline{\Omega}^1 \xrightarrow[d]{} \underline{\Omega}^2 \xrightarrow[d]{} \dots \xrightarrow[d]{} \underline{\Omega}^{k-1}$$

truncated at $k-1$ by $\Omega^\bullet(k)$, this complex is called the smooth Deligne complex $\Omega^\bullet(k)$.

Definition 2.2.15. Let M be a smooth manifold. The hypercohomology groups $H^p(M, \Omega^\bullet(p))$ are called smooth Deligne cohomology groups of M and denoted by $H^{p+1}_D(M, \Omega^\bullet)$.

We show that there are well defined morphisms

$$I\colon H^p_D(M, \Omega^\bullet(k)) \to H^p(M;\mathbb{Z}) \qquad R\colon H^p_D(M, \Omega^\bullet(k)) \to \Omega^p(M)$$

To define the morphism I, consider the morphism of complexes of sheaves

$$\begin{array}{ccccccccc}
\underline{\mathbb{C}}^* & \xrightarrow[dLog]{} & \underline{\Omega}^1 & \xrightarrow[d]{} & \underline{\Omega}^2 & \xrightarrow[d]{} & \dots & \xrightarrow[d]{} & \underline{\Omega}^{k-1} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & & & \downarrow \\
\underline{\mathbb{C}}^* & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0
\end{array}$$

Passing to hypercohomology we obtain the induced morphism $\pi\colon H^p(M, \Omega^\bullet(p)) \to H^p(M, \underline{\mathbb{C}}^*)$. However, $H^p(M, \underline{\mathbb{C}}^*) \cong H^{p+1}(M, \underline{\mathbb{Z}})$, for $p > 0$. To see this consider the exponential sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \underline{\mathbb{C}} \xrightarrow{exp} \underline{\mathbb{C}}^* \longrightarrow 0.$$

Hence, by the long exact sequence induced in sheaf cohomology and the fact that $H^p(M, \underline{\mathbb{C}}) = 0$[3], we obtain the claim. Next, consider the identification $H^{p+1}(M, \underline{\mathbb{Z}}) \cong H^{p+1}(M; \mathbb{Z})$, between sheaf cohomology over the sheaf $\underline{\mathbb{Z}}$ and singular cohomology with

[3]This is consequence of the existence partitions of unity for $\underline{\mathbb{C}}$, that is, the sheaf $\underline{\mathbb{C}}$ is fine.

integral coefficients. Finally, composition of π with the previous identifications defines the morphism $I\colon H^p_D(M,\Omega^\bullet)\to H^p(M;\mathbb{Z})$, for $p>1$.

Using a good cover $\{U_\alpha\}$ of M and the fact that sheaf cohomology is isomorphic to Čech cohomology for paracompact spaces, we may give a more explicit description of the morphism I. Let $(g_{\alpha_0,\dots,\alpha_{p-1}}, A^1_{\alpha_0,\dots,\alpha_{p-2}},\dots,A^{p-1}_{\alpha_0})$ be a representative of a smooth Deligne class. By definition $D(g_{\alpha_0,\dots,\alpha_{p-1}}, A^1_{\alpha_0,\dots,\alpha_{p-2}},\dots,A^{p-1}_{\alpha_0})=0$ and in particular $\delta(g_{\alpha_0,\dots,\alpha_{p-1}})=1$ on the Čech complex associated to the sheaf $\underline{\mathbb{C}}^*$. Since we are using a good cover, we may choose a principal branch of $Log(g_{\alpha_0,\dots,\alpha_{p-1}})$ over $U_{\alpha_0,\dots,\alpha_{p-1}}$ and by the connecting homomorphism of the exponential sequence we may define

$$f_{\alpha_0,\dots,\alpha_p}=\frac{1}{2\pi i}\left(-Log(g_{\alpha_1,\dots,\alpha_p})+\cdots+(-1)^{p-1}Log(g_{\alpha_0,\dots,\alpha_{p-1}})\right).$$

This clearly defines a Čech p-cocycle over the sheaf $\mathbb{Z}$. So

$$I(g_{\alpha_0,\dots,\alpha_{p-1}}, A^1_{\alpha_0,\dots,\alpha_{p-2}},\dots,A^{p-1}_{\alpha_0})=[f_{\alpha_0,\dots,\alpha_p}]$$

Now consider the morphism of complexes of sheaves given by

$$\begin{array}{ccccccccccc}
\underline{\mathbb{C}}^* & \xrightarrow[dLog]{} & \underline{\Omega}^1 & \xrightarrow[d]{} & \underline{\Omega}^2 & \xrightarrow[d]{} & \dots & \xrightarrow[d]{} & \underline{\Omega}^{k-2} & \xrightarrow[d]{} & \underline{\Omega}^{k-1}\\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle d}\\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & \underline{\Omega}^k_{cl}
\end{array}$$

passing to hypercohomology it induces the curvature morphism $R\colon H^k_D(M,\Omega^\bullet)\to\Omega^k_{cl}(M)$. As we did above, we can give an explicit description of the morphism R in terms of Čech cohomology. Fix a good cover $\{U_\alpha\}$ for M and consider a representative $(g_{\alpha_0,\dots,\alpha_{k-1}}, A^1_{\alpha_0,\dots,\alpha_{k-2}},\dots,A^{k-1}_{\alpha_0})$ of a smooth Deligne class. The morphism of complexes defining R yields the association

$$(g_{\alpha_0,\dots,\alpha_{k-1}}, A^1_{\alpha_0,\dots,\alpha_{k-2}},\dots,A^{k-1}_{\alpha_0})\mapsto dA^{k-1}_{\alpha_0}.$$

By hypothesis, we have that $D(g_{\alpha_0,\dots,\alpha_{k-1}}, A^1_{\alpha_0,\dots,\alpha_{k-2}},\dots,A^{k-1}_{\alpha_0})=0$ and in particular

$$\delta A^{k-1}_{\alpha_0}=\pm dA^{k-2}_{\alpha_0,\alpha_1}$$

it follows that $\delta dA^{k-1}_{\alpha_0}=d\delta A^{k-1}_{\alpha_0}=d^2A^{k-2}_{\alpha_0,\alpha_1}=0$. Hence $dA^{k-1}_{\alpha_0}$ defines a Čech 0-cocycle, that is, a globally defined k-form $F\in\Omega^k_{cl}(M)$.

Ring structure

When compared with the Cheeger-Simons characters, the set of Deligne cohomology groups $H^\bullet_D(M,\Omega^\bullet)$ have an easily defined ring structure. The multiplicative structure will be induced by the product $\cup\colon E^p(U)\times E^q(U)\to E^{p+q}(U)$ given by

$$x \cup y = \begin{cases} x \cdot y & \text{if } deg(x) = 0; \\ x \wedge dy & \text{if } deg(x) > 0 \text{ and } deg(y) = q-1; \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that the structure thus obtained is compatible with the morphisms of characteristic class and curvature, so it defines a morphism of rings.

Equivalence with Cheeger-Simons

It is clear that both the Deligne cohomology and the Cheeger-Simons characters yield differential refinements of singular cohomology, in fact, with minor modifications we may define the Deligne smooth complex over *real* valued differential forms instead of complex ones and a completely parallel argument would produce a differential refinement whose elements have real curvature. Then a natural question comes up: What is the relation between the Cheeger-Simons and Deligne differential refinements?

Theorem 2.2.16. The group of differential characters of degree k is canonically isomorphic to the Deligne cohomology group of degree k.

For a proof of this result see [14] and [15].

2.3 Differential K-theory

In the preceding sections we saw two different models for a differential extension of singular cohomology, we also remarked that both models are canonically isomorphic. It is natural to ask if there are similar extensions for other cohomological theories. An interesting case is that of K-theory. For K-theory there are various models that yield a differential extension. In this section we will give a basic review of one of them, namely, the construction via vector bundles with connection, this model is known as the Freed-Lott model. A similar model was proposed by Simons-Sullivan (see [36]), nevertheless we choose the Freed-Lott model for it behaves better with respect to the curvature morphism. For a brief account about other models for differential K-theory see [11]. Throughout this section we will restrict ourselves to the category of compact smooth manifolds.

2.3.1 The construction

In order to carry out the construction we recall briefly some properties of the Chern-Simons class. Fix a complex vector bundle $p\colon E \to M$ and two connections defined over E, say ∇ and ∇'. To each connection we associate its Chern character $ch(\nabla) \in H^{even}(M;\mathbb{R})$, given by $Tr(exp(\Omega/2\pi i))$, where Ω is the curvature associated to ∇. We define the Chern-Simons class associated to ∇ and ∇' as the class $CS(\nabla,\nabla') \in \Omega^{odd}(M)/Im(d)$, up to an exact one, such that

$$ch(\nabla) - ch(\nabla') = dCS(\nabla,\nabla').$$

We give a more explicit description of the Chern-Simons class. Consider the projection map $\pi\colon I\times M \to M$ and the induced bundle $\pi^*E \to I\times M$. On π^*E we define the connection $\tilde{\nabla}$, such that for a section $(a,X) \in \Gamma(T(I\times M))$ and a vector field $V \in \Gamma(\pi^*E)$ we have

$$\tilde{\nabla}_{(a,X)_{(t,p)}}(V) = t\nabla_{X_p}(V) + (1-t)\nabla'_{X_p}(V) + a\partial_t V \tag{2-3}$$

then the class $CS(\nabla,\nabla')$ is given by

$$CS(\nabla,\nabla') = \int_0^1 ch(\tilde{\nabla}) \mod Im(d).$$

The class CS has the following properties

- $CS(\nabla,\nabla') = -CS(\nabla',\nabla)$;
- $CS(\nabla,\nabla') + CS(\nabla',\nabla'') = CS(\nabla,\nabla'')$;
- let E and F be bundles over M, with connections ∇ and ∇', respectively, $\overline{\nabla}$ and $\overline{\nabla}'$. Then for the bundle $E\oplus F$ with connections $\nabla\oplus\overline{\nabla}$ and $\nabla'\oplus\overline{\nabla}'$ we have $CS(\nabla\oplus\overline{\nabla},\nabla'\oplus\overline{\nabla}') = CS(\nabla,\nabla') + CS(\overline{\nabla},\overline{\nabla}')$;
- if F and E are vector bundles over M, $\Phi: F \to E$ is a isomorphism of vector bundles over M, ∇ and ∇' are connections over E then $CS(\Phi^*\nabla,\Phi^*\nabla') = CS(\nabla,\nabla')$.

We are ready to describe our model for differential K-theory. Our building blocks will be hermitian complex vector bundles with connection plus a differential information. Explicitly,

Definition 2.3.1. Let M be a smooth manifold. A *differential vector bundle* over M is a quadruple (E,h,∇,ω) such that

- E is a complex vector bundle over M,
- h is a hermitian metric on E,
- ∇ is a connection on E compatible with the metric,
- ω is a class in $\Omega^{odd}(M)/Im(d)$.

As in the classic construction of K-theory, an equivalence relation between differential vector bundles is defined so that objects representing the "same" information are identified. We say that two differential vector bundles (E,h,∇,ω) and (E',h',∇',ω') are equivalent if there is an isomorphism of bundles $\phi\colon (E,h) \to (E',h')$ such that

$$\omega - \omega' = CS(\nabla, \phi^*\nabla'),$$

in particular, the bundles have the same fibrewise dimension.

The relation thus defined is an equivalence relation. Indeed, by properties of the class CS we have that $CS(\nabla, \nabla) = 0$, hence the relation is reflexive. Since, $CS(\nabla, \nabla') = -CS(\nabla', \nabla)$ and $CS(\Phi^*\nabla, \Phi^*\nabla') = CS(\nabla, \nabla')$ we have that $\omega - \omega' = CS(\nabla, \phi^*\nabla')$ is equivalent to $\omega' - \omega = CS(\nabla', (\phi^*)^{-1}\nabla)$, so the relation is symmetric. Finally, if $(E, h, \nabla, \omega) \sim (E', h', \nabla', \omega')$ and $(E', h', \nabla', \omega') \sim (E'', h'', \nabla'', \omega'')$, through isomorphisms $\phi\colon (E, h) \to (E', h')$ and $\phi'\colon (E', h') \to (E'', h'')$ then

$$\begin{aligned}
\omega - \omega'' =&(\omega - \omega') + (\omega' - \omega'') \\
=&CS(\nabla, \phi^*\nabla') + CS(\nabla', \phi'^*\nabla'') \\
=&CS(\nabla, \phi^*\nabla') + CS(\phi^*\nabla', \phi^*\phi'^*\nabla'') \\
=&CS(\nabla, (\phi' \circ \phi)^*\nabla''),
\end{aligned}$$

thus the relation is transitive.

The set of differential vector bundles comes with a naturally defined semigroup structure. In fact, for differential bundles (E, h, ∇, ω) and $(E', h', \nabla', \omega')$, we define their sum as

$$(E, h, \nabla, \omega) \oplus (E', h', \nabla', \omega') = (E \oplus E', h \oplus h', \nabla \oplus \nabla', \omega + \omega').$$

This operation induces a semigroup structure on the set of equivalence classes defined above. Now, following Grothendieck, we give a construction that produces a group out of a semigroup. The construction is a as follows: given two pairs of equivalence classes of differential vector bundles $([(E, h, \nabla, \omega)], [(F, f, \Lambda, \sigma)])$ and $([(E', h', \nabla', \omega')], [(F', f', \Lambda', \sigma')])$, we say that they are equivalent if there is a differential vector bundle class $[(G, g, \Gamma, \eta)]$ such that

$$[(E, h, \nabla, \omega)] \oplus [(F', f', \Lambda', \sigma')] \oplus [(G, g, \Gamma, \eta)] = [(E', h', \nabla', \omega')] \oplus [(F, f, \Lambda, \sigma)] \oplus [(G, g, \Gamma, \eta)]$$

We call the set of equivalence classes of pairs of classes of differential vector bundles the *differential K-theory group of M* and denote it by $\hat{K}(M)$. To ease the notation, the class of the pair $[[(E, h, \nabla, \omega)], [(F, f, \Lambda, \sigma)]]$ will be denoted by $((E, h, \nabla, \omega), (F, f, \Lambda, \sigma))$ or $(E, h, \nabla, \omega) - (F, f, \Lambda, \sigma)$[4], whenever there is not risk of confusion. Observe that there is a canonical map of the semigroup of classes of differential vector bundles into $\hat{K}(M)$ given by $[(E, h, \nabla, \omega)] \mapsto ((E, h, \nabla, \omega), (0, 0, 0, 0))$, again to ease the notation we will denote the latter simply by (E, h, ∇, ω). With respect to this map we will say that two classes of differential vector bundles are stably equivalent if they have the same image in $\hat{K}(M)$, that is, two classes $[(E, h, \nabla, \omega)]$ and $[(F, f, \Lambda, \sigma)]$ are stably equivalent if there is a differential vector bundle (G, g, Γ, η) such that

[4]This notation comes from the usual construction of the integers using pairs of natural numbers.

$$[(E,h,\nabla,\omega)]\oplus[(G,g,\Gamma,\eta)]=[(F,f,\Lambda,\sigma)]\oplus[(G,g,\Gamma,\eta)]$$

As we observed above, $\hat{K}(M)$ is in fact a group under the induced sum of pairs

$$((E,h,\nabla,\omega),(F,f,\Lambda,\sigma))\oplus((E',h',\nabla',\omega'),(F',f',\Lambda',\sigma'))=\\ ((E\oplus E',h\oplus h',\nabla\oplus\nabla',\omega+\omega'),(F\oplus F',f\oplus f',\Lambda\oplus\Lambda',\sigma+\sigma')).$$

Indeed, it is clear that the class $((E,h,\nabla,\omega),(E,h,\nabla,\omega))$ represents the identity element of the group for any differential vector bundle (E,h,∇,ω) , while the inverse of the class $((E,h,\nabla,\omega),(F,f,\Lambda,\sigma))$ is given by $((F,f,\Lambda,\sigma),(E,h,\nabla,\omega))$.

Let M be a compact space. By a classical result [1, 1.4.14], for a given complex bundle E over M there is a complex bundle G over M, such that $E\oplus G$ is trivial. Hence, if we have a differential class $(E,h,\nabla,\omega)-(F,f,\Lambda,\sigma)$ then there is a differential vector bundle $(G,g,\Gamma,-\sigma)$ such that

$$(F,f,\Lambda,\sigma)\oplus(G,g,\Gamma,-\sigma)\sim(\underline{n},f',\Lambda',0).$$

Thus an arbitrary element in $\hat{K}(M)$ can be represented as

$$(E,h,\nabla,\omega)-(\underline{n},f,\Lambda,0)$$

where $ch(\Lambda)$ is necessarily exact.

As in the case of Cheeger-Simons characters and Deligne cohomology, from every class in $\hat{K}(M)$ we may extract cohomological and differential information, thus inducing morphisms into the respective groups. Using the same notation of the previous cases we define the morphisms

$$I\colon\hat{K}(M)\to K(M)$$

given by

$$(E,h,\nabla,\omega)-(\underline{n},f,\Lambda,0)\mapsto E-\underline{n}.$$

The curvature morphism

$$R\colon\hat{K}(M)\to\Omega^{even}(M)$$

given by

$$(E,h,\nabla,\omega)-(\underline{n},f,\Lambda,0)\mapsto ch(\nabla)-ch(\Lambda)-d\omega.$$

And the morphism

$$a\colon\Omega^{odd}(M)/Im(d)\to\hat{K}(M)$$

given by

$$\omega \mapsto (0,0,0,-\omega).$$

Before we proceed to establish the principal properties of the morphisms just introduced, we recall briefly the group $K^{-1}(M)$ and the odd Chern character. The group $K^{-1}(M)$ is classically defined as the K-theory group corresponding to SM, the suspension of M. Alternatively, using the exact sequence of pairs

$$\cdots \longrightarrow K(S^1 \times M, M) \longrightarrow \tilde{K}(S^1 \times M) \xrightarrow{i^*} \tilde{K}(M) \longrightarrow \cdots$$

and the fact that the projection onto M is a left inverse to the inclusion of M into $S^1 \times M$, we obtain a decomposition

$$\tilde{K}(X) \oplus Ker(i^*) \cong \tilde{K}(S^1 \times M) \cong \tilde{K}(X) \oplus K(S^1 \times M, M) = \tilde{K}(X) \oplus K^{-1}(M)$$

So we identify $K^{-1}(M) \cong Ker(i^*)$. About the odd Chern character, we recall that for a class $E - F \in K(S^1 \times M)$ such that $E - F \in Ker(i^*)$, that is, $E - F$ represents an element in $K^{-1}(M)$, we have

$$ch^{-1}(E - F) = \int_{S^1} (ch\nabla - ch\nabla') \tag{2-4}$$

where ∇ and ∇' are connections on E and F, respectively.

Let (F, h, ∇) be a fixed vector bundle with metric h and compatible connection ∇, and denote by $Aut(F, h)$ the family of all isometries of (F, h). For each $\phi \in Aut(F, h)$ we obtain an element $CS(\nabla, \phi^*\nabla) \in \Omega^{odd}(M)/Im(d)$. It is easy to see that the element $CS(\nabla, \phi^*\nabla)$ is independent of ∇, for

$$\begin{aligned} CS(\nabla, \phi^*\nabla) =& CS(\nabla, \nabla') + CS(\nabla', \phi^*\nabla) \\ =& CS(\nabla, \nabla') + CS(\nabla', \phi^*\nabla') + CS(\phi^*\nabla', \phi^*\nabla) \\ =& CS(\nabla, \nabla') + CS(\nabla', \phi^*\nabla') + CS(\nabla', \nabla) \\ =& CS(\nabla, \nabla') + CS(\nabla', \phi^*\nabla') - CS(\nabla, \nabla') \\ =& CS(\nabla', \phi^*\nabla') \end{aligned}$$

Using the observation above we may define the function

$$\Theta_{(F,k)} \colon Aut(F, k) \to \Omega^{odd}(M)/Im(d).$$

Lemma 2.3.2. Let

$$\Theta \colon \bigcup_{(F,k)} Aut(F, k) \to \Omega^{odd}(M)/Im(d)$$

be the function that assigns to each $\phi \in Aut(F, k)$ the form $\Theta_{(F,k)}(\phi)$. Then

$$Im(\Theta) = Im(ch^{-1}).$$

Proof. Let $CS(\nabla,\phi^*\nabla)\in Im(\Theta)$, for a given bundle $p\colon F\to M$ with connection ∇. Consider the bundle $p'\colon p^*F\to I\times M$ with connection $\tilde{\nabla}$ given by (2-3); using the isometry ϕ we may identify $p^*F|_{\{1\}\times M}$ and $p^*F|_{\{0\}\times M}$ to obtain a space E that in fact defines a bundle $q\colon E\to S^1\times M$. Likewise, the connection $\tilde{\nabla}$ glues naturally to a connection ∇_E on E, in fact, for $\pi_{I,S^1}\colon I\times M\to S^1\times M$ the identification map, we have $\pi^*_{I,S^1}\nabla_E\simeq\tilde{\nabla}$. Hence by definition we have

$$CS(\nabla,\phi^*\nabla)=\int_0^1 ch(\tilde{\nabla})=\int_0^1\pi^*_{I,S^1}ch\nabla_E=\int_{S^1}ch\nabla_E.$$

Now consider a similar construction with $\phi=\mathrm{id}$. Call the resulting total space by E_{id} and the connection by $\tilde{\nabla}_{\mathrm{id}}$. It is easy to see that $\tilde{\nabla}_{\mathrm{id}}=i^*\nabla$, where $i\colon M\to S^1\times M$ is the inclusion. It follows that $\int_{S^1}ch(\tilde{\nabla}_{\mathrm{id}})=0$ and so by (2-4)

$$CS(\nabla,\phi^*\nabla)=\int_{S^1}ch\nabla_E=\int_{S^1}(ch\nabla_E-ch\nabla_{\mathrm{id}})=ch^{-1}(E-E_{\mathrm{id}}),$$

it follows that $Im(\Theta)\subseteq Im(ch^{-1})$.

Conversely, consider a class $E-\underline{n}\in K^{-1}(M)$, that is, a class $E-\underline{n}\in\tilde{K}(S^1\times M)$ such that $i^*(E-\underline{n})=0$, for $i^*\colon\tilde{K}(S^1\times M)\to\tilde{K}(M)$. In particular, we have that $E|_M-\underline{n}=0$ or equivalently that exists $\underline{m}$ such that $E|_M\oplus\underline{m}\simeq\underline{n}\oplus\underline{m}$. Summing $\underline{m}$ to both terms, we may as well suppose that the bundles E and $\underline{n}$ are such that $E|_M\simeq\underline{n}$. Now consider the bundle $\pi^*_{I,S^1}E\to I\times M$. By homotopy properties this bundle is of the form $\pi_I^*G\to I\times M$, for a bundle $G\to M$ and the projection map $\pi_I\colon I\times M\to M$. However, by the observations made above $\pi^*_{I,S^1}E|_{\{0\}\times X}\simeq\underline{n}$, hence $\pi^*_{I,S^1}E\simeq\underline{n}$. Let us denote by $\Upsilon\colon\pi^*_{I,S^1}E\to\underline{n}$ the isomorphism relating both bundles, then we have an isomorphism $\upsilon\colon\underline{n}\to\underline{n}$ defined by restricting Υ to the boundaries, that is, $\Upsilon_1\circ\Upsilon_0^{-1}$. Then identifying boundaries through υ we obtain a bundle $\underline{n}_\upsilon$ over $S^1\times M$, such that $E\simeq\underline{n}_\upsilon$ and so $E-\underline{n}=\underline{n}_\upsilon-\underline{n}$. It follows that

$$ch^{-1}(E-\underline{n})=ch^{-1}(\underline{n}_\upsilon-\underline{n})=CS(\nabla,\upsilon^*\nabla)$$

for any connection ∇ on $\underline{n}$. Hence $Im(ch^{-1})\subseteq Im(\Theta)$. □

Proposition 2.3.3. The morphisms I, R and a hold

1. $dR\circ R=ch\circ I$, where dR is the morphism that takes a closed form to its de Rham class and ch is the Chern character in K-theory;

2. $R\circ a=d$;

3. the sequence

$$K^{-1}(M)\xrightarrow{ch^{-1}}\frac{\Omega^{odd}(M)}{Im(d)}\xrightarrow{a}\hat{K}(M)\xrightarrow{I}K(M)\longrightarrow 0$$

is exact.

Proof. 1. Let $\alpha = (E, h, \nabla, \omega) - (\underline{n}, e, can, 0) \in \hat{K}(M)$. By definition we have $R(\alpha) = ch(\nabla) - d\omega$, so $dR \circ R(\alpha) = [ch(\nabla)] = ch(E)$, the last equality being consequence of de Rham theorem. On the other hand, $I(\alpha) = E - \underline{n}$, thus $ch \circ I(\alpha) = ch(E)$. The result follows.

2. Take $\omega \in \Omega^{odd}(M)/Im(d)$, then $a(\omega) = (0, 0, 0, -\omega)$ and $R \circ a(\omega) = d\omega$.

3. That I is surjective is clear. To prove the exactness at $\Omega^{odd}(M)/Im(d)$, using lemma 2.3.2, it is enough to note that for $\omega \in \Omega^{odd}(M)/Im(d)$ we have $a(\omega) = 0$ if and only if $\omega = CS(\nabla, \phi^*\nabla)$, for some vector bundle F with connection ∇ and some isometry $\phi\colon F \to F$. Let ω be such that $(0, 0, 0, \omega) = (0, 0, 0, 0)$ in $\hat{K}(M)$, that is, there is a differential vector bundle, that we may as well assume to be of the form $(\underline{n}, g, \Gamma, \sigma)$, such that $(\underline{n}, g, \Gamma, \omega + \sigma) \sim (\underline{n}, g, \Gamma, \sigma)$. This is equivalent to

$$\omega \in CS(\Gamma, \phi^*\Gamma)$$

for an appropriated ϕ. On the other direction, if $\omega = CS(\nabla, \phi^*\nabla)$ for a given vector bundle F with connection ∇, then it is clear that $(0, 0, 0, -\omega) = (0, 0, 0, 0)$ in $\hat{K}(M)$. In fact,

$$\omega = CS(\nabla, \phi^*\nabla)$$

implies that

$$(F, g, \nabla, \sigma) \sim (F, g, \nabla, \sigma - \omega)$$

or equivalently

$$(0, 0, 0, 0) \oplus (F, g, \nabla, \sigma) \sim (0, 0, 0, -\omega) \oplus (F, g, \nabla, \sigma).$$

About the exactness at $\hat{K}(M)$, it is clear that $I \circ a = 0$. In the other direction, if $I((E, h, \nabla, \omega) - (\underline{n}, f, \Lambda, 0)) = 0$, then exists a bundle G such that $E \oplus G \cong \underline{n} \oplus G$. This is equivalent to say that there exist trivial bundles $\underline{p}$ and $\underline{m}$ such that $E \oplus \underline{p} \cong \underline{m}$, hence the class $(E, h, \nabla, \omega) - (\underline{n}, f, \Lambda, 0)$ is the same as $(\underline{m}, h', \nabla', \omega) - (\underline{m}, h', \nabla', 0)$. From this observation it follows directly that $(\underline{m}, h', \nabla', \omega) - (\underline{m}, h', \nabla', 0) = a(-\omega)$. □

Let us denote by $\mathbb{T}^n$ the n-torus, that is, the n-fold product of S^1, and by i_j the inclusions

$$\begin{aligned} i_j\colon \mathbb{T}^{n-1} \times M &\hookrightarrow \mathbb{T}^n \times M \\ (t_1, \ldots, t_{n-1}, m) &\mapsto (t_1, \ldots, t_{j-1}, 1, t_j, \ldots, t_{n-1}, m). \end{aligned}$$

Also, we will denote by $\mathfrak{k}^\bullet_{\mathbb{R}}$, the K-theory ring of the point with real coefficients, that is, $\mathfrak{k}^{2n}_{\mathbb{R}} \cong \mathbb{R}$ and $\mathfrak{k}^{2n+1}_{\mathbb{R}} = 0$. With these conventions we have $\Omega^{2n}(M; \mathfrak{k}^\bullet_{\mathbb{R}}) \cong \Omega^{even}(M)$, whereas $\Omega^{2n+1}(M; \mathfrak{k}^\bullet_{\mathbb{R}}) \cong \Omega^{odd}(M)$.

Now to define the differential groups for all negative degrees we observe that in the topological case the following relation holds

$$K^{-n}(M) \cong \bigcap_j Ker(i_j^*\colon K(\mathbb{T}^n \times M) \to K(\mathbb{T}^{n-1} \times M)). \tag{2-5}$$

So its generalization to the differential case takes the form:

Definition 2.3.4. We define $\hat{K}^{-n}(M)$ as the set of elements $\alpha \in \hat{K}(\mathbb{T}^n \times M)$ such that

- $\alpha \in \bigcap_j Ker(i_j^*\colon \hat{K}(\mathbb{T}^n \times M) \to \hat{K}(\mathbb{T}^{n-1} \times M))$,
- exists $\omega \in \Omega^{-n}(M; \mathfrak{k}_{\mathbb{R}}^\bullet)$ such that

  $$R(\alpha) = dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \omega, \tag{2-6}$$

  where $\pi_M\colon \mathbb{T}^n \times M \to M$ is the projection.

Using (2-5) we have the obvious extension of the morphism I

$$\begin{aligned} I^{-n}\colon \hat{K}^{-n}(M) &\to K^{-n}(M) \\ (E, h, \nabla, \omega) - (\underline{n}, f, \Gamma, 0) &\mapsto E - \underline{n}. \end{aligned}$$

We define the curvature morphism in degree $-n$ as

$$\begin{aligned} R^{-n}\colon \hat{K}^{-n}(M) &\to \Omega^{-n}(M; \mathfrak{k}_{\mathbb{R}}^\bullet) \\ \alpha &\mapsto \int_{\mathbb{T}^n} R(\alpha). \end{aligned}$$

Note that by (2-6) this is equivalent to take $R^{-n}(\alpha) = \omega$. Finally the generalization of the morphism a is given as

$$\begin{aligned} a^{-n}\colon \Omega^{-n-1}(M; \mathfrak{k})_{\mathbb{R}}^\bullet / Im(d) &\to \hat{K}^{-n}(M) \\ \omega &\mapsto (0, 0, 0, (-1)^{n+1} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \omega). \end{aligned}$$

As expected, the morphisms thus defined preserve the properties established in proposition 2.3.3 for the 0-degree case.

Proposition 2.3.5. The morphisms I^{-n}, R^{-n} and a^{-n} hold

1. $dR \circ R^{-n} = ch^{-n} \circ I^{-n}$ where dR is the morphism that takes a closed form to its de Rham class and ch^{-n} is the Chern character on degree $-n$ for k-theory;
2. $R^{-n} \circ a^{-n} = d$;

3. the sequence

$$K^{-n-1}(M) \xrightarrow{ch^{-n-1}} \frac{\Omega^{-n-1}(M;\mathfrak{k}^\bullet_{\mathbb{R}})}{Im(d)} \xrightarrow{a^{-n}} \hat{K}^{-n}(M) \xrightarrow{I^{-n}} K^{-n}(M) \longrightarrow 0$$

is exact.

Proof. 1. Take $\alpha \in \hat{K}^{-n}(M)$, by definition $R^{-n}(\alpha) = \int_{\mathbb{T}^n} R(\alpha)$, so

$$\begin{aligned} dR \circ R^{-n}(\alpha) =& \int_{\mathbb{T}^n} dR(R(\alpha)) \\ =& \int_{\mathbb{T}^n} ch(I(\alpha)) \\ =& ch^{-n}\left(\int_{\mathbb{T}^n} I(\alpha)\right) \\ =& ch^{-n} \circ I^{-n}(\alpha). \end{aligned}$$

2. Let $\omega \in \Omega^{-n-1}(M;\mathfrak{k}^\bullet_{\mathbb{R}})$, then $a^{-n}(\omega) = (0,0,0,(-1)^{n+1}dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega)$. Hence

$$\begin{aligned} R^{-n} \circ a^{-n}(\omega) =& R^{-n}(0,0,0,(-1)^{n+1}dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega) \\ =& \int_{\mathbb{T}^n} R(0,0,0,(-1)^{n+1}dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega) \\ =& \int_{\mathbb{T}^n} -d((-1)^{n+1}dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega) \\ =& \int_{\mathbb{T}^n} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* d\omega \\ =& d\omega. \end{aligned}$$

3. Exactness at $\Omega^{-n-1}(M;\mathfrak{k}^\bullet_{\mathbb{R}})/Im(d)$ is clear once we note that $a^{-n}(\omega) = 0$ if and only if $a(dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega) = 0$ in $\hat{K}(S^1 \times M)$. By proposition 2.3.3, this happens if and only if $dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\omega \in Im(ch^{-1})$, which is equivalent to $\omega \in Im(ch^{-n-1})$.

   It is clear that $I^{-n} \circ a^{-n} = 0$. Conversely, suppose that $\alpha \in \hat{K}^{-n}(M)$ is such that $I^{-n}(\alpha) = 0$, then as a class in $\hat{K}(\mathbb{T}^n \times M)$ we have that $I(\alpha) = 0$, so by item 2 of proposition 2.3.3 it follows that $\alpha = a(\omega)$. By condition (2-6) and item 1 in proposition 2.3.3, we have that

$$d\omega = R \circ a(\omega) = R(\alpha) = dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* d\rho.$$

   Hence $\omega = \eta + (-1)^n dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^*\rho$, where $d\eta = 0$. Since $\alpha \in Ker(i_n^*)$, the commutativity of a and i_n^* implies that $a(i_n^*\eta) = 0$. Define $\eta' = \eta - \pi_n^* \circ i_n^*\eta$, where

$\pi_j \colon \mathbb{T}^n \times M \to \mathbb{T}^{n-1} \times M$ is the projection of all but the j-th component

$$\begin{aligned} \pi_j \colon \mathbb{T}^n \times M &\to \mathbb{T}^{n-1} \times M \\ (t_1, \dots, t_n, m) &\mapsto (t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n-1}, m). \end{aligned} \tag{2-7}$$

Then setting $\omega' = \eta' + (-1)^n dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \rho$, we have that $a(\omega') = a(\omega) = \alpha$ and $i_n^* \eta' = 0$, in particular we obtain that $\eta' = dt_n \wedge \pi_n^* \sigma$ and $\omega' = -dt_n \wedge \pi_n^* \overline{\omega}$. Repeating successively the argument above for i_j^* with $j = n-1, n-2, \dots, 1$, we obtain a $\omega^{(n)} = (-1)^n dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \rho$, such that $a^{-n}(\rho) = a(\omega^{(n)}) = \alpha$.

The morphism I^{-n} is surjective. Indeed, consider a class $E - F \in K(\mathbb{T}^n \times M)$, such that $E - F \in Ker(i_j^* \colon K(\mathbb{T}^n \times M) \to K(\mathbb{T}^{n-1} \times M))$, for $j = 1, \dots, n$. By proposition 2.3.3, there is a differential class $\alpha \in \hat{K}(\mathbb{T}^n \times M)$, such that $I(\alpha) = E - F$. We may suppose that $\alpha = (E, h, \nabla, 0) - (F, f, \nabla', 0)$, so under the de Rham morphism we have that $[ch(\nabla) - ch(\nabla')] = ch(E - F)$. From the naturality of ch, it follows that $[ch(i_j^* \nabla) - ch(i_j^* \nabla')] = 0$, for each $j = 1, \dots, n$. Hence $ch(\nabla) - ch(\nabla')$ has the form $d\omega + dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \rho$. We consider the class $\alpha' = (E, h, \nabla, \omega) - (F, f, \nabla', 0)$; its curvature is given by

$$R(\alpha') = ch(\nabla) - ch(\nabla') - d\omega = dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* \rho,$$

so it holds the condition (2-6) on the definition of $\hat{K}^{-n}(M)$. In order to obtain a class that holds the first condition on $\hat{K}^{-n}(M)$, we will add an appropriated class. Since $i_n^*(E - F) = 0$, we have that $I \circ i_n^*(\alpha) = 0$, hence exists ξ such that $a(\xi) = i_n^* \alpha$ and $d\xi = 0$. Note that for $\pi_n \colon \mathbb{T}^n \times M \to \mathbb{T}^{n-1} \times M$, as in (2-7), we have $\pi_n \circ i_n = \mathrm{id}$, therefore

$$\begin{aligned} i_n^*(\alpha - a(\pi_n^* \xi)) &= a(\alpha) - a(i_n^* \pi_n^* \xi) \\ &= a(\alpha) - a(\alpha) \\ &= 0. \end{aligned} \tag{2-8}$$

Let us denote $\alpha' = \alpha - a(\pi_n^* \xi)$. Clearly $I(\alpha') = I(\alpha) = E - F$, so $i_{n-1}^* \circ I(\alpha') = 0$. As above we have that exists ξ', such that $a(\xi') = i_{n-1}^* \alpha'$, $d\xi' = 0$ and, since $i_n^* \alpha' = 0$, also $i'^*_{n-1} \alpha' = 0$, for $i'_{n-1} : \mathbb{T}^{n-2} \times M \to \mathbb{T}^{n-1} \times M$ given by $(t_1, \dots, t_{n-2}, m) \mapsto (t_1, \dots, t_{n-2}, 1, m)$. We consider $\alpha'' = \alpha' - a(\pi_{n-1}^* \xi')$, where $\pi_{n-1} \colon \mathbb{T}^n \times M \to \mathbb{T}^{n-1} \times M$ is given by (2-7), with $\pi_{n-1} \circ i_{n-1} = \mathrm{id}$. A similar calculation as in (2-8) shows that $i_{n-1}^* \alpha'' = 0$, on the other hand $i'^*_{n-1} \alpha' = 0$ implies that $i_n^* \alpha'' = 0$. Proceeding successively in the same way, we find $\alpha^{(n)}$ such that $i_j^* \alpha''^{(n)} = 0$, for all $j = 1, \dots, n$ and such that $I^{-n} \alpha^{(n)} = E - F$.

□

2.3.2 Ring Structure

There is a naturally defined product of classes of differential vector bundles. In fact, for (E, h, ∇, ω) and $(E', h', \nabla', \omega')$, we define

$$(E, h, \nabla, \omega)(E', h', \nabla', \omega') = (E \otimes E', h \otimes h', \nabla \otimes \nabla', \omega \wedge (ch\nabla' - d\omega') + (ch\nabla - d\omega) \wedge \omega' + \omega \wedge d\omega')$$

This product induces a ring structure

$$\begin{aligned} \hat{K}(M) \times \hat{K}(M) &\to \hat{K}(M) \\ (\alpha, \beta) &\mapsto \alpha \cdot \beta, \end{aligned}$$

on $\hat{K}(M)$. Obviously, $I(\alpha \cdot \beta) = I(\alpha)I(\beta)$ for classes α, $\beta \in \hat{K}(M)$. Similarly, we have $R(\alpha \cdot \beta) = R(\alpha) \wedge R(\beta)$. Indeed, suppose $\alpha = (E, h, \nabla, \omega) - (F, f, \Gamma, 0)$ and $\beta = (E', h', \nabla', \omega') - (F', f', \Gamma', 0)$, then

$$\begin{aligned} \alpha \cdot \beta = (E \otimes E' \oplus F \otimes F', \nabla \otimes \nabla' \oplus \Gamma \otimes \Gamma', \omega \wedge (ch\nabla' - d\omega') + (ch\nabla - d\omega) \wedge \omega' + \omega \wedge d\omega') \\ -(E \otimes F' \oplus F \otimes E', \nabla \otimes \Gamma' \oplus \Gamma \otimes \nabla', \omega \wedge ch\Gamma' + ch\Gamma \wedge \omega'). \end{aligned}$$

Hence

$$\begin{aligned} R(\alpha \cdot \beta) =& ch(\nabla \otimes \nabla' \oplus \Gamma \otimes \Gamma') - ch(\nabla \otimes \Gamma' \oplus \Gamma \otimes \nabla') - d\omega \wedge (ch\nabla' - d\omega') \\ & - (ch\nabla - d\omega) \wedge d\omega' - d\omega \wedge d\omega' + d\omega \wedge ch\Gamma' + ch\Gamma \wedge d\omega' \\ =& (ch\nabla - ch\Gamma - d\omega) \wedge (ch\nabla' - ch\Gamma' - d\omega') \\ =& R(\alpha) \wedge R(\beta). \end{aligned}$$

From this internal product we can easily define an external product $\hat{K}(M) \times \hat{K}(N) \to \hat{K}(M \times N)$. In fact, for $\alpha \in \hat{K}(M)$ and $\beta \in \hat{K}(N)$ we define their product as

$$\alpha \boxtimes \beta := \pi_1^*(\alpha) \cdot \pi_2^*(\beta),$$

where $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ are the natural projections.

Using the external product we can extend the multiplicative structure to all non-positive degrees, yielding a family of morphisms

$$\boxtimes \colon \hat{K}^{-m}(M) \times \hat{K}^{-n}(N) \to \hat{K}^{-n-m}(M \times N)$$

Explicitly, consider $\alpha \in \hat{K}^{-m}(M)$ and $\beta \in \hat{K}^{-n}(N)$, then as elements of $\hat{K}(\mathbb{T}^m \times M)$ and $\hat{K}(\mathbb{T}^n \times N)$, respectively, we have

$$\alpha \boxtimes_0 \beta \in \hat{K}(\mathbb{T}^m \times M \times \mathbb{T}^n \times N),$$

where $\boxtimes_0$ is the external product as defined above. Set $\varphi \colon \mathbb{T}^{m+n} \times M \times N \to \mathbb{T}^m \times M \times \mathbb{T}^n \times N$, the natural diffeomorphism, so that $\varphi^* \colon \hat{K}(\mathbb{T}^m \times M \times \mathbb{T}^n \times N) \to \hat{K}(\mathbb{T}^{m+n} \times M \times N)$. Then, we define the external product as

$$\alpha \boxtimes \beta = (-1)^{mn} \varphi^*(\alpha \boxtimes_0 \beta).$$

It is easy to prove that $\alpha \boxtimes \beta$ holds both conditions on the definition of $\hat{K}^{-m-n}(M \times N)$.

The product thus defined is also compatible with the morphism a, that is, for $\alpha \in \hat{K}^{-n}(M)$ and $\omega \in \Omega^{-m}(M; \mathfrak{k}^\bullet_{\mathbb{R}})$ the product has the property

$$a^{-m}(\omega) \cdot \alpha = a^{-m-n}(\omega \wedge R^{-n}\alpha)$$

Indeed,

$$\begin{aligned} a^{-m}(\omega) \cdot \alpha =& (0, 0, 0, (-1)^{m+1} dt_1 \wedge \cdots \wedge dt_m \wedge \pi_M^* \omega) \cdot \alpha \\ =& (0, 0, 0, (-1)^{mn+m+1} dt_1 \wedge \cdots \wedge dt_m \wedge \pi_M^* \omega \wedge dt_{m+1} \wedge \cdots \wedge dt_{m+n} \wedge \pi_M^* R^{-n}\alpha) \\ =& (0, 0, 0, (-1)^{n+m+1} dt_1 \wedge \cdots \wedge dt_{m+n} \wedge \pi_M^* \omega \wedge \pi_M^* R^{-n}\alpha) \\ =& (0, 0, 0, (-1)^{n+m+1} dt_1 \wedge \cdots \wedge dt_{m+n} \wedge \pi_M^* (\omega \wedge R^{-n}\alpha)) \\ =& a^{-n-m}(\omega \wedge R^{-n}\alpha). \end{aligned}$$

2.3.3 Integration

In order to define S^1-integration for all non-positive degrees in $\hat{K}^{-m}(M)$, we first define it for degree zero and then extend it to all non-positive degrees using the relation $\hat{K}^{-m}(M) \subseteq \hat{K}(\mathbb{T}^m \times M)$. In degree zero the integration map should take a class $\alpha \in \hat{K}(S^1 \times M)$ to a class $\int_{S^1} \alpha \in \hat{K}^{-1}(M) \subseteq \hat{K}(S^1 \times M)$. We set $\int_{S^1} \alpha = \alpha$, if α is already in $\hat{K}^{-1}(M)$. In order to define the integral for an arbitrary element we consider

$$\alpha' := \alpha - \pi_1^* i_1^* \alpha$$

where $i_1 \colon M \to S^1 \times M$ is the natural inclusion and $\pi_1 \colon S^1 \times M \to M$ is the canonical projection. Note that $\alpha' \in Ker(i_1^*)$, in particular $i_1^* \circ I(\alpha') = 0$, therefore its image in the de Rham cohomology has the form $[dt \wedge \pi_1^* \omega]$ and $R(\alpha') = d\eta + dt \wedge \pi_1^* \omega$. Thus we can ensure that α' holds the first condition defining $\hat{K}^{-1}(M)$. From the relation $R(\alpha') = d\eta + dt \wedge \pi_1^* \omega$ follows that

$$di_1^*(\eta) = i_1^* d\eta = i_1^* \left(R(\alpha') - dt \wedge \pi_1^* \omega\right) = R(i_1^* \alpha') = 0.$$

Now, to transform α' into a class that holds the second condition we consider $\alpha'' = \alpha' - a(\eta')$, where the form η' holds $\eta' = \eta - \pi_1^* i_1^* \eta$. Note that $i_1^*(\eta') = 0$, for $\pi_1 \circ i_1 = \mathrm{id}$; hence we have also $i_1^*(\alpha'') = 0$. The class α'' also has the second condition, indeed

$$R(\alpha'') = R(\alpha' - a(\eta')) = d\eta + dt \wedge \pi_1^* \omega - d\eta' = dt \wedge \pi_1^* \omega + d\pi_1^* i_1^* \eta = dt \wedge \pi_1^* \omega.$$

Thus $\alpha'' \in \hat{K}^{-1}(M)$. So for an arbitrary class $\alpha \in \hat{K}(S^1 \times M)$ we define

$$\begin{aligned} \int_{S^1} \alpha :=& \alpha - \pi_1^* i_1^* \alpha - a(\eta') + \int_{S^1} a(\eta') \\ =& \int_{S^1} \alpha'' + \int_{S^1} a(\eta'), \end{aligned}$$

where

$$\int_{S^1} a(\eta') := -a^{-1}\left(\int_{S^1} \eta'\right) = a\left(dt \wedge \pi_1^*\left(\int_{S^1} \eta'\right)\right).$$

We still have to show that this definition is independent of the choice of η'. Let ρ be another form such that

$$R(\alpha - \pi_1^* i_1^* \alpha - a(\eta' + \rho)) = dt \wedge \pi_1^* \omega'; \qquad i_1^*(\eta' + \rho) = 0$$

then $d(\rho) = dt \wedge \pi_1^* \mu$ and $i_1^* \rho = 0$. Hence, if S^1-integration is to be independent of choice of η', we should have

$$\alpha - \pi_1^* i_1^* \alpha - a(\eta') + \int_{S^1} a(\eta') = \alpha - \pi_1^* i_1^* \alpha - a(\eta') - a(\rho) + \int_{S^1} a(\eta') + \int_{S^1} a(\rho),$$

or equivalently

$$a(\rho) = \int_{S^1} a(\rho) = a\left(dt \wedge \pi_1^*\left(\int_{S^1} \rho\right)\right).$$

From $d\rho = dt \wedge \pi_1^* \mu$ it follows that, as a de-Rham cohomology class, $[dt \wedge \pi_1^* \mu] = 0$. Since S^1-integration defines an isomorphism, with inverse $dt \wedge \pi_1^*$, between de-Rham cohomology of M and $Ker(i_1^*)$, we have $[\mu] = 0$ and $\mu = -d\tau$. It follows that $d\rho = d(dt \wedge \pi_1^* \tau)$ and $\rho = dt \wedge \pi_1^* \tau + \varphi$, where φ is a closed form.

Observe that

$$\int_{S^1} \rho = \int_{S^1} (dt \wedge \pi_1^* \tau + \varphi) = \tau + \int_{S^1} \varphi,$$

hence

$$dt \wedge \pi_1^*\left(\int_{S^1} \rho\right) = dt \wedge \pi_1^* \tau + dt \wedge \pi_1^*\left(\int_{S^1} \varphi\right).$$

Since $i_1^* \rho = 0$, we have $i_1^* \varphi = 0$ and $dt \wedge \pi_1^*\left(\int_{S^1} \varphi\right) = \varphi + d\beta$. Thus

$$dt \wedge \pi_1^*\left(\int_{S^1} \rho\right) = dt \wedge \pi_1^* \tau + \varphi + d\beta = \rho + d\beta.$$

After applying the morphism a to both sides of the last equation and recalling that $a \circ d = 0$, we obtain

$$a\left(dt \wedge \pi_1^*\left(\int_{S^1} \rho\right)\right) = a(\rho).$$

Thus we obtain a well defined morphism on degree 0. In order to extend the definition for all non-positive degrees we consider the generalization

$$\int_{S^1} : \hat{K}^{-n}(S^1 \times M) \to \hat{K}^{-n-1}(M),$$

given by

$$\int_{S^1} \alpha := \alpha - \pi_{n+1}^* i_{n+1}^* \alpha - a^{-n}(\eta) + a^{-n}\left(dt_{n+1} \wedge \pi_{n+1}^*\left(\int_{S^1} \eta\right)\right).$$

The following proposition says that this definition yields a morphism with natural geometric properties and compatible with the morphisms I, R and a.

Proposition 2.3.6. The morphism of S^1-integration has the following properties

- the diagram

$$\begin{tikzcd}
\frac{\Omega^{-n-1}(S^1\times M;\mathfrak{k}^\bullet_{\mathbb{R}})}{im(d)} \arrow[r, "a^{-n}"] \arrow[d, "\int_{S^1}"] & \hat{K}^{-n}(S^1\times M) \arrow[r, "R^{-n}"] \arrow[d, "\int_{S^1}"'] \arrow[rr, bend left, "I^{-n}"] & \Omega^{-n}(S^1\times M) \arrow[d, "\int_{S^1}"] & K^{-n}(S^1\times M) \arrow[d, "\int_{S^1}"] \\
\frac{\Omega^{-n-2}(M;\mathfrak{k}^\bullet_{\mathbb{R}})}{im(d)} \arrow[r, "-a^{-n-1}"'] & \hat{K}^{-n-1}(M) \arrow[r, "R^{-n-1}"'] \arrow[rr, bend right, "I^{-n-1}"'] & \Omega^{-n-1}(M;\mathfrak{k}^\bullet_{\mathbb{R}}) & K^{-n-1}(M)
\end{tikzcd}$$

  is commutative;

- $\int_{S^1}\circ\pi_M^* = 0$;
- let $t\colon S^1\to S^1$ be the map given by $e^{2\pi is}\mapsto e^{-2\pi is}$. Then, for the map $t\times\mathrm{id}\colon S^1\times M\to S^1\times M$, we have
$$\int_{S^1}\circ(t\times\mathrm{id})^* = -\int_{S^1}.$$

2.3.4 Periodicity

The differential extension defined above inherits the periodic property of topological K-theory, that is, $K^{-n}(M)\cong K^{-n-2}(M)$. To see this, recall that for topological K-theory we take a generator $\kappa-1\in\tilde{K}(\mathbb{T}^2)$ and define

$$\begin{aligned} K^{-n}(M) &\xrightarrow{\cong} K^{-n-2}(M)\\ \alpha &\mapsto (\kappa-1)\boxtimes\alpha. \end{aligned}$$

Thus for the differential case we may lift $\kappa-1$ to an appropriated differential class $\hat{\kappa}-1\in\hat{K}(\mathbb{T}^2)$, so that we also have an isomorphism via exterior multiplication. Such element is given as the element $\hat{\kappa}-1\in\hat{K}(\mathbb{T}^2)$ with $I(\hat{\kappa}-1)=\kappa-1$ and representing a class in $\hat{K}^{-2}(*)$ with curvature 1, that is, $\hat{\kappa}-1\in Ker(i_1^*)\cap Ker(i_2^*)$ and $R(\hat{\kappa}-1)=dt_1\wedge dt_2$. We claim that multiplication by $\hat{\kappa}-1$ yields an isomorphism $\hat{K}^{-n}(M)\cong\hat{K}^{-n-2}(M)$:

- It is surjective. Indeed, given a class $\beta\in\hat{K}^{-n-2}(M)$ with $I^{-n-2}(\beta)=\tilde{\beta}\in K^{-n-2}(M)$, the topological periodicity gives a class $\tilde{\alpha}\in K^{-n}(M)$ such that $(\kappa-1)\boxtimes\tilde{\alpha}=\tilde{\beta}$. Since

$\beta \in \hat{K}^{-n-2}(M)$, we have that $R(\beta) = dt_1 \wedge \cdots \wedge dt_{n+2} \wedge \pi_M^* R^{-n-2}(\beta)$; hence we choose a lifting $\alpha \in \hat{K}^{-n}(M)$ of $\tilde{\alpha}$ so that $R(\alpha) = dt_1 \wedge \cdots \wedge dt_n \wedge \pi_M^* R^{-n-2}(\beta)$, that is, $R^{-n}(\alpha) = R^{-n-2}(\beta)$. Observe that

$$\begin{aligned} I^{-n-2}((\hat{\kappa}-1) \boxtimes \alpha) =& (\kappa - 1) \boxtimes \tilde{\alpha} \\ =& \tilde{\beta} \\ =& I^{-n-2}(\beta). \end{aligned}$$

So $\beta = (\hat{\kappa}-1) \boxtimes \alpha + a(\rho)$. Note also that $\beta \in Ker(i_j^*)$, for $j = 1, \ldots, n+2$ and by construction the same holds for $(\hat{\kappa}-1) \boxtimes \alpha$, it follows that $a(\rho) \in Ker(i_j^*)$, for $j = 1, \ldots, n+2$. Therefore $\rho = (-1)^n dt_1 \ldots dt_{n+2} \wedge \pi_M^* \omega + d\xi$ and $\beta = (\hat{\kappa}-1) \boxtimes (\alpha + a^{-n}(\rho))$, the claim follows.

- It is injective. Suppose that $(\hat{\kappa}-1) \boxtimes \alpha = 0$. Then after applying I^{-n-2}, we have $(\kappa - 1) \boxtimes I^{-n}(\alpha) = 0$, so by the topological Bott isomorphism we have $I^{-n}(\alpha) = 0$. By proposition 2.3.5, it follows that $\alpha = a^{-n}(\omega)$. By the multiplicative property of the curvature, we have

$$\begin{aligned} R^{-n}(\alpha) =& 1 \cdot R^{-n}(\alpha) \\ =& R^{-2}(\hat{\kappa}-1) \wedge R^{-n}(\alpha) \\ =& R^{-n-2}((\hat{\kappa}-1) \boxtimes \alpha) \\ =& 0. \end{aligned}$$

  So $d\omega = R^{-n}(\alpha) = 0$ and ω represents a de-Rham class. Observe that

$$\begin{aligned} a^{-n-2}(\omega) =& a^{-n}(dt_1 \wedge dt_2 \wedge \omega) \\ =& (\hat{\kappa}-1) \boxtimes \alpha \\ =& 0. \end{aligned}$$

  Hence ω is in the image of the Chern character and $\alpha = a^{-n}(\omega) = 0$, by proposition 2.3.5.

As in the topological case, the periodicity morphism on non-positive degrees yields a natural definition of differential K-theory for positive degrees. Concretely, we set $\hat{K}^n(M) := \hat{K}^{-n}(M)$, whereas the morphisms $I^n \colon \hat{K}^n(M) \to K^n(M)$, $R^n \colon \hat{K}^n(M) \to \Omega^n(M; \mathfrak{k}_{\mathbb{R}}^{\bullet})$ and $a^n \colon \Omega^{n-1}(M; \mathfrak{k}_{\mathbb{R}}^{\bullet})/Im(d) \to \hat{K}^n(M)$ are defined using topological periodicity and the natural isomorphism $\Omega^n(M; \mathfrak{k}_{\mathbb{R}}^{\bullet}) \cong \Omega^{-n}(M; \mathfrak{k}_{\mathbb{R}}^{\bullet})$, respectively. The product and S^1-integration are defined using the identification $\hat{K}^n(M) = \hat{K}^{-n}(M)$ and appropriated periodicity isomorphisms, that is, for $\alpha^n \in \hat{K}^n(M)$, we consider the corresponding (unique) $\alpha^{-n} \in \hat{K}^{-n}(M)$ representing α^n and take

$$\textstyle\int_{S^1} \alpha^n := (B^{-n+1})^{-1} \left(\int_{S^1} \alpha^{-n} \right) \quad ; \quad \alpha^n \boxtimes \beta^{-m} := (\alpha^{-n} \boxtimes \beta^{-m})^{n+m},$$

where $B^{-n+1} \colon \hat{K}^{-n+1}(M) \to \hat{K}^{-n-1}(M)$ is the Bott periodicity.

2.4 Axiomatic Framework

As we saw on the models described in the previous sections, there are a set of common features that one should expect to be displayed by any differential extension of an arbitrary cohomology theory. In this section we define an axiomatic framework that represents the principal features of a differential extension.

We begin by fixing some notation. For a generic cohomology theory $E^\bullet$, we will denote the cohomology groups of the point, $E^\bullet(*)$, as the $\mathbb{Z}$-graded group $\mathfrak{E}^\bullet$ and its tensor product with $\mathbb{R}$, $\mathfrak{E}^\bullet \otimes \mathbb{R}$, by $\mathfrak{E}^\bullet_\mathbb{R}$. We will denote by $\mathcal{M}$ the category with objects smooth manifolds and morphisms the smooth maps between smooth manifolds. Also, we will denote by $\mathcal{A}_\mathbb{Z}$ the category of $\mathbb{Z}$-graded abelian groups and by $\mathcal{R}_\mathbb{Z}$ the category of $\mathbb{Z}$-graded rings.

Definition 2.4.1. Let $E^\bullet$ be a cohomology theory. A *differential extension* of $E^\bullet$ is a contravariant functor

$$\hat{E}^\bullet \colon \mathcal{M} \to \mathcal{A}_\mathbb{Z}$$

together with natural transformations

$$I \colon \hat{E}^\bullet \to E^\bullet; \qquad R \colon \hat{E}^\bullet \to \Omega^\bullet_{cl}(-;\mathfrak{E}^\bullet_\mathbb{R}); \qquad a \colon \Omega^\bullet(-;\mathfrak{E}^\bullet_\mathbb{R})/Im(d) \to \hat{E}^\bullet$$

such that

1. $$R \circ a = d, \tag{2-9}$$

2. for each $M \in \mathcal{M}$, the diagram

$$\begin{tikzcd} \hat{E}^n(M) \arrow[r, "I"] \arrow[d, "R"'] & E^n(M) \arrow[d, "ch"] \\ \Omega^n_{cl}(M;\mathfrak{E}^\bullet_\mathbb{R}) \arrow[r, "dR"'] & H^n_{dR}(M;\mathfrak{E}^\bullet_\mathbb{R}) \end{tikzcd}$$

is commutative,

3. for each $M \in \mathcal{M}$, the sequence

$$E^{n-1}(M) \xrightarrow{ch} \frac{\Omega^{n-1}(M;\mathfrak{E}^\bullet_\mathbb{R})}{Im(d)} \xrightarrow{a} \hat{E}^n(M) \xrightarrow{I} E^n(M) \longrightarrow 0 \tag{2-10}$$

is exact.

With the notation above we will denote a given extension as a quadruple $(\hat{E}, I, R, a)$, when there is no risk of confusion.

It is known that most cohomology theories are enriched with a ring structure and the differential extensions described in the previous sections show that it is in fact possible to endow a given extension with a multiplicative structure compatible with the base cohomology theory and with the differential data. These properties make the following definition natural.

Definition 2.4.2. A *multiplicative differential extension* of E is a differential extension $(\hat{E}, I, R, a)$ such that

- $\hat{E}^\bullet$ takes values in $\mathcal{R}_\mathbb{Z}$,
- the morphism R is multiplicative, that is, for all $M \in \mathcal{M}$ and all α, $\beta \in \hat{E}^\bullet(M)$ we have $R(\alpha \cdot \beta) = R(\alpha) \wedge R(\beta)$,
- the morphism R is multiplicative, that is, for all $M \in \mathcal{M}$ and all α, $\beta \in \hat{E}^\bullet(M)$ we have $I(\alpha \cdot \beta) = I(\alpha)I(\beta)$,
- for all $\alpha \in \hat{E}^\bullet(M)$ and $\omega \in \Omega^\bullet(M; \mathfrak{E}^\bullet_\mathbb{R})/Im(d)$, the identity
$$\alpha \cdot a(\omega) = a(R(\alpha) \wedge \omega)$$
holds true.

In the same vein, the models above show that there is a natural way to characterize an integration map for a given differential extension. First, we introduce some useful notations. Let $F\colon \mathcal{M} \to \mathcal{A}$ be an arbitrary functor. We will denote by SF the functor that to every manifold $M \in \mathcal{M}$ associates $F(S \times M) \in \mathcal{A}$ and to every smooth map of manifolds $f\colon M \to N$ associates the morphism $F(\mathrm{id} \times f)$, where $\mathrm{id} \times f : S^1 \times M \to S^1 \times N$ is the map given by $(s, m) \mapsto (s, f(m))$. Also, we represent the elements of S^1 as the points in the complex plain of the form $e^{\pi i r}$. Finally, we denote by $\pi\colon S^1 \times M \to M$ the usual projection map.

Definition 2.4.3. Let $E^\bullet$ be a cohomology theory. A *differential extension with integration* of $E^\bullet$ is a differential extension $(\hat{E}, I, R, a)$ together with a natural transformation
$$\int : S\hat{E}^{\bullet+1} \to \hat{E}^\bullet$$
such that

- $\int \circ (t \times \mathrm{id})^* = -\int$, with $t\colon S^1 \to S^1$ the conjugation map,
- $\int \circ \pi^* = 0$,

- the diagram

$$\begin{tikzcd}
S\Omega^\bullet(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[r, "a"] \arrow[d, "\int"] & S\hat{E}^{\bullet+1}(M) \arrow[r, "R"] \arrow[d, "\int"'] \arrow[rr, bend left, "I"] & S\Omega^\bullet_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[d, "\int"] & SE^{\bullet+1}(M) \arrow[d, "\int"] \\
\Omega^{\bullet-1}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[r, "a"'] & \hat{E}^\bullet(M) \arrow[r, "R"'] \arrow[rr, bend right, "I"'] & \Omega^\bullet_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) & E^\bullet(M)
\end{tikzcd}$$

commutes.

Remark 2.4.4. For a cohomology theory the integration map may be described explicitly in terms of the suspension isomorphism. Consider the embedding $i\colon M \hookrightarrow S^1 \times M$, where we take S^1 with base point 1. Observe that the canonical projection $\pi\colon S^1 \times M \to M$ is a left inverse of i. The long exact sequence of the pair $(S^1 \times M, M)$ and $\pi \circ i = \mathrm{id}$ give the split exact sequence

$$0 \longrightarrow E^{\bullet+1}(S^1 \times M, M) \xrightarrow{q} E^{\bullet+1}(S^1 \times M) \xrightarrow{i^*} E^{\bullet+1}(M) \longrightarrow 0$$

Hence we have a splitting

$$E^{\bullet+1}(S^1 \times M) \cong \pi^* E^{\bullet+1}(M) \oplus Ker(i^*).$$

In particular, the elements of $Ker(i^*)$ have the form $\alpha - \pi^* i^* \alpha$. Clearly we have a canonical isomorphism $Ker(i^*) \cong E^{\bullet+1}(S^1 \times M, M) \cong \tilde{E}^{\bullet+1}(\Sigma M_+)$, so we identify $Ker(i^*)$ with $\tilde{E}^{\bullet+1}(\Sigma M_+)$. Denoting by $s\colon E^\bullet(M) \to \tilde{E}^{\bullet+1}(\Sigma M_+)$ the suspension isomorphism we have that integration is defined as:

$$\int \colon E^{\bullet+1}(S^1 \times M) \to E^\bullet(M)$$
$$\alpha \mapsto s^{-1}(\alpha - \pi^* i^* \alpha).$$

In this way it is easy to see that the integration is a natural transformation, $\int \circ s$ is the identity map on $E^\bullet(M)$ and it also holds the properties $\int \circ \pi^* = 0$ and $\int \circ (t \times \mathrm{id})^* = -\int$.

A byproduct of the use of non-homotopical data, i.e. differential information, to enrich a cohomological theory is the loss of homotopical invariance. Explicitly, we have

Proposition 2.4.5. Let $E^\bullet$ be a cohomology theory and $(\hat{E}, I, R, a)$ a differential extension. For $\alpha \in \hat{E}^\bullet(I \times M)$ we have

$$i_1^*\alpha - i_0^*\alpha = a\left(\int_0^1 R(\alpha)\right), \tag{2-11}$$

where $i_j\colon M \to I \times M$ is defined by $m \mapsto (j, m)$.

Proof. We have that both $\pi \circ i_0$ and $\pi \circ i_1$ coincide with the identity and that $i_0 \circ \pi$ is homotopic to the identity. Define $\alpha_0 = i_0^* \alpha$. Observe that $I(\pi^* \alpha_0) = I(\alpha)$, for $I(\pi^* \alpha_0) = I(\pi^* i_0^* \alpha) = \pi^* i_0^* I(\alpha)$. Hence, by the exact sequence (2-10), it follows that there is a ω such that $\alpha = \pi^* \alpha_0 + a(\omega)$ and, by (2-9), we have $R(\alpha) = \pi^* R(\alpha_0) + d\omega$. Taking the difference of the pullbacks of α we have

$$\begin{aligned} i_1^*\alpha - i_0^*\alpha =& i_1^*\pi^*\alpha_0 + a(i_1^*\omega) - i_0^*\pi^*\alpha_0 - a(i_0^*\omega) \\ =& a(i_1^*\omega - i_0^*\omega) = a\left(\int_{\partial I} \omega\right) = a\left(\int_I d\omega\right) \\ =& a\left(\int_I (R(\alpha) - \pi^* R(\alpha_0))\right) \\ =& a\left(\int_I R(\alpha)\right). \end{aligned}$$

□

2.5 Uniqueness of Differential Extensions

So far we have seen three models of differential extensions for two different cohomology theories; two out of the three models refine singular cohomology and what's more important both refinements are equivalent (proposition 2.2.16). Then it is natural to ask if this situation also occurs for other cohomology theories. First we need to be precise as to what it means when we say that two models extending the same cohomology theory are equivalent, then we can settle the problem of uniqueness of the extension. In this section we follow [10]. The following definitions deal with the first question.

Definition 2.5.1. Let $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ be two differential extensions of $E^\bullet$. A *natural transformation of differential extensions* is a natural transformation of $\mathbb{Z}$-graded abelian group valued functors $\Phi\colon \hat{E}^\bullet \to \hat{E'}^\bullet$, such that for every manifold M the diagram

$$\begin{tikzcd} \Omega^{\bullet-1}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[r, "a"] \arrow[d, "\mathrm{id}"] & \hat{E}^\bullet(M) \arrow[r, "R"] \arrow[d, "\Phi"'] \arrow[rr, bend left, "I"] & \Omega^\bullet_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[d, "\mathrm{id}"] & E^\bullet(M) \arrow[d, "\mathrm{id}"] \\ \Omega^{\bullet-1}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \arrow[r, "a'"'] & \hat{E'}^\bullet(M) \arrow[r, "R'"'] \arrow[rr, bend right, "I'"'] & \Omega^\bullet_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) & E^\bullet(M) \end{tikzcd}$$

commutes.

Definition 2.5.2. Let $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ be two multiplicative differential extensions of $E^\bullet$. A *natural transformation of multiplicative differential extensions* is a natural transformation $\Phi\colon \hat{E}^\bullet \to \hat{E'}^\bullet$ of $\mathbb{Z}$-graded ring valued functors.

Definition 2.5.3. Let $(\hat{E}, R, I, a, \int)$ and $(\hat{E}', R', I', a', \int')$ be two differential extensions with integration of $E^\bullet$. A *natural transformation of differential extensions with integration* is a natural transformation of differential extensions $\Phi\colon \hat{E}^\bullet \to \hat{E'}^\bullet$, such that for every manifold M the diagram

$$\begin{array}{ccc} S\hat{E}^{\bullet+1}(M) & \xrightarrow{\Phi} & S\hat{E'}^{\bullet+1}(M) \\ {\scriptstyle\int}\big\downarrow & & \big\downarrow{\scriptstyle\int'} \\ \hat{E}^\bullet(M) & \xrightarrow{\quad\Phi\quad} & \hat{E'}^\bullet(M) \end{array}$$

commutes.

So we will say that two differential extensions $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ are equivalent if there is a natural transformation of differential extensions $\Phi\colon \hat{E}^\bullet \to \hat{E'}^\bullet$, such that Φ is a natural equivalence of functors.

Now we show that differential extensions with integration are essentially unique for topological cohomology theories holding some mild conditions. First we recall some basic facts and state some useful results. Given a cohomology theory $E^\bullet$, we assume that it is represented by an Ω-spectrum $\{\mathbb{E}_n, e_n, \epsilon_n\}$, that is, a family of based topological spaces $(\mathbb{E}_n, e_n)$, usually CW spaces, and a family of maps

$$\epsilon_n\colon \sum \mathbb{E}_n \to \mathbb{E}_{n+1},$$

called structure maps, such that the adjoint maps

$$\epsilon_n^{adj}\colon \mathbb{E}_n \to \Omega\mathbb{E}_{n+1}$$

define homeomorphisms and so that for a space pointed space X we have $E^n(X) = [X, \mathbb{E}_n]$. The idea is to use this generic representation to define a natural transformation between two extensions of a cohomological theory. As they are, spectra are not suited to work within our smooth framework, nevertheless we have

Proposition 2.5.4 ([10] Prop. 2.1)**.** Let $\mathbf{E}$ be a connected pointed topological space. If $\mathbf{E}$ is simply connected and $\pi_k(\mathbf{E})$ is finitely generated for all $k \geq 2$, then there exist a sequence of compact pointed manifolds with boundary $(\mathcal{E}_i)_{i\in\mathbb{N}}$ together with pointed maps $\kappa_i\colon \mathcal{E}_i \to \mathcal{E}_{i+1}$, $x_i\colon \mathcal{E} \to \mathbf{E}$ for all $i \in \mathbb{N}$ such that

1. $\mathcal{E}_i$ is homotopy equivalent to an i-dimensional CW-complex,

2. the map x_i is i-connected,

3. $\kappa_i\colon \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ is an embedding of a submanifold,

4. the diagram
$$\begin{tikzcd} \mathcal{E}_i \arrow[rr, "\kappa_i"] \arrow[dr, "x_i"'] & & \mathcal{E}_{i+1} \arrow[dl, "x_{i+1}"] \\ & \mathbf{E} & \end{tikzcd}$$
commutes,

5. for all finite dimensional pointed CW-complexes X the induced map
$$colim([X, \mathcal{E}_i]) \to [X, \mathbf{E}]$$
is an isomorphism.

Proposition 2.5.5 ([10] Prop. 2.3)**.** Let $\mathbf{E}$ be a topological space with countably many connected components such that the groups $\pi_k(\mathbf{E}, x)$ are countably generated for all $k \geq 1$. Then there exist a sequence of pointed manifolds $(\mathcal{E}_i)_{i \in \mathbb{N}}$ together with pointed maps $\kappa_i \colon \mathcal{E}_i \to \mathcal{E}_{i+1}$, $x_i \colon \mathcal{E} \to \mathbf{E}$ for all $i \in \mathbb{N}$ such that

1. $\mathcal{E}_i$ is homotopy equivalent to an i-dimensional CW-complex,

2. the map x_i is i-connected,

3. $\kappa_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ is an embedding of a submanifold,

4. the diagram
$$\begin{tikzcd} \mathcal{E}_i \arrow[rr, "\kappa_i"] \arrow[dr, "x_i"'] & & \mathcal{E}_{i+1} \arrow[dl, "x_{i+1}"] \\ & \mathbf{E} & \end{tikzcd}$$
commutes,

5. for all finite dimensional pointed CW-complexes X the induced map
$$colim([X, \mathcal{E}_i]) \to [X, \mathbf{E}] \tag{2-12}$$
is an isomorphism.

Thus if we fix an element $\mathbb{E}_n$ of the spectrum, we can approximate it through a sequence of smooth spaces $(\mathcal{E}_i)$, so that these smooth spaces retain the information about the cohomology theory $E^\bullet$. We also need a way to represent coherently the invariants of the spectrum.

Proposition 2.5.6 ([10] Prop. 2.5)**.** Let $\mathbf{E}$ and $(\mathcal{E}_i)$ be as in proposition 2.5.5. Then for $u \in H^\bullet(\mathbf{E}; \mathfrak{E}^\bullet_\mathbb{R})$ there exists a sequence of forms $\omega_i \in \Omega^\bullet_{cl}(\mathcal{E}_i, \mathfrak{E}^\bullet_\mathbb{R})$, such that $dR(\omega_i) = x_i^* u$ and $\kappa_i^* \omega_{i+1} = \omega_i$ for all $i \geq 0$.

Proposition 2.5.7 ([10] Prop. 2.6)**.** There is a sequence $\hat{u}_i \in \hat{E}^\bullet(\mathcal{E}_i)$ such that $R(\hat{u}_i) = \omega_i$, $I(\hat{u}_i) = x_i^* u$ and $\kappa_i^* \hat{u}_{i+1} = \hat{u}_i$ for all $i \geq 0$.

We are ready to define a natural transformation for two given differential extensions $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ of the cohomology theory $E^\bullet$. We fix a Ω-spectrum $(\mathbb{E}_i, e_i, \epsilon_i)$ representing $E^\bullet$.

Remark 2.5.8. In order to use the previous results we will assume either of the following conditions:

- $E^{k-1}(pt) = \pi_1(\mathbb{E}_k) = 0$, the abelian groups $E^m(pt)$ are finitely generated for all $m \leq k$, and the smooth extensions are defined on the category of compact manifolds, or
- the smooth extensions are defined on the category of all manifolds and the abelian groups $E^m(pt)$ are countably generated for all $m \leq k$.

Let us fix an element $\mathbb{E}_n$ in the spectrum and an approximation through manifolds $(\mathcal{E}_i, x_i, \kappa_i)$, as defined above. Let $u \in E^n(\mathbb{E}_n)$ be the tautological class represented by the identity map $u\colon \mathbb{E}_n \to \mathbb{E}_n$. By propositions 2.5.6 and 2.5.7, there exist sequences $\omega_i \in \Omega^n_{cl}(\mathcal{E}_i, \mathfrak{E}^\bullet_{\mathbb{R}})$, $\hat{u}_i \in \hat{E}^n(\mathcal{E}_i)$ and $\hat{u}'_i \in \hat{E'}^n(\mathcal{E}_i)$, such that

- $dR(\omega_i) = ch(x_i^* u)$,
- $R(\hat{u}_i) = \omega_i = R'(\hat{u}'_i)$,
- $I(\hat{u}_i) = x_i^* u = I'(\hat{u}'_i)$,
- $\kappa_i^* \hat{u}_{i+1} = \hat{u}_i$,
- $\kappa_i^* \hat{u}'_{i+1} = \hat{u}'_i$.

Observe that for an arbitrary $\hat{\alpha} \in \hat{E}^n(M)$, exists a map $f : M \to \mathbb{E}_n$ such that $I(\hat{\alpha}) = [f] = f^* u$, where u is the tautological class as defined above. By (2-12), exists $f_i \in [M, \mathcal{E}_i]$ such that $f = x_i \circ f_i$, hence

$$I(\hat{\alpha}) = f^* u = f_i^* x_i^* u = f_i^* I(\hat{u}_i) = I(f_i^* \hat{u}_i).$$

Finally, by (2-10) exists a unique form $\rho \in \Omega^{n-1}(M; \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch)$, such that

$$\hat{\alpha} = f_i^* \hat{u}_i + a(\rho).$$

Using this representation we define the map

$$\begin{aligned} \Phi\colon \hat{E}^n(M) &\to \hat{E'}^n(M) \\ \hat{\alpha} &\mapsto f_i^* \hat{u}'_i + a'(\rho) \end{aligned} \tag{2-13}$$

We still have to show that this map is a well defined natural transformation of differential extensions.

Proposition 2.5.9. The map Φ is well defined.

Proof. We only need to show that the value of Φ is independent of the choice of the function f_i. Suppose there are two functions $f_i\colon M \to \mathcal{E}_i$ and $f_j\colon M \to \mathcal{E}_j$ such that

$$x_j \circ f_j = f = x_i \circ f_i.$$

By properties of the colimit (2-12), we have that exists $l \geq i, j$, such that $\kappa_l^i \circ f_i = \kappa_l^j \circ f_j$ are homotopic. Hence there is no loss of generality if we consider instead two homotopic maps $f_i\colon M \to \mathcal{E}_i$ and $\tilde{f}_i\colon M \to \mathcal{E}_i$ such that $x_i \circ f_i = x_i \circ \tilde{f}_i = f$. Now by (2-11), we have that $f_i^*\hat{u}_i - \tilde{f}_i^*\hat{u}_i = a\left(\int_I F^*\omega\right)$, where $F\colon I \times M \to \mathcal{E}_i$ defines a homotopy between f and $\tilde{f}$, and $\int_I F^*\omega$ defines a unique class in $\Omega^{n-1}(M; \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch)$. Suppose then that for a given $\hat{\alpha}$ we have

$$f_i^*\hat{u}_i + a(\rho) = \hat{\alpha} = \tilde{f}_i^*\hat{u}_i + a(\tilde{\rho}).$$

By the injectivity of a in $\Omega^{n-1}(M; \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch)$, it follows that

$$\tilde{\rho} = \rho + \int_I F^*\omega.$$

Then using the definition of Φ in both representations of $\hat{\alpha}$ we have

$$\begin{aligned}
\Phi(f_i^*\hat{u}_i + a(\rho)) =& f_i^*\hat{u}_i' + a'(\rho)\\
=& \tilde{f}_i^*\hat{u}_i' + a'\left(\int_I F^*\omega\right) + a'(\rho)\\
=& \tilde{f}_i^*\hat{u}_i' + a'(\tilde{\rho})\\
=& \Phi(\tilde{f}_i^*\hat{u}_i + a(\tilde{\rho})).
\end{aligned}$$

□

Proposition 2.5.10. The map Φ as the following properties

1. $R' \circ \Phi = R$,
2. $I' \circ \Phi = I$,
3. $\Phi \circ a = a'$,
4. Φ is natural.

Proof. About 1. note that

$$\begin{aligned}
R' \circ \Phi(\hat{\alpha}) =& R' \circ \Phi(f_i^*\hat{u}_i + a(\rho))\\
=& R'(f_i^*\hat{u}_i' + a'(\rho)) = f_i^*\omega_i + d\rho\\
=& R(f^*\hat{u}_i + a(\rho)) = R(\hat{\alpha}).
\end{aligned}$$

A similar calculation shows

$$\begin{aligned} I' \circ \Phi(\hat{\alpha}) =& I' \circ \Phi(f_i^* \hat{u}_i + a(\rho)) \\ =& I'(f_i^* \hat{u}_i' + a'(\rho)) = f_i^* u_i \\ =& I(f^* \hat{u}_i + a(\rho)) = I(\hat{\alpha}). \end{aligned}$$

For property 3. we have obviously $\Phi \circ a(\eta) = a'(\eta)$.

Finally, consider a smooth map of manifolds $g \colon M \to N$ and $\hat{\alpha} \in \hat{E}^n(N)$. Suppose that $\hat{\alpha} = f_i^* \hat{u}_i + a(\rho)$, then $g^* \hat{\alpha} = g^* f_i^* \hat{u}_i + a(g^* \rho)$ and $\Phi(g^* \hat{\alpha}) = g^* f_i^* \hat{u}_i' + a'(g^* \rho)$. On the other hand $g^* \Phi(\hat{\alpha}) = g^*(f_i^* \hat{u}_i' + a'(\rho))$, the result follows. □

Remark 2.5.11. The properties above imply that Φ is part of a morphism of the exact sequences (2-10) associated to both extensions. In particular we have that $Ker(a) = Ker(a')$.

So far we have a well defined natural set-map between two differential extensions, however in order to define a morphism of differential extensions we still need to show that Φ defines a *natural transformation between groups.* A priori the map Φ has the form

$$\Phi(\hat{\alpha} + \hat{\beta}) = \Phi(\hat{\alpha}) + \Phi(\hat{\beta}) + \hat{B}(\hat{\alpha}, \hat{\beta}),$$

where

$$\hat{B} \colon \hat{E}^n(M) \times \hat{E}^n(M) \to \hat{E}^n(M)$$

is a map such that

- $\hat{B}(\hat{\alpha}, \hat{\beta} + \hat{\gamma}) + \hat{B}(\hat{\beta}, \hat{\gamma}) = \hat{B}(\hat{\alpha}, \hat{\beta}) + \hat{B}(\hat{\alpha} + \hat{\beta}, \hat{\gamma})$,
- $\hat{B}(\hat{\alpha}, \hat{\beta}) = \hat{B}(\hat{\beta}, \hat{\alpha})$,
- $\hat{B}(\hat{\alpha}, a(\rho)) = 0$,
- $R'(\hat{B}(\hat{\alpha}, \hat{\beta})) = 0$,
- $I'(\hat{B}(\hat{\alpha}, \hat{\beta})) = 0$.

In particular, Φ factors through a map

$$\tilde{B} \colon E^n(M) \times E^n(M) \to H^{n-1}(M, \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch). \tag{2-14}$$

So if we are to get a morphism of groups, the morphism $\tilde{B}$ should vanish identically. To obtain such a property we impose some mild conditions on the topological theory $E^\bullet$.

Lemma 2.5.12 ([10] Lemma 3.8)**.** Let X be a CW-complex such that $\pi_{2i+1}(X) \otimes \mathbb{Q} = 0$ for $i = 0, \dots, n$. Then $H_{2i+1}(X; \mathbb{Q}) = 0$ for $i = 0, \dots, n$.

Definition 2.5.13. We say that a cohomology theory $E^\bullet$ is rationally even if $E^m(pt) \otimes \mathbb{Q} = 0$ for all $m \in \mathbb{Z}$ odd.

Lemma 2.5.14. Let $(\mathbb{E}_k, e_k, \epsilon_k)$ be a spectrum representing a rationally even cohomology theory $E^\bullet$ and $(\mathcal{E}_i, x_i, \kappa_i)$ a system of manifolds approximating the term $\mathbb{E}_n$ of the spectrum. Then for every $i \in \mathbb{N}$ and all $r \geq i+1$ we have

$$(\kappa_i^{i+r} \times \kappa_i^{i+r})^* H^{n-1}(\mathcal{E}_{i+r} \times \mathcal{E}_{i+r}; \mathfrak{E}^\bullet_{\mathbb{R}}) = 0.$$

Proof. By hypothesis the map $x_{i+r} \colon \mathcal{E}_{i+r} \to \mathbb{E}_n$ is $2i+1$-connected for $r \geq i+1$ (propositions 2.5.4 and 2.5.5), hence $\pi_k(\mathcal{E}_{i+r}) \cong \pi_k(\mathbb{E}_n) \cong E^{n-k}(pt)$, for all $k \leq 2i$. Since $E^\bullet$ is rationally even, we have that $\pi_{2l-1}(\mathcal{E}_{i+r}) \otimes \mathbb{Q} = 0$ and $\pi_{2l-1}(\mathcal{E}_{i+r} \times \mathcal{E}_{i+r}) \otimes \mathbb{Q} = 0$, for $2l-1 \leq 2i$. Now using lemma 2.5.12, we obtain $H^{2l-1}(\mathcal{E}_{i+r} \otimes \mathcal{E}_{i+r}; \mathbb{Q}) = 0$ for all $2l-1 \leq 2i$. Finally, since $\mathfrak{E}^{odd}_{\mathbb{R}} = 0$ it follows that

$$\begin{aligned}(\kappa_i^{i+r} \times \kappa_i^{i+r})^* H^{n-1}(\mathcal{E}_{i+r} \times \mathcal{E}_{i+r}; \mathfrak{E}^\bullet_{\mathbb{R}}) \cong& (\kappa_i^{i+r} \times \kappa_i^{i+r})^* \bigoplus_{2l-1 \leq 2i} H^{2l-1}(\mathcal{E}_{i+r} \times \mathcal{E}_{i+r}; \mathfrak{E}^{n-2l}_{\mathbb{R}}) \\ =&0\end{aligned}$$

□

Recall that we are considering a fixed component $\mathbb{E}_n$ of the spectrum and a fixed approximation $(\mathcal{E}_i, x_i, \kappa_i)$. The map $(2-14)$ induces a system of elements

$$B_i \in H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i, \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch),$$

where

$$B_i = \tilde{B}(\pi_1^* x_i^* u, \pi_2^* x_i^* u).$$

Taking the inverse limit of this systems yields an element

$$B \in \varprojlim_i H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i, \mathfrak{E}^\bullet_{\mathbb{R}})/Im(ch).$$

Lemma 2.5.15. We have $B = 0$.

Proof. Consider the family of exact sequences

$$0 \longrightarrow ch(E^{n-1}(\mathcal{E}_i \times \mathcal{E}_i)) \longrightarrow H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}^\bullet_{\mathbb{R}}) \longrightarrow \frac{H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}^\bullet_{\mathbb{R}})}{ch(E^{n-1}(\mathcal{E}_i \times \mathcal{E}_i))} \longrightarrow 0.$$

Taking the term-wise inverse limit we obtain

$$\varprojlim_i (H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}^\bullet_{\mathbb{R}})) \longrightarrow \varprojlim_i \left(\frac{H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}^\bullet_{\mathbb{R}})}{ch(E^{n-1}(\mathcal{E}_i \times \mathcal{E}_i))} \right) \longrightarrow {\varprojlim_i}^1 (ch(E^{n-1}(\mathcal{E}_i \times \mathcal{E}_i))) . \tag{2-15}$$

Note that $ch(E^{n-1}(\mathcal{E}_i \times \mathcal{E}_i)) \subseteq H^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}^\bullet_{\mathbb{R}})$, hence, by lemma 2.5.14, the first and third terms on the sequence above vanish. The result follows. □

Remark 2.5.16. From the preceding lemma it follows directly that $\Phi(\pi_1^*\hat{u}_i + \pi_2^*\hat{u}_i) - \Phi(\pi_1^*\hat{u}_i) - \Phi(\pi_2^*\hat{u}_i) = \tilde{B}(\pi_1^* x_i^* u, \pi_2^* x_i^* u) = 0$. We use this special case to show that $\hat{B} = 0$, so that Φ is a morphism of groups.

Proposition 2.5.17. Let $n \in \mathbb{Z}^{even}$. If $E^\bullet$ is a rationally even cohomology theory and one of the conditions on remark 2.5.8 is satisfied, then the transformation $\Phi\colon \hat{E}^n \to \hat{E}'^n$ is additive.

Proof. Let $\mathbb{E}_n$ and $\{\mathcal{E}_i\}_{i\in\mathbb{N}}$, as in lemma 2.5.14. Consider $\hat{\alpha}$, $\hat{\beta} \in \hat{E}^n$. We can take j big enough and $f_{\hat{\alpha}}\colon M \to \mathcal{E}_j$, $f_{\hat{\beta}}\colon M \to \mathcal{E}_j$ such that $I(\hat{\alpha}) = f_{\hat{\alpha}}^* x_j^* u$ and $I(\hat{\beta}) = f_{\hat{\beta}}^* x_j^* u$. Fixing $k \geq j$ and $\mu\colon \mathcal{E}_j \times \mathcal{E}_j \to \mathcal{E}_k$, such that $\mu^* x_k^* u = \pi_1^* x_j^* u + \pi_2^* x_j^* u$, we define $f_{\hat{\alpha}+\hat{\beta}} \coloneqq \mu \circ (f_{\hat{\alpha}}, f_{\hat{\beta}})$. Observe that

$$\begin{aligned} f_{\hat{\alpha}+\hat{\beta}}^* x_k^* u =& (f_{\hat{\alpha}}, f_{\hat{\beta}})^* \mu^* x_k^* u \\ =& (f_{\hat{\alpha}}, f_{\hat{\beta}})^* (\pi_1^* x_j^* u + \pi_2^* x_j^* u) \\ =& f_{\hat{\alpha}}^* x_j^* u + f_{\hat{\beta}}^* x_j^* u = I(\hat{\alpha} + \hat{\beta}). \end{aligned}$$

Now choose $\eta \in \Omega^{n-1}(\mathcal{E}_i \times \mathcal{E}_i; \mathfrak{E}_{\mathbb{R}}^\bullet)$ and $\omega_{\hat{\alpha}}$, $\omega_{\hat{\beta}}$, $\omega_{\hat{\alpha}+\hat{\beta}} \in \Omega^{n-1}(M; \mathfrak{E}_{\mathbb{R}}^\bullet)$ such that

$$\begin{aligned} \mu^*(\hat{u}_k) + a(\eta) =& \pi_1^* \hat{u}_j + \pi_2^* \hat{u}_j, \\ f_{\hat{\alpha}}^* \hat{u}_j + a(\omega_{\hat{\alpha}}) =& \hat{\alpha}, \\ f_{\hat{\beta}}^* \hat{u}_j + a(\omega_{\hat{\beta}}) =& \hat{\beta}, \\ f_{\hat{\alpha}+\hat{\beta}}^* \hat{u}_k + a(\omega_{\hat{\alpha}+\hat{\beta}}) =& \hat{\alpha} + \hat{\beta}. \end{aligned}$$

From this it follows that $\Phi(\hat{\alpha}) = f_{\hat{\alpha}}^* \hat{u}'_j + a'(\omega_{\hat{\alpha}})$, $\Phi(\hat{\beta}) = f_{\hat{\beta}}^* \hat{u}'_j + a'(\omega_{\hat{\beta}})$ and $\Phi(\hat{\alpha} + \hat{\beta}) = f_{\hat{\alpha}+\hat{\beta}}^* \hat{u}'_k + a'(\omega_{\hat{\alpha}+\hat{\beta}})$. Hence

$$\begin{aligned} 0 =& (\hat{\alpha} + \hat{\beta}) - \hat{\alpha} - \hat{\beta} \\ =& f_{\hat{\alpha}+\hat{\beta}}^* \hat{u}_k + a(\omega_{\hat{\alpha}+\hat{\beta}}) - f_{\hat{\alpha}}^* \hat{u}_j - a(\omega_{\hat{\alpha}}) - f_{\hat{\beta}}^* \hat{u}_j - a(\omega_{\hat{\beta}}) \\ =& a(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + (f_{\hat{\alpha}}, f_{\hat{\beta}})^* \mu^* \hat{u}_k - f_{\hat{\alpha}}^* \hat{u}_j - f_{\hat{\beta}}^* \hat{u}_j \\ =& a(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + (f_{\hat{\alpha}}, f_{\hat{\beta}})^* (\pi_1^* \hat{u}_j + \pi_2^* \hat{u}_j - a(\eta)) \\ & \qquad - f_{\hat{\alpha}}^* \hat{u}_j - f_{\hat{\beta}}^* \hat{u}_j \\ =& a(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + f_{\hat{\alpha}}^*(\hat{u}_j) + f_{\hat{\beta}}^*(\hat{u}_j) - (f_{\hat{\alpha}}, f_{\hat{\beta}})^* a(\eta) \\ & \qquad - f_{\hat{\alpha}}^* \hat{u}_j - f_{\hat{\beta}}^* \hat{u}_j \\ =& a(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) - (f_{\hat{\alpha}}, f_{\hat{\beta}})^* a(\eta) \\ =& a(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}} - (f_{\hat{\alpha}}, f_{\hat{\beta}})^* \eta). \end{aligned}$$

On the other hand applying the morphism Φ we have

$$
\begin{aligned}
\Phi(\hat{\alpha}+\hat{\beta})-\Phi(\hat{\alpha})-\Phi(\hat{\beta}) =& f^*_{\hat{\alpha}+\hat{\beta}}\hat{u}'_k + a'(\omega_{\hat{\alpha}+\hat{\beta}}) - f^*_{\hat{\alpha}}\hat{u}'_j - a'(\omega_{\hat{\alpha}}) - f^*_{\hat{\beta}}\hat{u}'_j - a'(\omega_{\hat{\beta}})\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + (f_{\hat{\alpha}}, f_{\hat{\beta}})^*\mu^*\hat{u}'_k - f^*_{\hat{\alpha}}\hat{u}'_j - f^*_{\hat{\beta}}\hat{u}'_j\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + (f_{\hat{\alpha}}, f_{\hat{\beta}})^*(\Phi(\pi_1^*\hat{u}_j + \pi_2^*\hat{u}_j) - a'(\eta))\\
& \qquad - f^*_{\hat{\alpha}}\hat{u}'_j - f^*_{\hat{\beta}}\hat{u}'_j\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + (f_{\hat{\alpha}}, f_{\hat{\beta}})^*(\Phi(\pi_1^*\hat{u}_j) + \Phi(\pi_2^*\hat{u}_j) - a'(\eta))\\
& \qquad - f^*_{\hat{\alpha}}\hat{u}'_j - f^*_{\hat{\beta}}\hat{u}'_j\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) + f^*_{\hat{\alpha}}(\hat{u}'_j) + f^*_{\hat{\beta}}(\hat{u}'_j) - (f_{\hat{\alpha}}, f_{\hat{\beta}})^*a'(\eta)\\
& \qquad - f^*_{\hat{\alpha}}\hat{u}'_j - f^*_{\hat{\beta}}\hat{u}'_j\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}}) - (f_{\hat{\alpha}}, f_{\hat{\beta}})^*a'(\eta)\\
=& a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}} - (f_{\hat{\alpha}}, f_{\hat{\beta}})^*\eta).
\end{aligned}
$$

By remark 2.5.11, it follows that $a'(\omega_{\hat{\alpha}+\hat{\beta}} - \omega_{\hat{\alpha}} - \omega_{\hat{\beta}} - (f_{\hat{\alpha}}, f_{\hat{\beta}})^*\eta) = 0$, so $\Phi(\hat{\alpha}+\hat{\beta}) = \Phi(\hat{\alpha}) + \Phi(\hat{\beta})$. □

Hence, we have a well defined natural transformation between differential extensions in even degrees, the next step is to extend the morphism to odd degrees. First we establish a useful lemma.

Lemma 2.5.18. For $\hat{\alpha} \in \hat{E}^\bullet(M)$, exists a class $\hat{\beta} \in \hat{E}^{\bullet+1}(S^1 \times M)$ such that $\int \hat{\beta} = \hat{\alpha}$, $R(\hat{\beta}) = dt \wedge \pi^* R(\hat{\alpha})$ and $I(\hat{\beta}) = s(I(\hat{\alpha}))$.

Proof. Indeed, by remark 2.4.4, we have that $s(I(\hat{\alpha})) \in E^{\bullet+1}(S^1 \times M)$ and by surjectivity of I there always exists a lifting of $s(I(\hat{\alpha}))$ to a differential class $\hat{\beta} \in \hat{E}^{\bullet+1}(S^1 \times M)$ such that $I(\hat{\beta}) = s(I(\hat{\alpha}))$. Clearly we have $\int I(\hat{\beta}) = I(\hat{\alpha})$, hence $\hat{\alpha} = a(\omega) + \int \hat{\beta}$. From this, it follows that

$$R(\hat{\alpha}) = d\omega + \int R(\hat{\beta}). \tag{2-16}$$

Observe that $\int dt \wedge \pi^* R(\hat{\alpha}) = R(\hat{\alpha})$, thus $\int dt \wedge \pi^* R(\hat{\alpha}) = d\omega + \int R(\hat{\beta})$. Passing to de Rham cohomology we obtain

$$\int [dt \wedge \pi^* R(\hat{\alpha})] = \int [R(\hat{\beta})];$$

however, $dt \wedge \pi^* R(\hat{\alpha}) = d(t\pi^* R(\hat{\alpha}))$, so $\int [R(\hat{\beta})] = 0$ and using remark 2.4.4 again, we see that $[R(\hat{\beta})] = [\pi^* \eta]$.

On the other hand, by definition of $\hat{\beta}$ and the axioms for a differential extension we have $[R(\hat{\beta})] = ch \circ I(\hat{\beta}) = ch(s(I(\hat{\alpha})))$, it follows that $[R(\hat{\beta})] = 0$ and $R(\hat{\beta}) = dt \wedge \pi^* R(\hat{\alpha}) + d\rho$, so combining with (2-16) we have $d\omega = d \int \rho$ and

$$\int \rho = \omega + \eta$$

where $d\eta = 0$. We define $\hat{\beta}' := \hat{\beta} - a(\rho - dt \wedge \pi^*\eta)$. Clearly we have

$$R(\hat{\beta}') = R(\hat{\beta}) - d\rho = dt \wedge \pi^* R(\hat{\alpha}) + d\rho - d\rho = dt \wedge \pi^* R(\hat{\alpha}),$$

and

$$\begin{aligned}\int \hat{\beta}' =& \int \Big(\hat{\beta} - a(\rho - dt \wedge \pi^*\eta)\Big)\\ =&\hat{\alpha} - a(\omega) - a(\int \rho) + a(\int dt \wedge \pi^*\eta)\\ =&\hat{\alpha} - a(\omega) - a(\omega + \eta) + a(\eta)\\ =&\hat{\alpha}.\end{aligned}$$

□

Theorem 2.5.19 ([10] Thm. 3.10)**.** Let E be a rationally even generalized cohomology theory which is represented by a spectrum $\mathbf{E}$. Let $(\hat{E}, R, I, \int)$ and $(\hat{E}', R', I', \int')$ be two differential extensions with integration. We assume that either the extensions are defined on the category of all smooth manifolds and the groups $E^m(pt)$ are countably generated for all $m \in \mathbb{Z}$, or that $E^m(pt) = 0$ for all $m \in \mathbb{Z}^{odd}$ and E^m is finitely generated for $m \in \mathbb{Z}^{even}$. Then there is a unique natural isomorphism

$$\Phi\colon \hat{E} \to \hat{E}'$$

of differential extensions with integration.

Proof. Proposition 2.5.17 says that morphism (2-13) is a well defined natural transformation between differential extensions for all even degrees. Now we extend this morphism to all odd degrees using integration.

Let $\hat{\alpha} \in \hat{E}^{2k-1}(M)$. By lemma 2.5.18 there exists $\hat{\beta} \in \hat{E}^{2k}(S^1 \times M)$ such that $R(\hat{\beta}) = dt \wedge \pi^* R(\hat{\alpha})$, $\int \hat{\beta} = \hat{\alpha}$ and $I(\hat{\beta}) = s(I(\hat{\alpha}))$. We define

$$\Phi(\hat{\alpha}) = \int' \Phi(\hat{\beta}).$$

This gives a well defined morphism. Indeed, suppose $\hat{\beta}'$ is another class satisfying the conditions above. In particular we have $\hat{\beta}' - \hat{\beta} = a(\omega)$ and $\int a(\omega) = 0$. By the additivity of Φ for even degrees it follows

$$\int' \Phi(\hat{\beta}') = \int' \Phi(\hat{\beta} + a(\omega)) = \int' \Phi(\hat{\beta}) + \int' a'(\omega) = \int' \Phi(\hat{\beta}).$$

About the naturality of Φ, let us consider $f\colon N \to M$ a smooth map of manifolds and $f^*\hat{\alpha} \in \hat{E}^{2k-1}(N)$. By naturality of integration we have that if $\hat{\beta} \in \hat{E}^{\bullet+1}(S^1 \times M)$ is such that $\int \hat{\beta} = \hat{\alpha}$, then $\int (\mathrm{id} \times f)^* \hat{\beta} = f^*\hat{\alpha}$ and also holds the other properties of lemma 2.5.18. Since we already have naturality in even degree, we obtain

$$\Phi(f^*\hat{\alpha}) = \int' \Phi((\mathrm{id} \times f)^*\hat{\beta}) = \int' (\mathrm{id} \times f)^* \Phi(\hat{\beta}) = f^* \int' \Phi(\hat{\beta}) = f^*\Phi(\hat{\alpha}).$$

The morphism thus defined is an isomorphism. To see this it is enough to note that for even degrees the morphism $\Phi^{-1}(f^*\hat{u}'_i + a'(\omega)) = f^*\hat{u}_i + a(\omega)$ yields the inverse of Φ and for odd degrees the morphism defined by $\Phi^{-1}(\hat{\alpha}') = \int \Phi^{-1}(\hat{\beta}')$ gives the inverse.

Finally, about uniqueness, we note that if in even degrees there were another morphism Γ, compatible with the morphisms R, I and a, then the difference $\Phi - \Gamma$ would factor through a morphism

$$\Delta : E^{2k}(M) \to H^{2k-1}(M; \mathfrak{E}^{\bullet}_{\mathbb{R}})/Im(ch).$$

Hence by an argument similar to that of lemma 2.5.14, we would get $\Delta = 0$, thus Φ is uniquely defined in even degrees. The uniqueness on odd degrees follows directly by uniqueness on even degrees and the construction. □

The existence of a multiplicative structure on a differential extension also induces an integration map for the extension, so the uniqueness result extends to multiplicative differential extensions for suitable cohomology theories.

Proposition 2.5.20 ([10] Cor. 4.3)**.** If $(\hat{E}, R, I, a)$ is a multiplicative differential extension of a cohomology theory $E^{\bullet}$ such that $E^{-1}(pt)$ is a torsion group, then there is a canonical choice of an integration.

Theorem 2.5.21 ([10] Cor. 4.4)**.** Let $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ be two multiplicative extensions of a rationally even cohomology theory $E^{\bullet}$. We assume that either both extensions are defined on the category of all smooth manifolds and the groups $E^n(pt)$ are countably generated for all $n \in \mathbb{Z}$, or they are defined on the category of compact manifolds, $E^n(pt) = 0$ for all $n \in \mathbb{Z}^{odd}$ and $E^n(pt)$ is countably generated for all $n \in \mathbb{Z}^{even}$. Then there is a unique natural isomorphism between these differential extensions preserving the canonical integration.

2.6 Existence

We have already seen models of differential extensions for two different cohomology theories, namely singular cohomology with integral coefficients and complex K-theory; at this point it is natural to ask whether is possible to obtain differential extensions for any given cohomology theory. In this section we present a model that gives an affirmative answer to this question. The contents of this section is based essentially on [21].

Let us fix a cohomology theory $E^{\bullet}$ and a spectrum $\{\mathbb{E}_n, e_n, \epsilon_n\}$ representing it (see the beginning of the previous section). Additionally, we fix a family of fundamental cocycles $\iota_n \in Z^n(\mathbb{E}_n; \mathfrak{E}^{\bullet}_{\mathbb{R}})$, such that

$$\iota_{n-1} = \int_{S^1} \epsilon^*_{n-1} \iota_n$$

and such that they represent the Chern character $ch : E^\bullet(-) \to H^\bullet_{dR}(-;\mathfrak{E}^\bullet_{\mathbb{R}})$, that is, for a manifold M and a class $[f] \in [M, \mathbb{E}_n] \cong E^n(M)$, the relation

$$ch([f]) = f^*\iota_n$$

holds.

Definition 2.6.1. Let M be a smooth manifold. A *differential function* of degree n is a triple (f, h, ω), where

- $f \colon M \to \mathbb{E}_n$,
- $\omega \in \Omega^n_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}})$,
- $h \in C^{n-1}_{sm}(M;\mathfrak{E}^\bullet_{\mathbb{R}})$

and such that

$$\delta h = \omega - f^*\iota_n. \tag{2-17}$$

As they are, relation (2-17) shows that the set of differential functions has redundant information; in order to cut down this excess we set the following definition:

Definition 2.6.2. Let M be a smooth manifold. Given two differential functions of degree n (f_1, h_1, ω_1) and (f_0, h_0, ω_0), we say that they are equivalent, $(f_1, h_1, \omega_1) \sim (f_0, h_0, \omega_0)$, if:

- $\omega_0 = \omega_1$,
- exists a homotopy $F \colon I \times M \to E_n$, such that $F_0 = f_0$ and $F_1 = f_1$,
- exists $H \in C^{n-1}_{cl}(I \times M, \mathfrak{E}^\bullet_{\mathbb{R}})$, such that $i_0^*H = h_0$ and $i_1^*H = h_1$, and
- $(F, H, \pi^*\omega)$ is a differential function of degree n over $I \times M$.

Definition 2.6.3. Let M be a smooth manifold. We define the *differential cohomology group* of degree n, denoted by $\hat{E}^n(M)$, as the set of equivalence classes of differential functions of degree n.

With this structure in place, the following definitions are rather natural.

Definition 2.6.4. Let M be a smooth manifold. We define the *characteristic class* morphism

$$\begin{aligned} I^n \colon \hat{E}^n(M) &\to E^n(M) \\ [(f, h, \omega)] &\mapsto [f]. \end{aligned}$$

Definition 2.6.5. Let M be a smooth manifold. We define the *curvature* morphism

$$\begin{aligned} R^n \colon \hat{E}^n(M) &\to \Omega^n_{cl}(M;\mathfrak{E}^\bullet_{\mathbb{R}}) \\ [(f, h, \omega)] &\mapsto \omega. \end{aligned}$$

Definition 2.6.6. Let M be a smooth manifold. We define the morphism

$$\begin{aligned} a^n \colon \Omega^{n-1}(M; \mathfrak{E}^\bullet_{\mathbb{R}})/Im(d) &\to \hat{E}^n(M) \\ \omega &\mapsto [(const, \omega, d\omega)]. \end{aligned}$$

It is easy to check that this structure holds the first two axioms for a differential extension of $E^\bullet$ (definition 2.4.1). Indeed, by definition we have

$$R \circ a(\omega) = R([const, \omega, d\omega]) = d\omega.$$

The second axiom follows directly by condition (2-17) for differential functions and the calculation

$$\begin{aligned} [R([f, h, \omega])] &= [\omega] \\ &= [f^* \iota_n] \\ &= ch([f]) \\ &= ch \circ I([f, h, \omega]). \end{aligned}$$

For the third axiom the argument is a bit more elaborate, we refer to [39] Theorem. 2.6 for a complete discussion.

3 Relative Differential Cohomology

3.1 Relative Cheeger-Simons Characters

3.1.1 Definition and Morphisms

The relative framework for differential characters is defined by a natural generalization of the objects involved in the definition of the absolute case. Our goal is to define a sensible generalization of the absolute case for the category $\mathcal{M}_2$ of smooth maps between manifolds. First we review the definition of singular homology and de Rham cohomology in the relative case.

Let $\varphi : A \to X$ be an object in Top_2. The singular relative chain complex associated to φ is defined as

$$C_k(\varphi) \coloneqq C_k(X) \oplus C_{k-1}(A),$$

with differential

$$\begin{aligned} \partial : C_k(\varphi) &\to C_{k-1}(\varphi) \\ (\sigma, \rho) &\mapsto (\partial\sigma + \varphi_*\rho, -\partial\rho). \end{aligned} \tag{3-1}$$

This complex is also called the mapping cone complex of φ. When the map φ is the inclusion, the homology groups $H_k(\varphi)$ coincide with the relative groups $H_k(X, A; \mathbb{Z})$, this follows by the long exact sequence of the pair (X, A), the five lemma and the long exact sequence induced by the short exact sequence

$$0 \longrightarrow C_k(X) \xrightarrow{\ i\ } C_k(\varphi) \xrightarrow{\ p\ } C_{k-1}(A) \longrightarrow 0, \tag{3-2}$$

where $i(\sigma) = (\sigma, 0)$, $p(\sigma, \rho) = \rho$ and $C_{k-1}(A)$ has the differential with opposite sign [28].

In a similar fashion, we define the relative de Rham complex by

$$\Omega^k(\varphi) \coloneqq \Omega^k(X) \oplus \Omega^{k-1}(A).$$

In the cohomological case the differential is defined as

$$\begin{aligned} d : \Omega^k(\varphi) &\to \Omega^{k+1}(\varphi) \\ (\omega, \eta) &\mapsto (d\omega, \varphi^*\omega - d\eta). \end{aligned}$$

Definition 3.1.1. Let $\varphi : A \to X$ be a smooth map of manifolds. The set of relative differential characters of φ of degree k is defined as

$$\hat{H}^k(\varphi) = \{h \in Hom(Z_{k-1}(\varphi) \to \mathbb{R}/\mathbb{Z}) \quad | \quad h \circ \partial \in \Omega^k_0(\varphi)\}.$$

As in the absolute case, the condition $h \circ \partial \in \Omega_0^k(\varphi)$ means that h restricted to boundaries is represented by an element of $\Omega_0^k(\varphi)$, the group of closed relative forms with integral periods. Explicitly, the condition says that there are a pair of forms $(\omega, \eta) \in \Omega^k(\varphi)$ such that

$$h \circ \partial(\sigma, \rho) = \int_{(\sigma,\rho)} (\omega, \eta) \mod \mathbb{Z} = (\int_\sigma \omega + \int_\rho \eta) \mod \mathbb{Z}$$

for all $(\sigma, \rho) \in C_k(\varphi)$.

We also define the morphisms

$$\begin{aligned} R : \hat{H}^k(\varphi) &\to \Omega_0^k(\varphi) \\ h &\mapsto (\omega, \eta) \end{aligned}$$

and

$$\begin{aligned} I : \hat{H}^k(\varphi) &\to H^k(\varphi; \mathbb{Z}) \\ h &\mapsto \int (\omega, \eta) - \tilde{h} \circ \partial, \end{aligned}$$

where $\tilde{h}$ is a real lift of h.

There are also natural generalizations of the morphisms a and b of the absolute case. They are given by

$$\begin{aligned} a : \frac{\Omega^{k-1}(\varphi)}{\Omega_0^{k-1}(\varphi)} &\to \hat{H}^k(\varphi) \\ [(\omega, \eta)] &\mapsto h_{(\omega,\eta)} \end{aligned}$$

where $h_{(\omega,\eta)}(\sigma, \rho) = \int_{(\sigma,\rho)} (\omega, \eta) \mod \mathbb{Z}$, for all $(\sigma, \rho) \in Z_{k-1}(\varphi)$ and

$$\begin{aligned} b : H^{k-1}(\varphi; \mathbb{R}/\mathbb{Z}) &\to \hat{H}^k(\varphi) \\ [c] &\mapsto h_c \end{aligned}$$

where $h_c(\sigma, \rho) = c(\sigma, \rho)$, for all $(\sigma, \rho) \in Z_{k-1}(\varphi)$.

3.1.2 Properties of the Relative Characters

Given the construction of the relative characters it is not surprise to see that the key features of the absolute characters are also present in the relative case.

Proposition 3.1.2. The group of relative Cheeger-Simons characters holds

- for any $\varphi \in \mathcal{M}_2$, the diagram

$$\begin{tikzcd} \hat{H}^k(\varphi) \arrow[r, "I"] \arrow[d, "R"'] & H^k(\varphi; \mathbb{Z}) \arrow[d, "ch"] \\ \Omega_0^k(\varphi) \arrow[r, "dR"'] & H^k_{dR}(\varphi) \end{tikzcd}$$

  commutes,

- $R \circ a = d$,
- for any $\varphi \in \mathcal{M}_2$, the sequences

$$0 \longrightarrow \frac{\Omega^{k-1}(\varphi)}{\Omega_0^{k-1}(\varphi)} \xrightarrow{\quad a \quad} \hat{H}^k(\varphi) \xrightarrow{\quad I \quad} H^k(\varphi;\mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^{k-1}(\varphi;\mathbb{R}/\mathbb{Z}) \xrightarrow{\quad b \quad} \hat{H}^k(\varphi) \xrightarrow{\quad R \quad} \Omega_0^k(\varphi) \longrightarrow 0$$

are exact.

Proof. The proof of the first two properties follows a completely analogous argument as the one given on the absolute case. About the third property, the same ideas of the absolute case together with the observation that

$$\begin{aligned}
\left(\int(\omega,\eta)\right)\circ\partial(\sigma,\rho) &= \int_{(\partial\sigma+\varphi_*\rho,-\partial\rho)}(\omega,\eta)\\
&= \int_{\partial\sigma+\varphi_*\rho}\omega - \int_{\partial\rho}\eta\\
&= \int_{\partial\sigma}\omega + \int_{\varphi_*\rho}\omega - \int_{\partial\rho}\eta\\
&= \int_{\sigma}d\omega + \int_{\rho}\varphi^*\omega - d\eta\\
&= \int_{(\sigma,\rho)}(d\omega,\varphi^*\omega - d\eta)\\
&= \int_{(\sigma,\rho)}d(\omega,\eta).
\end{aligned}$$

yield the result. □

A mayor theoretical and computational tool in cohomology theory is the long exact sequence associated to a pair of spaces. Naturally, one would want to have such property on our model of relative characters. However, even though our model of relative characters is based on relative singular chains, we do not recover completely the long exact sequence of the topological theory; in spite of that, we still obtain a family of long exact sequences relating the differential and the topological groups.

Proposition 3.1.3. Let $\varphi \in \mathcal{M}_2$. Then the sequence

$$
\begin{tikzcd}
\ldots \arrow[r] & \hat{H}^{k-1}_{\mathrm{fl}}(\varphi) \arrow[r, "i^*"] & \hat{H}^{k-1}_{\mathrm{fl}}(X) \arrow[r, "\varphi^*"] & \hat{H}^{k-1}(A) \arrow[dll, "\beta"'] & & \\
& & \hat{H}^k(\varphi) \arrow[r, "i^*"] & \hat{H}^k(X) \arrow[r, "\varphi^* \circ ch"] & H^k(A;\mathbb{Z}) \arrow[r, "\beta"] & \ldots
\end{tikzcd}
$$

is exact, for $k \geq 2$.

Proof. As we will see later, it is enough to show that the diagram

$$
\begin{tikzcd}
\hat{H}^k(\varphi) \arrow[r, "i^*"] \arrow[d, "cov"] & \hat{H}^k(X) \arrow[d, "\varphi^*"] \\
\Omega^{k-1}(A) \arrow[r, "a"] & \hat{H}^k(A)
\end{tikzcd}
$$

commutes. Consider a relative differential character $f : Z_{k-1}(\varphi) \to \Re/\mathbb{Z}$. Using the sequence (3-2) and the definition of the relative boundary operator (3-1), we have that for $\sigma \in Z_{n-1}(A)$ it holds

$$i \circ \varphi_*(\sigma) = (\varphi_* \sigma, 0) = \partial(0, \sigma)$$

Hence the character $\varphi^* \circ i^*(f) \in \hat{H}^k(A)$ acts as

$$
\begin{aligned}
\varphi^* \circ i^*(f)(\sigma) =& f(i \circ \varphi_*(\sigma)) \\
=& f \circ \partial(0, \sigma) \\
=& \int_\sigma \eta \\
=& a(\eta) \\
=& a(cov(f)).
\end{aligned}
$$

□

4 Relative differential extension

4.1 Axioms for relative differential cohomology

We are going to state the axioms of relative differential cohomology. When we use the expression "relative cohomology", we mean that we are considering the cohomology groups of any map of spaces, not necessarily an embedding. Thus, we start with a brief review of the axioms of (topological) cohomology for maps.

4.1.1 Relative cohomology

Let $\mathcal{C}$ be the category of spaces with the homotopy type of a CW-complex or of a finite CW-complex. We call $\mathcal{C}_+$ the category whose objects are the ones of $\mathcal{C}$ with a marked point, and whose morphisms are the continuous functions that respect the marked points. Taking the quotient of the morphisms of $\mathcal{C}$ and $\mathcal{C}_+$ by homotopy (relative to the marked point in $\mathcal{C}_+$), we get the categories $\mathcal{HC}$ and $\mathcal{HC}_+$. Moreover, we denote by $\mathcal{C}_2$ the category of morphism of $\mathcal{C}$, defined in the following way:

- an object of $\mathcal{C}_2$ is a morphism $\rho\colon A \to X$ of $\mathcal{C}$ (i.e. a continuous function between objects of $\mathcal{C}$);
- given two objects $\eta\colon B \to Y$ and $\rho\colon A \to X$, a morphism between η and ρ is a pair of continuous functions $f\colon Y \to X$ and $g\colon B \to A$, making the following diagram commutative:

$$\begin{tikzcd} B \arrow[r, "\eta"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ A \arrow[r, "\rho"] & X. \end{tikzcd} \tag{4-1}$$

We set $I := [0, 1]$ and we call $\mathrm{id}_I\colon I \to I$ the identity map. A *homotopy* between two morphisms $(f_0, g_0), (f_1, g_1)\colon \eta \to \rho$ is a morphism $(F, G)\colon \eta \times \mathrm{id}_I \to \rho$, such that, for $i = 0, 1$, we have $(F|_{X\times\{i\}}, G|_{A\times\{i\}}) = (f_i, g_i)$. Taking the quotient of the morphisms of $\mathcal{C}_2$ by homotopy we define the category $\mathcal{HC}_2$. There are the following natural embeddings of categories:

- $\mathcal{C} \hookrightarrow \mathcal{C}_+$ and $\mathcal{HC} \hookrightarrow \mathcal{HC}_+$, defined identifying an object X with (X_+, ∞), where $X_+ = X \sqcup \{\infty\}$;

- $\mathcal{C}_+ \hookrightarrow \mathcal{C}_2$ and $\mathcal{HC}_+ \hookrightarrow \mathcal{HC}_2$, defined identifying the object (X, x_0) with the morphism $\rho\colon pt \to X$ such that $\rho(pt) = x_0$;

- by composition, we get the embeddings $\mathcal{C} \hookrightarrow \mathcal{C}_2$ and $\mathcal{HC} \hookrightarrow \mathcal{HC}_2$; we can also define these embeddings identifying X with the empty function $\emptyset \to X$, if we consider the empty set as a manifold.

Finally, there are two natural functors $\Pi\colon \mathcal{C}_2 \to \mathcal{C}$ and $\Pi\colon \mathcal{HC}_2 \to \mathcal{HC}$, defined in the following way: if $\rho\colon A \to X$ is an object, then $\Pi(\rho) = A$; if $\eta\colon B \to Y$ is another object and $(f, g)\colon \rho \to \eta$ is a morphism, then $\Pi(f, g) = g$.

We call $\mathcal{A}_{\mathbb{Z}}$ the category of $\mathbb{Z}$-graded abelian groups. A *cohomology theory* on $\mathcal{C}_2$ is defined by a functor $h^\bullet\colon \mathcal{HC}_2 \to \mathcal{A}_{\mathbb{Z}}$ and a morphism of functors $\beta^\bullet\colon h^\bullet \circ \Pi \to h^{\bullet+1}$, satisfying the following axioms:

1. *Long exact sequence:* the functor $h^\bullet$ and the morphism of functors $\beta^\bullet$ define a functor from $\mathcal{HC}_2$ to the category of long exact sequences of abelian groups, that assigns to an object $\rho\colon A \to X$ the sequence:

$$\cdots \longrightarrow h^n(\rho) \xrightarrow{\pi^*} h^n(X) \xrightarrow{\rho^*} h^n(A) \xrightarrow{\beta} h^{n+1}(\rho) \longrightarrow \cdots$$

   (π being the natural morphism from $\emptyset \to X$ to $\rho\colon A \to X$) and to a morphism the corresponding morphism of exact sequences.

2. *Excision:* if $i\colon Z \hookrightarrow A$ and $j\colon A \hookrightarrow X$ are embeddings such that the closure of $j(i(Z))$ is contained in the interior of $j(A)$, then the morphism

$$\begin{tikzcd} A \setminus i(Z) \arrow[r, "j'"] \arrow[d, "\iota'"'] & X \setminus j(i(Z)) \arrow[d, "\iota"] \\ A \arrow[r, "j"] & X \end{tikzcd}$$

   induces an isomorphism between $h^\bullet(j)$ and $h^\bullet(j')$.

If the objects of $\mathcal{C}$ have the homotopy type of a finite CW-complex this is enough, otherwise we must add the multiplicativity axiom [29].

Such a definition of cohomology theory is equivalent to the usual one on pairs of spaces or on spaces with a marked point. In fact, starting from a reduced cohomology theory on $\mathcal{HC}_+$, the cohomology groups of a morphism $\rho\colon A \to X$ are defined as the reduced ones of the cone $C(\rho) := X \sqcup_A CA$, and the axioms are satisfied. Vice-versa, if we start from the axioms on the category $\mathcal{HC}_2$, we can prove that $h^\bullet(\rho)$ is naturally isomorphic to $\tilde{h}^\bullet(C(\rho))$, hence the theory is the unique possible extension to $\mathcal{HC}_2$ of a reduced cohomology theory on $\mathcal{HC}_+$. In fact, we consider the cylinder $\mathrm{Cyl}(\rho) := X \sqcup_A \mathrm{Cyl}(A)$ and the following commutative diagram:

$$\begin{tikzcd} \{*\} \arrow[r, hook, "v"] & C(\rho) \\ A \arrow[u, "p'"] \arrow[r, hook, "i_1"] \arrow[d, "\mathrm{id}"'] & \mathrm{Cyl}(\rho) \arrow[u, "p"'] \arrow[d, "\pi"] \\ A \arrow[r, "\rho"] & X. \end{tikzcd} \tag{4-2}$$

The point $\{*\}$ is the vertex of the cone. The projection π shrinks the cylinder of A on the base and the projection p collapses the upper base of the cylinder to the vertex of the cone. Finally, the embedding i_1 sends A to the upper base of the cylinder. Since i_1 is a cofibration, the projection p, that collapses A to a point, induces an isomorphism in relative cohomology $h^\bullet(i_1) \simeq \tilde{h}^\bullet(C(\rho))$. Moreover, since π is a homotopy equivalence, the pair (π, id) induces, by the five lemma applied to the corresponding long exact sequences, an isomorphism $h^\bullet(i_1) \simeq h^\bullet(\rho)$. Composing the two isomorphisms we get $h^\bullet(\rho) \simeq \tilde{h}^\bullet(C(\rho))$. Such an isomorphism is natural. In fact, given two maps $\rho\colon A \to X$ and $\eta\colon B \to Y$ and a morphism $(k,h)\colon \rho \to \eta$, from the induced morphism between the two diagrams (4-2) of ρ and η, we see that the following diagram commutes:

$$\begin{tikzcd} h^\bullet(\eta) \arrow[r, "(k,h)^*"] \arrow[d, "\simeq"'] & h^\bullet(\rho) \arrow[d, "\simeq"] \\ \tilde{h}^\bullet(C(\eta)) \arrow[r, "C(k,h)^*"] & \tilde{h}^\bullet(C(\rho)). \end{tikzcd} \tag{4-3}$$

In particular, if $C(k,h)$ is a homotopy equivalence, then $(k,h)^*$ is an isomorphism, even if (k,h) is not a homotopy equivalence in the category $\mathcal{C}_2$.

In order to introduce products, we call $\mathcal{R}_\mathbb{Z}$ the category of $\mathbb{Z}$-graded commutative rings. There is a natural forgetful functor $\mathcal{R}_\mathbb{Z} \to \mathcal{A}_\mathbb{Z}$, that we apply when needed, without writing it explicitly. The cohomology theory $h^\bullet$ is called *multiplicative* if it can be refined to a functor $h^\bullet\colon \mathcal{HC}_2 \to \mathcal{R}_\mathbb{Z}$, in such a way that the product satisfies a suitable compatibility condition with the morphisms $\beta^\bullet$. The isomorphism $h^\bullet(\rho) \simeq \tilde{h}^\bullet(C(\rho))$ is a *ring* isomorphism, hence the product in relative cohomology is canonically induced by the one on the corresponding reduced cohomology theory.

Finally, given a morphism $\rho\colon A \to X$, the group $h^\bullet(\rho)$ has a natural right module structure over $h^\bullet(X)$:

$$\cdot : h^\bullet(\rho) \otimes_\mathbb{Z} h^\bullet(X) \to h^\bullet(\rho) \tag{4-4}$$

defined as follows. We compute the product $h^\bullet(\rho) \otimes h^\bullet(X) \simeq \tilde{h}^\bullet(C(\rho)) \otimes \tilde{h}^\bullet(X_+) \to \tilde{h}^\bullet(C(\rho) \wedge X_+) \simeq \tilde{h}^\bullet(C(\rho \times \mathrm{id}_X)) \simeq h^\bullet(\rho \times \mathrm{id}_X)$. Then we apply the pull-back via the diagonal morphism

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "(\mathrm{id}_A, \rho)"'] & X \arrow[d, "\Delta_X"] \\ A \times X \arrow[r, "\rho \times \mathrm{id}_X"] & X \times X. \end{tikzcd}$$

We could construct (4-4) directly from the axioms, without passing through the cone of ρ, but it would be a little bit longer.

4.1.2 Fibre-wise integration and Stokes theorem

Given a smooth map of manifolds $\rho\colon A \to X$, we call $\Omega^\bullet(\rho)$ the cochain complex $\Omega^\bullet(X) \times \Omega^{\bullet-1}(A)$ with coboundary $d(\omega, \eta) = (d\omega, \rho^*\omega - d\eta)$. We get the following short exact sequence of chain complexes:

$$0 \longrightarrow (\Omega^{\bullet-1}(A), -d^{\bullet-1}) \xrightarrow{i} (\Omega^\bullet(\rho), d^\bullet) \xrightarrow{\pi} (\Omega^\bullet(X), d^\bullet) \longrightarrow 0, \tag{4-5}$$

where $i(\chi) = (0, \chi)$ and $\pi(\omega, \chi) = \omega$. Let us fix the following data:

- a smooth map $\rho\colon A \to X$ between compact manifolds, possibly with boundary (the map is not necessarily neat);
- two neat proper submersions $f\colon Y \to X$ and $g\colon B \to A$ with n-dimensional compact oriented fibres;
- a morphism of fibre bundles $\tilde{\rho}\colon B \to Y$ covering ρ and inducing a diffeomorphism in each fibre;[1]
- an orientation of the bundle f, inducing an orientation of g.

We get the following diagram:

$$\begin{tikzcd} B \arrow[r, "\tilde{\rho}"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ A \arrow[r, "\rho"] & X. \end{tikzcd} \tag{4-6}$$

We define the fibre-wise integration of a relative form $(\omega, \eta) \in \Omega^\bullet(\tilde{\rho})$ in the following way:

$$\int_{\tilde{\rho}/\rho} (\omega, \eta) := \left(\int_{Y/X} \omega, \int_{B/A} \eta \right). \tag{4-7}$$

If the fibres of f and g have boundary and $\tilde{\rho}$ is neat, we call $\partial f\colon \partial Y \to X$ and $\partial g\colon \partial B \to A$ the fibre bundles induced by the restrictions of f and g to the boundary of the total space; moreover, we call $\partial\tilde{\rho}\colon \partial B \to \partial Y$ the restriction of $\tilde{\rho}$ to the boundary. We get a diagram analogous to (4-6). The following relative version of Stokes theorem holds [4, formula (82) p. 165]:[2]

$$(-1)^n d \int_{\tilde{\rho}/\rho} (\omega, \eta) = \int_{\tilde{\rho}/\rho} d(\omega, \eta) - \int_{\partial\tilde{\rho}/\rho} (\omega, \eta). \tag{4-8}$$

[1] Such a morphism is equivalent to a bundle isomorphism between B and $\rho^* X$.

[2] In [4] a different convention on signs is used. Here we assume that, if vol_f is a form on Y, restricting to a volume form on each fibre of f, then $\int_{Y/X} \mathrm{vol}_f \wedge f^*\omega = \omega$. In [4] they assume that $\int_{Y/X} f^*\omega \wedge \mathrm{vol}_f = \omega$.

Finally, we remark that, given a smooth map $\rho\colon A \to X$, the complex $\Omega^\bullet(\rho)$ has a natural right module structure over $\Omega^\bullet(X)$, defined by:

$$(\omega, \eta) \wedge \xi := (\omega \wedge \xi, \eta \wedge \rho^*\xi). \tag{4-9}$$

We get correctly that $d((\omega, \eta) \wedge \xi) = d(\omega, \eta) \wedge \xi + (-1)^{|\omega|}(\omega, \eta) \wedge d\xi$.

4.1.3 Differential extension

Let $\mathcal{M}$ be the category of smooth manifolds or of smooth compact manifolds (even with boundary), and let $\mathcal{A}_\mathbb{Z}$ be the category of $\mathbb{Z}$-graded abelian groups. We consider a cohomology theory $h^\bullet$, defined on a category including $\mathcal{M}$. We use the following notation:

$$\mathfrak{h}^\bullet := h^\bullet(\{pt\}) \qquad\qquad \mathfrak{h}^\bullet_\mathbb{R} := \mathfrak{h}^\bullet \otimes_\mathbb{Z} \mathbb{R}.$$

We consider the category $\mathcal{M}_2$ of morphisms of $\mathcal{M}$. For any object $\rho\colon A \to X$ of $\mathcal{M}$, we call $\mathrm{ch}\colon h^\bullet(\rho) \to H^\bullet_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_\mathbb{R})$ the generalized Chern character [21, sec. 4.8 p. 47]. We follow [10, sec. 1], adapting the construction to the relative case.

Definition 4.1.1. A *relative differential extension* of $h^\bullet$ is a functor $\hat{h}^\bullet\colon \mathcal{M}_2^{\mathrm{op}} \to \mathcal{A}_\mathbb{Z}$, together with the following natural transformations of $\mathcal{A}_\mathbb{Z}$-valued functors:

- $I\colon \hat{h}^\bullet(\rho) \to h^\bullet(\rho)$;
- $R\colon \hat{h}^\bullet(\rho) \to \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_\mathbb{R})$, called *curvature*;
- $a\colon \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_\mathbb{R})/\mathrm{Im}(d) \to \hat{h}^\bullet(\rho)$,

such that:

(A1) $R \circ a = d$;

(A2) the following diagram is commutative:

$$\begin{tikzcd} \hat{h}^\bullet(\rho) \arrow[r, "I"] \arrow[d, "R"'] & h^\bullet(\rho) \arrow[d, "\mathrm{ch}"] \\ \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_\mathbb{R}) \arrow[r, "dR"] & H^\bullet_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_\mathbb{R}); \end{tikzcd} \tag{4-10}$$

(A3) the following sequence is exact:

$$h^{\bullet-1}(\rho) \xrightarrow{\mathrm{ch}} \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_\mathbb{R})/\mathrm{Im}(d) \xrightarrow{a} \hat{h}^\bullet(\rho) \xrightarrow{I} h^\bullet(\rho) \longrightarrow 0; \tag{4-11}$$

(A4) calling $\mathrm{cov}(\rho)$ the second component of the curvature $R(\rho)$ and π the natural morphism from $\emptyset \to X$ to $\rho\colon A \to X$, the following diagram is commutative:

$$\begin{tikzcd} \hat{h}^\bullet(\rho) \arrow[r, "\pi^*"] \arrow[d, "\mathrm{cov}"'] & \hat{h}^\bullet(X) \arrow[d, "\rho^*"] \\ \Omega^{\bullet-1}(A) \arrow[r, "a"] & \hat{h}^\bullet(A). \end{tikzcd}$$

We also call $\hat{h}^\bullet$ *relative differential cohomology theory.*

A class $\hat{\alpha} \in \hat{h}^n(\rho)$ is *flat* when $R(\hat{\alpha}) = 0$. Considering flat classes, we get the functor $\hat{h}^\bullet_{\mathrm{fl}}\colon \mathcal{M}_2 \to \mathcal{A}_{\mathbb{Z}}$. Thus, we get the following commutative hexagon:

$$\begin{tikzcd} & \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[dr, "dR"] & \\ H^{\bullet-1}_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[ur] \arrow[dr, "a"'] & & \hat{h}^\bullet(\rho) \arrow[ur, "R"] \arrow[dr, "I"] & & H^\bullet_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}). \\ & \hat{h}^\bullet_{\mathrm{fl}}(\rho) \arrow[ur, hook] \arrow[rr, "I"] & & h^\bullet(\rho) \arrow[ur, "\mathrm{ch}"'] & \end{tikzcd} \tag{4-12}$$

We call $\Omega^\bullet_{\mathrm{ch}}(\rho)$ the following sub-group of $\Omega^\bullet_{\mathrm{cl}}(\rho)$. A closed relative form (ω, η) belongs to $\Omega^\bullet_{\mathrm{ch}}(\rho)$ if and only if the cohomology class $[(\omega, \eta)] \in H^\bullet_{dR}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$ lies in the image of the Chern character $\mathrm{ch}\colon h^\bullet(\rho) \to H^\bullet_{dR}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$.

Lemma 4.1.2. The group $\Omega^\bullet_{\mathrm{ch}}(\rho)$ is the image of the curvature functor R, thus we have the following exact sequence:

$$0 \longrightarrow \hat{h}^\bullet_{\mathrm{fl}}(\rho) \longrightarrow \hat{h}^\bullet(\rho) \xrightarrow{R} \Omega^\bullet_{\mathrm{ch}}(\rho) \longrightarrow 0. \tag{4-13}$$

Proof. It immediately follows from diagram (4-10) that the image of R is contained in $\Omega^\bullet_{\mathrm{ch}}(\rho)$. Given a form $(\omega, \eta) \in \Omega^\bullet_{\mathrm{ch}}(\rho)$, let $\alpha \in h^\bullet(\rho)$ be a class such that $\mathrm{ch}(\alpha) = [(\omega, \eta)]$. Because of the exact sequence (4-11), the morphism I is surjective, hence there exists $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ such that $I(\hat{\alpha}) = \alpha$. It follows from diagram (4-10) that $[R(\hat{\alpha})] = [(\omega, \eta)]$, thus there exists $(\alpha, \beta) \in \Omega^{\bullet-1}(\rho)$ such that $R(\hat{\alpha}) = (\omega, \eta) + d(\alpha, \beta)$. Then $R(\hat{\alpha} - a(\alpha, \beta)) = R(\hat{\alpha}) - d(\alpha, \beta) = (\omega, \eta)$. □

Lemma 4.1.3. The following long exact sequence holds:

$$\cdots \longrightarrow h^\bullet(\rho) \xrightarrow{\mathrm{ch}} H^\bullet_{dR}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \xrightarrow{a} \hat{h}^{\bullet+1}_{\mathrm{fl}}(\rho) \xrightarrow{I} h^{\bullet+1}(\rho) \longrightarrow \cdots . \tag{4-14}$$

Proof. It easily follows from the axioms (A1) and (A3) of definition 4.1.1. □

We use the following notation: if $\rho\colon A \to X$ is a map, we set

$$\rho_I := \mathrm{id}_I \times \rho\colon I \times A \to I \times X. \tag{4-15}$$

The inclusions $i_0, i_1\colon X \to I \times X$ and $j_0, j_1\colon A \to I \times A$ induce the following morphisms between ρ and ρ_I:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, hook, "j_0"'] & X \arrow[d, hook, "i_0"] \\ I \times A \arrow[r, "\rho_I"] & I \times X \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, hook, "j_1"'] & X \arrow[d, hook, "i_1"] \\ I \times A \arrow[r, "\rho_I"] & I \times X. \end{tikzcd}$$

We set $\iota_0 := (i_0, j_0)$ and $\iota_1 := (i_1, j_1)$. Analogously, the projections $\pi_X\colon I \times X \to X$ and $\pi_A\colon I \times A \to A$ induce the morphism $(\pi_X, \pi_A)\colon \rho_I \to \rho$. We set $\pi := (\pi_X, \pi_A)$.

Lemma 4.1.4 (Homotopy formula)**.** If $\hat{\alpha} \in \hat{h}^\bullet(\rho_I)$, we have (using the notation of formula (4-7)):

$$\iota_1^*\hat{\alpha} - \iota_0^*\hat{\alpha} = a\left(\int_{\rho_I/\rho} R(\hat{\alpha})\right). \tag{4-16}$$

Proof. Since $\iota_0 \circ \pi\colon \rho_I \to \rho_I$ is homotopic to the identity of ρ_I in the category $\mathcal{M}_2$, we have that $I(\hat{\alpha}) = \pi^*\iota_0^* I(\hat{\alpha})$. We set $\hat{\alpha}_0 := \iota_0^*\hat{\alpha}$, so that $I(\hat{\alpha}) = \pi^* I(\hat{\alpha}_0)$. It follows that $\hat{\alpha} = \pi^*(\hat{\alpha}_0) + a(\omega, \eta)$, therefore $\iota_0^*\hat{\alpha} = \hat{\alpha}_0 + a(\iota_0^*(\omega, \eta))$ and $\iota_1^*\hat{\alpha} = \hat{\alpha}_0 + a(\iota_1^*(\omega, \eta))$. Hence:

$$\begin{aligned} \iota_1^*\hat{\alpha} - \iota_0^*\hat{\alpha} &= a(\iota_1^*(\omega,\eta) - \iota_0^*(\omega,\eta)) = a\left(\int_{\partial\rho_I/\rho} (\omega,\eta)\right) \\ &\overset{(4\text{-}8)}{=} a\left(\int_{\rho_I/\rho} d(\omega,\eta) + d\int_{\rho_I/\rho} (\omega,\eta)\right). \end{aligned}$$

Applying a to an exact form we get 0, thus the term $d\int_{\rho_I/\rho}(\omega,\eta)$ can be cut. Moreover, $R(\hat{\alpha}) = \pi^*R(\hat{\alpha}) + d(\omega,\eta)$ and $\int_{\rho_I/\rho} \pi^*R(\hat{\alpha}) = 0$, hence we get the result. ☐

Corollary 4.1.5. Let $\rho\colon A \to X$, $\eta\colon B \to Y$ be two objects of $\mathcal{M}_2$ and let $(f_0, g_0), (f_1, g_1)\colon \eta \to \rho$ be two morphisms. If $(F, G)\colon \eta \times \mathrm{id}_I \to \rho$ is a homotopy between (f_0, g_0) and (f_1, g_1), then, for any $\hat{\alpha} \in \hat{h}^\bullet(\rho)$, we have:

$$(f_1, g_1)^*\hat{\alpha} - (f_0, g_0)^*\hat{\alpha} = \int_{\eta_I/\eta} (F, G)^* R(\hat{\alpha}). \tag{4-17}$$

Proof. The result follows replacing $\hat{\alpha}$ by $(F, G)^*\hat{\alpha}$ in formula (4-16). ☐

Remark 4.1.6. Thanks to the previous corollary, the flat theory is a homotopy-invariant functor. From the exact sequence (4-14), it is easy to prove that it also satisfies excision and multiplicativity with respect to the disjoint union. In fact, both hold for $h^\bullet$ and $H^\bullet_{\mathrm{dR}}$, since they are cohomology theories, thus it is enough to apply the five lemma.

We briefly recall some basic facts about S^1-integration. Given a space A and fixing a marked point on S^1, we get a natural embedding $i_1\colon A \to S^1 \times A$ and a natural projection $\pi_1\colon S^1 \times A \to A$. Since π_1 is a retraction with right inverse i_1, we have the following split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & h^\bullet(i_1) \arrow[r, "\pi^*"] & h^\bullet(S^1 \times A) \arrow[r, "i_1^*"] \arrow[l, bend left, "\xi"] & h^\bullet(A) \arrow[r] \arrow[l, bend left, "\pi_1^*"] & 0. \end{tikzcd} \tag{4-18}$$

Here π is the natural morphism from $\emptyset \to S^1 \times A$ to i_1 and $\xi(\alpha) = (\pi^*)^{-1}(\alpha - \pi_1^* i_1^* \alpha)$. Moreover, we have the following isomorphism:

$$s\colon h^\bullet(i_1) \simeq \tilde{h}^\bullet(\Sigma(A_+)) \simeq \tilde{h}^{\bullet-1}(A_+) \simeq h^{\bullet-1}(A). \tag{4-19}$$

Thanks to this picture, we can define the following integration map:

$$\begin{aligned} \int_{S^1}\colon h^{\bullet+1}(S^1 \times A) &\to h^\bullet(A) \\ \alpha &\mapsto s \circ \xi(\alpha). \end{aligned} \tag{4-20}$$

We also have the ordinary integration map on differential forms $\int_{S^1}\colon S\Omega^{\bullet+1}(A) \to \Omega^\bullet(A)$. If we apply it to closed forms, we get a well-defined integration map in de-Rham cohomology, coinciding with (4-20).

A similar construction holds in the relative case. Given a morphism $\rho\colon A \to X$, we set $S\rho := \mathrm{id}_{S^1} \times \rho\colon S^1 \times A \to S^1 \times X$. Fixing a marked point on S^1, we get a natural embedding $i_1\colon \rho \to S\rho$ and a natural projection $\pi_1\colon S\rho \to \rho$. We define the groups $h^\bullet(i_1)$ as follows: we consider the induced embedding $i_1'\colon C(\rho) \to C(S\rho)$ and we set $h^\bullet(i_1) := h^\bullet(i_1') \simeq \tilde{h}^\bullet(C(S\rho)/C(\rho))$. Since the induced map $\pi_1'\colon C(S\rho) \to C(\rho)$ is a retraction with right inverse i_1', we have the following split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & h^\bullet(i_1) \arrow[r, "\pi^*"] & h^\bullet(S\rho) \arrow[r, "i_1^*"] \arrow[l, bend left, "\xi"] & h^\bullet(\rho) \arrow[r] \arrow[l, bend left, "\pi_1^*"] & 0. \end{tikzcd}$$

Here $\pi\colon C(S\rho) \to C(S\rho)/C(\rho)$ and $\xi(\alpha) = (\pi^*)^{-1}(\alpha - \pi_1^* i_1^* \alpha)$. Moreover, we have the following isomorphism:

$$s\colon h^\bullet(i_1) \simeq \tilde{h}^\bullet(C(S\rho)/C(\rho)) \simeq \tilde{h}^\bullet(\Sigma(C(\rho)) \simeq \tilde{h}^{\bullet-1}(C(\rho)) \simeq h^{\bullet-1}(\rho).$$

Thanks to this picture, we can define the following integration map:

$$\begin{aligned} \int_{S^1}\colon h^{\bullet+1}(S\rho) &\to h^\bullet(\rho) \\ \alpha &\mapsto s \circ \xi(\alpha). \end{aligned} \tag{4-21}$$

We also have the ordinary integration map on differential forms $\int_{S^1}\colon \Omega^{\bullet+1}(S\rho) \to \Omega^\bullet(\rho)$ defined by $\int_{S^1}(\omega, \eta) := \big(\int_{S^1} \omega, \int_{S^1} \eta\big)$.[3] If we apply it to closed relative forms, we get a

[3] Again we use the convention that, if $\pi_1\colon S^1 \times X \to X$ is the projection, then $\int_{S^1} dt \wedge \pi_1^* \eta = \eta$. This implies that $d \int_{S^1} \omega = -\int_{S^1} d\omega$.

well-defined integration map in de-Rham cohomology, coinciding with the one constructed as (4-21). Of course (4-20) is a particular case of (4-21).

Notation 4.1.7. Given a functor $\mathcal{F}\colon \mathcal{M}_2 \to \mathcal{C}$, for any category $\mathcal{C}$, we define the functor $S\mathcal{F}\colon \mathcal{M}_2 \to \mathcal{C}$ by $S\mathcal{F}(\rho) := \mathcal{F}(S\rho)$ on objects and $S\mathcal{F}(f,g) := \mathcal{F}(Sf, Sg)$ on morphisms. Moreover, given a morphism $\rho\colon A \to X$ and a morphism $t\colon S^1 \to S^1$, we denote by $t_{\#}\rho\colon \mathrm{id}_{S^1} \times \rho \to \mathrm{id}_{S^1} \times \rho$ the morphism $(t \times \mathrm{id}_X, t \times \mathrm{id}_A)$.

Definition 4.1.8. A *relative differential extension with integration* of $h^\bullet$ is a relative differential extension $(\hat{h}^\bullet, I, R, a)$ together with a natural transformation:

$$\int_{S^1} : S\hat{h}^{\bullet+1} \to \hat{h}^\bullet,$$

such that:

- $\int_{S^1} \circ (t_{\#}\rho)^* = -\int_{S^1}$, where $t : S^1 \to S^1$ is defined by $t(e^{i\theta}) := e^{-i\theta}$;
- $\int_{S^1} \circ p^* = 0$, where $p\colon \mathrm{id}_{S^1} \times \rho \to \rho$ is the projection;
- the following diagram is commutative:

$$\begin{tikzcd}
S\Omega^\bullet(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "Sa"] \arrow[d, "-\int_{S^1}"] & S\hat{h}^{\bullet+1}(\rho) \arrow[r, "SI"] \arrow[d, "\int_{S^1}"] \arrow[rr, bend left, "SR"] & Sh^{\bullet+1}(\rho) \arrow[d, "\int_{S^1}"] & S\Omega^{\bullet+1}_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[d, "\int_{S^1}"] \\
\Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] & \hat{h}^\bullet(\rho) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & h^\bullet(\rho) & \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}),
\end{tikzcd} \tag{4-22}$$

  where the first and last vertical arrows are defined by $\int_{S^1}(\omega, \eta) := \left(\int_{S^1}\omega, \int_{S^1}\eta\right)$ and the third one by (4-21).

Let us consider a differential extension with integration $(\hat{h}^\bullet, I, R, a, \int_{S^1})$. It is shown in [10, pp. 27-32] that, if $h^\bullet$ is rationally even (i.e., $\mathfrak{h}^{2k+1}_{\mathbb{R}} = 0$ for every $k \in \mathbb{Z}$), $\mathfrak{h}^k$ is countably generated for every $k \in \mathbb{Z}$ and $\mathcal{M}$ is the category of all smooth manifolds, there is an isomorphism of functors $\hat{h}^\bullet_{\mathrm{fl}}(\,\cdot\,) \simeq h^{\bullet-1}(\,\cdot\,; \mathbb{R}/\mathbb{Z})$. If $\mathcal{M}$ is the category of compact manifolds, we must require that $\mathfrak{h}^{2k+1} = 0$ and $\mathfrak{h}^{2k}$ is finitely generated for every $k \in \mathbb{Z}$. We will show that the same result holds in the relative case. Under these hypotheses, the commutative hexagon (4-12) becomes:

$$\begin{tikzcd}
& \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[dr, "dR"] & \\
H^{\bullet-1}_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[ur] \arrow[dr] & & \hat{h}^\bullet(\rho) \arrow[ur, "R"] \arrow[dr, "I"] & & H^\bullet_{\mathrm{dR}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}), \\
& h^{\bullet-1}(\rho; \mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[ur, hook] & & h^\bullet(\rho) \arrow[ur, "\mathrm{ch}"] &
\end{tikzcd} \tag{4-23}$$

where Bock, in the last line, is the Bockstein map of the long exact sequence induced by the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

Finally, we introduce products, thus we suppose that $h^\bullet$ is a *multiplicative* cohomology theory.

Definition 4.1.9. A *multiplicative relative differential extension* of $h^\bullet$ is a relative differential extension $(\hat{h}^\bullet, I, R, a)$ such that, for any map $\rho\colon A \to X$, there is a natural right $\hat{h}^\bullet(X)$-module structure on $\hat{h}^\bullet(\rho)$, in such a way that:

- $I(\hat{\alpha} \cdot \hat{\beta}) = I(\hat{\alpha}) \cdot I(\hat{\beta})$, using (4-4) on the r.h.s.;
- $R(\hat{\alpha} \cdot \hat{\beta}) = R(\hat{\alpha}) \wedge R(\hat{\beta})$, using (4-9) on the r.h.s.;
- $\hat{\alpha} \cdot a(\omega) = a(R(\hat{\alpha}) \wedge \omega)$ for every $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ and $\omega \in \Omega^\bullet(X; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$;
- $a(\omega, \eta) \cdot \hat{\alpha} = a((\omega, \eta) \wedge R(\hat{\alpha}))$ for every $(\omega, \eta) \in \Omega^\bullet(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$ and $\hat{\alpha} \in \hat{h}^\bullet(X)$.

In the following we consider multiplicative differential extensions with integration.

4.1.4 Parallel classes

A class $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ is called *parallel* if $\mathrm{cov}(\hat{\alpha}) = 0$ (we recall that cov is the second component of the curvature). We denote by $\hat{h}^\bullet_{\mathrm{par}}(\rho)$ the sub-group of parallel classes. Moreover, we use the following notation:

- $\Omega^\bullet_0(\rho)$ is the sub-group of $\Omega^\bullet(X)$ containing the forms ω on X such that $\rho^*\omega = 0$;
- $\Omega^\bullet_{\mathrm{cl},0}(\rho)$ is the intersection between $\Omega^\bullet_0(\rho)$ and $\Omega^\bullet_{\mathrm{cl}}(X)$;
- $\Omega^\bullet_{\mathrm{ch},0}(\rho)$ is the subgroup of $\Omega^\bullet_{\mathrm{cl},0}(\rho)$ containing the forms ω such that the relative cohomology class $[(\omega, 0)]$ belongs to the image of the Chern character.

If $(\omega, 0)$ is the curvature of a parallel class, then $\omega \in \Omega^\bullet_{\mathrm{ch},0}(\rho)$. We get the functor $\hat{h}^\bullet_{\mathrm{par}}\colon \mathcal{M}_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$, together with the following natural transformations of $\mathcal{A}_{\mathbb{Z}}$-valued functors:

- $I'\colon \hat{h}^\bullet_{\mathrm{par}}(\rho) \to h^\bullet(\rho)$, which is the restriction of the functor I on $\hat{h}^\bullet(\rho)$;
- $R'\colon \hat{h}^\bullet_{\mathrm{par}}(\rho) \to \Omega_{\mathrm{cl},0}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$, which is the first component of the curvature R;
- $a'\colon \Omega^{\bullet-1}_0(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{h}^\bullet_{\mathrm{par}}(\rho)$, defined by $a'(\omega) := a(\omega, 0)$.

Parallel classes are well-behaved when ρ is a closed embedding. In this case they satisfy four properties analogous to axioms (A1)–(A4) in definition 4.1.1, as the next theorem shows.

Theorem 4.1.10. Let $\mathcal{M}'_2$ be the full sub-category of $\mathcal{M}_2$, whose objects are closed embeddings. The functor $\hat{h}^\bullet_{\mathrm{par}}\colon \mathcal{M}_2'^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$ satisfies the statements (A′1)–(A′4), obtained from axioms (A1)–(A4) in definition 4.1.1, with the following replacements:

- $\hat{h}^\bullet$ by $\hat{h}^\bullet_{\mathrm{par}}$;
- I, R and a by I', R' and a';
- $\Omega^\bullet$ and $\Omega^\bullet_{\mathrm{cl}}$ by $\Omega^\bullet_0$ and $\Omega^\bullet_{\mathrm{cl},0}$.

In particular, (A′4) is the statement $\rho^* \circ \pi^* = 0$. Moreover, if the functor $\hat{h}^\bullet$ admits S^1-integration or it is multiplicative, the same holds for $\hat{h}^\bullet_{\mathrm{par}}$, with the analogous axioms.

Proof. It is easy to show that (A′1), (A′2) and (A′4) are just a particular case of axioms (A1), (A2) and (A4) (actually, they hold even if ρ is not a closed embedding). We only have to prove that (A′3) holds. The fact that, in the sequence obtained from (4-11), the composition of two consecutive morphisms vanishes is again a particular case of the general statement. Let us fix $\alpha \in h^\bullet(\rho)$. Because of the exactness of (4-11), there exists a class $\hat{\alpha}' \in \hat{h}^\bullet(\rho)$ such that $I(\hat{\alpha}') = \alpha$ and $R(\hat{\alpha}') = (\omega, \eta)$. Since ρ is a closed embedding, we can extend η to a form $\tilde{\eta}$ on the whole X, thus we set $\hat{\alpha} := \hat{\alpha}' - a(\tilde{\eta}, 0)$. Therefore $I(\hat{\alpha}) = I(\hat{\alpha}') - 0 = \alpha$ and $R(\hat{\alpha}) = (\omega, \eta) - (d\tilde{\eta}, \eta) = (\omega - d\tilde{\eta}, 0)$. It follows that $\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{par}}(\rho)$ and $I(\hat{\alpha}) = \alpha$, hence I is surjective. Let us fix $\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{par}}(\rho)$ such that $I(\hat{\alpha}) = 0$. Because of the exactness of (4-11), there exists a form $(\theta, \chi) \in \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$ such that $a(\theta, \chi) = \hat{\alpha}$. Since ρ is a closed embedding, we can extend χ to a form $\tilde{\chi}$ on the whole X, thus $\hat{\alpha} = a(\theta, \chi) - a(d(\tilde{\chi}, 0)) = a((\theta, \chi) - (d\tilde{\chi}, \chi)) = a(\theta - d\tilde{\chi}, 0) = a'(\theta - d\tilde{\chi})$. Finally, the exactness in the second position follows from the one of (4-11), since the first group remains unchanged. The axioms of S^1-integration and multiplicativity easily restricts to parallel classes (even without assuming that ρ is a closed embedding). □

From the exact sequence (4-13) we can immediately deduce the following one, for every morphism ρ:

$$0 \longrightarrow \hat{h}^\bullet_{\mathrm{fl}}(\rho) \longrightarrow \hat{h}^\bullet_{\mathrm{par}}(\rho) \xrightarrow{R'} \Omega^\bullet_{\mathrm{ch},0}(\rho) \longrightarrow 0. \tag{4-24}$$

The next lemma shows that the groups of parallel classes satisfy excision as the topological ones.

Lemma 4.1.11. If $i\colon Z \hookrightarrow A$ and $j\colon A \hookrightarrow X$ are embeddings such that the closure of $j(i(Z))$ is contained in the interior of $j(A)$, then the morphism

$$\begin{tikzcd} A \setminus i(Z) \arrow[r, "j'"] \arrow[d, "\iota'"'] & X \setminus j(i(Z)) \arrow[d, "\iota"] \\ A \arrow[r, "j"] & X \end{tikzcd}$$

induces an isomorphism between $\hat{h}^\bullet_{\mathrm{par}}(j)$ and $\hat{h}^\bullet_{\mathrm{par}}(j')$.

Proof. The morphism (ι, ι') induces the following morphism of exact sequences of the form (4-24):

$$\begin{tikzcd} 0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(j) \arrow[r] \arrow[d] & \hat{h}^{\bullet}_{\mathrm{par}}(j) \arrow[r, "R"] \arrow[d] & \Omega^{\bullet}_{\mathrm{ch},0}(j) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(j') \arrow[r] & \hat{h}^{\bullet}_{\mathrm{par}}(j') \arrow[r, "R"] & \Omega^{\bullet}_{\mathrm{ch},0}(j') \arrow[r] & 0. \end{tikzcd}$$

The left arrow is an isomorphism by the excision property of $\hat{h}^{\bullet}_{\mathrm{fl}}$ (remark 4.1.6). We now prove that the right one is an isomorphism too, hence the result follows from the five lemma. We identify A with $j(A)$ and Z with $i(Z)$ and $j(i(Z))$. The group of closed forms $\Omega^{\bullet}_{\mathrm{cl},0}(j)$ contains the forms $\omega \in \Omega^{\bullet}_{\mathrm{cl}}(X)$ such that $\omega|_A = 0$. Similarly, the group of closed forms $\Omega^{\bullet}_{\mathrm{cl},0}(j')$ contains the forms $\omega \in \Omega^{\bullet}_{\mathrm{cl}}(X \setminus Z)$ such that $\omega|_{A\setminus Z} = 0$. It is clear that the pull-back $(\iota, \iota')^* \colon \Omega^{\bullet}_{\mathrm{cl},0}(j) \to \Omega^{\bullet}_{\mathrm{cl},0}(j')$ is an isomorphism, inducing the excision isomorphism in de-Rham cohomology. We have to show that $(\iota, \iota')^*(\Omega^{\bullet}_{\mathrm{ch},0}(j)) = \Omega^{\bullet}_{\mathrm{ch},0}(j')$. This is a consequence of the commutativity of the following diagram:

$$\begin{tikzcd} h^{\bullet}(j) \arrow[r, "\mathrm{ch}"] \arrow[d] & H^{\bullet}_{\mathrm{dR}}(j; h^{\bullet}_{\mathbb{R}}) \arrow[d] & \Omega^{\bullet}_{\mathrm{cl},0}(j) \arrow[l, "\mathrm{dR}"'] \arrow[d] \\ h^{\bullet}(j') \arrow[r, "\mathrm{ch}"] & H^{\bullet}_{\mathrm{dR}}(j'; h^{\bullet}_{\mathbb{R}}) & \Omega^{\bullet}_{\mathrm{cl},0}(j'). \arrow[l, "\mathrm{dR}"'] \end{tikzcd}$$

The left vertical arrow is an isomorphism too, by excision. □

The following lemma will be useful in the construction of the long exact sequence of $\hat{h}^{\bullet}$.

Lemma 4.1.12. Let A be a smooth manifold, $\pi_1 \colon S^1 \times A \to A$ the natural projection and $i_1 \colon A \to S^1 \times A$ the natural closed embedding, defined marking a point on S^1. We call $\pi^* \colon \hat{h}^{\bullet}(i_1) \to \hat{h}^{\bullet}(S^1 \times A)$ the morphism analogous to the one appearing in the sequence (4-18). For every $\hat{\alpha} \in \hat{h}^{\bullet}(A)$ there exists a unique class $\hat{\beta} \in \hat{h}^{\bullet+1}_{\mathrm{par}}(i_1)$ such that $\int_{S^1} \pi^*\hat{\beta} = \hat{\alpha}$ and $R'(\hat{\beta}) = dt \wedge \pi_1^* R(\hat{\alpha})$.

Proof. Applying the isomorphism (4-19), we set $\beta := s^{-1}(\alpha) \in h^{\bullet+1}(i_1)$. It follows that $\int_{S^1} \pi^*\beta = s \circ \xi \circ \pi^*(\beta) = s(\beta) = \alpha$, the integral being defined by (4-20). We choose any parallel differential refinement $\hat{\beta}' \in \hat{h}^{\bullet+1}_{\mathrm{par}}(i_1)$ such that $I(\hat{\beta}') = \beta$; this is possible because of property (A′3) of theorem 4.1.10 (in particular, because of the surjectivity of I). From the commutativity of diagram (4-22), we get that

$$\int_{S^1} \pi^*\hat{\beta}' = \hat{\alpha} + a(\chi) \tag{4-25}$$

for a suitable form $\chi \in \Omega^{\bullet-1}(A; \mathfrak{h}^{\bullet}_{\mathbb{R}})$. We set $R(\hat{\beta}') = (\omega, 0)$ and $R(\hat{\alpha}) = \bar{\omega}$. It follows from (4-25) that

$$\int_{S^1} \omega = \bar{\omega} + d\chi, \tag{4-26}$$

thus, in de-Rham cohomology, $s([\omega, 0]) = [\bar{\omega}]$. Since also $s([dt \wedge \pi_1^*\bar{\omega}, 0]) = [\bar{\omega}]$ and s is an isomorphism, we have that $[\omega, 0] = [dt \wedge \pi_1^*\bar{\omega}, 0]$, thus there exists $\nu \in \Omega_0^\bullet(i_1)$ such that

$$\omega = dt \wedge \pi_1^*\bar{\omega} + d\nu. \tag{4-27}$$

Joining (4-26) and (4-27) we see that $-d\int_{S^1}\nu = d\chi$, thus

$$\int_{S^1}\nu = -\chi + \lambda, \quad d\lambda = 0. \tag{4-28}$$

We set $\hat{\beta} := \hat{\beta}' - a(\nu) + a(dt \wedge \pi_1^*\lambda)$.[4] It follows that $\int_{S^1}\pi^*\hat{\beta} = \hat{\alpha}$ and $R'(\hat{\beta}) = dt \wedge \pi_1^*\bar{\omega}$. Such a class is unique: if we choose another one, the difference is a flat class $\hat{u} \in \hat{h}^n_{\mathrm{fl}}(i_1)$ such that $\int_{S^1}\pi^*\hat{u} = 0$, i.e., $s(\hat{u}) = 0$, hence $\hat{u} = 0$ (because of the five lemma, s is an isomorphism for the flat theory too). □

4.1.5 Long exact sequences

Let us fix a differential cohomology theory $\hat{h}^\bullet$ with S^1-integration (it can be multiplicative or not). Considering the flat theory, we have the following exact sequence for every map $\rho\colon A \to X$:

$$\cdots \longrightarrow \hat{h}^\bullet_{\mathrm{fl}}(\rho) \longrightarrow \hat{h}^\bullet_{\mathrm{fl}}(X) \longrightarrow \hat{h}^\bullet_{\mathrm{fl}}(A) \longrightarrow \hat{h}^{\bullet+1}_{\mathrm{fl}}(\rho) \longrightarrow \cdots. \tag{4-29}$$

The morphisms involved are $\rho^*\colon \hat{h}^\bullet_{\mathrm{fl}}(X) \to \hat{h}^\bullet_{\mathrm{fl}}(A)$ and $\pi^*\colon \hat{h}^\bullet_{\mathrm{fl}}(\rho) \to \hat{h}^\bullet_{\mathrm{fl}}(X)$, where π is the natural morphism from $\emptyset \to X$ to ρ, and the Bockstein map $\mathrm{Bock}\colon \hat{h}^\bullet_{\mathrm{fl}}(A) \to \hat{h}^{\bullet+1}_{\mathrm{fl}}(\rho)$, that we are going to construct in following paragraphs.

Remark 4.1.13. The sequence (4-29), together with remark 4.1.6, shows that, if $\hat{h}^\bullet$ is a relative differential cohomology theory with S^1-integration, then the flat theory $\hat{h}^\bullet_{\mathrm{fl}}$ is a cohomology theory on $\mathcal{M}_2$.

Considering the whole groups $\hat{h}^\bullet$, we get long exact sequences of the following form:

$$\begin{aligned} \cdots &\longrightarrow \hat{h}^{\bullet-1}_{\mathrm{fl}}(\rho) \longrightarrow \hat{h}^{\bullet-1}_{\mathrm{fl}}(X) \longrightarrow \hat{h}^{\bullet-1}(A) \\ &\longrightarrow \hat{h}^\bullet(\rho) \longrightarrow \hat{h}^\bullet(X) \longrightarrow h^\bullet(A) \\ &\longrightarrow h^{\bullet+1}(\rho) \longrightarrow h^{\bullet+1}(X) \longrightarrow h^{\bullet+1}(A) \longrightarrow \cdots. \end{aligned} \tag{4-30}$$

The first line is a left-infinite part of (4-29), except for the last morphism, which is the composition between $\rho^*\colon \hat{h}^{\bullet-1}_{\mathrm{fl}}(X) \to \hat{h}^{\bullet-1}_{\mathrm{fl}}(A)$ and the inclusion $\hat{h}^{\bullet-1}_{\mathrm{fl}}(A) \hookrightarrow \hat{h}^{\bullet-1}(A)$. Similarly, from $h^\bullet(A)$ on we just have a part of the long exact sequence of the topological theory. The morphism $\hat{h}^\bullet(X) \to h^\bullet(A)$ is the composition between $\rho^*\colon \hat{h}^\bullet(X) \to \hat{h}^\bullet(A)$ and $I\colon \hat{h}^\bullet(A) \to h^\bullet(A)$. We will define the Bockstein map $\mathrm{Bock}\colon \hat{h}^{\bullet-1}(A) \to \hat{h}^\bullet(\rho)$ in the

[4] We recall that there is a minus sign in the left vertical arrow of diagram (4-22).

following. We remark that (4-30) represents a family of exact sequences, since we are free to decide at which degree we put the group $\hat{h}^{\bullet-1}(A)$ instead of the flat one.

Finally, considering parallel classes, if ρ is a *closed embedding* we get long exact sequences of the following form:

$$\begin{aligned} \cdots \longrightarrow \hat{h}^{\bullet-1}_{\mathrm{fl}}(\rho) \longrightarrow \hat{h}^{\bullet-1}_{\mathrm{fl}}(X) \longrightarrow \hat{h}^{\bullet-1}_{\mathrm{fl}}(A) \\ \longrightarrow \hat{h}^{\bullet}_{\mathrm{par}}(\rho) \longrightarrow \hat{h}^{\bullet}(X) \longrightarrow \hat{h}^{\bullet}(A) \\ \longrightarrow h^{\bullet+1}(\rho) \longrightarrow h^{\bullet+1}(X) \longrightarrow h^{\bullet+1}(A) \longrightarrow \cdots . \end{aligned} \tag{4-31}$$

For a generic map ρ, we have to stop at $\hat{h}^{\bullet}(A)$ and cut the third line. The Bockstein map in the first line is the composition of the one of the flat theory with the embedding $\hat{h}^{\bullet}_{\mathrm{fl}}(\rho) \hookrightarrow \hat{h}^{\bullet}_{\mathrm{par}}(\rho)$. The Bockstein map in the second line is the composition of the projection $\hat{h}^{\bullet}(A) \to h^{\bullet}(A)$ with the Bockstein map of the sequence of $h^{\bullet}$.

Now we define the Bockstein map of (4-30). We do it in the six following steps.

(S1) Given $\hat{\alpha} \in \hat{h}^{\bullet-1}(A)$, thanks to lemma 4.1.12 there exists a unique class $\hat{\beta} \in \hat{h}^{\bullet}_{\mathrm{par}}(i_1)$ such that $\int_{S^1} \hat{\beta} = \hat{\alpha}$ and $R(\hat{\beta}) = dt \wedge \pi_1^* R(\hat{\alpha})$, where $\pi_1 \colon S^1 \times A \to A$ is the natural projection and $i_1 \colon A \hookrightarrow S^1 \times A$ is the natural embedding, defined marking a point on S^1.

(S2) Embedding S^1 in $\mathbb{C}$, we suppose that the marked point is 1. We have a natural projection $\bar{p} \colon (I, \{0,1\}) \to (S^1, \{1\})$, defined by $t \mapsto e^{2\pi i t}$, inducing the projection $p = \bar{p} \times \mathrm{id}_A \colon (I \times A, \{0,1\} \times A) \to (S^1 \times A, \{1\} \times A)$, that can be thought of as a morphism $p \colon i_{0,1} \to i_1$ between the embeddings $i_{0,1} \colon A \sqcup A \hookrightarrow I \times A$ and $\iota_1 \colon A \hookrightarrow S^1 \times A$. We get the class $p^* \hat{\beta} \in \hat{h}^{\bullet}_{\mathrm{par}}(i_{0,1})$.

(S3) We define the following class:[5]

$$\hat{\gamma} = p^* \hat{\beta} - a(t \cdot p^* \pi_1^* R(\hat{\alpha}), 0) \in \hat{h}^{\bullet}(i_{0,1}).$$

Since $R(\hat{\beta}) = (dt \wedge \pi_1^* R(\hat{\alpha}), 0)$ and $R \circ a(t \cdot p^* \pi_1^* R(\hat{\alpha}), 0) = (dt \wedge p^* \pi_1^* R(\hat{\alpha}), 0 \sqcup R(\hat{\alpha}))$, it follows that

$$R(\hat{\gamma}) = (0, 0 \sqcup -R(\hat{\alpha})). \tag{4-32}$$

For $\epsilon = 0, 1$, we call $j_\epsilon \colon A \to A \times I$ the embedding with image $A \times \{\epsilon\}$. Moreover, we call π the natural morphism from $\emptyset \to I \times A$ to $i_{0,1}$. It follows from axiom (A4) and formula (4-32) that $j_0^* \pi^* \hat{\gamma} = 0$ and $j_1^* \pi^* \hat{\gamma} = -a(R(\hat{\alpha}))$.

[5]In the following expression, in the occurrence of p^* within $a(\,\cdot\,)$, we are thinking of $p \colon I \times A \to S^1 \times A$, without the two subspaces.

(S4) Let us consider the following morphism:

$$\begin{tikzcd} A \sqcup A \arrow[r, "i_{0,1}"] \arrow[d, "\mathrm{id}"'] & A \times I \arrow[d, "\pi"] \\ A \sqcup A \arrow[r, "\mathrm{id}'"] & A \end{tikzcd} \tag{4-33}$$

where id' acts as the identity on both components of the domain. Let us call $\hat{h}^\bullet_0(\,\cdot\,)$ the sub-group of $\hat{h}^\bullet(\,\cdot\,)$ formed by classes such that the first component of the curvature is vanishing.[6] We get the following morphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \hat{h}^\bullet_{\mathrm{fl}}(i_{0,1}) \arrow[r] & \hat{h}^\bullet_0(i_{0,1}) \arrow[r, "R"] & \Omega^{\bullet-1}_{\mathrm{ch}}(A \sqcup A) \arrow[r] & 0 \\ 0 \arrow[r] & \hat{h}^\bullet_{\mathrm{fl}}(\mathrm{id}') \arrow[r] \arrow[u, "(\pi,\mathrm{id})^*"'] & \hat{h}^\bullet_0(\mathrm{id}') \arrow[r, "R"] \arrow[u, "(\pi,\mathrm{id})^*"'] & \Omega^{\bullet-1}_{\mathrm{ch}}(A \sqcup A) \arrow[r] \arrow[u] & 0. \end{tikzcd} \tag{4-34}$$

We start proving that the left arrow is an isomorphism. Note that the vertical maps of diagram (4-33) are homotopy equivalences, so they induce isomorphisms in the topological theory $h^\bullet$. Thus, using the long exact sequences associated to the horizontal maps and applying the five lemma, we see that $(\pi, \mathrm{id})^* \colon h^\bullet(\mathrm{id}') \to h^\bullet(i_{0,1})$ is an isomorphism too. The same holds about the de-Rham theory, hence, applying again the five lemma to the exact sequence (4-14), we conclude that $(\pi, \mathrm{id})^* \colon \hat{h}^\bullet_{\mathrm{fl}}(\mathrm{id}') \to \hat{h}^\bullet_{\mathrm{fl}}(i_{0,1})$ is an isomorphism. This implies that the central arrow of diagram (4-34) is an isomorphism too, again because of the five lemma. Thus, we get a unique class $\hat{\delta} \in \hat{h}^\bullet_0(\mathrm{id}')$ whose pull-back is $\hat{\gamma}$. By construction $R(\hat{\delta}) = (0, 0 \sqcup -R(\hat{\alpha}))$, thus, if we pull $\hat{\delta}$ back to $A \sqcup A$, it vanishes on the first component of $A \sqcup A$.

(S5) Let us consider the following morphism:

$$\begin{tikzcd} A \sqcup A \arrow[r, "\mathrm{id}'"] \arrow[d, "\rho''"'] & A \arrow[d, "\rho"] \\ X \sqcup A \arrow[r, "\rho'"] & X \end{tikzcd} \tag{4-35}$$

where ρ' acts as the identity on the first component of $X \sqcup A$ and as ρ on the second. Let us call $\hat{h}^\bullet_1(\mathrm{id}')$ the sub-group of $\hat{h}^\bullet_0(\mathrm{id}')$ formed by classes such that the second component of the curvature is vanishing on the first component of $A \sqcup A$. Similarly, let us call $\hat{h}^\bullet_1(\rho')$ the sub-group of $\hat{h}^\bullet_0(\rho')$ formed by classes such that the second component of the curvature is vanishing on the first component of $X \sqcup A$.

[6] For what we are saying in this paragraph, it would be enough to require that the first component of the curvature is the pull-back of a form on A.

We get the following morphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(\mathrm{id}') \arrow[r] & \hat{h}^{\bullet}_{1}(\mathrm{id}') \arrow[r, "R"] & \Omega^{\bullet-1}_{\mathrm{ch}}(A) \arrow[r] & 0 \\ 0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(\rho') \arrow[r] \arrow[u, "(\rho,\rho'')^*"] & \hat{h}^{\bullet}_{1}(\rho') \arrow[r, "R"] \arrow[u, "(\rho,\rho'')^*"] & \Omega^{\bullet-1}_{\mathrm{ch}}(A) \arrow[r] \arrow[u] & 0. \end{tikzcd} \tag{4-36}$$

The left arrow is an isomorphism. In fact, let us consider the mapping cones $C(\mathrm{id}')$ and $C(\rho'')$. The embeddings $CA \hookrightarrow C(\mathrm{id}')$ (A being the first component of $A \sqcup A$) and $CX \hookrightarrow C(\rho')$ are cofibrations, because their images are a deformation retract of a neighborhood. Thus, collapsing CA and CX to a point, we see that both $C(\mathrm{id}')$ and $C(\rho')$ are homotopically equivalent to the suspension $\Sigma(A_+)$, i.e., to the double cone of A with the two vertices identified. We get the following commutative diagram:

$$\begin{tikzcd} C(\mathrm{id}') \arrow[r, "\simeq"] \arrow[d, "(\rho,\rho'')"'] & C(\mathrm{id}')/CA \arrow[d, "\approx"] \\ C(\rho') \arrow[r, "\simeq"] & C(\rho')/CX \end{tikzcd} \tag{4-37}$$

where '$\simeq$' denotes a homotopy equivalence and '$\approx$' an homeomorphism. This implies that $(\rho,\rho'')^*\colon \tilde{h}^{\bullet}(C(\rho')) \to \tilde{h}^{\bullet}(C(\mathrm{id}'))$ is an isomorphism, being the composition of three isomorphisms, therefore $(\rho,\rho'')^*\colon h^{\bullet}(\rho') \to h^{\bullet}(\mathrm{id}')$ is an isomorphism too. The same holds for the de-Rham cohomology, thus, applying the five lemma to the exact sequence (4-14), we see that the left arrow of diagram (4-36) is an isomorphism. Again because of the five lemma, the central arrow of diagram (4-36) is an isomorphism too. Hence, we get a unique class $\hat{\varepsilon} \in \hat{h}^{\bullet}_{1}(\rho')$, whose pull-back is $\hat{\delta}$. By construction $R(\hat{\varepsilon}) = (0, 0 \sqcup -R(\hat{\alpha}))$, thus, if we pull $\hat{\delta}$ back to $X \sqcup A$, it vanishes on X.

(S6) Finally, let us consider the following morphism:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "\mathrm{id}"] \\ X \sqcup A \arrow[r, "\mathrm{id}\sqcup\rho"] & X. \end{tikzcd}$$

The pull-back of $\hat{\varepsilon}$ is a class $\hat{\mu} \in \hat{h}^{\bullet}(\rho)$ and we set $\mathrm{Bock}^{\bullet-1}(\hat{\alpha}) := \hat{\mu}$. It follows that $R(\hat{\mu}) = (0, -R(\hat{\alpha}))$.

This completes the construction of the sequence (4-30). By construction, we have that:

$$R \circ \mathrm{Bock}(\hat{\alpha}) = (0, -R(\hat{\alpha})). \tag{4-38}$$

The next lemma shows the behaviour of the Bockstein map on topologically trivial classes.

Lemma 4.1.14. The following formula holds:

$$\mathrm{Bock} \circ a(\eta) = a(0, \eta). \tag{4-39}$$

Proof. Let us set $\hat{\alpha} = a(\eta)$ is step (S1). It follows that $\hat{\beta} = -a(dt \wedge \pi_1^*\eta)$. In step (S3), setting $p_1 := \pi_1 \circ p$, we get:

$$\hat{\gamma} = -a(dt \wedge p_1^*\eta, 0) - a(t \cdot p_1^* d\eta, 0) = -a(d(t \cdot p_1^*\eta), 0) = a(0, 0 \sqcup \eta),$$

the last equality being due to the fact that $0 = a \circ d(t \cdot p_1^*\eta, 0) = a(d(t \cdot p_1^*\eta), 0 \sqcup \eta)$. Then is step (S4) $\hat{\delta} = a(0, 0 \sqcup \eta)$ and is step (S5) $\hat{\varepsilon} = a(0, 0 \sqcup \eta)$. Finally, in step (S6) we get $\hat{\mu} = a(0, \eta)$. □

Remark 4.1.15. Formulas (4-38) and (4-39) are coherent with the functoriality of the exact sequence with respect to (4-5), but a minus sign is necessary when a is acting on $\hat{h}^\bullet(A)$. In fact, the following diagram commutes:

$$\begin{tikzcd}
\Omega^{\bullet-1}(A) \arrow[r, "i"] \arrow[d, "-a"'] & \Omega^\bullet(\rho) \arrow[r, "\pi"] \arrow[d, "a"] & \Omega^\bullet(X) \arrow[d, "a"] \\
\hat{h}^\bullet(A) \arrow[r, "\mathrm{Bock}"] \arrow[d, "R"'] & \hat{h}^{\bullet+1}(\rho) \arrow[r, "\pi^*"] \arrow[d, "R"] & \hat{h}^{\bullet+1}(X) \arrow[d, "R"] \\
\Omega^\bullet(A) \arrow[r, "i"] & \Omega^{\bullet+1}(\rho) \arrow[r, "\pi"] & \Omega^\bullet(X).
\end{tikzcd}$$

Moreover, let us suppose that, in formula (4-39), $d\eta = 0$. Then $a(\eta)$ only depends on the de-Rham cohomology class $[\eta]$, hence we can write $a[\eta]$. The Bokstein map in the (topological) exact sequence of de-Rham cohomology is defined by $\mathrm{Bock}_{\mathrm{dR}}[\eta] = [0, \eta]$, coherently with (4-5), thus formula (4-39) becomes $\mathrm{Bock} \circ a[\eta] = a \circ \mathrm{Bock}_{\mathrm{dR}}[\eta]$, coherently with the functoriality of the Bockstein map.

If we consider the composition $\hat{h}^{\bullet-1}(X) \to \hat{h}^{\bullet-1}(A) \to \hat{h}^\bullet(\rho)$, in general it does not vanish. In fact, in (4-30) only the flat group $\hat{h}_{\mathrm{fl}}^{\bullet-1}(X)$ appears in this segment of the sequence. The next lemma shows the behaviour of the composition.

Lemma 4.1.16. The following formula holds:

$$\mathrm{Bock} \circ \rho^*(\hat{\beta}) = -a(R(\hat{\beta}), 0). \tag{4-40}$$

Proof. Let us consider the following morphism $\rho' := (\mathrm{id}_X, \rho) \colon \rho \to \mathrm{id}_X$:

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "\rho"'] & X \arrow[d, "\mathrm{id}_X"] \\
X \arrow[r, "\mathrm{id}_X"] & X.
\end{tikzcd}$$

We get the following diagram:

$$\begin{tikzcd}
\hat{h}^\bullet(X) \arrow[r, "\rho^*"] & \hat{h}^\bullet(A) \arrow[r, "\mathrm{Bock}"] & \hat{h}^\bullet(\rho) \\
\hat{h}^\bullet(X) \arrow[r, "\mathrm{id}"] \arrow[u, "\mathrm{id}"] & \hat{h}^\bullet(X) \arrow[r, "\mathrm{Bock}'"] \arrow[u, "\rho^*"'] & \hat{h}^\bullet(\mathrm{id}_X). \arrow[u, "\rho'^*"']
\end{tikzcd}$$

It follows that $\mathrm{Bock} \circ \rho^*(\hat{\beta}) = \rho'^* \circ \mathrm{Bock}'(\hat{\beta})$. Since $h^\bullet(\mathrm{id}_X) = 0$, because of the sequence (4-11) we have that $\hat{h}^\bullet(\mathrm{id}_X) \simeq \Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$, hence every element of $\hat{h}^\bullet(\mathrm{id}_X)$ is of the form $a(\omega, \eta)$. Moreover, $(\omega, \eta) - d(\eta, 0) = (\omega, \eta) - (d\eta, \eta) = (\omega - d\eta, 0)$, thus every element of $\hat{h}^\bullet(\mathrm{id}_X)$ is of the form $a(\omega, 0)$, therefore:

$$\begin{aligned}
R \circ \mathrm{Bock}'(\hat{\beta}) \; &= \; R \circ a(\omega, 0) = (d\omega, \omega) \\
R \circ \mathrm{Bock}'(\hat{\beta}) \; &\overset{(4\text{-}38)}{=} \; (0, -R(\hat{\beta})).
\end{aligned}$$

Comparing the second components we get $\omega = -R(\hat{\beta})$, hence $\mathrm{Bock}'(\hat{\beta}) = a(-R(\hat{\beta}), 0)$. It follows that $\mathrm{Bock} \circ \rho^*(\hat{\beta}) = \rho'^* \circ \mathrm{Bock}'(\hat{\beta}) = a(-R(\hat{\beta}), 0)$. □

Remark 4.1.17. Let us suppose that, in formula (4-40), $\hat{\beta} = a(\theta)$. Then we get $\mathrm{Bock} \circ a(\rho^*\theta) = -a(d\theta, 0)$. Because of formula (4-39) we have $\mathrm{Bock} \circ a(\rho^*\theta) = a(0, \rho^*\theta)$. The two results are coherent. In fact, $(d\theta, 0) + (0, \rho^*\theta) = (d\theta, \rho^*\theta) = d(\theta, 0)$, hence, since a vanishes on exact forms, we have that $a(d\theta, 0) + a(0, \rho^*\theta) = 0$.

The Bockstein map of (4-30) has been defined. The one of (4-29) coincides with the one of (4-30), applied to flat classes; it follows from formula (4-38) that the image of a flat class is flat. Finally, in the comments after the sequence (4-31), we have already shown how to define the corresponding Bockstein maps. It remains to prove the exactness of each sequence.

4.1.6 Exactness

We start from the exactness of (4-29).

Exactness in $\hat{h}^n_{\mathrm{fl}}(X)$. The fact that $\rho^* \circ \pi^* = 0$ is an easy consequence of axiom (A4) in definition 4.1.1. Let us consider a class $\hat{\alpha} \in \mathrm{Ker}(\rho^*)$. We set $\alpha = I(\hat{\alpha})$. Since $\rho^*\alpha = 0$, because of the exactness of the topological sequence, there exists a class $\beta \in h^\bullet(\rho)$ such that $\pi^*\beta = \alpha$. Let $\hat{\beta}' \in \hat{h}^\bullet(\rho)$ be any differential class such that $I(\hat{\beta}') = \beta$. It follows that $\pi^*\hat{\beta}' = \hat{\alpha} + a(\theta)$, being $\theta \in \Omega^{\bullet-1}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$, thus $R(\pi^*\hat{\beta}') = d\theta$, therefore there exists a *closed* form $\eta \in \Omega^{\bullet-1}(A; \mathfrak{h}^\bullet_{\mathbb{R}})$ such that $R(\hat{\beta}') = (d\theta, \rho^*\theta + \eta)$. Let us prove that the de-Rham class $[\eta]$ belongs to the image of the Chern character. In fact, we have that $\rho^*\pi^*\hat{\beta}' = \rho^*\hat{\alpha} + \rho^*a(\theta) = a(\rho^*\theta)$ and, by axiom (A4), $\rho^*\pi^*\hat{\beta}' = a(\mathrm{cov}(\hat{\beta}')) = a(\rho^*\theta + \eta) = a(\rho^*\theta) + a(\eta)$. It follows that $a(\eta) = 0$, hence $\eta \in \Omega^{\bullet-1}_{\mathrm{ch}}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$. This implies that there exists $\hat{\gamma} \in \hat{h}^{n-1}(A)$ such that $R(\hat{\gamma}) = \eta$, thus we set $\hat{\beta} := \hat{\beta}' - a(\theta, 0) + \mathrm{Bock}(\hat{\gamma})$. Because of the following remark 4.1.18, we have that $\pi^* \circ \mathrm{Bock} = 0$, therefore $\pi^*\hat{\beta} = (\hat{\alpha} + a(\theta)) - a(\theta) - 0 = \hat{\alpha}$ and $R(\hat{\beta}) = (d\theta, \rho^*\theta + \eta) - (d\theta, \rho^*\theta) - (0, \eta) = (0, 0)$.

Exactness in $\hat{h}^n_{\mathrm{fl}}(A)$. If $\hat{\beta} \in \hat{h}^\bullet_{\mathrm{fl}}(X)$, by formula (4-40) we have that $\mathrm{Bock} \circ \rho^*(\hat{\beta}) = 0$, thus $\mathrm{Bock} \circ \rho^* = 0$. Let us consider a flat class $\hat{\alpha} \in \mathrm{Ker}(\mathrm{Bock})$. Setting $\alpha := I(\hat{\alpha})$, we have that $\mathrm{Bock}(\alpha) = 0$, thus there exists $\beta \in h^\bullet(X)$ such that $\alpha = \rho^*\beta$. If $\hat{\beta}' \in \hat{h}^\bullet(X)$ is any differential refinement of β, there exists $\theta \in \Omega^{\bullet-1}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$ such that $\rho^*\hat{\beta}' = \hat{\alpha} + a(\theta)$. Applying formula (4-39) we get $\mathrm{Bock} \circ \rho^*(\hat{\beta}') = \mathrm{Bock} \circ a(\theta) = a(0, \theta)$ and applying formula (4-40) we get $\mathrm{Bock} \circ \rho^*(\hat{\beta}') = -a(R(\hat{\beta}', 0))$, thus $a(R(\hat{\beta}'), \theta) = 0$. It follows that $(R(\hat{\beta}'), \theta)$ represents a class belonging to the image of the Chern character, hence there exists a class $\hat{\gamma} \in \hat{h}^\bullet(\rho)$ such that $R(\hat{\gamma}) = (R(\hat{\beta}'), \theta)$. We set $\hat{\beta} := \hat{\beta}' - \pi^*\hat{\gamma}$. We get that $R(\hat{\beta}) = R(\hat{\beta}') - R(\hat{\beta}') = 0$ and $\rho^*\hat{\beta} = \hat{\alpha} + a(\theta) - a(\theta) = \hat{\alpha}$.

Exactness in $\hat{h}^n_{\mathrm{fl}}(\rho)$. It follows from the construction of the Bockstein map that $\pi^* \circ \mathrm{Bock} = 0$. In fact, if $\hat{\mu} = \mathrm{Bock}(\hat{\alpha})$, by the step (S6) we have that $\hat{\mu} = (\mathrm{id}, \mathrm{id})^*\hat{\varepsilon}$. The pull-back $\pi^*\hat{\mu}$ coincides with the pull-back of $\hat{\varepsilon}$ via the following composition:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\
\emptyset \arrow[r] \arrow[d] & X \sqcup A \arrow[d, "\mathrm{id} \sqcup \rho"] \\
A \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "\mathrm{id}"] \\
X \sqcup A \arrow[r, "\mathrm{id} \sqcup \rho"] & X.
\end{tikzcd}$$

The last map provides the pull-back from $\hat{\varepsilon}$ to $\hat{\mu}$ and the composition of the first two coincides with π. The composition of the last two morphism coincides with the following:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \sqcup A \arrow[d, "\mathrm{id} \sqcup \rho"] \\
X \sqcup A \arrow[r, "\mathrm{id} \sqcup \rho"] & X.
\end{tikzcd}$$

By axiom (A4), the pull-back of $\hat{\varepsilon}$ is equal to $a(\mathrm{cov}(\hat{\varepsilon}))$. Since we start from a flat class $\hat{\alpha}$, we have that $\mathrm{cov}(\hat{\varepsilon}) = 0$, thus the pull-back vanishes.

Remark 4.1.18. Even if $\hat{\alpha}$ is not flat, since $\hat{\varepsilon} \in \hat{h}^\bullet_1(\mathrm{id} \sqcup \rho)$ by construction, it follows that $\mathrm{cov}(\hat{\varepsilon})$ vanishes on X, thus the pull-back to $\emptyset \to X$ is 0. This proves that $\pi^* \circ \mathrm{Bock} = 0$ in (4-29) and in (4-30).

Let us consider a flat class $\hat{\mu} \in \mathrm{Ker}(\pi^*)$. It is enough to prove that there exists a flat class $\hat{\varepsilon} \in \hat{h}^\bullet_{\mathrm{fl}}(\mathrm{id} \sqcup \rho)$ such that $(\mathrm{id}, \mathrm{id})^*\hat{\varepsilon} = \hat{\mu}$. In fact, dealing with flat classes, all of the steps (S1)-(S5) consist in the application of an isomorphism, thus, starting from $\hat{\varepsilon}$, we get a class $\hat{\alpha} \in \hat{h}^{\bullet-1}_{\mathrm{fl}}(A)$ such that $\mathrm{Bock}(\hat{\alpha}) = \hat{\mu}$. Applying the steps analogous to (S1)-(S5) to the topological class α, we get the Bockstein map of the topological exact sequence, hence, in particular, we get a class $\varepsilon \in h^\bullet(\mathrm{id} \sqcup \rho)$ such that $(\mathrm{id}, \mathrm{id})^*\varepsilon = \mu := I(\hat{\mu})$. Let $\hat{\varepsilon}'' \in \hat{h}^\bullet(\mathrm{id} \sqcup \rho)$ be any differential class such that $I(\hat{\varepsilon}'') = \varepsilon$. It follows that there exists a relative form $(\theta, \eta) \in \Omega^{\bullet-1}(\rho)$ such that $(\mathrm{id}, \mathrm{id})^*\hat{\varepsilon}'' = \hat{\mu} + a(\theta, \eta)$. We set $\hat{\varepsilon}' := \hat{\varepsilon}'' - a(\theta, 0 \sqcup \eta)$, so that $(\mathrm{id}, \mathrm{id})^*\hat{\varepsilon}' = (\hat{\mu} + a(\theta, \eta)) - a(\theta, \eta) = \hat{\mu}$. Now we have to reach a flat class with the same pull-back of $\hat{\varepsilon}'$. We have that $(\mathrm{id}, \mathrm{id})^* R(\hat{\varepsilon}') = R(\hat{\mu}) = 0$, thus there exists a form $\chi \in \Omega^{\bullet-1}(X)$ such that $R(\hat{\varepsilon}') = (0, \chi \sqcup 0)$. Let us show that the de-Rham class $[\chi]$ belongs to the image of the Chern character. We consider the pull-back of $\hat{\varepsilon}'$ via the following composition:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \sqcup A \arrow[d, "\mathrm{id}\sqcup\rho"] \\
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "\mathrm{id}"] \\
X \sqcup A \arrow[r, "\mathrm{id}\sqcup\rho"] & X.
\end{tikzcd}$$

The pull-back of $\hat{\varepsilon}'$ from the last to the third line is $\hat{\mu}$ and the pull-back to the second line is $\pi^*(\hat{\mu})$, that vanishes by hypothesis. Thus, the overall pull-back is 0. On the other side, the pull-back from the forth to the first line, by the axiom (A4), is $a(\mathrm{cov}(\hat{\varepsilon}')) = a(\chi) \sqcup 0$, thus $a(\chi) = 0$. This shows that $\chi \in \Omega^{\bullet-1}_{\mathrm{ch}}(X)$, thus there exists a class $\hat{\gamma} \in \hat{h}^{\bullet-1}(X)$ such that $R(\hat{\gamma}) = \chi$. Considering the sequence (4-30) associated to the last line of the previous diagram, we get the class $\mathrm{Bock}(\hat{\gamma} \sqcup 0) \in \hat{h}^\bullet(\mathrm{id} \sqcup \rho)$ and we set $\hat{\varepsilon} := \hat{\varepsilon}' + \mathrm{Bock}(\hat{\gamma} \sqcup 0)$. It follows that $R(\hat{\varepsilon}) = (0, \chi \sqcup 0) + (0, -R(\hat{\gamma} \sqcup 0)) = 0$. Moreover, because of the naturality of the Bockstein map, $(\mathrm{id}, \mathrm{id})^*\mathrm{Bock}(\hat{\gamma} \sqcup 0) = \mathrm{Bock}(\mathrm{id}^*(\hat{\gamma} \sqcup 0)) = \mathrm{Bock}(0) = 0$, thus $(\mathrm{id}, \mathrm{id})^*\hat{\varepsilon} = \hat{\mu} - 0 = \hat{\mu}$.

About (4-30), we must prove the exactness from $\hat{h}^{\bullet-1}_{\mathrm{fl}}(X)$ to $h^\bullet(A)$. Actually, the exactness in $\hat{h}^{\bullet-1}_{\mathrm{fl}}(X)$ easily follows from the embedding $\hat{h}^{\bullet-1}_{\mathrm{fl}}(A) \hookrightarrow \hat{h}^{\bullet-1}(A)$ and the exactness of (4-29). Similarly, the exactness in $h^\bullet(A)$ easily follows from the surjectivity of $I\colon \hat{h}^\bullet(X) \to h^\bullet(X)$ and the exactness of the sequence associated to $h^\bullet$. Thus, there are three meaningful positions left.

Exactness in $\hat{h}^{n-1}(A)$. The composition $\mathrm{Bock} \circ \rho^*$, starting from $\hat{h}^\bullet_{\mathrm{fl}}(X)$, coincides with the one of the flat sequence, hence it vanishes. Given $\hat{\alpha} \in \hat{h}^{n-1}(A)$, if $\mathrm{Bock}^{n-1}(\hat{\alpha}) = 0$ then $R(\hat{\alpha}) = 0$, because of formula (4-38). Therefore the kernel of Bock^{n-1} is contained in the flat part $\hat{h}^{n-1}_{\mathrm{fl}}(A)$, hence the exactness follows from the one of (4-29).

Exactness in $\hat{h}^n(\rho)$. We have already proven that $\pi^* \circ \mathrm{Bock} = 0$ in remark 4.1.18. Let us consider $\hat{\mu} \in \hat{h}^n(\rho)$ such that $\pi^*\hat{\mu} = 0$. It follows that $R(\hat{\mu}) = (0, \eta)$. Moreover, $0 = \rho^*\pi^*\hat{\mu} = a(\eta)$, thus η represents a class belonging to the image the Chern character. It follows that there exists a class $\hat{\alpha} \in \hat{h}^{n-1}(A)$ such that $R(\hat{\alpha}) = -\eta$. Then, because of formula (4-38), $\mathrm{Bock}(\hat{\alpha}) = \hat{\mu} + \hat{\mu}'$, with $\hat{\mu}' \in \hat{h}^n_{\mathrm{fl}}(\rho)$. Since $0 = \pi^*\mathrm{Bock}(\hat{\alpha}) = \pi^*\hat{\mu}'$, because of the exactness of (4-29) there exists a class $\hat{\alpha}' \in \hat{h}^{n-1}_{\mathrm{fl}}(A)$ such that $\mathrm{Bock}(\hat{\alpha}') = \hat{\mu}'$. It follows that $\mathrm{Bock}(\hat{\alpha} - \hat{\alpha}') = \hat{\mu}$.

Exactness in $\hat{h}^n(X)$. The pull-back to A of a class in $\hat{h}^n(\rho)$ is topologically trivial because of the long exact sequence of $h^\bullet$ (or because of axiom (A4)). Let us fix a class $\hat{\nu} \in \hat{h}^n(X)$, such that $\rho^* I(\hat{\nu}) = 0$. It follows that $\rho^*\hat{\nu} = a(\eta)$ for a suitable $\eta \in \Omega^{\bullet-1}(A)$. We set $\omega := R(\hat{\nu})$. Then $\rho^*\omega = d\eta$, that is equivalent to $d(\omega, \eta) = 0$. Moreover:

$$\begin{aligned} &\mathrm{Bock} \circ \rho^*(\hat{\nu}) \overset{\text{(4-40)}}{=} -a(\omega, 0) \\ &\mathrm{Bock} \circ \rho^*(\hat{\nu}) = \mathrm{Bock} \circ a(\eta) \overset{\text{(4-39)}}{=} a(0, \eta). \end{aligned}$$

It follows that $a(\omega, \eta) = 0$, thus we can fix a class $\hat{\alpha}' \in \hat{h}^n(\rho)$ such that $R(\hat{\alpha}') = (\omega, \eta)$. It follows that $\pi^*(\hat{\alpha}') = \hat{\nu} + \hat{\nu}'$, with $\hat{\nu}'$ flat. Then:

$$\begin{aligned} &\rho^*\pi^*(\hat{\alpha}') = a(\eta) + \rho^*\hat{\nu}' \\ &\rho^*\pi^*(\hat{\alpha}') \overset{\text{(A4)}}{=} a \circ \mathrm{cov}(\hat{\alpha}') = a(\eta). \end{aligned}$$

Thus $\rho^*\hat{\nu}' = 0$ and, by the exactness of the flat sequence, we can find $\hat{\alpha}''$ such that $\pi^*\hat{\alpha}'' = \hat{\nu}'$. Setting $\hat{\alpha} := \hat{\alpha}' - \hat{\alpha}''$, we get that $\pi^*\hat{\alpha} = \hat{\nu}$.

About (4-31), we must prove the exactness from $\hat{h}^{\bullet-1}_{\mathrm{fl}}(A)$ to $h^{\bullet+1}(\rho)$. Actually, the exactness in $\hat{h}^{\bullet-1}_{\mathrm{fl}}(A)$ easily follows from the embedding $\hat{h}^\bullet_{\mathrm{fl}}(\rho) \hookrightarrow \hat{h}^\bullet_{\mathrm{par}}(\rho)$ and the exactness of (4-29). Similarly, the exactness in $h^{\bullet+1}(\rho)$ easily follows from the surjective map $I \colon \hat{h}^\bullet(A) \to h^\bullet(A)$ and the exactness of the sequence associated to $h^\bullet$. Thus, there are three meaningful positions left.

Exactness in $\hat{h}^\bullet_{\mathrm{par}}(\rho)$. If a class belongs to the image of the Bockstein map, it follows from the exact sequence of the flat theory that its pull-back to X vanishes. Vice-versa, let us consider $\hat{\mu} \in \hat{h}^\bullet(\rho)_{\mathrm{par}}$ such that $\pi^*\hat{\mu} = 0$. It follows that $R(\hat{\mu}) = (0, 0)$, thus the class is flat. By the sequence of the flat theory, we can find a pre-image via the Bockstein morphism.

Exactness in $\hat{h}^n(X)$. The pull-back to A of a class in $\hat{h}^\bullet_{\mathrm{par}}(\rho)$ vanishes because of the axiom (A4). Vice-versa, let us fix a class $\hat{\nu} \in \hat{h}^\bullet(X)$ such that $\rho^*\hat{\nu} = 0$. Since, in particular, $\rho^* I(\hat{\nu}) = 0$, by the exactness of (4-30) there exists $\hat{\alpha}' \in \hat{h}^\bullet(\rho)$ such that $\pi^*\hat{\alpha}' = \hat{\nu}$. We set $(\omega, \eta) := R(\hat{\alpha}')$, thus $\omega = R(\hat{\nu})$. We have that $0 = \rho^*\pi^*\hat{\alpha}' = a(\eta)$, thus there exists a class $\hat{\beta} \in h^{\bullet-1}(A)$ such that $R(\hat{\beta}) = \eta$. We set $\hat{\alpha} := \hat{\alpha}' + \mathrm{Bock}(\hat{\beta})$. Then, by formula (4-38), $R(\hat{\alpha}) = (\omega, \eta) + (0, -\eta) = (\omega, 0)$, thus $\hat{\alpha}$ is parallel, and, by the exactness of (4-30), $\pi^*\hat{\alpha} = \hat{\nu}$.

Exactness in $\hat{h}^\bullet(A)$. If a class $\hat{\alpha} \in \hat{h}^\bullet(A)$ is the pull-back of a class in X, it follows from the exact sequence of $h^\bullet$ that it lies in the kernel of the (topological) Bockstein map. Vice-versa, if it belongs to the kernel, by the exact sequence of $h^\bullet$ we can find a class $\hat{\beta} \in \hat{h}^n(X)$ such that $\rho^* I(\hat{\beta}) = I(\hat{\alpha})$, hence $\rho^*\hat{\beta} = \hat{\alpha} + a(\theta)$ for a suitable form $\theta \in h^{\bullet-1}(A)$. If ρ is a closed embedding, there exists a form $\xi \in h^{\bullet-1}(X)$ such that $\theta = \rho^*\xi$, thus $\rho^*(\hat{\beta} - a(\xi)) = \hat{\alpha}$.

4.1.7 Existence of the relative extension

Given a cohomology theory $h^\bullet$, there exists a relative differential extension, which is multiplicative if $h^\bullet$ is. This can be shown using the Hopkins-Singer model [33]. We briefly recall the construction and verify that it satisfies the axioms.

Definition 4.1.19. If X is a smooth manifold, Y a topological space, $V^\bullet$ a graded real vector space and $\kappa_n \in C^n(Y; V^\bullet)$ a real singular cocycle, a *differential function* from X to (Y, κ_n) is a triple (f, h, ω) such that:

- $f\colon X \to Y$ is a continuous function;
- $h \in C^{n-1}_{sm}(X; V^\bullet)$ ('sm' means smooth);
- $\omega \in \Omega^n_{\mathrm{cl}}(X; V^\bullet)$

satisfying, for $\chi\colon \Omega^\bullet(X; V^\bullet) \to C^\bullet(X; V^\bullet)$ the natural homomorphism:

$$\delta^{n-1} h = \chi^n(\omega) - f^* \kappa_n. \tag{4-41}$$

Moreover, a *homotopy between two differential functions* (f_0, h_0, ω) and (f_1, h_1, ω) is a differential function $(F, H, \pi^*\omega)\colon X \times I \to (Y, \kappa_n)$, such that F is a homotopy between f_0 and f_1, $H|_{X \times \{i\}} = h_i$ for $i = 0, 1$, and $\pi\colon X \times I \to X$ is the natural projection.

We represent a fixed cohomology theory $h^\bullet$ via an Ω-spectrum $(E_n, e_n, \varepsilon_n)$, where e_n is the marked point of E_n and $\varepsilon_n\colon (\Sigma E_n, \Sigma e_n) \to (E_{n+1}, e_{n+1})$ is the structure map, whose adjoint $\tilde{\varepsilon}_n\colon E_n \to \Omega_{e_{n+1}} E_{n+1}$ is a homeomorphism. We also fix real singular cocycles $\iota_n \in C^n(E_n, e_n, \mathfrak{h}^\bullet_{\mathbb{R}})$ representing the Chern character of $h^\bullet$, such that $\iota_{n-1} = \int_{S^1} \varepsilon_n^* \iota_n$ [39].

Definition 4.1.20. Given a differential function $(f, h, \omega)\colon X \to (E_n, \iota_n)$, a *strong topological trivialization* of (f, h, ω) is a homotopy $(F, H, \pi^*\omega)\colon X \times I \to (E_n, \iota_n)$ between (f, h, ω) and a function of the form $(c_{e_n}, \chi(\eta), d\eta)$, where c_{e_n} is the constant function with value e_n and $\eta \in \Omega^{n-1}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$.

Let us consider a smooth function between manifolds $\rho\colon A \to X$. We define the cylinder $\mathrm{Cyl}(\rho) := X \sqcup (A \times I)/\sim$, where $(a, 0) \sim \rho(a)$. We consider the following natural maps:

- $\iota_{\mathrm{Cyl}(\rho)}\colon \mathrm{Cyl}(\rho) \to X \times I$, $x \mapsto (x, 0)$, $[(a, t)] \mapsto (\rho(a), t)$;
- $\iota_{\mathrm{Cyl}(A)}\colon \mathrm{Cyl}(A) \to \mathrm{Cyl}(\rho)$, $(a, t) \mapsto [(a, t)]$;
- $\iota_A\colon A \to \mathrm{Cyl}(\rho)$, $a \mapsto [(a, 0)]$;
- $\iota'_A\colon A \to \mathrm{Cyl}(\rho)$, $a \mapsto (a, 1)$;
- $\pi_A\colon I \times A \to A$, $(t, a) \mapsto a$.

In general $\mathrm{Cyl}(\rho)$ is not a manifold, nevertheless we will deal with differential functions $(f, h, \omega)\colon \mathrm{Cyl}(\rho) \to (E_n, \iota_n)$, defined in the following way:

- $f\colon \mathrm{Cyl}(\rho) \to E_n$ is a continuous function.
- $\omega \in \Omega^n_{cl}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$, and it defines a smooth cocycle $\chi^n(\omega)$ on $\mathrm{Cyl}(\rho)$ as follows. Let us consider the pull-back $\pi_X^*\omega$ on $X \times I$. A simplex $\sigma\colon \Delta^n \to \mathrm{Cyl}(\rho)$ is defined to be smooth if and only if the composition $\iota_{\mathrm{Cyl}(\rho)} \circ \sigma\colon \Delta^n \to X \times I$ is. The smooth cochain $\chi^n(\omega)$ on $\mathrm{Cyl}(\rho)$ is defined by $\chi^n(\omega)(\sigma) := \chi^n(\pi_X^*\omega)(\iota_{\mathrm{Cyl}(\rho)} \circ \sigma)$.
- $h \in C^{n-1}_{sm}(\mathrm{Cyl}(\rho); \mathfrak{h}^\bullet_{\mathbb{R}})$ and it satisfies $\delta^{n-1}h = \chi^n(\omega) - f^*\iota_n$.

Definition 4.1.21. The group $\hat{h}^n(\rho)$ contains the equivalence classes $[(f, h, \omega, \eta)]$, where:

- $(f, h, \omega)\colon \mathrm{Cyl}(\rho) \to (E_n, \iota_n)$ is a differential function such that $\iota^*_{\mathrm{Cyl}(A)}(f, h, \omega)$ is a strong topological trivialization of $\iota_A^*(f, h, \omega)$ verifying the relation $(\iota'_A)^*(f, h, \omega) = (c_{e_n}, \chi(\eta), d\eta)$;
- (f, h, ω, η) is equivalent to (g, k, ω, η) if the differential functions (f, h, ω) and (g, k, ω) are homotopic relatively to the upper base of the cylinder. This means that a homotopy $(F, H, \pi^*\omega)\colon \mathrm{Cyl}(\rho_I) \to (E_n, \iota_n)$ between the two functions[7] is required to satisfy $(\iota'_A)^*_I(F, H, \pi^*\omega) = \pi_A^*(\iota'_A)^*(f, h, \omega)$.

We set:

$$I[(f, h, \omega)] := [f] \qquad\qquad R[(f, h, \omega, \eta)] := (\omega, \eta), \tag{4-42}$$

being $[f] \in [(\mathrm{Cyl}(\rho), A \times \{1\}), (E_n, e_n)] \simeq h^n(\rho)$. Moreover, we define the map a in the following way. Given a form $\omega \in \Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}})$, we set $\tilde{\omega} := \pi_A^*\rho^*\omega \in \Omega^n(\mathrm{Cyl}(A); \mathfrak{h}^\bullet_{\mathbb{R}})$ and, given a form $\eta \in \Omega^{n-1}(A; \mathfrak{h}^\bullet_{\mathbb{R}})$, we set $\tilde{\eta} := \pi_A^*\eta \in \Omega^{n-1}(\mathrm{Cyl}(A); \mathfrak{h}^\bullet_{\mathbb{R}})$. We define the smooth singular cochain $\chi^n(\omega, \eta) \in C^n_{sm}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$ as follows. We fix a real number $\varepsilon \in (0, 1)$ and we take a smooth non-descreasing function $\theta\colon I \to I$ such that $\theta(t) = 0$ for $t \leq \varepsilon$ and $\theta(1) = 1$. We set the open cover $\{U, W\}$ of $\mathrm{Cyl}(\rho)$ defined by $U = A \times (\frac{\varepsilon}{3}, 1]$ and $W = A \times [0, \frac{\varepsilon}{2}) \sqcup_\rho X$. For each smooth chain $\sigma\colon \Delta^n \to \mathrm{Cyl}(\rho)$, we take the iterated barycentric subdivision so that the image of each sub-chain is contained in U or in W; then, for each small chain σ', we set

$$\chi^n(\omega, \eta)(\sigma') = \begin{cases} \chi^n\big(\tilde{\omega} - d(\theta(t) \cdot \tilde{\eta})\big)(\sigma') & \text{if } \sigma' \subset U \\ \chi^n(\omega)(\pi_X \circ \sigma') & \text{if } \sigma' \subset W \end{cases}$$

where $\pi_X\colon W \to X$ is the natural projection defined by $[a, t] \mapsto \rho(a)$ and $[x] \mapsto x$. Note that the morphism is well defined for $\sigma' \subset U \cap W$, since $\theta(t) = 0$ for $t \leq \varepsilon$. Finally, we define

$$\begin{aligned} a\colon \Omega^{n-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) &\to \hat{h}^n(\rho) \\ [(\omega, \eta)] &\mapsto [(c_{e_n}, \chi^{n-1}(\omega, \eta), d\omega, \rho^*\omega - d\eta)]. \end{aligned} \tag{4-43}$$

[7] We defined ρ_I in (4-15). Since I is (locally) compact, $I \times \mathrm{Cyl}(\rho)$ is canonically homeomorphic to $\mathrm{Cyl}(\rho_I)$. We will apply this homeomorphism when necessary, without stating it explicitly.

The cochain $\chi^{n-1}(\omega,\eta)$ depends on the choice of the function θ, but the equivalence class $[(c_{e_n}, \chi^{n-1}(\omega,\eta), d\omega, \rho^*\omega - d\eta)]$ does not, since two different functions θ lead to homotopic representatives.

We remark that, given two maps $\rho\colon A \to X$ and $\eta\colon B \to Y$ and a morphism $(\varphi,\psi)\colon \rho \to \eta$, there is a natural induced map $(\varphi,\psi)\colon \mathrm{Cyl}(\rho) \to \mathrm{Cyl}(\eta)$, $x \mapsto \varphi(x)$, $[(a,t)] \mapsto [(\psi(a),t)]$. We define the pull-back $(\varphi,\psi)^*[(f,h,\omega,\eta)] := [(f\circ\varphi, \varphi^*h, \varphi^*\omega, \psi^*\eta)]$.

Let us verify that axioms (A1)–(A4) of definition 4.1.1 hold. The first one is a direct consequence of the definitions (4-43) and (4-42). About axiom (A4), we have that

$$\rho^*\pi^*[(f,h,\omega,\eta)] = \rho^*[(f|_X, h|_X, \omega)] = \iota_A^*[(f,h,\omega)] \overset{(*)}{=} a(\eta) = a \circ \mathrm{cov}([(f,h,\omega,\eta)]),$$

the equality $(*)$ being due to the fact that, by definition, $\iota^*_{\mathrm{Cyl}(A)}(f,h,\omega)$ is a homotopy between $\iota_A^*(f,h,\omega)$ and $a(\eta) = (c_{e_n}, \chi(\eta), d\eta)$. In order to verify (A2), we observe that a representative (f,h,ω,η) of an element of $\hat{h}^n(\rho)$, as defined in 4.1.21, can be described in the following equivalent way:

- $f\colon (\mathrm{Cyl}(\rho), A\times\{1\}) \to (E_n, e_n)$ is a map of pairs;
- $h \in C^{n-1}(\mathrm{Cyl}(\rho); \mathfrak{h}^\bullet_{\mathbb{R}})$;
- $(\omega,\eta) \in \Omega^n_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$;
- $\delta^{n-1}(h,0) = (\chi^n(\omega), \chi^{n-1}(\eta)) - (f^*\iota_n, 0)$, the boundary δ being the one of the mapping cone complex $C^\bullet(\mathrm{Cyl}(\rho)) \oplus C^{\bullet-1}(A)$ associated to the embedding of the upper base $\iota'_A\colon A \to \mathrm{Cyl}(\rho)$.

The condition $\delta^{n-1}(h,0) = (\chi^n(\omega), \chi^{n-1}(\eta)) - (f^*\iota_n, 0)$ immediately implies that $[(\omega,\eta)]_{\mathrm{dR}} = [f^*\iota_n] = \mathrm{ch}[f]$, where f is a map of pairs. Finally, in order to show that axiom (A3) holds, the proof is similar to [39, Theorems 2.4 and 2.5], applied to the pair $(\mathrm{Cyl}(\rho), A\times\{1\})$.

This differential extension has a natural S^1-integration, defined integrating each component of the differntial function (f,h,ω,η) as in [39, Chapter 3], and, if $h^\bullet$ is mutiplicative, then $\hat{h}^\bullet$ is multiplicative too, the product being defined as in [39, Chapter 4].

4.1.8 Uniqueness

We are going to show that the uniqueness result of [10] holds even in the relative case. In particular, we extend to the relative case the construction of the morphism between any two differential extensions of the same cohomology theory and we prove that it induces a morphism between the corresponding long exact sequences. It will follow from the five lemma that the two relative extensions are isomorphic.

We use the results of [10, Section 2] about approximation of spectra through manifolds, assuming the same hypotheses therein. Let $h^\bullet$ be a cohomology theory and consider a spectrum $\{E_n, e_n\}_{n\in\mathbb{Z}}$ representing it. For a fixed $n \in \mathbb{Z}$ we take a sequence of pointed manifolds $\{\mathcal{E}_i, a_i\}_{i\in\mathbb{Z}}$ and two sequences of maps

$$x_i\colon (\mathcal{E}_i, a_i) \to (E_n, e_n) \qquad\qquad k_i\colon (\mathcal{E}_i, a_i) \to (\mathcal{E}_{i+1}, a_{i+1})$$

such that $x_i = x_{i+1} \circ k_i$. Moreover, for a given class $u \in h^\bullet(E_n, e_n)$, we fix a family of closed forms $\omega_i \in \Omega^\bullet(\mathcal{E}_i, a_i; \mathfrak{h}^\bullet_\mathbb{R})$ such that $\omega_i = k_i^*\omega_{i+1}$ and, in the reduced cohomology with marked point a_i, we have $\mathrm{dR}(\omega_i) = \mathrm{ch}(x_i^* u)$. Finally, we fix a family of differential classes $\hat{u}_i \in \hat{h}^\bullet(\mathcal{E}_i, a_i)$ such that $I(\hat{u}_i) = x_i^*(u)$ and $R(\hat{u}_i) = \omega_i$.

Let $(\hat{h}^\bullet, I, R, a)$ and $(\hat{h}'^\bullet, I', R', a')$ be two differential extensions of $h^\bullet$; for any $\hat{v} \in \hat{h}^n(\rho)$, with $\rho\colon A \to X$, there exists a morphism in the category $\mathcal{C}_2$

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & X \arrow[d, "f"] \\ e_n \arrow[r, hook, "i_{e_n}"] & E_n \end{tikzcd}$$

such that $I(\hat{v})$ is represented by the homotopy class $[f] \in [\rho, i_{e_n}]$. It follows that $I(\hat{v}) = f^*(u)$, where $u \in h^n(i_{e_n})$ is the tautological class represented by the identity map of (E_n, e_n). By the approximation lemmas, there exist a based manifold $(\mathcal{E}_i, a_i)$ and a map

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & X \arrow[d, "f_i"] \\ a_i \arrow[r, hook, "i_{a_i}"] & \mathcal{E}_i \end{tikzcd}$$

such that $f = x_i \circ f_i$. Note that $I(\hat{v}) = f^*(u) = f_i^* x_i^*(u) = f_i^*(I(\hat{u}_i)) = I(f_i^*\hat{u}_i)$, hence we have

$$\hat{v} = f_i^*\hat{u} + a(\zeta, \nu)$$

for a unique $(\zeta, \nu) \in \Omega^{n-1}(\rho; \mathfrak{h}^\bullet_\mathbb{R})/\mathrm{Im}(\mathrm{ch})$. Repeating the same construction for the extension $\hat{h}'^\bullet$, we define:

$$\begin{aligned} \Phi\colon \hat{h}^\bullet(\rho) &\to \hat{h}'^\bullet(\rho) \\ f_i^*\hat{u} + a(\zeta, \nu) &\mapsto f_i^*\hat{u}'_i + a'(\zeta, \nu). \end{aligned}$$

In order to show that Φ is well defined, we must verify that it is independent of the choice of the functions f_i. Fix $\hat{v} \in \hat{h}^n(\rho)$. As in the absolute case, we may reduce the problem to the case of two homotopic functions $f_i, \tilde{f}_i\colon \rho \to (\mathcal{E}_i, a_i)$ such that $I(\hat{v}) = f_i^* x_i^*(u) = \tilde{f}_i^* x_i^*(u)$; take the homotopy to be $F\colon \rho_I \to (\mathcal{E}_i, a_i)$. To each f_j and $\tilde{f}_j$ associate as above the forms (ζ, ν) and $(\tilde{\zeta}, \tilde{\nu})$ and the morphisms Φ and $\tilde{\Phi}$ respectively. Define $(\alpha, \beta) = \int_{\rho_I/\rho} F^*(\omega_i, 0)$. Note that $F^*(\omega_i, 0) = R(F^*\hat{u}_i)$, so by the homotopy formula we obtain

$$f_i^*\hat{u}_i - \tilde{f}_i^*\hat{u}_i = a(\alpha, \beta) \qquad\qquad f_i^*\hat{u'}_i - \tilde{f}_i^*\hat{u'}_i = a'(\alpha, \beta).$$

Since $\hat{v} = f_i^*\hat{u}' + a'(\zeta, \nu) = \tilde{f}_i^*\hat{u}' + a'(\tilde{\zeta}, \tilde{\nu})$, the homotopy formula also implies that $a'(\tilde{\zeta}, \tilde{\nu}) = a'(\zeta, \nu) + a'(\alpha, \beta)$, thus:

$$\Phi(\hat{v}) = f_i^*\hat{u}_i' + a'(\zeta, \nu) = f_i^*\hat{u}_i' + a'(\tilde{\zeta}, \tilde{\nu}) - a'(\alpha, \beta) = a'(\tilde{\zeta}, \tilde{\nu}) + \tilde{f}_i^*\hat{u}_i' = \tilde{\Phi}(\hat{v}).$$

In order to show that Φ induces a morphism of long exact sequences it is enough to show that Φ commutes with the Bockstein map, for it is already a natural transformation.

4.1.9 Differential cohomology with compact support

We are going to define the compactly-supported version of differential cohomology. This has been done about ordinary differential cohomology in [5], using the language of Cheeger-Simons characters. Here we generalize the construction to any cohomology theory, within the axiomatic setting. Given a smooth manifold X, we denote by $\mathcal{K}_X$ the directed set formed by the compact subsets of X, the partial ordering being given by set inclusion. We think of $\mathcal{K}_X$ as a category, whose objects are the compact subsets of X and such that the set of morphisms from K to H contains one element if $K \subset H$ and is empty otherwise. There is a natural functor $\mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{M}_2^{\mathrm{op}}$, assigning to an object K the open embedding $i_K \colon X \setminus K \hookrightarrow X$ and to a morphism $K \subset H$ the natural morphism $i_{KH} \colon i_H \to i_K$ defined by the following diagram:

$$\begin{tikzcd} X \setminus H \arrow[r, hook, "i_H"] \arrow[d, hook] & X \arrow[d, equal] \\ X \setminus K \arrow[r, hook, "i_K"] & X. \end{tikzcd}$$

Given a cohomology theory $h^\bullet$ and a differential extension $\hat{h}^\bullet \colon \mathcal{M}_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$, the corresponding compactly-supported differential extension $\hat{h}^\bullet_{\mathrm{cpt}}(X)$ is the colimit of the composition functor $\hat{h}^\bullet_{\mathrm{par}} \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}}$:

$$\hat{h}^\bullet_{\mathrm{cpt}}(X) := \mathrm{colim}\big(\, \hat{h}^\bullet_{\mathrm{par}} \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \,\big). \tag{4-44}$$

Since $\hat{h}^\bullet_{\mathrm{par}}$ and $\mathfrak{C}_X$ are both contravariant, the composition is covariant. Concretely, an element $\hat{\alpha}_{\mathrm{cpt}} \in \hat{h}^\bullet_{\mathrm{cpt}}(X)$ is an equivalence class $\hat{\alpha}_{\mathrm{cpt}} = [\hat{\alpha}]$, represented by a parallel class $\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K)$, K being a compact subset of X. The colimit is taken over the groups $\hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K)$, where, if $K \subset H$, the corresponding morphism in the direct system is the pull-back $i^*_{KH} \colon \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K) \to \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus H)$.

We have defined the group associated to a manifold X. We can extend this definition to the category $\mathcal{M}'$ whose objects are smooth manifolds (the same of the category $\mathcal{M}$) and whose morphisms are *open embeddings*. In fact, let us fix an open embedding $\iota \colon Y \hookrightarrow X$. For any compact subset $K \subset Y$, from the embedding of pairs $\iota_K \colon (Y, Y \setminus K) \hookrightarrow (X, X \setminus \iota(K))$, we get the induced morphism $\iota_K^* \colon \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus \iota(K)) \to \hat{h}^\bullet_{\mathrm{par}}(Y, Y \setminus K)$. By the excision property of parallel classes 4.1.11, it follows that ι_K^* is an isomorphism. If $K \subset H$, the following diagram commutes:

$$\begin{tikzcd}
\hat{h}^\bullet_{\mathrm{par}}(Y, Y \setminus K) \arrow[r, "(\iota_K^*)^{-1}"] \arrow[d, "i^*_{KH}"] & \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus \iota(K)) \arrow[d, "i^*_{KH}"] \\
\hat{h}^\bullet_{\mathrm{par}}(Y, Y \setminus H) \arrow[r, "(\iota_H^*)^{-1}"] & \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus \iota(H))
\end{tikzcd}$$

therefore we get an induced morphism between the colimits, i.e., $\iota_* \colon \hat{h}^\bullet_{\mathrm{cpt}}(Y) \to \hat{h}^\bullet_{\mathrm{cpt}}(X)$.

We can define as above the following functors:

$$\begin{aligned}
&h^\bullet_{\mathrm{cpt}}(X) := \operatorname{colim}\big(h^\bullet \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \big) && \Omega^\bullet_{\mathrm{cpt}}(X) := \operatorname{colim}\big(\Omega^\bullet \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \big) \\
&\Omega^\bullet_{\mathrm{cl,cpt}}(X) := \operatorname{colim}\big(\Omega^\bullet_{\mathrm{cl}} \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \big) && \Omega^\bullet_{\mathrm{ex,cpt}}(X) := \operatorname{colim}\big(\Omega^\bullet_{\mathrm{ex}} \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \big) \\
&\Omega^\bullet_{\mathrm{ch,cpt}}(X) := \operatorname{colim}\big(\Omega^\bullet_{\mathrm{ch}} \circ \mathfrak{C}_X \colon \mathcal{K}_X \to \mathcal{A}_{\mathbb{Z}} \big).
\end{aligned}$$

A relative form $\omega \in \Omega^\bullet(X, X \setminus K)$ is defined as a form $\omega \in \Omega^\bullet(X)$ whose restriction to $X \setminus K$ vanishes. Since the pull-back $i^*_{KH} \colon \Omega^\bullet(X, X \setminus K) \to \Omega^\bullet(X, X \setminus H)$ is injective, an element of $\Omega^\bullet_{\mathrm{cpt}}(X)$ is a form ω on X whose support is compact, i.e., such that ω vanishes on the complement of a compact subset of X. The same holds for closed forms. In the case of exact forms, an element of $\Omega^\bullet_{\mathrm{ex,cpt}}(X)$ is a form ω on X such that there exist a form *with compact support* $\eta \in \Omega^{\bullet-1}_{\mathrm{cpt}}(X)$ satisfying the identity $\omega = d\eta$. Finally, an element of $\Omega^\bullet_{\mathrm{ch,cpt}}(X)$ is a form ω on X whose support K is compact and such that the *relative* de-Rham class $[\omega] \in H^\bullet_{\mathrm{dR}}(X, X \setminus K)$ belongs to the image of the Chern character. We also recall that colimit on abelian groups is an exact functor, hence the colimit of a quotient is the quocient of the colimits. For example, $H^\bullet_{\mathrm{dR,cpt}}(X) = \Omega^\bullet_{\mathrm{cl,cpt}}(X)/\Omega^\bullet_{\mathrm{ex,cpt}}(X)$.

We easily get the following natural transformations of $\mathcal{A}_{\mathbb{Z}}$-valued functors:

- $I_{\mathrm{cpt}} \colon \hat{h}^\bullet_{\mathrm{cpt}}(X) \to h^\bullet_{\mathrm{cpt}}(X)$;
- $R_{\mathrm{cpt}} \colon \hat{h}^\bullet_{\mathrm{cpt}}(X) \to \Omega^\bullet_{\mathrm{cl,cpt}}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$;
- $a_{\mathrm{cpt}} \colon \Omega^{\bullet-1}_{\mathrm{cpt}}(X; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{h}^\bullet_{\mathrm{cpt}}(X)$.

These transformations satisfy axioms analogous to (A1)–(A3) of definition 4.1.1, replacing the functors and the natural transformations involved with the corresponding compactly-supported version. The proof of (A1) and (A2) is straightforward and the proof of (A3) is analogous to [5, Theorem 4.2].

If $\hat{h}^\bullet$ is multiplicative, the module structure stated in definition 4.1.9 induces the following natural module structure:

$$\begin{aligned}
\cdot \colon \hat{h}^n(X) \times \hat{h}^m_{\mathrm{cpt}}(X) &\to \hat{h}^{n+m}_{\mathrm{cpt}}(X) \\
(\hat{\alpha}, [\hat{\beta}]) &\mapsto [\hat{\alpha} \cdot \hat{\beta}].
\end{aligned}$$

Here naturality consists in the commutativity of the following diagram for any open embedding $\iota\colon Y \to X$:

$$\begin{tikzcd}
\hat{h}^n(X) \times \hat{h}^m_{\mathrm{cpt}}(Y) \arrow[r, "\iota^* \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \iota_*"'] & \hat{h}^n(Y) \times \hat{h}^m_{\mathrm{cpt}}(Y) \arrow[r] & \hat{h}^{n+m}_{\mathrm{cpt}}(Y) \arrow[d, "\iota_*"] \\
\hat{h}^n(X) \times \hat{h}^m_{\mathrm{cpt}}(X) \arrow[rr] & & \hat{h}^{n+m}_{\mathrm{cpt}}(X).
\end{tikzcd}$$

Finally, we have the following natural homomorphism, which in general is neither injective nor surjective:

$$S\colon \hat{h}^\bullet_{\mathrm{cpt}}(X) \to \hat{h}^\bullet(X)$$
$$[\hat{\alpha}] \mapsto \pi_X^* \hat{\alpha},$$

where $\pi_X\colon (X, \emptyset) \hookrightarrow (X, X \setminus K)$ is the natural inclusion of pairs and the naturality consists in the commutativity of the following diagram for any open embedding $\iota\colon Y \to X$:

$$\begin{tikzcd}
\hat{h}^\bullet_{\mathrm{cpt}}(Y) \arrow[r, "S"] \arrow[d, "\iota_*"'] & \hat{h}^\bullet(Y) \\
\hat{h}^\bullet_{\mathrm{cpt}}(X) \arrow[r, "S"] & \hat{h}^\bullet(X). \arrow[u, "\iota^*"']
\end{tikzcd}$$

4.2 Orientation and integration

Following [8, sec. 4.8-4.10], we briefly recall the topological notions of orientation and integration and we extend them to the differential case, as we did in [34, chap. 3]. We will use the expression “compact manifold” to indicate a compact smooth manifold with corners (in particular, with or without boundary). All of the statements can be easily generalized removing the compactness hypothesis, but this is the only case we will need.

4.2.1 Topological orientation of a vector bundle

Let X be a compact manifold and $\pi\colon E \to X$ a real vector bundle of rank n. The bundle E is *orientable* with respect to a multiplicative cohomology theory $h^\bullet$ if there exists a *Thom class* $u \in h^n_{\mathrm{cpt}}(E)$ [31, p. 253]. We define the *Thom isomorphism* $T\colon h^\bullet(X) \to h^{\bullet+n}_{\mathrm{cpt}}(E)$, $\alpha \mapsto u \cdot \pi^*\alpha$, and we call *integration map* its inverse $\int_{E/X}\colon h^\bullet_{\mathrm{cpt}}(E) \to h^{\bullet-n}(X)$, $u \cdot \pi^*\alpha \mapsto \alpha$. If the characteristic of $\mathfrak{h}^\bullet$ is 0, the n-degree component of $\mathrm{ch}\, u$ defines an orientation of E in the usual sense, hence it is possible to integrate a compactly-supported form fibre-wise. We define the *Todd class* $\mathrm{Td}(u) := \int_{E/X} \mathrm{ch}\, u \in H^0_{\mathrm{dR}}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$. The following formula holds:

$$\int_{E/X} \mathrm{ch}\, \alpha = \mathrm{Td}(u) \cdot \left(\mathrm{ch} \int_{E/X} \alpha \right). \tag{4-45}$$

Lemma 4.2.1 (2x3 principle)**.** Given two bundles $E, F \to X$, we call $p_E\colon E \oplus F \to E$ and $p_F\colon E \oplus F \to F$ the natural projections. Let (u, v, w) be a triple of Thom classes on E, F and $E \oplus F$ respectively, such that $w = p_E^* u \cdot p_F^* v$. Two elements of such a triple uniquely determine the third one.

For the proof see [31, prop. 1.10 p. 307].

4.2.2 Topological orientation of smooth maps

Definition 4.2.2. A *representative of an $h^\bullet$-orientation* of a smooth neat map between compact manifolds $f\colon Y \to X$ is the datum of:

- a neat embedding $\iota\colon Y \hookrightarrow X \times \mathbb{R}^N$, for any $N \in \mathbb{N}$, such that $\pi_X \circ \iota = f$;
- a Thom class u of the normal bundle $N_{\iota(Y)}(X \times \mathbb{R}^N)$;
- a diffeomorphism $\varphi\colon N_{\iota(Y)}(X \times \mathbb{R}^N) \to U$, for U a neat tubular neighbourhood of $\iota(Y)$ in $X \times \mathbb{R}^N$.

We now introduce a suitable equivalence relation among representatives of orientations. Let us consider a representative (J, U, Φ) of an $h^\bullet$-orientation of $\mathrm{id} \times f : I \times Y \to I \times X$ and a neighbourhood $V \subset I$ of $\{0, 1\}$. We say that the representative is *proper on* V if a vector $(x, v)_{(t,y)} \in N_{\iota(V \times Y)}(V \times X \times \mathbb{R}^N)_{\iota(t,y)}$ is sent by Φ to a point $\Phi((x, v)_{(t,y)}) \in V \times X \times \mathbb{R}^N$ whose first component is t. This means that the following diagram commutes:

$$\begin{tikzcd} N_{\iota(V \times Y)}(V \times X \times \mathbb{R}^N) \arrow[r, "\Phi"] \arrow[d, "\pi_N"'] & U \arrow[d, "\pi_I"] \\ \iota(V \times Y) \arrow[r, "\pi_I"] & I. \end{tikzcd} \tag{4-46}$$

In this case, calling $f_0 := \mathrm{id}_{\{0\}} \times f$ and $f_1 := \mathrm{id}_{\{1\}} \times f$, we can define the restrictions $(J, U, \Phi)|_{f_0}$ and $(J, U, \Phi)|_{f_1}$.

Definition 4.2.3. A *homotopy* between two representatives (ι, u, φ) and (ι', u', φ') of an $h^\bullet$-orientation of $f\colon Y \to X$ is a representative (J, U, Φ) of an $h^\bullet$-orientation of $\mathrm{id} \times f\colon I \times Y \to I \times X$, such that:

- (J, U, Φ) is proper over a neighbourhood $V \subset I$ of $\{0, 1\}$;
- $(J, U, \Phi)|_{f_0} = (\iota, u, \varphi)$ and $(J, U, \Phi)|_{f_1} = (\iota', u', \varphi')$.

On the trivial bundle $X \times \mathbb{R}^N$ there is a canonical Thom class, defined in the following way. On $pt \times \mathbb{R}^N$, whose compactification is $pt \times S^N$, we put the class $u_0 \in \tilde{h}^N(S^N)$ corresponding to the suspension of $1 \in \mathfrak{h}^0$. Then, we put on $X \times \mathbb{R}^N$ the class $\pi_{\mathbb{R}^N}^* u_0$.

Definition 4.2.4. Let us consider a representative (ι, u, φ), with $\iota : Y \hookrightarrow X \times \mathbb{R}^N$.

- For any $L \in \mathbb{N}$, we define $\iota' \colon Y \hookrightarrow X \times \mathbb{R}^{N+L}$ by $\iota'(y) := (\iota(y), 0)$. Then $N_{\iota'(Y)}(X \times \mathbb{R}^{N+L}) \simeq N_{\iota(Y)}(X \times \mathbb{R}^N) \oplus (\iota(Y) \times \mathbb{R}^L)$.

- We put the canonical orientation u_0 on the trivial bundle $\iota(Y) \times \mathbb{R}^L$, and the orientation u' induced by u and u_0 on $N_{\iota'(Y)}(X \times \mathbb{R}^{N+L})$.

- Finally, for $v_y \in N_{\iota(Y)}(X \times \mathbb{R}^N)$ and $w \in \mathbb{R}^L$, we define $\varphi'(v_y, w) := (\varphi(v_y), w) \in X \times \mathbb{R}^{N+L}$.

The representative (ι', u', φ') is called *equivalent by stabilization* to (ι, u, φ).

Definition 4.2.5. A $h^\bullet$*-orientation* on $f \colon Y \to X$ is an equivalence class $[\iota, u, \varphi]$ of representatives, up to the equivalence relation generated by homotopy and stabilization.

Because of the uniqueness up to homotopy and stabilization of the tubular neighbourhood and of the diffeomorphism with the normal bundle, the class $[\iota, u, \varphi]$ does not depend on φ, hence we denote it by $[\iota, u]$. Moreover, any two embeddings ι and ι' become equivalent by homotopy and stabilization, therefore the meaningful datum is u.

Remark 4.2.6. If X and Y are manifolds with boundary, an orientation on $f \colon Y \to X$ canonically induces an orientation on $\partial f := f|_{\partial Y} \colon \partial Y \to \partial X$. In fact, fixing a representative (ι, u, φ) for f, by neatness ι restricts to $\iota' \colon \partial X \hookrightarrow \partial Y \times \mathbb{R}^N$. The normal bundle and the tubular neighbourhood, being neat, restrict to the boundary too, hence we get a representative (ι', u', φ') for ∂f. Any homotopy of representatives, being neat, determines a homotopy on the boundary, therefore the resulting orientation of ∂f is well-defined. A similar remark holds when X and Y have corners, but we need to be more careful in defining ∂f. We omit the details, since they are irrelevant for the present paper.

Definition 4.2.7. Let $f \colon Y \to X$ and $g \colon X \to W$ be $h^\bullet$-oriented neat maps, with orientations $[\iota, u]$ and $[\kappa, v]$, where $\iota \colon Y \hookrightarrow X \times \mathbb{R}^N$ and $\kappa \colon X \hookrightarrow W \times \mathbb{R}^L$. There is a naturally induced $h^\bullet$-orientation on $g \circ f \colon Y \to W$, that we denote by $[\kappa, v][\iota, u]$, defined in the following way:

- we choose the embedding $\xi = (\kappa, \mathrm{id}_{\mathbb{R}^N}) \circ \iota \colon Y \hookrightarrow W \times \mathbb{R}^{L+N}$;

- on the normal bundle $N_{\xi(Y)}(W \times \mathbb{R}^{L+N}) \simeq N_{\iota(Y)}(X \times \mathbb{R}^L) \oplus N_{\kappa(X) \times \mathbb{R}^L}(W \times \mathbb{R}^{L+N})|_{\xi(Y)} \simeq N_{\iota(Y)}(X \times \mathbb{R}^L) \oplus (\pi_N^* N_{\kappa(X)} W \times \mathbb{R}^L)|_{\xi(Y)}$, for $\pi_L \colon \mathbb{R}^{L+N} \to \mathbb{R}^N$, we put the Thom class w induced from the ones on $N_{\iota(Y)}(X \times \mathbb{R}^L)$ and $N_{\kappa(X)}(W \times \mathbb{R}^N)$.

We set $[\kappa, v][\iota, u] := [\xi, w]$.

The following lemma is a consequence of lemma 4.2.1 and of the uniqueness up to homotopy and stabilization of the embedding ι.

Lemma 4.2.8 (2x3 principle)**.** Let $f\colon Y \to X$ and $g\colon X \to W$ be $h^\bullet$-oriented neat maps, with orientations $[\iota, u]$ and $[\kappa, v]$, and let $[\xi, w] := [\kappa, v][\iota, u]$ be the orientation induced on $g \circ f$. Two elements of the triple $([\iota, u], [\kappa, v], [\xi, w])$ uniquely determine the third one.

For the proof see [23, theorem 5.24 p. 233]. Finally, let us consider two smooth neat maps $f, g\colon Y \to X$, with representatives (ι, u, φ) and (ι', u', φ') respectively. A *homotopy* between (ι, u, φ) and (ι', u', φ') is defined as in 4.2.3, replacing $\mathrm{id} \times f$ with a smooth neat homotopy $F\colon I \times Y \to I \times X$ between f and g.[8] The existence of such a homotopy only depends on the equivalence classes $[\iota, u]$ and $[\iota', u']$, therefore we can give the following definition.

Definition 4.2.9. Two smooth neat $h^\bullet$-oriented maps $f, g\colon Y \to X$ are *homotopic as $h^\bullet$-oriented maps* if there exists a homotopy between any two representatives of the orientations of f and g.

Remark 4.2.10. We remark that, since a homotopy must be neat from $I \times Y$ to $I \times X$ by definition, it restricts to the boundary, thus it is a homotopy of maps of pairs $f, g\colon (Y, \partial Y) \to (X, \partial X)$. In particular, it induces a homotopy between ∂f and ∂g.

4.2.3 Topological orientation of smooth manifolds

In this subsection we discuss separately the cases of manifolds without boundary, with boundary and (partially) with corners.

Definition 4.2.11. An $h^\bullet$-orientation of a manifold without boundary X is an $h^\bullet$-orientation of the map $p_X\colon X \to \{pt\}$.

By definition, giving an orientation to p_X means fixing an orientation u on the (stable) normal bundle of X; when u has been fixed, we set $\mathrm{Td}(X) := \mathrm{Td}(u)$.

Given a manifold with boundary X, we recall that a *defining function for the boundary* is a smooth neat map $\Phi\colon X \to I$ such that $\partial X = \Phi^{-1}\{0\}$ (by neatness, it follows that $\Phi^{-1}\{1\} = \emptyset$).

Definition 4.2.12. An *$h^\bullet$-orientation* on a smooth manifold with boundary X is a homotopy class of $h^\bullet$-oriented defining functions for the boundary (see def. 4.2.9).

It easy to verify that any two defining functions are neatly homotopic, therefore the only meaningful datum is again the Thom class u.

Remark 4.2.13. We set $\mathbb{H}^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N \geq 0\}$ (it is the local model of an n-dimensional manifold with boundary). Definition 4.2.12 is equivalent to fixing a neat embedding $\iota\colon X \hookrightarrow \mathbb{H}^N$, a Thom class on the normal bundle and a difeomorphism with a

[8] A homotopy is usually defined as a function $F'\colon I \times Y \to X$, but we consider the function $F\colon I \times Y \to I \times X$, $(t, y) \mapsto (t, F'(t, y))$.

neat tubular neighbourhood, up to homotopy and stabilization. In fact, if we fix an $h^\bullet$-orientation $[\iota, u, \varphi]$ of a defining map $\Phi\colon X \to I$, following definition 4.2.12, we have that $\iota\colon X \hookrightarrow I \times \mathbb{R}^N$. Since $\Phi^{-1}\{1\} = \emptyset$, the image of ι is contained in $[0,1) \times \mathbb{R}^N \simeq \mathbb{H}^{N+1}$. This confirms that definition 4.2.12 is natural.

Remark 4.2.14. It follows from remark 4.2.6 that an orientation on a manifold with boundary canonically induces an orientation on the boundary. In particular, let us fix a defining function $\Phi\colon X \to I$ and an orientation $[\iota, u]$, with $\iota\colon X \hookrightarrow I \times \mathbb{R}^N$. We call $i_{\partial X}\colon \partial X \hookrightarrow X$ the natural embedding and we set $\iota' := \iota \circ i_{\partial X}\colon \partial X \hookrightarrow \{0\} \times \mathbb{R}^N$ and $u' := u|_{\partial X}$. We get the orientation $[\iota', u']$ of ∂X.

Remark 4.2.15. If we apply definition 4.2.12 to a manifold without boundary (which is a particular case of a manifold with boundary), we get a function $\Phi\colon X \to I$ whose image is contained in $(0,1)$, the latter being diffeomorphic to $\mathbb{R}$. A representative (ι, u, φ) of an orientation of Φ is constructed from the embedding $\iota\colon X \hookrightarrow (0,1) \times \mathbb{R}^N \simeq \{pt\} \times \mathbb{R}^{N+1}$, therefore it can be thought of as a representative of an orientation of $p_X\colon X \to \{pt\}$. Any two such defining functions are homotopic, $(0,1)$ being contractible, and a homotopy between them determines a homotopy of representatives of an orientation of $p_X\colon X \to \{pt\}$. This shows that definition 4.2.11 is (equivalent to) a particular case of definition 4.2.12.

With respect to manifold with corners, we just consider the following case, that will be useful in order to define the generalized Cheeger-Simons characters.

Definition 4.2.16. A *manifold with split boundary* is a triple of manifolds (X, M, N) such that:

- X is a manifold with corners and M and N are manifolds with boundary;
- $\partial X = M \cup N$, M and N being embedded sub-manifolds (not neat in general) of ∂X of codimension 0;
- $\partial M = \partial N = M \cap N$;
- $\{\text{corners of } X\} \subset M \cap N$.

A *defining function for the boundary* of (X, M, N) is a smooth neat map $\Phi\colon X \to I \times I$ such that $M = \Phi^{-1}(I \times \{0\})$ and $N = \Phi^{-1}(\{0\} \times I)$ (by neatness, it follows that $\Phi^{-1}(I \times \{1\}) = \Phi^{-1}(\{1\} \times I) = \emptyset$). The definition of *$h^\bullet$-orientation* is analogous to 4.2.12. Remark 4.2.13 keeps on holding, replacing $\mathbb{H}^N$ by $\mathbb{H}^{N,2} := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_{N-1}, x_N \geq 0\}$. Remark 4.2.14 holds in the sense that an orientation of (X, M, N) induces an orientation of M and one of N, with defining functions (up to homotopy) $\Phi_M := \Phi|_M\colon M \to I \times \{0\} \approx I$ and $\Phi_N := \Phi|_N\colon N \to \{0\} \times I \approx I$ respectively. Finally, remark 4.2.15 holds too, in the sense that, setting $N = \emptyset$, we recover the notion of orientation for a manifold with boundary.

4.2.4 Topological integration

Let $f\colon Y \to X$ be a neat map. If we fix a representative (ι, u, φ) of an orientation of f, the *Gysin map* $f_!\colon h^\bullet(Y) \to h^{\bullet - n}(X)$, for $n = \dim Y - \dim X$, is defined as:

$$f_!(\alpha) = \int_{\mathbb{R}^N} i_* \varphi_*(u \cdot \pi^* \alpha), \tag{4-47}$$

i being the natural inclusion of the tubular neighbourhood $i\colon U \hookrightarrow X \times \mathbb{R}^N$, inducing a push-forward in compactly-supported cohomology. The Gysin map $f_!$ only depends on the $h^\bullet$-orientation $[\iota, u]$, not on the specific representative ([23, theorem 5.24 p. 233], [8, sec. 4.9]). If Y and X are oriented, because of the 2x3 principle a map $f\colon Y \to X$ inherits an orientation, hence the Gysin map is well-defined.

Theorem 4.2.17. Let $f\colon Y \to X$ be a neat $h^\bullet$-oriented map of compact manifolds.

- The Gysin map $f_!$ only depends on the homotopy class of f as an $h^\bullet$-oriented map.
- The Gysin map is a morphism of $h^\bullet(X)$-modules, i.e., given $\alpha \in h^\bullet(Y)$ and $\beta \in h^\bullet(X)$:

$$f_!(\alpha \cdot f^* \beta) = f_! \alpha \cdot \beta.$$

- Given another neat $h^\bullet$-oriented map $g\colon Z \to Y$ and endowing $f \circ g$ of the naturally induced orientation (def. 4.2.7), we have $(f \circ g)_! = f_! \circ g_!$.

For the proof see [23, theorem 5.24 p. 233]. If X and Y are manifolds with boundary, considering remark 4.2.6, one has, for every $\alpha \in h^\bullet(Y)$:

$$(\partial f)_!(\alpha|_{\partial Y}) = (f_! \alpha)|_{\partial X}. \tag{4-48}$$

Such a formula is due to the fact that all the structures involved in the definition of $(\partial f)_!$ are the restrictions to the boundary of the corresponding structures for $f_!$. A similar result holds when X and Y have corners.

4.2.5 Differential orientation of a vector bundle

If we consider a differential refinement $\hat{h}^\bullet$ of $h^\bullet$, in order to orient a vector bundle one just has to refine a Thom class u to a *differential Thom class*.

Definition 4.2.18. Let $\hat{h}^\bullet$ be a multiplicative differential extension of $h^\bullet$. A *differential Thom class* of E is a compactly supported class $\hat{u} \in \hat{h}^n_{\mathrm{cpt}}(E)$ such that $I(\hat{u}) \in h^n_{\mathrm{cpt}}(E)$ is a Thom class for $h^\bullet$.

Using the product $\hat{h}^\bullet_{\mathrm{cpt}}(E) \otimes_{\mathbb{Z}} \hat{h}^\bullet(E) \to \hat{h}^\bullet_{\mathrm{cpt}}(E)$, we define the differential Thom morphism, which is not surjective any more, as $\hat{\alpha} \mapsto \hat{u} \cdot \pi^* \hat{\alpha}$. We define the *Todd class* $\mathrm{Td}(\hat{u}) := \int_{E/X} R(\hat{u}) \in \Omega^0_{\mathrm{cl}}(X; \mathfrak{h}^\bullet_{\mathbb{R}})$. It follows that $I(\mathrm{Td}(\hat{u})) = \mathrm{Td}(I(\hat{u}))$.

Definition 4.2.19. Let $\pi_X \colon I \times X \to X$ be the natural projection and $i_0, i_1 \colon X \to I \times X$ the natural embeddings. Two differential Thom classes $\hat{u}, \hat{u}' \in \hat{h}^n_{\mathrm{cpt}}(E)$ are *homotopic* if there exists a Thom class $\hat{U} \in \hat{h}^n_{\mathrm{cpt}}(\pi_X^* E)$ such that $i_0^* \hat{U} = \hat{u}$, $i_1^* \hat{U} = \hat{u}'$ and $\mathrm{Td}(\hat{U}) = \pi_X^* \mathrm{Td}(\hat{u})$.

Lemma 4.2.20 (2x3 principle)**.** Given two bundles $E, F \to X$, with projections $p_E \colon E \oplus F \to E$ and $p_F \colon E \oplus F \to F$, we consider a triple $(\hat{u}, \hat{v}, \hat{w})$ of differential Thom classes on E, F and $E \oplus F$ respectively, such that $\hat{w}$ is homotopic to $p_E^* \hat{u} \cdot p_F^* \hat{v}$. Two elements of such a triple uniquely determine the third one up to homotopy.

Lemma 4.2.21. On the trivial bundle $X \times \mathbb{R}^N$ there is a canonical homotopy class of differential Thom classes, refining the canonical topological one.

For the proofs see [8, prob. 4.187] and [34, cor. 3.19].

4.2.6 Differential orientation of smooth maps

We define a *representative of an $\hat{h}^\bullet$-orientation of f* as in definition 4.2.2, but considering a differential Thom class. Fixing such a representative $(\iota, \hat{u}, \varphi)$, the Gysin map $f_! \colon \hat{h}^\bullet(Y) \to \hat{h}^{\bullet - n}(X)$ is well-defined via formula (4-47) applied to differential classes. Moreover, we have the following natural map on differential forms, called *curvature map*:

$$\begin{aligned} R_{(\iota, \hat{u}, \varphi)} \colon \Omega^\bullet(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^{\bullet - n}(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \\ \omega &\mapsto \int_{X \times \mathbb{R}^N / X} i_* \varphi_* (R(\hat{u}) \wedge \pi^* \omega). \end{aligned} \tag{4-49}$$

The following definition is analogous to 4.2.3, but it takes into account the curvature map.

Definition 4.2.22. A *homotopy* between two representatives $(\iota, \hat{u}, \varphi)$ and $(\iota', \hat{u}', \varphi')$ of an $\hat{h}^\bullet$-orientation of $f \colon Y \to X$ is a representative $(J, \hat{U}, \Phi)$ of an $\hat{h}^\bullet$-orientation of $\mathrm{id} \times f \colon I \times Y \to I \times X$, such that:

- $(J, I(\hat{U}), \Phi)$ is proper over a neighbourhood $V \subset I$ of $\{0, 1\}$;
- $(J, \hat{U}, \Phi)|_{f_0} = (\iota, \hat{u}, \varphi)$ e $(J, \hat{U}, \Phi)|_{f_1} = (\iota', \hat{u}', \varphi')$;
- $\pi_X^* \circ R_{(\iota, \hat{u}, \varphi)} = R_{(J, \hat{U}, \Phi)} \circ \pi_Y^*$.

In particular, it follows that $R_{(\iota, \hat{u}, \varphi)} = R_{(\iota', \hat{u}', \varphi')}$. Thanks to lemma 4.2.21, we define the equivalence of representatives up to stabilization as in the topological framework (def. 4.2.4).

Definition 4.2.23. An *$\hat{h}^\bullet$-orientation* on $f \colon Y \to X$ is an equivalence class $[\iota, \hat{u}, \varphi]$ of representatives, up to the equivalence relation generated by homotopy and stabilization.

Remark 4.2.24. By construction the curvature map (4-49) only depends on the orientation, not on the specific representative, therefore we will denote it by $R_{[\iota, \hat{u}, \varphi]}$.

Remark 4.2.6 keeps on holding. Now we need to extend to the differential setting the fundamental properties of topological orientation, in particular definition 4.2.7 and lemma 4.2.8. This can be done adding the following hypothesis, that will force us to focus on submersions. Let us consider a vector $v_y \in N_{\iota(Y)}(X \times \mathbb{R}^N)_{\iota(y)}$. It is sent by φ, as defined in 4.2.2, to a point $\varphi(v_y) \in X \times \mathbb{R}^N$. If f is a submersion, we can require that the first component of $\varphi(v_y)$ is $f(y)$. This means that the following diagram commutes:

$$\begin{tikzcd} N_{\iota(Y)}(X \times \mathbb{R}^N) \arrow[r, "\varphi"] \arrow[d, "\pi_N"'] & U \arrow[d, "\pi_X"] \\ \iota(Y) \arrow[r, "\pi_X"] & X. \end{tikzcd} \tag{4-50}$$

Definition 4.2.25. A representative of an $\hat{h}^\bullet$-orientation of a smooth neat map $f\colon Y \to X$ is *proper* if diagram (4-50) commutes.[9]

Lemma 4.2.26. If $(\iota, \hat{u}, \varphi)$ is proper, then:

$$R_{[\iota,\hat{u},\varphi]}(\omega) = \int_{Y/X} \mathrm{Td}(\hat{u}) \wedge \omega. \tag{4-51}$$

Corollary 4.2.27. Let $(\iota, \hat{u}, \varphi)$ and $(\iota, \hat{u}', \varphi')$ be two *proper* representatives of an $\hat{h}^\bullet$-orientation of a smooth neat map $f\colon Y \to X$, such that $\hat{u}$ and $\hat{u}'$ are homotopic as differential Thom classes. Then the two representatives are homotopic (independently of φ and φ'), thus $[\iota, \hat{u}, \varphi] = [\iota, \hat{u}', \varphi']$.

Lemma 4.2.28. Let $f\colon Y \to X$ be a neat submersion. For any neat embedding $\iota\colon Y \hookrightarrow X \times \mathbb{R}^N$ and any differential Thom class $\hat{u}$ of the normal bundle, there exists a *proper* representative $(\iota, \hat{u}, \varphi)$ of an $\hat{h}^\bullet$-orientation of f.

Because of lemma 4.2.28 and corollary 4.2.27, given a neat submersion $f\colon Y \to X$, a neat embedding $\iota\colon Y \hookrightarrow X \times \mathbb{R}^N$ and any differential Thom class $\hat{u}$, the $\hat{h}^\bullet$-orientation $[\iota, \hat{u}]$ is well-defined, extending $(\iota, \hat{u})$ to any proper representative $(\iota, \hat{u}, \varphi)$.[10] The orientation $[\iota, \hat{u}]$ only depends on the homotopy class of $\hat{u}$. Moreover, if $f\colon Y \to X$ and $g\colon X \to W$ are $\hat{h}^\bullet$-oriented neat submersions, there is a naturally induced $\hat{h}^\bullet$-orientation on $g \circ f\colon Y \to W$, defined as in 4.2.7. The following lemma is a consequence of lemma 4.2.20 and of the uniqueness up to homotopy and stabilization of the embedding ι.

Lemma 4.2.29 (2x3 principle)**.** Let $f\colon Y \to X$ and $g\colon X \to W$ be $\hat{h}^\bullet$-oriented neat submersions, with orientations $[\iota, \hat{u}]$ and $[\kappa, \hat{v}]$, and let $[\xi, \hat{w}]$ be the orientation induced on $g \circ f$. Two elements of the triple $([\iota, \hat{u}], [\kappa, \hat{v}], [\xi, \hat{w}])$ uniquely determine the third one.

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[9] The same definition could be given for a representative of an $h^\bullet$-orientation, but it is more relevant in the differential framework.

[10] If we start from a non-proper representative $(\iota, \hat{u}, \varphi)$, we do not know if the corresponding orientation always admits a proper representative.

Finally, definition 4.2.9 can be easily adapted to the differential framework, considering a smooth neat homotopy with a differential orientation. When such a definition holds, two maps $f, g\colon Y \to X$ are *homotopic as $\hat{h}^\bullet$-oriented maps*.

4.2.7 Differential orientation of smooth manifolds

We define the notion of differential orientation of a manifold without boundary as in the topological case (def. 4.2.11); when the orientation $\hat{u}$ of the stable normal bundle has been fixed, we set $\mathrm{Td}(X) := \mathrm{Td}(\hat{u})$. When X has a boundary, we have to take into account that a defining map for the boundary is not a submersion in general, therefore we cannot apply many results cited above. For this reason, we slightly modify the definition of orientation. Following definition (4-49), the curvature map should be $\omega \mapsto \int_{I\times\mathbb{R}^N/I} i_*\varphi_*(R(\hat{u}) \wedge \pi^*\omega)$, but we also integrate on I the result:

$$\begin{aligned} R^{\partial}_{(\iota,\hat{u},\varphi)}\colon \Omega^\bullet(X;\mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^{\bullet-n}(pt;\mathfrak{h}^\bullet_{\mathbb{R}}) \\ \omega &\mapsto \int_0^1 \int_{I\times\mathbb{R}^N/I} i_*\varphi_*(R(\hat{u}) \wedge \pi^*\omega). \end{aligned} \tag{4-52}$$

Definition 4.2.30. An $\hat{h}^\bullet$*-orientation* on a smooth manifold with boundary X is a homotopy class of $\hat{h}^\bullet$-oriented defining functions for the boundary, considering the curvature map (4-52) in the definition of homotopy.

This means that the curvature map from X to the point must be constant along the homotopy, not the one from X to I, as would follow from the definition without replacing the curvature map. The double integral in (4-52) is equivalent to the integral on the whole $I \times \mathbb{R}^N$. Considering remark 4.2.13, we are just integrating on $\mathbb{H}^{N+1}$. It follows that:

$$R^{\partial}_{(\iota,\hat{u},\varphi)}(\omega) = \int_{N_{\iota(X)}\mathbb{H}^{N+1}} R(\hat{u}) \wedge \pi^*\omega = \int_X \left(\int_{N_{\iota(X)}\mathbb{H}^{N+1}/X} R(\hat{u})\right) \wedge \omega) = \int_X \mathrm{Td}(X) \wedge \omega. \tag{4-53}$$

This result is analogous to formula (4-51), therefore we can state the following corollary, analogous to 4.2.27.

Corollary 4.2.31. Let $(\iota, \hat{u}, \varphi)$ e $(\iota, \hat{u}', \varphi')$ be two representatives of an $\hat{h}^\bullet$-orientation of the defining function $\Phi\colon X \to I$, such that $\hat{u}$ and $\hat{u}'$ are homotopic as differential Thom classes. Then the two representatives are homotopic (independently of φ and φ'), thus $[\iota, \hat{u}, \varphi] = [\iota, \hat{u}', \varphi']$.

It follows that an orientation of X only depends on ι and $\hat{u}$, therefore an orientation on a neat submersion $f\colon Y \to X$ and an orientation on X induce an orientation on Y by definition 4.2.7. Because of corollary 4.2.27 and the uniqueness up to homotopy and stabilization of the embedding ι, we get the following lemma, analogous to 4.2.29.

Lemma 4.2.32 (2x3 principle)**.** Let $f\colon Y \to X$ be a neat submersions between manifolds with boundary. Let $[\iota, \hat{u}]$ be an orientation of f, $[\kappa, \hat{v}]$ an orientation of X, and let $[\xi, \hat{w}]$ be the orientation induced on Y. Two elements of the triple $([\iota, \hat{u}], [\kappa, \hat{v}], [\xi, \hat{w}])$ uniquely determine the third one.

With this definition of the curvature map, remark 4.2.15 extends to the differential setting, i.e., an orientation of a manifold without boundary can be thought of as a particular case of an orientation of a manifold with boundary. This confirms the naturality of the definition. As well, remark 4.2.15 keeps on holding. Finally, in the case of manifolds with split boundary, we define a $\hat{h}^\bullet$-orientation in the same way, the curvature map (4-52) being defined integrating over $I \times I$.

4.2.8 Differential integration

The Gysin map $f_!\colon \hat{h}^\bullet(Y) \to \hat{h}^{\bullet-n}(X)$, for $n = \dim Y - \dim X$, is defined similarly to (4-47), starting from a representative of an $\hat{h}^\bullet$-orientation:

$$f_!(\hat{\alpha}) = \int_{\mathbb{R}^N} i_* \varphi_*(\hat{u} \cdot \pi^* \hat{\alpha}). \tag{4-54}$$

The integration map $\int_{\mathbb{R}^N}\colon \hat{h}^{\bullet+N}_{\mathrm{cpt}}(X \times \mathbb{R}^N) \to \hat{h}^\bullet(X)$ is defined as follows. The open embedding $j\colon \mathbb{R}^N \hookrightarrow (S^1)^N$, defined through the embedding $\mathbb{R} \hookrightarrow \mathbb{R}^+ \simeq S^1$ in each coordinate, induces the push-forward $(\mathrm{id} \times j)_*\colon \hat{h}^\bullet_{\mathrm{cpt}}(X \times \mathbb{R}^N) \to \hat{h}^\bullet(X \times (S^1)^N)$, thus we define

$$\int_{\mathbb{R}^N} \hat{\alpha} := \int_{S^1} \cdots \int_{S^1} (\mathrm{id} \times j)_* \hat{\alpha}. \tag{4-55}$$

It is easy to prove from the axioms that:

$$R(f_!\hat{\alpha}) = R_{(\iota,\hat{u},\varphi)}(R(\hat{\alpha})) \qquad f_! a(\omega) = a(R_{(\iota,\hat{u},\varphi)}(\omega)),$$

thus the following diagram commutes:

$$\begin{tikzcd}
\Omega^{\bullet-1}(Y;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[d, "R_{(\iota,\hat{u},\varphi)}"] & \hat{h}^\bullet(Y) \arrow[r, "I"] \arrow[d, "f_!"] \arrow[rr, bend left, "R"] & h^\bullet(Y) \arrow[d, "f_!"] & \Omega^\bullet_{\mathrm{cl}}(Y;\mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[d, "R_{(\iota,\hat{u},\varphi)}"] \\
\Omega^{\bullet-n-1}(X;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] & \hat{h}^{\bullet-n}(X) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & h^{\bullet-n}(X) & \Omega^{\bullet-n}_{\mathrm{cl}}(X;\mathfrak{h}^\bullet_{\mathbb{R}}).
\end{tikzcd} \tag{4-56}$$

As a consequence of formula (4-16), $f_!$ only depends on the $\hat{h}^\bullet$-orientation of f, not on the specific representative [8, sec. 4.10]. We now consider a submersion $f\colon Y \to X$. In this case the Gysin map provides a good notion of integration.

Theorem 4.2.33. Let $f\colon Y \to X$ be a neat $\hat{h}^\bullet$-oriented submersion between compact manifolds.

- The Gysin map $f_!$ only depends on the homotopy class of f as an $\hat{h}^\bullet$-oriented map.
- The Gysin map is a morphism of $\hat{h}^\bullet(X)$-modules, i.e., given $\hat{\alpha} \in h^\bullet(Y)$ and $\hat{\beta} \in \hat{h}^\bullet(X)$:
$$f_!(\hat{\alpha} \cdot f^*\hat{\beta}) = f_!\hat{\alpha} \cdot \hat{\beta}.$$
- Given another neat $\hat{h}^\bullet$-oriented map $g\colon Z \to Y$ and endowing $f \circ g$ of the naturally induced orientation (def. 4.2.7), we have $(f \circ g)_! = f_! \circ g_!$.
- We have that:
$$R(f_!\hat{\alpha}) = \int_{Y/X} \mathrm{Td}(\hat{u}) \wedge R(\hat{\alpha}) \qquad f_!(a(\omega)) = a\left(\int_{Y/X} \mathrm{Td}(\hat{u}) \wedge \omega\right). \tag{4-57}$$

Equations (4-57) follows from formula (4-51) and the commutativity of diagram (4-56). Moreover, formula (4-48) keeps on holding.

Remark 4.2.34. Let us consider a submersion $f\colon Y \to X$ between $\hat{h}^\bullet$-oriented manifolds. If X and Y have no boundary, since $p_Y = p_X \circ f$, it follows from lemma 4.2.29 that f inherits a unique orientation from the ones of X and Y. Hence, the integration map $f_!\colon \hat{h}^\bullet(Y) \to \hat{h}^{\bullet-n}(X)$ is well-defined for submersions between compact $\hat{h}^\bullet$-oriented manifolds without boundary. If X and Y have boundary, the same result follows from 4.2.32.

4.2.9 Flat classes

The Gysin map $f_!\colon \hat{h}^\bullet(Y) \to \hat{h}^{\bullet-n}(X)$, defined in the previous section, depends on the $\hat{h}^\bullet$-orientation of f, but, if we restrict it to flat classes, it only depends on the topological $h^\bullet$-orientation. In fact, $\hat{h}^\bullet_{\mathrm{fl}}(X)$ has a natural graded-module structure over $h^\bullet(X)$, i.e., the product $h^\bullet(X) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{\mathrm{fl}}(X) \to \hat{h}^\bullet_{\mathrm{fl}}(X)$ is well-defined. This can be easily proven in the two following steps.

- The product of differential classes $\hat{h}^\bullet(X) \otimes \hat{h}^\bullet(X) \to \hat{h}^\bullet(X)$ restricts to the product $\hat{h}^\bullet(X) \otimes \hat{h}^\bullet_{\mathrm{fl}}(X) \to \hat{h}^\bullet_{\mathrm{fl}}(X)$, since, the curvature being multiplicative, if one of the two factors has vanishing curvature, also the result has.
- The product $\hat{\alpha} \cdot \hat{\beta}$, when $\hat{\beta}$ is flat, only depends on $I(\hat{\alpha})$. In fact, if $I(\hat{\alpha}) = 0$, then $\hat{\alpha} = a(\omega)$. Because of definition 4.1.9, we have $a(\omega) \cdot \hat{\beta} = a(\omega \wedge R(\hat{\beta})) = a(0) = 0$.

We can show in the same way that also the product $\hat{h}^\bullet_{\mathrm{cpt}}(E) \otimes_{\mathbb{Z}} \hat{h}^\bullet(E) \to \hat{h}^\bullet_{\mathrm{cpt}}(E)$ can be refined to $h^\bullet_{\mathrm{cpt}}(E) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{\mathrm{fl}}(E) \to \hat{h}^\bullet_{\mathrm{fl,cpt}}(E)$, therefore, given a real vector bundle $\pi\colon E \to X$ of rank n with (topological) Thom class u, we define the Thom isomorphism:
$$\begin{aligned} T_{\mathrm{fl}}\colon \hat{h}^\bullet_{\mathrm{fl}}(X) &\to \hat{h}^{\bullet+n}_{\mathrm{fl,cpt}}(E) \\ \hat{\alpha} &\mapsto u \cdot \pi^*\hat{\alpha}. \end{aligned}$$

From this it easily follows that the Gysin map $f_!$, when applied to a flat class, only depends on the topological orientation of f. Lemma 4.2.17 keeps on holding, with the same proof (for any f, not necessarily a submersion). With respect to the commutativity of diagram (4-56) (that, in the case of a submersion, leads to the last item of theorem 4.2.33), obviously the behaviour of the curvature is trivial in the flat case. About the map a, the commutativity of the diagram is equivalent to the following lemma (and, in the case of a submersion, to the right-hand side of equation (4-57)).

Lemma 4.2.35. Given a $h^\bullet$-oriented smooth neat map $f\colon Y \to X$ and a class $\theta \in H^{\bullet-1}_{\mathrm{dR}}(Y; \mathfrak{h}^\bullet_{\mathbb{R}})$, we have:

$$f_!(a(\theta)) = a(f_!(\mathrm{Td}(u) \wedge \theta)).$$

Equivalently, for any $\alpha \in h^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R}$:[11]

$$f_!(a(\mathrm{ch}\alpha)) = a(\mathrm{ch}(f_!\alpha)). \tag{4-58}$$

Proof. Let us consider a differential Thom class $\hat{u}$ of $N_{\iota(Y)}(X \times \mathbb{R}^N)$ refining the orientation u induced by the ones of X and Y. We have:

$$\begin{aligned} f_!(a(\theta)) &= i_*\varphi_*(\hat{u} \cdot \pi^* a(\theta)) = i_*\varphi_*(a([R(\hat{u})] \wedge \pi^*\theta)) = a(i_*\varphi_*(\mathrm{ch}\, u \wedge \pi^*\theta)) \\ &= a(i_*\varphi_*(\mathrm{ch}^{(n)} u \wedge \pi^*(\mathrm{Td}(u) \wedge \theta)) = a(f_!(\mathrm{Td}(u) \wedge \theta)). \end{aligned}$$

Formula (4-58) follows from the Grothendieck-Riemann-Roch theorem. □

Corollary 4.2.36. The Gysin map associated to a $h^\bullet$-oriented smooth map $f\colon Y \to X$ induces the following morphism of exact sequences of $\mathfrak{h}^\bullet$-modules:

$$\begin{tikzcd} \cdots \arrow[r] & h^\bullet(Y) \arrow[r] \arrow[d] & h^\bullet(Y) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] \arrow[d] & \hat{h}^{\bullet+1}_{\mathrm{fl}}(Y) \arrow[r] \arrow[d] & h^{\bullet+1}(Y) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & h^\bullet(X) \arrow[r] & h^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] & \hat{h}^{\bullet+1}_{\mathrm{fl}}(X) \arrow[r] & h^{\bullet+1}(X) \arrow[r] & \cdots \end{tikzcd}$$

where the map $h^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \hat{h}^{\bullet+1}_{\mathrm{fl}}(X)$ is defined by $\alpha \mapsto a(\mathrm{ch}\alpha)$.

4.3 Relative integration and integration to the point

We are going to show that the Gysin map, both in the topological and in the differential case, can also be defined for classes relative to the boundary.

[11] In equation (4-58) we are considering the Chern character as defined on $h^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R}$, in which case it is an isomorphism. If we consider it as defined on $h^\bullet(X)$, then $a(\mathrm{ch}\alpha) = 0$, and formula (4-58) implies coherently that $f_!(a(\mathrm{ch}\alpha)) = 0$.

4.3.1 Relative Thom (iso)morphism

If $\pi\colon E \to X$ is a real vector bundle of rank n and $A \subset X$ is a topological subspace, fixing a Thom class u of E we also get the relative version of the Thom isomorphism, i.e., $T\colon h^{\bullet}(X, A) \to h^{\bullet+n}_{\mathrm{cpt}}(E, E|_A)$, $\alpha \mapsto u \cdot \pi^* \alpha$, using the natural module structure of the relative cohomology over the absolute one (in this case $h^{\bullet}_{\mathrm{cpt}}(E, E|_A)$ is a module over $h^{\bullet}_{\mathrm{cpt}}(E)$). When the bundle is smooth and A is a closed submanifold of X, the same construction holds in the differential setting, getting the relative Thom morphism $T\colon \hat{h}^{\bullet}(X, A) \to \hat{h}^{\bullet+n}_{\mathrm{cpt}}(E, E|_A)$, $\hat{\alpha} \mapsto \hat{u} \cdot \pi^* \hat{\alpha}$. In this section we will apply such an (iso)morphism to the following particular case: $\pi\colon E \to X$ is a smooth vector bundle, X being a manifold with boundary, and $A = \partial X$. It follows that $E|_A = \partial E$, therefore we get the Thom (iso)morphism for classes relative to the boundary, i.e., $T\colon h^{\bullet}(X, \partial X) \to h^{\bullet+n}_{\mathrm{cpt}}(E, \partial E)$ and $T\colon \hat{h}^{\bullet}(X, \partial X) \to \hat{h}^{\bullet+n}_{\mathrm{cpt}}(E, \partial E)$.

4.3.2 Relative topological integration

Let $f\colon Y \to X$ be a smooth neat map and let us fix a representative of an $h^{\bullet}$-orientation (ι, u, φ). We define the Gysin map on classes relative to the boundary:

$$f_{!!}\colon h^{\bullet}(Y, \partial Y) \to h^{\bullet-n}(X, \partial X) \tag{4-59}$$

where $n = \dim Y - \dim X$. The definition is analogous to (4-47), but applying the *relative* Thom isomorphism on the normal bundle. Even in this case the map $f_{!!}$ only depends on the orientation $[\iota, u]$, not on the specific representative. With the same technique of [23, Prop. 5.24] one can prove the following theorem.

Theorem 4.3.1. Let $f\colon Y \to X$ be a neat $h^{\bullet}$-oriented map of compact manifolds.

- The Gysin map $f_{!!}$ only depends on the homotopy class of f as an $h^{\bullet}$-oriented map (see remark 4.2.10).
- The Gysin map is a morphism of $h^{\bullet}(X, \partial X)$-modules, i.e., given $\alpha \in h^{\bullet}(Y, \partial Y)$ and $\beta \in h^{\bullet}(X, \partial X)$:
$$f_{!!}(\alpha \cdot f^* \beta) = f_{!!}\alpha \cdot \beta.$$
- Given $\alpha \in h^{\bullet}(Y)$ and $\beta \in \hat{h}^{\bullet}(X, \partial X)$:
$$f_{!!}(\alpha \cdot f^* \beta) = f_{!}\alpha \cdot \beta.$$
- Given another neat $h^{\bullet}$-oriented map $g\colon Z \to Y$ and endowing $f \circ g$ of the naturally induced orientation (def. 4.2.7), we have $(f \circ g)_{!!} = f_{!!} \circ g_{!!}$.

We remark that, if $L_X\colon h^{\bullet}(X) \to h_{n-\bullet}(X, \partial X)$ is the Lefschetz duality [37], then $f_! = L_X^{-1} \circ f_* \circ L_Y$. If we consider the duality in the form $L'_X\colon h^{\bullet}(X, \partial X) \to h_{n-\bullet}(X)$, then $f_{!!} = {L'_X}^{-1} \circ f_* \circ L'_Y$.

4.3.3 Relative differential integration

The Gysin map $f_{!!}\colon \hat{h}^\bullet(Y,\partial Y) \to \hat{h}^{\bullet-n}(X,\partial X)$, for $n = \dim Y - \dim X$, is defined similarly to (4-47), starting from a representative of an $\hat{h}^\bullet$-orientation. As a consequence of formula (4-16), it only depends on the corresponding orientation. Considering the following version of diagram (4-6):

$$\begin{tikzcd} \partial X \times \mathbb{R}^N \arrow[r, hook] \arrow[d, "\pi_X|_{\partial X \times \mathbb{R}^N}"'] & X \times \mathbb{R}^N \arrow[d, "\pi_X"] \\ \partial X \arrow[r, hook] & X \end{tikzcd}$$

and applying formulas (4-7) and (4-9), we define the *relative curvature map*:

$$\begin{aligned} R^{rel}_{(\iota,\hat{u},\varphi)}\colon \Omega^\bullet(Y,\partial Y;\mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^{\bullet-n}(X,\partial X;\mathfrak{h}^\bullet_{\mathbb{R}}) \\ (\omega,\rho) &\mapsto \int_{X\times\mathbb{R}^N/X} i_*\varphi_*(R(\hat{u}) \wedge \pi^*(\omega,\rho)). \end{aligned} \tag{4-60}$$

Calling $\partial(\iota,\hat{u},\varphi)$ the representative induced on the boundary by $(\iota,\hat{u},\varphi)$, it follows that

$$R^{rel}_{(\iota,\hat{u},\varphi)}(\omega,\rho) = \big(R_{(\iota,\hat{u},\varphi)}(\omega), R_{\partial(\iota,\hat{u},\varphi)}(\rho)\big). \tag{4-61}$$

It is easy to prove from the axioms that:

$$R(f_{!!}\hat{\alpha}) = R^{rel}_{(\iota,\hat{u},\varphi)}(R(\hat{\alpha})) \qquad f_{!!}a(\omega,\rho) = a(R^{rel}_{(\iota,\hat{u},\varphi)}(\omega,\rho)),$$

thus the following diagram commutes:

$$\begin{tikzcd} \Omega^{\bullet-1}(Y,\partial Y;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[d, "R^{rel}_{(\iota,\hat{u},\varphi)}"] & \hat{h}^\bullet(Y,\partial Y) \arrow[r, "I"] \arrow[d, "f_{!!}"] \arrow[rr, bend left, "R"] & h^\bullet(Y,\partial Y) \arrow[d, "f_{!!}"] & \Omega^\bullet_{\mathrm{cl}}(Y,\partial Y;\mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[d, "R^{rel}_{(\iota,\hat{u},\varphi)}"] \\ \Omega^{\bullet-n-1}(X,\partial X;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] & \hat{h}^{\bullet-n}(X,\partial X) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & h^{\bullet-n}(X,\partial X) & \Omega^{\bullet-n}_{\mathrm{cl}}(X,\partial X;\mathfrak{h}^\bullet_{\mathbb{R}}). \end{tikzcd} \tag{4-62}$$

We now consider a submersion $f\colon Y \to X$, choosing proper representatives of orientations.

Theorem 4.3.2. Let $f\colon Y \to X$ be a neat $\hat{h}^\bullet$-oriented submersion between compact manifolds.

- The Gysin map $f_{!!}$ only depends on the homotopy class of f as an $\hat{h}^\bullet$-oriented map (see remark 4.2.10).

- The Gysin map is a morphism of $\hat{h}^\bullet(X,\partial X)$-modules, i.e., given $\hat{\alpha} \in h^\bullet(Y,\partial Y)$ and $\hat{\beta} \in \hat{h}^\bullet(X,\partial X)$:
$$f_{!!}(\hat{\alpha} \cdot f^*\hat{\beta}) = f_{!!}\hat{\alpha} \cdot \hat{\beta}.$$

- Given $\hat{\alpha} \in h^\bullet(Y)$ and $\hat{\beta} \in \hat{h}^\bullet(X, \partial X)$:
$$f_{!!}(\hat{\alpha} \cdot f^*\hat{\beta}) = f_!\hat{\alpha} \cdot \hat{\beta}.$$
- Given another neat $\hat{h}^\bullet$-oriented map $g\colon Z \to Y$ and endowing $f \circ g$ of the naturally induced orientation (def. 4.2.7), we have $(f \circ g)_{!!} = f_{!!} \circ g_{!!}$.
- Considering the following version of diagram (4-6):
$$\begin{tikzcd} \partial Y \arrow[r, hook] \arrow[d, "f|_{\partial Y}"'] & Y \arrow[d, "f"] \\ \partial X \arrow[r, hook] & X \end{tikzcd}$$
and applying formulas (4-7) and (4-9), we have:
$$R(f_{!!}\hat{\alpha}) = \int_{Y/X} \mathrm{Td}(\hat{u}) \wedge R(\hat{\alpha}) \qquad f_{!!}a(\omega, \eta) = a\biggl(\int_{Y/X} \mathrm{Td}(\hat{u}) \wedge (\omega, \eta)\biggr). \tag{4-63}$$

Equations (4-63) follows from formula (4-51) (in the relative setting) and the commutativity of diagram (4-62).

4.3.4 Flat classes

The relative Gysin map $f_{!!}\colon \hat{h}^\bullet(Y, \partial Y) \to \hat{h}^{\bullet-n}(X, \partial X)$, defined in the previous section, depends on the $\hat{h}^\bullet$-orientation of f, but, if we restrict it to flat classes, it only depends on the topological $h^\bullet$-orientation. The reason is the same of section 4.2.9, applying the relative Thom isomorphism $T_{\mathrm{fl}}\colon \hat{h}^\bullet_{\mathrm{fl}}(X, \partial X) \to \hat{h}^{\bullet+n}_{\mathrm{fl,cpt}}(E, \partial E)$. Lemma 4.3.1 keeps on holding (for any f, not necessarily a submersion). The relative versions of lemma 4.2.35 (in the case of a submersion, the right-hand side of equation (4-63)) and corollary 4.2.36 hold with the same proof.

4.3.5 Integration to the point - Manifolds without boundary

If X is an $h^\bullet$-oriented manifold of dimension n without boundary, the integration $(p_X)_!\colon h^\bullet(X) \to \mathfrak{h}^{\bullet-n}$ is well-defined applying (4-47). The same holds about the differential extension, defining $(p_X)_!\colon \hat{h}^\bullet(X) \to \hat{\mathfrak{h}}^{\bullet-n}$ through (4-54). Since p_X is a submersion, formula (4-57) becomes the following in this case:
$$R((p_X)_!(\hat{\alpha})) = \int_X \mathrm{Td}(X) \wedge R(\hat{\alpha}) \qquad (p_X)_!(a(\omega)) = a\biggl(\int_X \mathrm{Td}(\hat{u}) \wedge \omega\biggr). \tag{4-64}$$
As a particular case, X can be the boundary of another manifold. Equivalently, we consider a manifold with non-empty boundary X and the integration to the point $(p_{\partial X})_!$. We start from the following preliminary lemma, then we will show the behaviour of $(p_{\partial X})_!$.

Lemma 4.3.3. Let X be a $\hat{h}^\bullet$-oriented manifold with non-empty boundary and $\Phi\colon X \to I$ a defining function for the boundary, as a part of the orientation of X (see def. 4.2.12). For any $\hat{\alpha} \in \hat{h}^\bullet(X)$, we have:

$$\int_0^1 R(\Phi_!\hat{\alpha}) = \int_X \mathrm{Td}(X) \wedge R(\hat{\alpha}). \tag{4-65}$$

Proof. Let $(\iota, \hat{u}, \varphi)$ be any representative of the orientation of Φ. Because of the commutativity of diagram (4-56), we have that $R(\Phi_!\hat{\alpha}) = R_{(\iota,\hat{u},\varphi)}(R(\hat{\alpha}))$. It follows from definition (4-52) that $\int_0^1 R(\Phi_!\hat{\alpha}) = R^\partial_{(\iota,\hat{u},\varphi)}(R(\hat{\alpha}))$, hence the result follows from formula (4-53). □

Theorem 4.3.4. Let X be a $\hat{h}^\bullet$-oriented manifold with non-empty boundary. For any $\hat{\alpha} \in \hat{h}^\bullet(X)$, considering the induced $\hat{h}^\bullet$-orientation on ∂X, we have:

$$(p_{\partial X})_!(\hat{\alpha}|_{\partial X}) = -a\bigg(\int_X \mathrm{Td}(X) \wedge R(\hat{\alpha})\bigg). \tag{4-66}$$

In particular, in the topological framework, $(p_{\partial X})_!(\alpha|_{\partial X}) = 0$.

Proof. Let Φ be a defining function for the boundary, as a part of the orientation of X. Since $\Phi^{-1}\{1\} = \emptyset$, the map $\partial\Phi\colon \partial X \to \partial I$ can be identified with $p_{\partial X}\colon \partial X \to \{0\}$. Thanks to formula (4-48), one has $(p_{\partial X})_!(\hat{\alpha}|_{\partial X}) = (\Phi_!\hat{\alpha})|_{\{0\}}$. Since $(\Phi_!\hat{\alpha})|_{\{1\}} = 0$, because $\Phi^{-1}(1) = \emptyset$, from the homotopy formula (4-16) we have:

$$(p_{\partial X})_!(\hat{\alpha}|_{\partial X}) = -\big((\Phi_!\hat{\alpha})|_{\{1\}} - (\Phi_!\hat{\alpha})|_{\{0\}}\big) \overset{(4\text{-}16)}{=} -a\bigg(\int_I R(\Phi_!\hat{\alpha})\bigg).$$

The result follows from formula (4-65). □

4.3.6 Integration to the point - Manifolds with boundary

When X has a boundary, neither of the two Gysin maps $(p_X)_!$ and $(p_X)_{!!}$ is well defined, since p_X is not neat. Nevertheless, we can define the integration map to the point for classes relative to the boundary, that we denote anyway by $(p_X)_{!!}$, in the following way. In the topological framework, we set:

$$\begin{aligned} (p_X)_{!!}\colon h^\bullet(X, \partial X) &\to \mathfrak{h}^{\bullet-n} \\ \alpha &\mapsto \int_{S^1} \Phi_{!!}(\alpha), \end{aligned} \tag{4-67}$$

where the map $\Phi\colon (X, \partial X) \to (I, \partial I)$ is provided by the orientation of X (see def. 4.2.12) and the integration over S^1 is defined as follows. Since

$$h^{\bullet+1-n}(I, \partial I) \simeq h^{\bullet+1-n}(I/\partial I, \partial I/\partial I) \simeq h^{\bullet+1-n}(S^1, *), \tag{4-68}$$

'$*$' being a marked point on S^1, we apply the suspension isomorphism

$$\int_{S^1}\colon h^{\bullet+1-n}(S^1, *) \xrightarrow{\simeq} h^{\bullet-n}(S^0, **) \simeq \mathfrak{h}^{\bullet-n},$$

'$**$' being a marked point on S^0.

In order to define the integration map for differential classes, we can apply a formula analogous to (4-67), but we have to define the integration over S^1, since the isomorphism (4-68) does not apply any more. We do it through the following two lemmas.

Lemma 4.3.5. We have the following isomorphism:

$$\begin{aligned}\Psi\colon \hat{h}^\bullet(I,\partial I) &\xrightarrow{\simeq} \hat{h}^\bullet_{\mathrm{par}}(I,\partial I)\oplus\Omega^{\bullet-1}(\partial I;\mathfrak{h}^\bullet_{\mathbb{R}})\\ \hat{\alpha} &\mapsto \big(\hat{\alpha}-a(t\eta_1+(1-t)\eta_0,0),\,\eta_0\sqcup\eta_1\big),\end{aligned}\tag{4-69}$$

where $\eta_0\sqcup\eta_1=\mathrm{cov}(\hat{\alpha})$.

Proof. The inverse isomorphism is defined by

$$(\hat{\beta},\eta_0\sqcup\eta_1)\mapsto\hat{\beta}+a(t\eta_1+(1-t)\eta_0,0).$$

□

Remark 4.3.6. The previous lemma can be interpreted in the following way. Let us consider the following short exact sequence:

$$0\longrightarrow\hat{h}^\bullet_{\mathrm{par}}(I,\partial I)\longrightarrow\hat{h}^\bullet(I,\partial I)\xrightarrow{\mathrm{cov}}\Omega^{\bullet-1}(\partial I;\mathfrak{h}^\bullet_{\mathbb{R}})\longrightarrow 0.$$

The map $\Omega^{\bullet-1}(\partial I;\mathfrak{h}^\bullet_{\mathbb{R}})\to\hat{h}^\bullet(I,\partial I)$, $\eta_0\sqcup\eta_1\mapsto a(t\eta_1+(1-t)\eta_0,0)$, is a splitting of the sequence.

We now show how to integrate a parallel class defined on $(I,\partial I)$. The idea is that, glueing the two boundary points, we get a class on $(S^1,*)$ as in the topological case. Nevertheless, we have to take care of the smoothness condition when glueing the two extrema, hence we need a class that vanishes not only on ∂I, but also in an open neighbourhood $[0,\varepsilon)\cup(1-\varepsilon,1]$. In order to achieve this condition, we consider a smooth function $\xi\colon(I,\partial I)\to(I,\partial I)$ such that $\xi|_{[0,\varepsilon)}=0$ and $\xi|_{(1-\varepsilon,1]}=1$.[12] Given $\hat{\alpha}\in\hat{h}^\bullet_{\mathrm{par}}(I,\partial I)$, we consider its pull-back $\xi^*\hat{\alpha}\in\hat{h}^\bullet_{\mathrm{par}}(I,\partial I)$. Thanks to the following lemma, the class $\xi^*\hat{\alpha}$ induces a well-defined class on $(S^1,*)$, that we can integrate, getting a class on the point. Finally, we will have to prove that the latter is independent of the choice of ξ.

Notation 4.3.7. We set $I':=[0,\varepsilon)\cup(1-\varepsilon,1]$ for a fixed ε. Moreover, we denote by $\pi\colon(I,\partial I)\to(S^1,*)$ the natural projection and we set $S':=\pi(I')$. We think of π as a map of pairs $\pi\colon(I,I')\to(S^1,S')$.

Lemma 4.3.8. The pull-back $\pi^*\colon\hat{h}^\bullet_{\mathrm{par}}(S^1,S')\to\hat{h}^\bullet_{\mathrm{par}}(I,I')$ is an isomorphism.

[12] It is natural to think of ξ as an increasing function, but we will see that it is not necessary, since in any case it is smoothly homotopic to the identity of I relatively to ∂I.

Proof. Given $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{par}}(I, I')$, we have to show that there exists a unique class $\hat{\beta} \in \hat{h}^{\bullet}_{\mathrm{par}}(S^1, S')$ such that $\pi^*(\hat{\beta}) = \hat{\alpha}$. We set $\alpha := I(\hat{\alpha})$. Since π^* is an isomorphism in (topological) cohomology, there exists a unique class $\beta \in h^{\bullet}(S^1, S')$ such that $\pi^*\beta = \alpha$. We choose any parallel differential refinement $\hat{\beta}' \in \hat{h}^{\bullet}_{\mathrm{par}}(S^1, S')$ of β. It follows that $\pi^*\hat{\beta}' = \hat{\alpha} + a(\eta, 0)$, with $\eta|_{I'} = 0$. There exists a unique form $\bar{\eta}$ on (S^1, S') such that $\pi^*\bar{\eta} = \eta$, thus we set $\hat{\beta} := \hat{\beta}' - a(\bar{\eta}, 0)$ and we get $\pi^*\hat{\beta} = \hat{\alpha}$. About the uniqueness, let us suppose that $\hat{\beta}''$ is another parallel class such that $\pi^*\hat{\beta}'' = \hat{\alpha}$. Then $\pi^*(\hat{\beta} - \hat{\beta}'') = 0$, thus, since π^* is an isomorphism in cohomology, $I(\hat{\beta} - \hat{\beta}'') = 0$. It follows that $\hat{\beta} - \hat{\beta}'' = a(\xi, 0)$, with $\pi^* a(\xi, 0) = 0$, thus $[(\pi^*\xi, 0)] = \mathrm{ch} u$. Since π^* is an isomorphism in cohomology, there exists v such that $u = \pi^* v$, hence $\pi^*[(\xi, 0)] = \pi^*(\mathrm{ch} v)$. Again since π^* is an isomorphism in cohomology, it follows that $[(\xi, 0)] = \mathrm{ch} v$, hence $a(\xi, 0) = 0$, therefore $\hat{\beta} = \hat{\beta}''$. □

Notation 4.3.9. Considering the statement of lemma 4.3.8, we set $\pi_* := (\pi^*)^{-1}$.

Summarizing, given a class $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{par}}(I, \partial I)$ and a smooth function $\xi\colon (I, \partial I) \to (I, \partial I)$ such that $\xi|_{[0,\varepsilon)} = 0$ and $\xi|_{(1-\varepsilon,1]} = 1$, we get $\xi^*\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{par}}(I, I')$. Applying lemma 4.3.8, we get $\pi_*\xi^*\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{par}}(S^1, S')$. We can think of $\pi_*\xi^*\hat{\alpha}$ as an absolute class on S^1, applying the pull-back with respect to the natural morphism $\mathrm{id}_{S^1}\colon (S^1, \emptyset) \to (S^1, S')$, therefore we can integrate $\pi_*\xi^*\hat{\alpha}$ on S^1. We get the following integration map:

$$\begin{aligned} \int_I \colon\ & \hat{h}^{\bullet}_{\mathrm{par}}(I, \partial I) \to \hat{\mathfrak{h}}^{\bullet-1} \\ & \hat{\alpha} \mapsto \int_{S^1} \pi_*\xi^*\hat{\alpha}. \end{aligned} \tag{4-70}$$

Lemma 4.3.10. The integration map (4-70) is independent of the choice of ξ.

Proof. Let us fix two maps ξ and ξ', vanishing on $[0, \varepsilon) \cup (1-\varepsilon, 1]$ and $[0, \varepsilon') \cup (1-\varepsilon', 1]$ respectively. We call $\int_I$ and $\int_I'$ the corresponding integration maps (4-70). We set $J := I = [0, 1]$, in order to distinguish the two components of $I \times I = I \times J$ (this notation will make clearer the fibre-wise integrations). Let us consider a homotopy $\Xi\colon (I, \partial I) \times J \to (I, \partial I)$ between them. By formula (4-17), thinking of ξ and ξ' as relative maps $(\xi, \xi|_{\partial I}), (\xi', \xi'|_{\partial I})\colon (I, \partial I) \to (I, \partial I)$, we get

$$\xi^*\hat{\alpha} - (\xi')^*\hat{\alpha} = a\bigg(\int_{I\times J/I} R(\Xi^*\hat{\alpha})\bigg).$$

It follows that:

$$\begin{aligned} \int_I' \hat{\alpha} - \int_I \hat{\alpha} &= \int_{S^1} \pi_*(\xi^*\hat{\alpha} - (\xi')^*\hat{\alpha}) = \int_{S^1} \pi_* a\bigg(\int_{I\times J/I} R(\Xi^*\hat{\alpha})\bigg) \\ &= a\bigg(\int_{S^1} \pi_* \int_{I\times J/I} R(\Xi^*\hat{\alpha})\bigg) = a\bigg(\int_{I\times J} \Xi^* R(\hat{\alpha})\bigg). \end{aligned}$$

Since $\hat{\alpha} \in \hat{h}^n_{\mathrm{par}}(I, \partial I)$, it follows that $R(\hat{\alpha}) \in \Omega^n(I; \mathfrak{h}^{\bullet}_{\mathbb{R}}) = \Omega^0(I; \mathfrak{h}^n_{\mathbb{R}}) \oplus \Omega^1(I; \mathfrak{h}^{n-1}_{\mathbb{R}})$. Integrating over $I \times J$, only the components of degree 2 or more are meaningful, therefore we get 0. This shows that $\int_I \hat{\alpha} = \int_I' \hat{\alpha}$. □

Now we can give the following definition, for an $\hat{h}^{\bullet}$-oriented manifold with boundary X of dimension n:

$$\begin{aligned}(p_X)_{!!}\colon \hat{h}^{\bullet}(X,\partial X) &\to \hat{\mathfrak{h}}^{\bullet-n}\\ \hat{\alpha} &\mapsto \int_I \Psi_1\big(\Phi_{!!}(\hat{\alpha})\big),\end{aligned} \tag{4-71}$$

Ψ_1 being the first component of the isomorphism (4-69). For any representative $(\iota,\hat{u},\varphi)$ of an orientation of Φ, we call $\partial(\iota,\hat{u},\varphi)$ the induced representative on ∂X and we define the following curvature map:

$$\begin{aligned}R^{pt}_{(\iota,\hat{u},\varphi)}\colon \Omega^{\bullet}(X,\partial X;\mathfrak{h}^{\bullet}_{\mathbb{R}}) &\to \Omega^{\bullet-n}(pt;\mathfrak{h}^{\bullet}_{\mathbb{R}})\\ (\omega,\rho) &\mapsto R^{\partial}_{(\iota,\hat{u},\varphi)}(\omega) + R_{\partial(\iota,\hat{u},\varphi)}(\rho).\end{aligned} \tag{4-72}$$

It follows from formulas (4-51) and (4-53) that:

$$R^{pt}_{(\iota,\hat{u},\varphi)}(\omega,\rho) = \int_X \mathrm{Td}(X)\wedge\omega + \int_{\partial X}\mathrm{Td}(\partial X)\wedge\rho. \tag{4-73}$$

Theorem 4.3.11. The following diagram is commutative:[13]

$$\begin{tikzcd}
\Omega^{\bullet-1}(X,\partial X;\mathfrak{h}^{\bullet}_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[d, "R^{pt}_{(\iota,\hat{u},\varphi)}"] & \hat{h}^{\bullet}(X,\partial X) \arrow[r, "I"] \arrow[d, "(p_X)_{!!}"] \arrow[rr, bend left, "R"] & h^{\bullet}(X,\partial X) \arrow[d, "(p_X)_{!!}"] & \Omega^{\bullet}_{\mathrm{cl}}(X,\partial X;\mathfrak{h}^{\bullet}_{\mathbb{R}}) \arrow[d, "R^{pt}_{(\iota,\hat{u},\varphi)}"] \\
\mathfrak{h}^{\bullet-n-1}_{\mathbb{R}} \arrow[r, "a"] & \hat{\mathfrak{h}}^{\bullet-n} \arrow[r, "I"] \arrow[rr, bend right, "R"'] & \mathfrak{h}^{\bullet-n} & \mathfrak{h}^{\bullet-n}_{\mathbb{R}}.
\end{tikzcd} \tag{4-74}$$

Proof. □

As a particular case, X can be one component of a manifold with split boundary. Equivalently, we consider a manifold with split boundary (X,M,N), with $M\neq\emptyset$, and the integration to the point $(p_M)_{!!}$. We set again $J := I$, in order to distinguish the two components of $I\times J = I\times I$. In order to achieve a result analogous to formula (4-66) in the relative case, we consider a defining function for the boundary $\Phi\colon X\to I\times J$, we call $\pi_J\colon I\times J\to J$ the projection and we set $\Phi' := \pi_J\circ\Phi\colon X\to J$. It follows that $\Phi'^{-1}\{0\} = M$, hence $\Phi'|_M = p_M$, and $\Phi'^{-1}\{1\}=\emptyset$. Of course Φ' is not neat, for the same reason why p_M is not.

Topologically, we can define the following integration map:

$$\begin{aligned}\Phi'_{!!}\colon h^{\bullet}(X,N) &\to h^{\bullet-n+1}(J)\\ \alpha &\mapsto \int_{S^1\times J/J}\Phi_{!!}(\alpha),\end{aligned} \tag{4-75}$$

[13] In the diagram, observe that $\Omega^n(pt;\mathfrak{h}^{\bullet}_{\mathbb{R}}) = \Omega^0(pt;\mathfrak{h}^n_{\mathbb{R}}) \simeq \mathfrak{h}^n_{\mathbb{R}}$. Moreover, every form on the point is closed and only the zero one (in any degree) is exact.

considering $\Phi\colon (X,N) \to (I \times J, \partial I \times J)$. The map $\Phi_{!!}\colon h^{\bullet}(X,N) \to h^{\bullet-n+2}(I \times J, \partial I \times J)$ is defined by the same construction of the relative Gysin map (4-59), using the Thom isomorphism relative to N. The integration over S^1 is defined observing that $h^{\bullet-n+2}(I \times J, \partial I \times J) \simeq h^{\bullet-n+2}(S^1 \times J, \{*\} \times J) \simeq \tilde{h}^{\bullet-n+2}(S^1 \wedge (J_+))$ and applying the suspension isomorphism $\tilde{h}^{\bullet-n+2}(S^1 \wedge (J_+)) \simeq \tilde{h}^{\bullet-n+1}(J_+) \simeq h^{\bullet-n+1}(J)$.

In order to define the analogous integration map in the differential framework, we consider the following isomorphism, analogous to (4-69):

$$\begin{aligned} \Psi\colon \hat{h}^{\bullet}(I \times J, \partial I \times J) &\xrightarrow{\simeq} \hat{h}^{\bullet}_{\mathrm{par}}(I \times J, \partial I \times J) \oplus \Omega^{\bullet-1}(\partial I \times J; \mathfrak{h}^{\bullet}_{\mathbb{R}}) \\ \hat{\alpha} &\mapsto \big(\hat{\alpha} - a(t\eta_1 + (1-t)\eta_0, 0),\, \eta_0 \sqcup \eta_1\big). \end{aligned} \tag{4-76}$$

Then we consider a smooth function $\xi\colon (I, \partial I) \to (I, \partial I)$ such that $\xi|_{[0,\varepsilon)} = 0$ and $\xi|_{(1-\varepsilon,1]} = 1$, and we think of it as a function $\xi\colon (I \times J, \partial I \times J) \to (I \times J, \partial I \times J)$, constant on J. Since lemma 4.3.8 keeps on holding, with respect to the pull-back $\pi^*\colon \hat{h}^{\bullet}_{\mathrm{par}}(S^1 \times J, S' \times J) \to \hat{h}^{\bullet}_{\mathrm{par}}(I \times J, I' \times J)$, we get the integration map:

$$\begin{aligned} \int_{I\times J/J}\colon \hat{h}^{\bullet}_{\mathrm{par}}(I \times J, \partial I \times J) &\to \hat{h}^{\bullet-1}(J) \\ \hat{\alpha} &\mapsto \int_{S^1} \pi_* \xi^* \hat{\alpha}. \end{aligned} \tag{4-77}$$

Therefore, we define:

$$\begin{aligned} \Phi'_{!!}\colon \hat{h}^{\bullet}(X,N) &\to \hat{h}^{\bullet-n+1}(J) \\ \hat{\alpha} &\mapsto \int_{I\times J/J} \Psi_1\big(\Phi_{!!}(\hat{\alpha})\big), \end{aligned} \tag{4-78}$$

Ψ_1 being the first component of the isomorphism (4-76).

Remark 4.3.12. The construction of $\Phi'_{!!}$, that we have shown, is completely analogous to the one of $(p_X)_{!!}$, but there is only one difference, concerning the proof of lemma 4.3.10. In order to show that the integration map is independent of ξ, let us consider a homotopy $\Xi\colon (I, \partial I) \times J' \to (I, \partial I)$ between ξ and ξ', inducing the homotopy $\Xi\colon (I, \partial I) \times J \times J' \to (I, \partial I) \times J$ which is constant along J. With the same proof we get $\int'_{I\times J/J} \hat{\alpha} - \int_{I\times J/J} \hat{\alpha} = a\big(\int_{I\times J\times J'/J} \Xi^* R(\hat{\alpha})\big)$. Now $R(\hat{\alpha})$, being defined on $I \times J$, has also a component of degree 2, that could be non-vanishing after integrating along J' and I. Nevertheless, such an integral is a 0-form, whose value at $t \in J$ is $\int_{I\times J'} \Xi_t^* R(\hat{\alpha}|_{I\times\{t\}})$. The restriction $R(\hat{\alpha}|_{I\times\{t\}})$ of the degree-2 component is a 2-form on $I \times \{t\}$, hence it vanishes.

It follows from the construction that

$$\Phi'_{!!}(\hat{\alpha})|_{\{0\}} = (p_M)_{!!}(\hat{\alpha}|_{(M,\partial M)}), \tag{4-79}$$

since all the tools and the operations involved in the definition of $\Phi'_{!!}$ restrict on M to the corresponding ones for $(p_M)_{!!}$.

Lemma 4.3.13. Let (X, M, N) be a $\hat{h}^\bullet$-oriented manifold with split boundary and $\Phi\colon X \to I \times J$ a defining function for the boundary, as a part of the orientation of X. We call R' and cov the two components of the curvature R. For any $\hat{\alpha} \in \hat{h}^\bullet(X, N)$, we have:

$$\int_J R(\Phi'_{!!}\hat{\alpha}) = \int_X \mathrm{Td}(X) \wedge R'(\hat{\alpha}) + \int_N \mathrm{Td}(N) \wedge \mathrm{cov}(\hat{\alpha}). \tag{4-80}$$

Proof. By formulas (4-78) and (4-77) we have that

$$\begin{aligned}\int_J R(\Phi'_{!!}\hat{\alpha}) &= \int_J R\Big(\int_{S^1} \pi_* \xi^* \Psi_1\big(\Phi_{!!}(\hat{\alpha})\big)\Big) = \int_J \int_{S^1} \pi_* \xi^* R\big(\Psi_1\big(\Phi_{!!}(\hat{\alpha})\big)\big) \\ &= \int_{I\times J} \xi^* R\big(\Psi_1\big(\Phi_{!!}(\hat{\alpha})\big)\big) \overset{(\star)}{=} \int_{I\times J} R\big(\Psi_1\big(\Phi_{!!}(\hat{\alpha})\big)\big).\end{aligned} \tag{4-81}$$

In order to prove the equality $(\star)$, it is enough to choose ξ as a diffeomorphism from $(\varepsilon, 1-\varepsilon)$ to $(0,1)$. When we apply Ψ_1, defined in formula (4-78), to $\Phi_{!!}(\hat{\alpha})$, we have that $\eta_1 = 0$, thus $\eta_0 = \mathrm{cov}(\Phi_{!!}(\hat{\alpha}))$. It follows that

$$\begin{aligned}&\Psi_1(\Phi_{!!}(\hat{\alpha})) = \Phi_{!!}(\hat{\alpha}) - a\big((1-t)\mathrm{cov}(\Phi_{!!}(\hat{\alpha})), 0\big) \\ &R\big(\Psi_1(\Phi_{!!}(\hat{\alpha}))\big) = R'\big(\Phi_{!!}(\hat{\alpha})\big) + dt \wedge \mathrm{cov}(\Phi_{!!}(\hat{\alpha})) - (1-t)d\mathrm{cov}(\Phi_{!!}(\hat{\alpha})) \\ &\int_{I\times J} R\big(\Psi_1(\Phi_{!!}(\hat{\alpha}))\big) = \int_{I\times J} R'\big(\Phi_{!!}(\hat{\alpha})\big) + \int_J \mathrm{cov}(\Phi_{!!}(\hat{\alpha})).\end{aligned} \tag{4-82}$$

The term $(1-t)d\mathrm{cov}(\Phi_{!!}(\hat{\alpha}))$ vanishes when integrated on I, since there is no dt component. Joining (4-81) and (4-82) we get:

$$\int_J R(\Phi'_{!!}\hat{\alpha}) = \int_{I\times J} R'\big(\Phi_{!!}(\hat{\alpha})\big) + \int_J \mathrm{cov}(\Phi_{!!}(\hat{\alpha})). \tag{4-83}$$

Let $(\iota, \hat{u}, \varphi)$ be any representative of the orientation of Φ. Because of the commutativity of diagram (4-62) and formula (4-61), on $(I \times J, \partial I \times J)$ we have that

$$R(\Phi_{!!}\hat{\alpha}) \overset{(4\text{-}62)}{=} R^{rel}_{(\iota,\hat{u},\varphi)}(R(\hat{\alpha})) \overset{(4\text{-}61)}{=} \big(R_{(\iota,\hat{u},\varphi)}(R'(\hat{\alpha})), R_{(\iota,\hat{u},\varphi)|_N}(\mathrm{cov}(\hat{\alpha}))\big).$$

Therefore:

$$\begin{aligned}&\int_{I\times J} R'\big(\Phi_{!!}(\hat{\alpha})\big) = \int_{I\times J} R_{(\iota,\hat{u},\varphi)}(R'(\hat{\alpha})) \overset{(\#)}{=} \int_X \mathrm{Td}(X) \wedge R'(\hat{\alpha}) \\ &\int_J \mathrm{cov}(\Phi_{!!}(\hat{\alpha})) = \int_J R_{(\iota,\hat{u},\varphi)|_N}(\mathrm{cov}(\hat{\alpha})) \overset{(4\text{-}52)}{=} R^{\partial}_{(\iota,\hat{u},\varphi)|_N}(\mathrm{cov}(\hat{\alpha})) \overset{(4\text{-}53)}{=} \int_N \mathrm{Td}(N) \wedge \mathrm{cov}(\hat{\alpha}).\end{aligned}$$

The equality $(\#)$ follows again from formula (4-53), adapted to the case of a manifold with split boundary. $\square$

Theorem 4.3.14. Let (X, M, N) be a $\hat{h}^\bullet$-oriented manifold with split boundary. For any $\hat{\alpha} \in \hat{h}^\bullet(X, N)$, considering the induced $\hat{h}^\bullet$-orientation on M, we have:

$$(p_M)_{!!}(\hat{\alpha}|_{(M,\partial M)}) = -a\bigg(\int_X \mathrm{Td}(X) \wedge R'(\hat{\alpha}) + \int_N \mathrm{Td}(N) \wedge \mathrm{cov}(\hat{\alpha})\bigg). \tag{4-84}$$

In particular, in the topological framework, $(p_M)_{!!}(\alpha|_{(M,\partial M)}) = 0$.

Proof. Let Φ be a defining function for the boundary, as a part of the orientation of X. The map $\Phi_M \colon M \to I$ can be identified with a defining function for the boundary of M. Since $(\Phi'_{!!}\hat{\alpha})|_{\{1\}} = 0$, because $\Phi^{-1}(I \times \{1\}) = \emptyset$, from formula (4-79) and the homotopy formula (4-16) we have:

$$(p_M)_{!!}(\hat{\alpha}|_{(M,\partial M)}) = -\big((\Phi'_{!!}\hat{\alpha})|_{\{1\}} - (\Phi'_{!!}\hat{\alpha})|_{\{0\}}\big) \overset{(4\text{-}16)}{=} -a\bigg(\int_J R(\Phi'_{!!}\hat{\alpha})\bigg).$$

The result follows from formula (4-80). □

Finally, we remark that, since the flat theory is a (topological) cohomology theory, we can integrate a flat class over the point just using (4-67). We get the integration map $(p_X)_{!!} \colon \hat{h}^\bullet_{\mathrm{fl}}(X, \partial X) \to \hat{\mathfrak{h}}^{\bullet-n}_{\mathrm{fl}}$, that only depends on the topological $h^\bullet$-orientation of X.

4.4 Flat pairing and generalized Cheeger-Simons characters

We are going to define the relative version of generalized Cheeger-Simons characters, starting from flat classes.

4.4.1 Relative homology

We extend to the relative case the geometric model for the dual homology theory $h_\bullet$, described in [22]. When we say "relative", we consider the cohomology of any smooth map, not necessarily the embedding of the boundary as in the previous section. The following definition generalizes the one given in [34].

Definition 4.4.1. Given a continuous map $\rho : A \to X$, between spaces having the homotopy type of a finite CW-complex, we define:

- the group of *n-precycles* as the free abelian group generated by the quintuples (M, u, α, f, g), with:
  - (M, u) a smooth compact manifold, possibly with boundary, with $h^\bullet$-orientation u, whose connected components $\{M_i\}$ have dimension $n + q_i$, with q_i arbitrary;

- $\alpha \in h^{\bullet}(M)$, such that $\alpha|_{M_i} \in h^{q_i}(M)$;
- $f\colon M \to X$ a continuous function;
- $g\colon \partial M \to A$ a continuous function such that $\rho \circ g = f|_{\partial M}$;

- the group of *n-cycles*, denoted by $z_n(\rho)$, as the quotient of the group of n-precycles by the free subgroup generated by elements of the form:
  - $(M, u, \alpha + \beta, f, g) - (M, u, \alpha, f, g) - (M, u, \beta, f, g)$;
  - $(M, u, \alpha, f, g) - (M_1, u|_{M_1}, \alpha|_{M_1}, f|_{M_1}, g|_{\partial M_1}) - (M_2, u|_{M_2}, \alpha|_{M_2}, f|_{M_2}, g|_{\partial M_2})$, for $M = M_1 \sqcup M_2$;
  - $(M, u, \varphi_! \alpha, f, g) - (N, v, \alpha, f \circ \varphi, g \circ \varphi|_{\partial N})$ for $\varphi\colon N \to M$ a neat submersion, oriented via the 2x3 principle, and $\varphi_!$ the Gysin map for *absolute* classes (α is not relative to the boundary);

- the group of *n-boundaries*, denoted by $b_n(\rho)$, as the subgroup of $z_n(\rho)$ generated by the cycles which are representable by a pre-cycle (M, u, α, f, g) such that there exists quintuple $((W, M, N), U, A, F, G)$, where (W, M, N) is a manifold with split boundary, U is an $h^{\bullet}$-orientation of W and $U|_M = u$, $A \in h^{\bullet}(W)$ such that $A|_M = \alpha$, $F\colon W \to X$ is a smooth map satisfying $F|_M = f$ and $G\colon N \to A$ is a smooth map satisfying $\rho \circ G = F|_N$ and $G|_{\partial N} = g$.

We define $h_n(\rho) := z_n(\rho)/b_n(\rho)$.

There is a natural map:

$$\begin{aligned} \xi^n\colon h^n(\rho) &\to \mathrm{Hom}_{\mathfrak{h}^{\bullet}}(h_{n-\bullet}(\rho), \mathfrak{h}^{\bullet}) \\ \alpha &\mapsto \big([M, u, \beta, f, g] \mapsto (p_M)_{!!}(\beta \cdot (f, g)^* \alpha)\big), \end{aligned} \tag{4-85}$$

where $p_M\colon M \to \{pt\}$ (see def. (4-71)) and (f, g) is the following morphism:

$$\begin{tikzcd} \partial M \arrow[r, hook, "\iota"] \arrow[d, "g"'] & M \arrow[d, "f"] \\ A \arrow[r, "\rho"] & X. \end{tikzcd}$$

In order to multiply β and $(f, g)^*\alpha$, we used the module structure (4-4), since β is an absolute class on M, while $(f, g)^*\alpha$ is relative to the boundary. We verify that (4-85) is well-defined. If we consider a neat submersion $\varphi\colon N \to M$ and two representatives $(M, u, \varphi_! \beta, f, g)$ and $(N, v, \beta, f \circ \varphi, g \circ \varphi|_{\partial N})$ of the homology class, we have:

$$\begin{aligned} \xi^n(\alpha)[N, v, \beta, f \circ \varphi, g \circ \varphi|_{\partial N}] &= (p_N)_{!!}(\beta \cdot (\varphi, \varphi|_{\partial N})^*(f, g)^*\alpha) \\ &= (p_M)_{!!}(\varphi, \varphi|_{\partial N})_{!!}(\beta \cdot (\varphi, \varphi|_{\partial N})^*(f, g)^*\alpha) \\ &= (p_M)_{!!}(\varphi_! \beta \cdot (f, g)^*\alpha) = \xi^n(\alpha)[M, u, \varphi_! \beta, f, g]. \end{aligned}$$

If $(M, u, \beta, f, g) = \partial((W, M, N), U, B, F, G)$, then, by theorem 4.3.14, $(p_M)_{!!}(\beta \cdot (f, g)^*\alpha) = 0$, thus $\xi^n(\alpha)$ is well-defined on homology classes. Finally, the image of α is a $\mathfrak{h}^\bullet$-module homomorphism, since, for $\gamma \in \mathfrak{h}^t$:

$$\begin{aligned}\xi^n(\alpha)([(M, u, \beta, f, g)] \cap \gamma) &= \xi^n(\alpha)[M, u, \beta \cdot (p_M)^*\gamma, f, g] = (p_M)_{!!}(\beta \cdot (f, g)^*\alpha \cdot (p_M)^*\gamma) \\ &= (p_M)_{!!}(\beta \cdot (f, g)^*\alpha) \cdot \gamma = \xi^n(\alpha)[M, u, \beta, f, g] \cdot \gamma.\end{aligned}$$

Tensorizing with $\mathbb{R}$, we get the isomorphism:

$$\xi^n_{\mathbb{R}} \colon h^n(\rho) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \mathfrak{h}^\bullet_{\mathbb{R}}). \tag{4-86}$$

Moreover, thanks to the structure of $h^\bullet$-module on $h^\bullet(\,\cdot\,; \mathbb{R}/\mathbb{Z})$, we get the following map:

$$\begin{aligned}\xi^n_{\mathbb{R}/\mathbb{Z}} \colon h^n(\rho; \mathbb{R}/\mathbb{Z}) &\to \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \mathfrak{h}^\bullet_{\mathbb{R}/\mathbb{Z}}) \\ \alpha &\mapsto \big([M, u, \beta, f, g] \mapsto (p_M)_{!!}(\beta \cdot (f, g)^*\alpha)\big).\end{aligned} \tag{4-87}$$

4.4.2 Flat pairing

We define the natural $\hat{\mathfrak{h}}^\bullet_{\mathrm{fl}}$-valued pairing for a map $\rho \colon A \to X$ between $\hat{h}^\bullet_{\mathrm{fl}}$ and $h_\bullet$, that, in the case of singular differential cohomology, reduces to the holonomy of a flat relative Deligne cohomology class. When $\hat{h}^\bullet_{\mathrm{fl}} \simeq h^\bullet(\,\cdot\,; \mathbb{R}/\mathbb{Z})$, such a pairing coincides with formula (4-87).

Definition 4.4.2. For $\rho \colon A \to X$ a smooth map (not necessarily neat), we have the following natural pairing:

$$\begin{aligned}\xi^n_{\mathrm{fl}} \colon \hat{h}^n_{\mathrm{fl}}(\rho) &\to \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \hat{\mathfrak{h}}^\bullet_{\mathrm{fl}}) \\ \hat{\alpha} &\mapsto \big([M, u, \beta, f, g] \mapsto (p_M)_{!!}(\beta \cdot (f, g)^*\hat{\alpha})\big).\end{aligned} \tag{4-88}$$

The invariance by $\mathfrak{h}^\bullet$ is defined by:

$$\xi^n_{\mathrm{fl}}(\hat{\alpha})([M, u, \beta, f, g] \cdot \gamma) = \xi^n_{\mathrm{fl}}(\hat{\alpha})([M, u, \beta, f, g]) \cdot \gamma. \tag{4-89}$$

In order to show that (4-88) is well-defined, i.e. that it does not depend on the representative (M, u, β, f, g), and that formula (4-89) holds, we apply the same argument used about (4-85).

Lemma 4.4.3. We have the following morphism of complexes of $\mathfrak{h}^\bullet$-modules (the lower one not being exact in general):

$$\begin{tikzcd}
\cdots \arrow[r, "r"] & h^n(\rho) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r, "a"] \arrow[d, "\xi^n_{\mathbb{R}}"] & \hat{h}^{n+1}_{\mathrm{fl}}(\rho) \arrow[r, "I"] \arrow[d, "\xi^{n+1}_{\mathrm{fl}}"] & h^{n+1}(\rho) \arrow[r, "r"] \arrow[d, "\xi^{n+1}"] & \cdots \\
\cdots \arrow[r, "r'"] & \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[r, "a'"] & \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n+1-\bullet}(\rho), \hat{\mathfrak{h}}^\bullet_{\mathrm{fl}}) \arrow[r, "I'"] & \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n+1-\bullet}(\rho), \mathfrak{h}^\bullet) \arrow[r, "r'"] & \cdots
\end{tikzcd}$$

Proof. We only have to prove the commutativity of the square under the map a. It easily follows from the fact that, for $\alpha \in h^\bullet(\rho) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\beta \in h^\bullet(X)$:

$$a(\mathrm{ch}\alpha) \cdot \beta = a(\mathrm{ch}(\alpha\beta)).$$

That's because, for any differential refinement $\hat{\beta}$ of β, we have $a(\mathrm{ch}\alpha) \cdot \hat{\beta} = a(\mathrm{ch}\alpha \cdot R(\hat{\beta})) = a(\mathrm{ch}\alpha \cdot \mathrm{ch}\beta) = a(\mathrm{ch}(\alpha\beta))$. □

We call $\mathfrak{h}^n_{\mathbb{Z}}$ the image of the Chern character $\mathrm{ch}\colon \mathfrak{h}^n \to H^n_{\mathrm{dR}}(pt; \mathfrak{h}^\bullet_{\mathbb{R}}) \simeq \mathfrak{h}^n_{\mathbb{R}}$, which coincides with $\alpha \mapsto \alpha \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 4.4.4. If $\mathfrak{h}^\bullet$ has no torsion, the pairing (4-88) is an isomorphism and $\hat{\mathfrak{h}}^\bullet_{\mathrm{fl}} \simeq \mathfrak{h}^{\bullet-1}_{\mathbb{R}} / \mathfrak{h}^{\bullet-1}_{\mathbb{Z}}$.

Proof. Same of [34, Theorem 5.5]. □

4.4.3 Homology via differential cycles

We can define pre-cycles, cycles and boundaries as in definition 4.4.1, but refining each orientation and each cohomology class to a differential one. We call $\hat{z}_n(\rho)$ and $\hat{b}_n(\rho)$ the corresponding groups of cycles and boundaries. It follows that $\hat{z}_n(\rho)$ is generated by classes of the form $[(M, \hat{u}, \hat{\alpha}, f, g)]$, and $\hat{b}_n(\rho)$ is generated by cycles with a representative such that $(M, \hat{u}, \hat{\alpha}, f, g) = \partial((W, M, N), \hat{U}, \hat{A}, F, G)$. We define $h'_n(\rho) := \hat{z}_n(\rho)/\hat{b}_n(\rho)$.

Theorem 4.4.5. The natural group morphism:

$$\begin{aligned} \Phi\colon h'_\bullet(\rho) &\to h_\bullet(\rho) \\ [(M, \hat{u}, \hat{\alpha}, f, g)] &\to [(M, I(\hat{u}), I(\hat{\alpha}), f, g)] \end{aligned}$$

is an isomorphism.

Proof. It follows from the same result about absolute classes [34, Theorem 6.2] and the five lemma applied to the long exact sequence in homology associated to ρ. Alternatively, one can adapt to the relative case the same proof of [34, Theorem 6.2]. □

4.4.4 Cheeger-Simons characters

The following definition generalizes to any cohomology theory the one of [6] and [32] (type II).

Definition 4.4.6. A *Cheeger-Simons differential $\hat{h}^\bullet$-character* of degree n on $\rho\colon A \to X$ is a triple $(\chi_n, \omega_n, \eta_{n-1})$, where:

$$\chi_n \in \mathrm{Hom}_{\hat{\mathfrak{h}}^\bullet}(\hat{z}_{n-\bullet}(\rho), \hat{\mathfrak{h}}^\bullet) \qquad (\omega_n, \eta_{n-1}) \in \Omega^n(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) \tag{4-90}$$

such that, if $(M, \hat{u}, \hat{\beta}, f, g) = \partial((W, M, N), \hat{U}, \hat{B}, F, G)$, then:

$$\chi_n[M, \hat{u}, \hat{\beta}, f, g] = -a\bigg(\int_W \mathrm{Td}(W) \wedge R(\hat{B}) \wedge F^*\omega_n + \int_N \mathrm{Td}(N) \wedge R(\hat{B}\vert_N) \wedge G^*\eta_{n-1}\bigg). \quad (4\text{-}91)$$

The $\hat{\mathfrak{h}}^\bullet$-invariance is defined by:

$$\chi_n(\hat{\alpha})([M, \hat{u}, \hat{\beta}, f, g] \cdot \hat{\gamma}) = \chi_n(\hat{\alpha})[M, \hat{u}, \hat{\beta}, f, g] \cdot \hat{\gamma}. \quad (4\text{-}92)$$

We denote by $\check{h}^n(\rho)$ the group of characters of degree n.

We briefly comment on formula (4-91). Let us suppose that $[M, \hat{u}, \hat{\beta}, f, g] \in \hat{z}_{n-k}(X)$ and that M is connected. Then $\dim(M) = n-k+q$ and $\hat{\beta} \in \hat{h}^q(M)$, hence $\dim(W) = n-k+q+1$ and $\hat{B} \in \hat{h}^q(W)$. Thus, in the r.h.s. of (4-91), we integrate on W a $\mathfrak{h}^\bullet_{\mathbb{R}}$-valued form of degree $0+q+n$, hence we get a form on the point of degree $q+n-(n-k+q+1) = k-1$. Applying a, we get a class belonging to $\hat{\mathfrak{h}}^k$, as desired.

Theorem 4.4.7. There is a natural graded-group morphism:

$$\begin{aligned} CS^\bullet_{\hat{h}}\colon \hat{h}^\bullet(\rho) &\to \check{h}^\bullet(\rho) \\ \hat{\alpha} &\mapsto (\chi, R(\hat{\alpha})), \end{aligned} \quad (4\text{-}93)$$

where χ is defined, for $[M, \hat{u}, \hat{\beta}, f, g] \in \hat{z}_{n-k}(\rho)$, by:

$$\chi[M, \hat{u}, \hat{\beta}, f, g] := (p_M)_{!!}(\hat{\beta} \cdot (f, g)^*\hat{\alpha}).$$

Proof. If we consider two representatives $(M, u, \varphi_!\beta, f, g)$ and $(N, v, \beta, f \circ \varphi, g \circ \varphi\vert_{\partial N})$ of the same homology class, we have:

$$\begin{aligned} \chi[N, \hat{v}, \hat{\beta}, f \circ \varphi, g \circ \varphi\vert_{\partial N}] &= (p_N)_{!!}(\hat{\beta} \cdot (\varphi, \varphi\vert_{\partial N})^*(f, g)^*\hat{\alpha}) \\ &= (p_M)_{!!}(\varphi, \varphi\vert_{\partial N})_{!!}(\hat{\beta} \cdot (\varphi, \varphi\vert_{\partial N})^*(f, g)^*\hat{\alpha}) \\ &= (p_M)_{!!}(\varphi_!\hat{\beta} \cdot (f, g)^*\hat{\alpha}) = \chi[M, \hat{u}, \varphi_!\hat{\beta}, f, g]. \end{aligned}$$

Let us now suppose that $(M, \hat{u}, \hat{\beta}, f, g) = \partial((W, M, N), \hat{U}, \hat{B}, F, G)$. From formula (4-84), replacing X by W and $\hat{\alpha}$ by $\hat{\beta} \cdot (f, g)^*\hat{\alpha}$, we get formula (4-91). Finally:

$$\begin{aligned} \chi([(M, \hat{u}, \hat{\beta}, f, g)] \cap \hat{\gamma}) &= \chi[M, \hat{u}, \hat{\beta} \cdot (p_M)^*\hat{\gamma}, f, g] = (p_M)_{!!}(\hat{\beta} \cdot (f, g)^*\hat{\alpha} \cdot (p_M)^*\hat{\gamma}) \\ &= (p_M)_{!!}(\hat{\beta} \cdot (f, g)^*\hat{\alpha}) \cdot \hat{\gamma} = \chi[M, \hat{u}, \hat{\beta}, f, g] \cdot \hat{\gamma}. \end{aligned} \qquad \square$$

The proof of the following theorem is straightforward from the previous definition.

Theorem 4.4.8. When $\hat{\alpha}$ is flat, the value of the associated Cheeger-Simons character over $[M, \hat{u}, \hat{\beta}, f, g]$ coincides with the value of (4-88) on the corresponding homology class.

Considering the pairing (4-88), we have the following embedding:

$$j\colon \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \hat{\mathfrak{h}}^\bullet_{\mathrm{fl}}) \hookrightarrow \check{h}^n(\rho).$$

In fact, a morphism $\varphi_n \in \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(\rho), \hat{\mathfrak{h}}^\bullet_{\mathrm{fl}})$ determines a unique morphism $\chi_n\colon \hat{z}_{n-\bullet}(\rho) \to \hat{\mathfrak{h}}^\bullet$ defined by $\chi_n[M, \hat{u}, \hat{\beta}, f, g] := \varphi_n[M, I(\hat{u}), I(\hat{\beta}), f, g]$, and we define $j(\varphi_n) := (\chi_n, 0, 0)$. It follows from formula (4-91) that the image of j is the subgroup of generalized Cheeger-Simons characters with vanishing curvature, that we call $\check{h}^n_{\mathrm{fl}}(\rho)$. Let us consider the embedding $i : \hat{h}^\bullet_{\mathrm{fl}}(\rho) \hookrightarrow \hat{h}^\bullet(\rho)$. The following diagram commutes:

$$\begin{tikzcd}
\hat{h}^n_{\mathrm{fl}}(\rho) \arrow[r, "\xi^n_{\mathrm{fl}}"] \arrow[d, hook, "i"'] & \mathrm{Hom}_{\mathfrak{h}^\bullet}(h_{n-\bullet}(X), \hat{\mathfrak{h}}^\bullet_{\mathrm{fl}}) \arrow[d, hook, "j"] \\
\hat{h}^n(\rho) \arrow[r, "CS^n_{\hat{h}}"] & \check{h}^n(\rho).
\end{tikzcd}$$

Therefore i restricts to the embedding $i'\colon \mathrm{Ker}(\xi^n_{\mathrm{fl}}) \hookrightarrow \mathrm{Ker}(CS^n_{\hat{h}})$, and j restricts to the embedding $j'\colon \mathrm{Im}(\xi^n_{\mathrm{fl}}) \hookrightarrow \mathrm{Im}(CS^n_{\hat{h}})$. Because of j and j' we can construct a morphism $a\colon \mathrm{Coker}(\xi^n_{\mathrm{fl}}) \to \mathrm{Coker}(CS^n_{\hat{h}})$. We now show that actually i' and a are isomorphisms.

Theorem 4.4.9. The following canonical isomorphisms hold:

$$\mathrm{Ker}(\xi^n_{\mathrm{fl}}) \simeq \mathrm{Ker}(CS^n_{\hat{h}}), \qquad \mathrm{Coker}(\xi^n_{\mathrm{fl}}) \simeq \mathrm{Coker}(CS^n_{\hat{h}}).$$

Proof. If $\hat{\alpha} \in \hat{h}^n(\rho)$ is not flat, then $CS^n_{\hat{h}}(\hat{\alpha}) \neq 0$, since $CS^n_{\hat{h}}(\hat{\alpha}) = (\chi_n, R(\hat{\alpha}))$ and $R(\hat{\alpha}) \neq 0$. Hence $\mathrm{Ker}(CS^n_{\hat{h}}) \subset \mathrm{Ker}(\xi^n_{\mathrm{fl}})$ and the equality follows. Moreover, $\check{h}^n_{\mathrm{fl}}(\rho) \cap \mathrm{Im}(CS^n_{\hat{h}}) = \mathrm{Im}(\xi^n_{\mathrm{fl}})$, hence $a\colon \mathrm{Coker}(\xi^n_{\mathrm{fl}}) \to \mathrm{Coker}(CS^n_{\hat{h}})$ is an embedding. If $(\chi_n, \omega_n, \eta_{n-1}) \in \check{h}^n(\rho)$, we consider a class $\hat{\alpha} \in \hat{h}^n(\rho)$ such that $R(\hat{\alpha}) = (\omega_n, \eta_{n-1})$, and we call $(\chi'_n, \omega_n, \eta_{n-1}) := CS_{\hat{h}}(\hat{\alpha})$. Then $(\chi'_n - \chi_n, 0, 0) \in \check{h}^n_{\mathrm{fl}}(\rho)$, and, in $\mathrm{Coker}(CS^n_{\hat{h}})$, one has $[(\chi_n, \omega_n, \eta_{n-1})] = [(\chi'_n - \chi_n, 0, 0)] \in \mathrm{Im}\, a$. Therefore a is also surjective. □

Corollary 4.4.10. If $\mathfrak{h}^\bullet$ has no torsion, (4-93) is an isomorphism.

Proof. It immediately follows from theorems 4.4.9 and 4.4.4. □

4.5 Integration relative to the boundary

Let us consider a smooth fibre bundle $f\colon Y \to X$, such that X is a manifold without boundary and Y with boundary. It follows that the typical fibre is a manifold with boundary M. Moreover, the restriction of f to the boundary, that we call $\partial f\colon \partial Y \to X$, is a fibre bundle too, with typical fibre ∂M. Of course f is not neat, therefore we cannot apply the integration map as previously defined, but we can define the following integration map for classes relative to the boundary:

$$f_{!!}\colon \hat{h}^\bullet(Y, \partial Y) \to \hat{h}^{\bullet - m}(X), \tag{4-94}$$

m being the dimension of M. When X is a point, we get (4-71) as a particular case. The map (4-94) generalizes to any cohomology theory the one described in [32].

4.5.1 Topological integration

Let us start with the notion of orientation. The idea is the following. We choose a neat embedding $\iota\colon Y \hookrightarrow X \times \mathbb{H}^N$, such that $\pi_X \circ \iota = f$ (restricting ι to the boundary, we get the embedding $\partial\iota\colon \partial Y \hookrightarrow X \times \mathbb{R}^{N-1}$). This is always possible: for example, we can choose a neat embedding $\kappa\colon Y \hookrightarrow \mathbb{H}^N$ and define $\iota := f \times \kappa$. Then we choose a Thom class on the normal bundle and a neat tubular neighbourhood, as always. We think of ι as a map to $X \times \mathbb{R}^{N-1} \times I$, through the embedding $[0, +\infty) \approx [0, 1) \subset I$. In this way, we can first integrate on $\mathbb{R}^{N-1}$, getting a class in $X \times I$, relative to $X \times \partial I$. This is equivalent to getting a class in $X \times S^1$, therefore we can integrate on S^1 and obtain the result.

Let us define this integration map using the same language of sections 4.2 and 4.3. Given a fibre bundle $f\colon Y \to X$, such that $\partial X = \emptyset$, a *defining function for the boundary* is a smooth neat map $\Phi\colon Y \to X \times I$ such that $\partial Y = \Phi^{-1}(X \times \{0\})$ (by neatness, it follows that $\Phi^{-1}\{1\} = \emptyset$). In particular, the restriction of Φ to a fibre $Y_x := \pi^{-1}\{x\}$ is a defining function for the boundary of Y_x.

Definition 4.5.1. An $h^\bullet$-*orientation* on $f\colon Y \to X$ is a homotopy class of $h^\bullet$-oriented defining functions for the boundary.

Remarks analogous to 4.2.13, 4.2.14 and 4.2.15 hold in this case. In particular, the remark analogous to 4.2.13 shows that the idea we sketched at the beginning of this section corresponds to definition 4.5.1. We set:

$$\begin{aligned} f_{!!}\colon h^\bullet(Y, \partial Y) &\to h^{\bullet-m}(X) \\ \alpha &\mapsto \int_{S^1} \Phi_{!!}(\alpha), \end{aligned} \tag{4-95}$$

where the map $\Phi\colon (Y, \partial Y) \to (X \times I, X \times \partial I)$ is provided by the orientation of f and the integration over S^1 is defined as follows. Since $h^{\bullet+1-m}(X \times I, X \times \partial I) \simeq h^{\bullet+1-m}(X \times S^1, X \times \{*\}) = \tilde{h}^{\bullet+1-m}(X_+ \wedge S^1)$, '$*$' being a marked point on S^1, we apply the suspension isomorphism $\tilde{h}^{\bullet+1-m}(X_+ \wedge S^1) \simeq \tilde{h}^{\bullet-m}(X_+) \simeq h^{\bullet-m}(X)$ and we get the result. The same construction holds for differential integration of flat classes.

4.5.2 Differential integration

We generalize the curvature map (4-52) in the following natural way:

$$\begin{aligned} R^{\partial}_{(\iota,\hat{u},\varphi)}\colon \Omega^\bullet(X; \mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^{\bullet-m}(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \\ \omega &\mapsto \int_{X\times\mathbb{R}^{N-1}\times I/X} i_*\varphi_*(R(\hat{u}) \wedge \pi^*\omega) \end{aligned} \tag{4-96}$$

and we define:

$$\begin{aligned} R^{pt}_{(\iota,\hat{u},\varphi)}\colon \Omega^\bullet(X, \partial X; \mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^{\bullet-m}(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \\ (\omega, \rho) &\mapsto R^{\partial}_{(\iota,\hat{u},\varphi)}(\omega) + R_{\partial(\iota,\hat{u},\varphi)}(\rho). \end{aligned} \tag{4-97}$$

Requiring that the orientation is proper, i.e. that the fibre of the normal bundle of $\iota(Y)$ in $\iota(y) = (x,t)$ is sent by φ to a subset of $\{x\} \times \mathbb{H}^n$, it follows from formulas analogous to (4-51) and (4-53) that:

$$R^{pt}_{(\iota,\hat{u},\varphi)}(\omega,\rho) = \int_{Y/X} \mathrm{Td}(\hat{u}) \wedge \omega + \int_{\partial Y/X} \mathrm{Td}(\hat{u}|_{\partial Y}) \wedge \rho. \tag{4-98}$$

Definition 4.5.2. An *$\hat{h}^\bullet$-orientation* on $f\colon Y \to X$ is a homotopy class of $\hat{h}^\bullet$-oriented defining functions for the boundary, considering the curvature map (4-96) (equivalently, (4-97)) in the definition of homotopy.

Corollary 4.2.31 and lemma 4.2.32 hold with the same proof. The isomorphism (4-69) keeps on holding, replacing $(I, \partial I)$ by $(X \times I, X \times \partial I)$, therefore we can integrate on $\mathbb{R}^N$ and apply the integration map (4-78). This defines (4-94).

4.6 Relative Thom isomorphism and Gysin map

Up to now we have generalized the Thom isomorphism and the Gysin map in various ways, but always considering classes relative to the boundary. Now we are going to define the Thom isomorphism and the Gysin map in the framework of relative cohomology associated to a map of spaces.

4.6.1 Topological preliminaries

We state some preliminary results about relative cohomology, completing what stated in section 4.1.1.

Suspension isomorphism

Given a map of spaces $\rho\colon A \to X$, that we extend to $\rho_+\colon A_+ \to X_+$, we consider the induced map on the suspensions $\Sigma\rho_+\colon \Sigma A_+ \to \Sigma X_+$. There are canonical suspension isomorphisms

$$h^\bullet(\rho) \simeq h^{\bullet+1}(\Sigma\rho_+) \tag{4-99}$$

that can be proven from the axioms or in the following way. We observe that:

$$h^\bullet(\rho) \simeq \tilde{h}^\bullet(C(\rho)) \simeq \tilde{h}^{\bullet+1}(\Sigma(C(\rho))). \tag{4-100}$$

The space $\Sigma(C(\rho))$ is naturally homeomorphic to the *reduced* cone $C(\Sigma\rho_+)$, the latter having the same homotopy type of the unreduced one, therefore we can complete (4-100) as follows:

$$\tilde{h}^{\bullet+1}(\Sigma(C(\rho))) \simeq \tilde{h}^{\bullet+1}(C(\Sigma\rho_+)) \simeq h^{\bullet+1}(\Sigma\rho_+). \tag{4-101}$$

Compact support

If $\rho\colon A \to X$ is a proper map, we define $h^{\bullet}_{\mathrm{cpt}}(\rho) := h^{\bullet}(\rho^{+})$, being $\rho^{+}\colon A^{+} \to X^{+}$ the extension of ρ to the one-point compactifications of A and X. Given a generic map between *compact* spaces $\rho\colon A \to X$, we define $\rho \times \mathrm{id}_{\mathbb{R}^n}\colon A \times \mathbb{R}^n \to X \times \mathbb{R}^n$ in the obvious way and we get the natural isomorphism

$$\int_{\mathbb{R}^n} : h^{\bullet+n}_{\mathrm{cpt}}(\rho \times \mathrm{id}_{\mathbb{R}^n}) \xrightarrow{\simeq} h^{\bullet}(\rho) \tag{4-102}$$

defined as follows. Since $(\rho \times \mathrm{id}_{\mathbb{R}^n})^{+} \simeq \Sigma^n \rho_{+}$, we apply the suspension isomorphism $h^{\bullet+n}_{\mathrm{cpt}}(\rho \times \mathrm{id}_{\mathbb{R}^n}) \simeq h^{\bullet+n}(\Sigma^n \rho_{+}) \simeq h^{\bullet}(\rho)$.

Long exact sequence of a triple

Given a couple of morphism $\eta\colon B \hookrightarrow A$ and $\rho\colon A \to X$, such that η and $\rho \circ \eta$ are embeddings, we get the following long exact sequence, generalizing the long exact sequence of a triple:

$$\cdots \longrightarrow h^n(\rho) \xrightarrow{(\mathrm{id},\eta)^*} h^n(\rho \circ \eta) \xrightarrow{(\rho,\mathrm{id})^*} h^n(\eta) \xrightarrow{\beta} h^{n+1}(\rho) \longrightarrow \cdots . \tag{4-103}$$

We have considered the natural morphisms $(\mathrm{id}, \eta)\colon \rho \circ \eta \to \rho$ and $(\rho, \mathrm{id})\colon \eta \to \rho \circ \eta$. Such a sequence can be deduced from the axioms or, when (A, B) is a CW-pair, from the long exact sequence of the pair $(C(\rho), C(\rho \circ \eta))$. In fact, since $\rho \circ \eta$ is an embedding, we have that $C(\rho \circ \eta) \subset C(\rho)$ and, since η is an embedding, $C(\rho)/C(\rho \circ \eta) \simeq \Sigma(A/B)$.

A simple lemma

Finally, we state a brief lemma, that will be used in the following. Let $\rho\colon A \to X$ be a continuous function and $i\colon B \hookrightarrow A$ a cofibration. We set $B' := \rho(B)$ and we call $\bar{\rho}\colon A/B \to X/B'$ the projection of ρ to the quotient. We get the following diagram:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ A/B \arrow[r, "\bar{\rho}"] & X/B'. \end{tikzcd}$$

Lemma 4.6.1. With the data introduced above, if $\rho|_B\colon B \to B'$ is a homeomorphism, then $(\pi, \pi')^*\colon h^{\bullet}(\bar{\rho}) \to h^{\bullet}(\rho)$ is an isomorphism.

Proof. Because of diagram (4-3), it is enough to prove that $C(\pi, \pi')^*\colon \tilde{h}^{\bullet}(C(\bar{\rho})) \to \tilde{h}^{\bullet}(C(\rho))$ is an isomorphism. We show that the map $C(\pi, \pi')\colon C(\rho) \to C(\bar{\rho})$ is a homotopy equivalence, thus the result follows. In fact, since $\rho|_B$ is a homeomorphism with the image, the cone $C(\rho|_B)$ is contractible. We consider the space $C'(\rho) := C(\rho)/C(\rho|_B)$, which is canonically homeomorphic to the quotient of $C(\bar{\rho})$ by the cone of the point $B/B \simeq B'/B'$. We get the following diagram:

$$
\begin{tikzcd}
C(\bar{\rho}) \arrow[rr, "C(\pi,\pi')"] \arrow[dr, "p'"'] & & C(\rho) \arrow[dl, "p"] \\
& C'(\rho). &
\end{tikzcd}
$$

The projections p and p' are both homotopy equivalences, since they collapse to a point a contractible sub-space, whose embedding is a cofibration. Thus, $C(\pi, \pi')$ is a homotopy equivalence too. $\square$

4.6.2 Push-outs and homotopy push-outs

Given two continuous functions $f\colon Z \to X$ and $g\colon Z \to Y$, we denote by $X \sqcup_Z Y$ the push-out and by $X \coprod_Z Y$ the homotopy push-out defined by f and g. We briefly recall the two constructions. The corresponding diagrams are the following:

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_X"] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "i_Y"] \arrow[drr, bend right, "k"'] & X \sqcup_Z Y \arrow[dr, dotted, "\varphi"] & \\
& & W
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow F"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "j_Y"] \arrow[drr, bend right, "k"'] & X \coprod_Z Y \arrow[dr, dotted, "\psi"] & \\
& & W.
\end{tikzcd}
\tag{4-104}
$$

In the first diagram, the push-out is the triple $(X \sqcup_Z Y, i_X, i_Y)$. Given h e k, there exists a unique function φ making the diagram commutative. In the second diagram, the homotopy push-out is the quadruple $(X \coprod_Z Y, j_X, j_Y, F)$. Given h, k and G there exists a triple (ψ, H, K), unique up to homotopy, making the diagram 2-commutative. The space $X \sqcup_Z Y$ is unique up to canonical homeomorphism and the space $X \coprod_Z Y$ is unique up to canonical homotopy equivalence, in both cases coherently with the maps involved in the definition. One canonical representative for the push-out is the following:

$$
X \sqcup_Z Y = X \sqcup Y/\sim,\ f(z) \sim g(z)\, \forall z \in Z \qquad i_X(x) = [x] \qquad i_Y(y) = [y]. \tag{4-105}
$$

One canonical representative for the homotopy push-out is the following:

$$
\begin{aligned}
&X \coprod_Z Y = X \sqcup (Z \times I) \sqcup Y/\sim,\ (z,0) \sim f(z),\ (z,1) \sim g(z)\ \forall z \in Z \qquad (4\text{-}106)\\
&j_X(x) = [x] \qquad j_Y(y) = [y] \qquad F(z,t) = [(z,t)].
\end{aligned}
$$

There is a natural map $p\colon X \coprod_Z Y \to X \sqcup_Z Y$ defined by the following diagram:

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow F"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "i_X"] \arrow[dr, phantom, "\Nearrow H"] & \\
Y \arrow[r, "j_Y"] \arrow[drr, bend right, "i_Y"'] & X \coprod_Z Y \arrow[dr, dotted, "p"] \arrow[d, phantom, "\Nearrow K"] & \\
& & X \sqcup_Z Y.
\end{tikzcd}
\tag{4-107}
$$

The homotopy G is trivial. With respect to the canonical representatives of $X \coprod_Z Y$ and $X \sqcup_Z Y$, the map p is defined by $[x] \mapsto [x]$, $[y] \mapsto [y]$ and $[(z,t)] \mapsto [f(z)] = [g(z)]$. The homotopies H and K are trivial too.

Lemma 4.6.2. If one of the two maps f and g in diagram (4-107) is a cofibration, then the map p appearing in the same diagram is a homotopy equivalence.

Proof. By [38, Prop. 5.3.2 p. 112] the push-out $X \sqcup_Z Y$ is a homotopy push-out too, hence $X \sqcup_Z Y$ and $X \coprod_Z Y$ have the same homotopy type. In particular, we consider the following two diagrams:

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow F"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "i_X"] \arrow[dddrr, bend left, "j_X"] & & \\
Y \arrow[r, "j_Y"] \arrow[drr, bend right, "i_Y"'] \arrow[ddrrr, bend right, "j_Y"'] & X \coprod_Z Y \arrow[dr, dotted, "p"] & & \\
& & X \sqcup_Z Y \arrow[dr, dotted, "q"] \arrow[r, phantom, "\Nearrow H"] \arrow[d, phantom, "\Nearrow K"] & \arrow[d, phantom, "\Nearrow F"] \\
& & & X \coprod_Z Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_X"] \arrow[ddr, bend left, "j_X"] \arrow[dddrr, bend left, "i_X"] \arrow[dr, phantom, "\Nearrow H"] & & \\
Y \arrow[r, "i_Y"] \arrow[drr, bend right, "j_Y"'] \arrow[ddrrr, bend right, "i_Y"'] & X \sqcup_Z Y \arrow[dr, dotted, "q"] \arrow[d, phantom, "\Nearrow K"] \arrow[r, phantom, "\Nearrow F"] & & \\
& & X \coprod_Z Y \arrow[dr, dotted, "p"] & \\
& & & X \sqcup_Z Y.
\end{tikzcd}
$$

Because of the uniqueness up to homotopy of the induced map, $q \circ p$ and $p \circ q$ are homotopic to the corresponding identities, hence p and q are homotopy equivalences. $\square$

Let us consider two maps $h\colon X \to W$ and $k\colon Y \to W$ such that $h \circ f = k \circ g$. In the first diagram of (4-104) the map φ is uniquely defined. If, in the second diagram of (4-104), we choose G to be the constant homotopy, the induced map ψ is $\varphi \circ p$:

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[r, "\psi"] \arrow[d, "p"'] & W \arrow[d, "\mathrm{id}"] \\
X \sqcup_Z Y \arrow[r, "\varphi"] & W.
\end{tikzcd}
\tag{4-108}
$$

Lemma 4.6.3. With the hypotheses stated above diagram (4-108), if one of the two maps f and g is a cofibration, the morphism

$$
(\mathrm{id}, p)_* \colon h^\bullet(\psi) \xrightarrow{\simeq} h^\bullet(\varphi) \tag{4-109}
$$

is a (canonical) isomorphism.

Proof. We show that the induced map of pointed spaces $C(\mathrm{id}, p)\colon C(\psi) \to C(\varphi)$ is a homotopy equivalence. In fact, from diagram (4-108) we get the following:

$$\begin{tikzcd} W \arrow[d, "\mathrm{id}"'] & X \coprod_Z Y \arrow[l, "\psi"'] \arrow[r, hook, "i"] \arrow[d, "p"] & C(X \coprod_Z Y) \arrow[d, "C(p)"] \\ W & X \sqcup_Z Y \arrow[l, "\varphi"'] \arrow[r, hook, "j"] & C(X \sqcup_Z Y). \end{tikzcd} \tag{4-110}$$

Since i and j are closed cofibrations and, because of lemma 4.6.2, the three vertical maps are homotopy equivalences, by [7, Theorem 7.5.7 p. 294] the induced map between the push-outs $W \sqcup_{X \coprod_Z Y} C(X \coprod_Z Y)$ and $W \sqcup_{X \sqcup_Z Y} C(X \sqcup_Z Y)$ is a homotopy equivalence. Such a map coincides with $C(\mathrm{id}, p)$. Since p and id are both homotopy equivalences, the commutativity of diagram (4-3) implies that the map (4-109) is an isomorphism. □

4.6.3 Product in relative cohomology

We call $\mathcal{R}_{\mathbb{Z}}$ be the category of $\mathbb{Z}$-graded commutative rings with unit. A *multiplicative cohomology theory* on $\mathcal{C}_2$ is defined by a functor $h^\bullet\colon \mathcal{HC}_2 \to \mathcal{R}_{\mathbb{Z}}$, refining a cohomology theory $h^\bullet\colon \mathcal{HC}_2 \to \mathcal{A}_{\mathbb{Z}}$, satisfying a suitable compatibility axiom with the Bockstein morphisms. We can define the exterior product $h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(\eta) \to h^\bullet(\rho \times \eta)$ by $(\alpha, \beta) \mapsto \pi_1^*\alpha \cdot \pi_2^*\beta$, being $\pi_1\colon \rho\times\eta \to \rho$ and $\pi_2\colon \rho\times\eta \to \eta$ the canonical projections. For pair of spaces, such a product corresponds to the exterior product $h^\bullet(X, A) \times h^\bullet(Y, B) \to h^\bullet(X \times Y, A \times B)$. Actually, for CW-pairs, we can also define the following product:

$$\cdot : h^\bullet(X, A) \times h^\bullet(Y, B) \to h^\bullet(X \times Y, (X \times B) \cup (A \times Y)). \tag{4-111}$$

In fact, since

$$X \times Y/((X \times B) \cup (A \times Y)) \simeq (X/A) \wedge (Y/B),$$

the product (4-111) is equivalent to $\tilde{h}^\bullet(X/A) \times \tilde{h}^\bullet(Y/B) \to \tilde{h}^\bullet(X/A \wedge Y/B)$, which is of course well-defined. We generalize (4-111) to generic morphisms. Given $\rho\colon A \to X$ and $\eta\colon B \to Y$, we define the function

$$\rho \wedge \eta\colon (X \times B) \coprod_{A\times B} (A \times Y) \to X \times Y \tag{4-112}$$

via the following homotopy push-out diagram:

$$\begin{tikzcd} A \times B \arrow[r, "\rho\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\eta"'] \arrow[dr, phantom, "\overset{\Phi}{\Rrightarrow}"] & X \times B \arrow[d, "j_{(X\times B)}"] \arrow[ddr, bend left, "\mathrm{id}\times\eta"] & \\ A \times Y \arrow[r, "j_{(A\times Y)}"] \arrow[drr, bend right, "\rho\times\mathrm{id}"'] & (X \times B) \coprod_{A\times B} (A \times Y) \arrow[dr, dotted, "\rho\wedge\eta"] & \\ & & X \times Y \end{tikzcd} \tag{4-113}$$

where, using the canonical representative (4-106):

- $(X \times B) \coprod_{A\times B}(A \times Y)$ is the space $(X \times B) \sqcup (A \times B \times I) \sqcup (A \times Y)/\sim$, identifying $(a, b, 1) \in A \times B \times I$ with $(a, \eta(b)) \in A \times Y$ and $(a, b, 0) \in A \times B \times I$ with $(\rho(a), b) \in X \times B$;
- $\Phi(a, b, t) := [(a, b, t)]$ and the homotopy between $(\mathrm{id}\times\eta)\circ(\rho\times\mathrm{id})$ and $(\rho\times\mathrm{id})\circ(\mathrm{id}\times\eta)$ is the identity;
- $\rho \wedge \eta$ is defined by $(x, b) \mapsto (x, \eta(b))$, $(a, y) \mapsto (\rho(a), y)$ and $(a, b, t) \mapsto (\rho(a), \eta(b))$;
- the two homotopies involving $\rho \wedge \eta$ are the identities.

We have a canonical homeomorphism $C(\rho\wedge\eta) \approx C(\rho)\wedge C(\eta)$, respecting the marked points. This means that the two functors from $\mathcal{C}_2 \times \mathcal{C}_2$ to $\mathcal{C}_+$, defined by $(\rho, \eta) \mapsto C(\rho \wedge \eta)$ and $(\rho, \eta) \mapsto C(\rho) \wedge C(\eta)$, are isomorphic. In fact, let us consider the function $\Theta\colon I \times I \to I \times I$ such that, for $0 \leq t \leq \frac{1}{2}$, $\Theta(t, u) = u(2t, 1)$ and, for $\frac{1}{2} \leq t \leq 1$, $\Theta(t, u) = u(1, 2 - 2t)$. The function Θ sends the base $I \times \{0\}$ to the origin and the upper side $I \times \{1\}$ to the upper and right sides $(I \times \{1\}) \cup (\{1\} \times I)$. The following function is a homeomorphism:

$$\begin{aligned}
\Psi\colon\ & C(\rho \wedge \eta) \to C(\rho) \wedge C(\eta)\\
& [(x, y)] \mapsto [(x, y)] \quad [(x, b, u)] \mapsto [x, (b, u)] \quad [(a, y, u)] \mapsto [(a, u), b]\\
& [(a, b, t, u)] \mapsto [(a, \Theta_1(t, u)), (b, \Theta_2(t, u))].
\end{aligned}$$

We define the exterior product

$$\times\colon h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(\eta) \to h^\bullet(\rho \wedge \eta) \tag{4-114}$$

composing the product $\tilde{h}^\bullet(C(\rho)) \times \tilde{h}^\bullet(C(\eta)) \to \tilde{h}^\bullet(C(\rho) \wedge C(\eta))$ with Ψ^*.

If one of the two maps, e.g. η, is a cofibration, we can apply the isomorphism (4-109) and replace the homotopy push-out by the push-out. In particular, we define the function

$$\rho\bar{\wedge}\eta\colon (X \times B) \sqcup_{A\times B} (A \times Y) \to X \times Y \tag{4-115}$$

replacing $(X \times B) \coprod_{A\times B}(A \times Y)$ by $(X \times B) \sqcup_{A\times B} (A \times Y)$ in diagram (4-113). It follows that $\rho\bar{\wedge}\eta$ is defined by $(x, b) \mapsto (x, \eta(b))$ and $(a, y) \mapsto (\rho(a), y)$. We get diagram (4-108) with the spaces we are considering now:

$$\begin{array}{ccc}
(X \times B) \coprod_{A\times B}(A \times Y) & \xrightarrow{\rho\wedge\eta} & X \times Y\\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{id}}\\
(X \times B) \sqcup_{A\times B} (A \times Y) & \xrightarrow{\rho\bar{\wedge}\eta} & X \times Y.
\end{array} \tag{4-116}$$

Composing with the isomorphism (4-109), we get the product

$$\times\colon h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(\eta) \to h^\bullet(\rho\bar{\wedge}\eta). \tag{4-117}$$

When both ρ and η are cofibrations, $\rho\bar{\wedge}\eta$ is the inclusion of $(A \times Y) \cup (X \times B)$ in $X \times Y$, hence we recover (4-114).

4.6.4 Relative Thom isomorphism

Let us fix the following data:

- a map $\rho\colon A \to X$ between CW-complexes;
- two vector bundles $\pi\colon E \to X$ and $\pi'\colon F \to A$ of rank n;
- a morphism of bundles $\tilde{\rho}\colon F \to E$ covering ρ and inducing an isomorphism in each fibre;[14]
- a Thom class u of E with respect to $h^\bullet$.

It follows that $\tilde{\rho}^* u$ is a Thom class of F. We construct the relative Thom isomorphism:

$$T\colon h^\bullet(\rho) \to h^{\bullet+n}_{\mathrm{cpt}}(\tilde{\rho}). \tag{4-118}$$

We call E_0 the sub-bundle of E containing the complement of the unit ball in each fibre, with respect to any fixed metric. We call F_0 the analogous sub-bundle of F, with respect to the pull-back metric via $\tilde{\rho}$. It follows that $F_0 = \tilde{\rho}^{-1}(E_0)$, hence $\tilde{\rho}(F_0) \subset E_0$. In particular, we can think of the Thom classes as $u \in h^n(E, E_0)$ and $\tilde{\rho}^*(u) \in h^n(F, F_0)$. Moreover, we consider the push-out $E_0 \sqcup_{F_0} F$ with respect to the inclusion $F_0 \hookrightarrow F$ and $\tilde{\rho}|_{F_0}$. There is a natural map $\bar{\rho}\colon E_0 \sqcup_{F_0} F \to E$ induced by the following push-out diagram:

$$\begin{tikzcd}
F_0 \arrow[r, "\tilde{\rho}|_{F_0}"] \arrow[d, hook] & E_0 \arrow[d] \arrow[ddr, hook, bend left] & \\
F \arrow[r] \arrow[drr, bend right, "\tilde{\rho}"'] & E_0 \sqcup_{F_0} F \arrow[dr, dotted, "\bar{\rho}"] & \\
& & E.
\end{tikzcd}$$

There is a canonical isomorphism $h^\bullet(\bar{\rho}) \simeq h^\bullet_{\mathrm{cpt}}(\tilde{\rho})$, due to the following diagram:

$$\begin{tikzcd}
F^+ \arrow[r, "\tilde{\rho}^+"] \arrow[d, "\pi'"'] & E^+ \arrow[d, "\pi"] \\
(F \setminus F_0)^+ \arrow[r, "\tilde{\rho}_0^+"] & (E \setminus E_0)^+ \\
E_0 \sqcup_{F_0} F \arrow[r, "\bar{\rho}"] \arrow[u, "p'"] & E. \arrow[u, "p"']
\end{tikzcd}$$

The map $\tilde{\rho}_0$ is the restriction of $\tilde{\rho}$. The maps π and π' collapse de complement of the unit ball to the infinity in each fibre. They are both homotopy equivalences, hence, by the five

[14] Such a morphism is what is called a bundle map in [30], and it is equivalent to an isomorphism between F and $\rho^* E$.

lemma, $(\pi, \pi')^* \colon h^\bullet_{\mathrm{cpt}}(\tilde{\rho}_0) \to h^\bullet_{\mathrm{cpt}}(\tilde{\rho})$ is an isomorphism. Finally, $(p, p')^* \colon h^\bullet_{\mathrm{cpt}}(\tilde{\rho}_0) \to h^\bullet(\bar{\rho})$ is an isomorphism because of lemma 4.6.1, where the space B of the lemma is the complement of $F \setminus F_0$ in $E_0 \sqcup_{F_0} F$. By composition, we get the isomorphism $h^\bullet(\bar{\rho}) \simeq h^\bullet_{\mathrm{cpt}}(\tilde{\rho})$.

Thus, we construct the isomorphism (4-118) in the following form:

$$T \colon h^\bullet(\rho) \to h^{\bullet+n}(\bar{\rho}). \tag{4-119}$$

In order to define (4-119), we need the following natural product:

$$h^\bullet(E, E_0) \otimes_{\mathbb{Z}} h^\bullet(\tilde{\rho}) \to h^\bullet(\bar{\rho}) \tag{4-120}$$

defined as follows. Starting from $\tilde{\rho} \colon F \to E$ and the embedding $\iota \colon E_0 \hookrightarrow E$, which is a closed cofibration, we apply the product (4-117), i.e. $h^\bullet(\iota) \otimes_{\mathbb{Z}} h^\bullet(\tilde{\rho}) \to h^\bullet(\iota \bar{\wedge} \tilde{\rho})$. The map $\iota \bar{\wedge} \tilde{\rho} \colon (E \times F) \sqcup_{E_0 \times F} (E_0 \times E) \to E \times E$ behaves in the following way: $(e, f) \mapsto (e, \tilde{\rho}(f))$ and $(e_0, e) \mapsto (e_0, e)$. Moreover, we define the map $\tilde{\rho}_1 \colon E_0 \sqcup_{F_0} F \to (E \times F) \sqcup_{E_0 \times F} (E_0 \times E)$ via the following push-out diagram:

$$\begin{tikzcd}
F_0 \arrow[r, "\tilde{\rho}|_{F_0}"] \arrow[d, hook] & E_0 \arrow[d] \arrow[ddr, bend left, "e_0 \mapsto (e_0, e_0)"] & \\
F \arrow[r] \arrow[drr, bend right, "f \mapsto (\tilde{\rho}(f), f)"'] & E_0 \sqcup_{F_0} F \arrow[dr, dotted, "\tilde{\rho}_1"] & \\
& & (E \times F) \sqcup_{E_0 \times F} (E_0 \times E).
\end{tikzcd}$$

It follows that $\tilde{\rho}_1(f) = (\tilde{\rho}(f), f)$ and $\tilde{\rho}_1(e_0) = (e_0, e_0)$. We get the following commutative diagram:

$$\begin{tikzcd}
E_0 \sqcup_{F_0} F \arrow[r, "\bar{\rho}"] \arrow[d, "\tilde{\rho}_1"'] & E \arrow[d, "\Delta"] \\
(E \times F) \sqcup_{E_0 \times F} (E_0 \times E) \arrow[r, "\iota \bar{\wedge} \tilde{\rho}"] & E \times E.
\end{tikzcd}$$

Here $\Delta \colon E \to E \times E$ is the diagonal map. The product (4-120) is defined as $(\alpha, \beta) \mapsto (\Delta, \tilde{\rho}_1)^*(\alpha \cdot \beta)$. Thanks to this product we can define the Thom isomorphism (4-119) by:

$$T(\alpha) := u \cdot (\pi, \pi')^* \alpha.$$

Let us show that T is an isomorphism. We consider the Thom isomorphisms associated to E, F, and ρ. The domains of such isomorphisms are $h^\bullet(X)$, $h^\bullet(Y)$ and $h^\bullet(\rho)$, that fit in the long exact sequence associated to ρ. The codomains are $h^\bullet(E, E_0)$, $h^\bullet(F, F_0)$ and $h^\bullet(\bar{\rho})$. We show that they also fit in a long exact sequence and we apply the five lemma. Such codomains can be described as $h^\bullet(\iota)$, $h^\bullet(j)$ and $h^\bullet(\bar{\rho})$, where $\iota \colon E_0 \hookrightarrow E$ and $j \colon F_0 \hookrightarrow F$ are the embeddings. The cohomology of j is naturally isomorphic to the one of the embedding $j' \colon E_0 \hookrightarrow E_0 \sqcup_E F$, defined by $e_0 \mapsto e_0$, because, being A and X CW-complexes, $h^\bullet(j) \simeq \tilde{h}^\bullet(F/F_0) \simeq \tilde{h}^\bullet((E_0 \sqcup_E F)/E_0) \simeq h^\bullet(j')$. Since $\bar{\rho} \circ j' = \iota$, we get the associated long exact sequence (4-103). Finally, we get the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & h^n(\rho) \arrow[r, "i^*"] \arrow[d, "T_{\bar{\rho}}"] & h^n(X) \arrow[r, "\rho^*"] \arrow[d, "T_E"] & h^n(A) \arrow[r, "\beta"] \arrow[d, "T_F"] & h^{n+1}(\rho) \arrow[r] \arrow[d, "T_{\bar{\rho}}"] & \cdots \\ \cdots \arrow[r] & h^n(\bar{\rho}) \arrow[r, "(\mathrm{id},j')^*"] & h^n(E, E_0) \arrow[r, "(\bar{\rho},\mathrm{id})^*"] & h^n(F, F_0) \arrow[r, "\beta'"] & h^{n+1}(\bar{\rho}) \arrow[r] & \cdots . \end{tikzcd}$$

It follows from the five lemma that $T_{\bar{\rho}}$ is an isomorphism.

4.6.5 Relative Gysin map

Let us fix the following data, analogous to the ones fixed in order to define the Thom isomorphism:

- a smooth map $\rho \colon A \to X$ between compact manifolds, possibly with boundary (the map is not necessarily neat);
- two neat proper submersions $f \colon Y \to X$ and $g \colon B \to A$ with n-dimensional compact fibres;
- a morphism of fibre bundles $\tilde{\rho} \colon B \to Y$ covering ρ and inducing a diffeomorphism in each fibre;[15]
- a proper representative (ι, u, φ) of an $h^\bullet$-orientation of f.

We get the following diagram:

$$\begin{tikzcd} B \arrow[r, "\tilde{\rho}"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ A \arrow[r, "\rho"] & X. \end{tikzcd} \tag{4-121}$$

We are going to define the relative Gysin map:

$$(f, g)_! \colon h^\bullet(\tilde{\rho}) \to h^{\bullet - n}(\rho). \tag{4-122}$$

Considering the embedding $\iota \colon Y \hookrightarrow X \times \mathbb{R}^n$, which is the first component of the fixed representative (ι, u, φ), we call $\kappa := \pi_{\mathbb{R}^N} \circ \iota$, so that $\iota(y) = (f(y), \kappa(y))$. It is easy to prove that the map $j \colon B \to A \times \mathbb{R}^N$, defined by $j(b) := (g(b), \kappa \circ \tilde{\rho}(b))$, is an injective immersion, hence, being B compact, an embedding. Thus, calling $\rho_1 := (\rho, \mathrm{id}) \colon A \times \mathbb{R}^N \to X \times \mathbb{R}^N$, we get the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\tilde{\rho}"] \arrow[d, hook, "j"'] & Y \arrow[d, hook, "\iota"] \\ A \times \mathbb{R}^N \arrow[r, "\rho_1"] & X \times \mathbb{R}^N. \end{tikzcd}$$

[15] Such a morphism is equivalent to a bundle isomorphism between B and $\rho^* X$.

Let us consider the map $d\rho_1\colon T(A \times \mathbb{R}^N) \to T(X \times \mathbb{R}^N)$. We have that $d\rho_1(dj(TB)) = d\iota(d\tilde{\rho}(TB)) \subset d\iota(TY)$, hence, passing to the quotient, we get a well defined bundle morphism $\tilde{\tilde{\rho}} := [d\rho_1]\colon N_{j(B)}(A \times \mathbb{R}^N) \to N_{\iota(Y)}(X \times \mathbb{R}^N)$. Let us show that it induces an isomorphism in each fibre. We call $N := N_{\iota(Y)}(X \times \mathbb{R}^N)$ and $N' := N_{j(B)}(A \times \mathbb{R}^N)$. Since g is a submersion, a vector $[(v, v')] \in N'_{j(b)}$ can always be represented by a couple $(0, w)$, with $w \in T\mathbb{R}^N$. We have that $d(\rho_1)_{j(b)}(0, w) = (0, w)$. If $(0, w) = d\iota(w') \in d\iota(TY)$, then $w' \in T(Y_{\rho\circ g(b)})$, since $df(w') = 0$, hence there exists a unique $w'' \in T(B_{g(b)})$ such that $d\tilde{\rho}(w'') = w'$. It follows that $dj(w'') = (dg(w''), d\kappa \circ d\tilde{\rho}(w'')) = (0, d\kappa(w')) = (0, w)$, thus $[(0, w)] = 0$. This shows that $\tilde{\tilde{\rho}}$ is injective in each fibre. Since the fibres of N and N' have the same dimension, it is an isomorphism in each fibre. Thus, we get the following diagram:

$$\begin{tikzcd} N' \arrow[r, "\tilde{\tilde{\rho}}"] \arrow[d, "\pi'"'] & N \arrow[d, "\pi"] \\ B \arrow[r, "\tilde{\rho}"] & Y \end{tikzcd}$$

and we can apply the relative Thom isomorphism $T\colon h^\bullet(\tilde{\rho}) \to h^{\bullet+N-n}_{\mathrm{cpt}}(\tilde{\tilde{\rho}})$. Let us now consider the tubular neighbourhood U of $\iota(Y)$. It follows that $V := \rho_1^{-1}(U)$ is a tubular neighborhood of B in $A \times \mathbb{R}^N$ and that the map $\psi\colon N' \to V$, defined by $v \mapsto (g \circ \pi'(v), \pi_{\mathbb{R}^N} \circ \varphi \circ \tilde{\tilde{\rho}}(v))$, is a diffeomorphism. We get the following commutative diagram:

$$\begin{tikzcd} N' \arrow[r, "\tilde{\tilde{\rho}}"] \arrow[d, "\psi"'] & N \arrow[d, "\varphi"] \\ V \arrow[r, "\rho_1|_V"] & U. \end{tikzcd}$$

Thus, given $\alpha \in h^\bullet(\tilde{\rho})$, we define (4-122) by:

$$(f, g)_!(\alpha) := \int_{\mathbb{R}^N} (i_U, i_V)_* (\varphi, \psi)_* T(\alpha)$$

where:

- $(\varphi, \psi)_*\colon h^{\bullet+N-n}_{\mathrm{cpt}}(\tilde{\tilde{\rho}}) \to h^{\bullet+N-n}_{\mathrm{cpt}}(\rho_1|_V)$ is the pull-back induced by $(\varphi^{-1}, \psi^{-1})$;
- $(i_U, i_V)_*\colon h^{\bullet+N-n}_{\mathrm{cpt}}(\rho_1|_V) \to h^{\bullet+N-n}_{\mathrm{cpt}}(\rho_1)$ is the push-forward in compactly-supported cohomology via the open embedding (i_U, i_V), i.e. the pull-back induced by the following morphism (ψ_U, ψ_V):

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\psi_V"'] & X \arrow[d, "\psi_U"] \\ V^+ \arrow[r, "(\rho_1|_V)^+"] & U^+ \end{tikzcd}$$

  where ψ_U and ψ_V fix the points of the corresponding tubular neighbourhood and send its complement to the point at infinity.

The same proofs of the absolute case show that the Gysin map $(f,g)_!$ only depends on the $h^\bullet$-orientation $[\iota,\varphi]$ of f, not on the specific representative ([23, theorem 5.24 p. 233], [8, sec. 4.9]). Moreover, the Gysin map is a morphism of $h^\bullet(X)$-moduless, i.e., for any $\alpha \in h^\bullet(\tilde{\rho})$ and $\beta \in h^\bullet(\rho)$ we have:

$$(f,g)_!(\alpha \cdot (f,g)^*\beta) = (f,g)_!(\alpha) \cdot \beta. \tag{4-123}$$

For the proof see [23, theorem 5.24 p. 233], adapted to the relative case. Moreover, using the module structure (4-4), for any $\alpha \in h^\bullet(\tilde{\rho})$ and $\beta \in h^\bullet(X)$ in diagram (4-121) we have:

$$(f,g)_!(\alpha \cdot f^*\beta) = (f,g)_!(\alpha) \cdot \beta. \tag{4-124}$$

In [8] and [34] we have shown that, given two $h^\bullet$-oriented maps $f_1 \colon Y \to X$ and $f_2 \colon X \to W$, with orientations $[\iota, u]$ and $[\kappa, v]$, there is a naturally induced $h^\bullet$-orientation on $f_2 \circ f_1 \colon Y \to W$, that we denote by $[\kappa, v][\iota, u]$, such that $(f_2 \circ f_1)_! = (f_2)_! \circ (f_1)_!$. Moreover, the 2x3 rule about the orientations of f_1, f_2 and $f_2 \circ f_1$ holds. The same considerations apply to the relative case, considering compositions of relative submersions, with the hypotheses stated at the beginning of this section.

Remark 4.6.4. We can relax the hypotheses introduced above in order to the define the Gysin map. In particular, it is enough that f is a submersion in a neighbourhood of the image $\rho(A)$. In fact, in this case, we can anyway choose the tubular neighbourhoods and the corresponding diffeomorphisms coherently for f and g, following the previous constructions.

Remark 4.6.5. The construction of the relative Gysin map may hold even in more general cases, without assuming that f and g are submersions at all. For example, given two manifolds Y and X and two closed sub-manifolds $B \subset Y$ and $A \subset X$, we can consider an embedding of pairs $i \colon (Y,B) \hookrightarrow (X,A)$, i.e. an embedding $i \colon Y \hookrightarrow X$ such that $i(B) \subset A$. In this case, if we are able to choose a tubular neighbourhood U of $i(Y)$ and a diffeomorphism $\varphi \colon N_{i(Y)}X \to U$, in such a way that they restrict to a tubular neighbourhood U' of $i(B)$ and to a diffeomorphism $\varphi' \colon N_{i(B)}A \to U'$, then the Gysin map $(i, i|_B)_!$ is well-defined. The point is that such condition is not verified in general.

We can apply the remark 4.6.5 to the following case.

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