SPARSE 1D DISCRETE DIRAC OPERATORS I: QUANTUM TRANSPORT

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Abstract. Some dynamical lower bounds for one-dimensional discrete Dirac operators with different classes of sparse potentials are presented, and the particular role of the particle mass is emphasized.

Keywords. *Dirac operators, sparse potentials, quantum transport.*

1. Introduction

In this paper we consider discrete Dirac operators

$$\mathbf{D}(m,c) := \mathbf{D}_0(m,c) + V\mathrm{I}_2 = \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix} + V\mathrm{I}_2, \qquad (1)$$

with boundary conditions so that (1) is self-adjoint, acting in $\ell^2(\mathbb{N}, \mathbb{C}^2)$, where $c > 0$ represents the speed of light, $m \geq 0$ the mass of the particle, I_2 is the 2×2 identity matrix and D is the finite difference operator defined by $(D\varphi)(n) = \varphi(n+1) - \varphi(n)$, with adjoint $(D^*\varphi)(n) = \varphi(n-1) - \varphi(n)$. The potential $V : \mathbb{N} \to \mathbb{R}$ is assumed to be polynomially bounded, that is, there exist constants $a, b > 0$ such that

$$|V(n)| \leq a(1+n^2)^{b/2}\ , \quad \forall n \in \mathbb{N} = \{1, 2, 3, \cdots\}. \qquad (2)$$

Model (1) was introduced in [10, 11] as a relativistic version of the more common tight-binding Schrödinger operator, and it was further studied in [3, 18, 19].

The goal of the present paper is to present some lower bounds of the dynamics generated by $\mathbf{D}(m,c)$ with sparse potentials (see ahead for precise statements); due to the particular role played by the zero mass case (i.e., $m = 0$) in inducing quantum transport for (1), at least when the potential is given by i.i.d. random variables with a Bernoulli law [11], we will pay special attention to how the dynamical exponents depend on the mass m for sparse potentials.

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There has been some interest in sparse potentials in the context of Schrödinger operators (see the recent paper [4] and references there in), mainly in obtaining precise spectral properties and dynamics. This work is the first one of a series of three papers devoted to the Dirac operator with sparse potentials; with another collaborator, in the second paper [5] we intend to discuss spectral type and its Hausdorff dimensional properties (under suitable sparseness conditions), and the third one will translate some of the results to the usual continuum Dirac operator.

The dynamics is dictated by the Dirac equation

$$i\frac{\partial}{\partial t}\psi_t = \mathbf{D}(m,c)\psi_t$$

with suitable initial condition $\psi_0 \in \ell^2(\mathbb{N},\mathbb{C}^2)$. We will present lower bounds for the averaged p-th moments at time $T>0$ defined by

$$M_m(p,f,T) := \frac{2}{T}\int_0^\infty e^{-2t/T}\left\|\langle X\rangle^{p/2} e^{-it\mathbf{D}(m,c)} f(\mathbf{D}(m,c))\delta_1^+\right\|^2 dt, \tag{3}$$

where $0 \le f \in C_0^\infty(I)$ is an infinitely differentiable function with compact support in the open interval I, δ_1^+ is the element of the canonical basis of $\ell^2(\mathbb{N},\mathbb{C}^2)$ equal to $\begin{pmatrix}1\\0\end{pmatrix}$ at $n=1$ and $\begin{pmatrix}0\\0\end{pmatrix}$ elsewhere, and X is the position operator $(\langle X\rangle\psi)(n) = \left(1+n^2\right)^{1/2}\psi(n)$, for ψ in its domain. Think of the open interval I as a particle energy restriction.

To investigate the polynomial behaviour of $M_m(p,f,T)$ as function of time T, we make use of the usual lower and upper transport exponents given by, respectively,

$$\beta_m^-(p,f) = \liminf_{T\to\infty}\frac{\log M_m(p,f,T)}{p\log T}\ ,\quad \beta_m^+(p,f) = \limsup_{T\to\infty}\frac{\log M_m(p,f,T)}{p\log T},$$

and to obtain transport rates nearby a given energy level, we follow [13] and introduce the local transport exponents

$$\beta_m^\pm(p,E) := \inf_{I\ni E}\sup_{f\in C_0^\infty(I)}\beta_m^\pm(p,f). \tag{4}$$

In the last years, problems of quantum transport through dynamical lower bounds have been of great interest, mainly for 1D Schrödinger operators. For instance, transport has been proven through the existence of critical energies for random palindrome [6] and (more generally) random polymer models [16], and Bernoulli-Dirac models [11, 18]. In [8] the authors have developed a general method which allows one to derive dynamical lower bounds from upper bounds on the growth of norms of transfer matrices, with applications to some substitution and prime models. An extension to 1D continuous Schrödinger operators, with application to the continuous Bernoulli-Anderson model, was presented in [7]. An approach to quasi-ballistic dynamics, for both discrete Dirac and Schrödinger operators, with potentials along some dynamical systems, has recently been obtained in [12].

Another method to obtain quantum transport from upper bounds on transfer matrices was proposed in [13], with applications to Schrödinger operators with random decaying potentials, discrete sparse potentials and to the Almost-Mathieu model. In this paper we will try to follow the method proposed in [13] with an extension of its ideas to the discrete Dirac model (1), particularly we will track the dependence of the exponents $\beta_m^{\pm}$ on the mass, including the pure relativistic possibility $m = 0$.

The general idea to get lower bounds are as follows. Let $\Phi_m(E, N, 1)$ denote the transfer matrices (see Section 2); if the norms of those matrices are polynomially bounded, i.e., $\|\Phi_m(E, N, 1)\| \leq C(E, m)N^{\gamma_m(E)}$, with $\gamma_m(E) < \infty$, on a bounded set S of positive Lebesgue measure of energies E, and for all N large enough, then one gets for all $p > 0$,

$$\beta_m^-(p, E) \geq 1 - \frac{2\gamma_m(E)}{p} \; . \tag{5}$$

A similar bound follows for the upper exponents $\beta_m^+(p, E)$ if, for some subsequence N_i, one checks $\|\Phi_m(E, N_i, 1)\| \leq C(E, m)N_i^{\gamma_m(E)}$, with $\gamma_m(E) < \infty$, for all $E \in S$.

We shall apply (5) to three class of sparse potentials of the form $V = \sum_{n=1}^{\infty} h_n \delta_{x_n}$, where $x_n > 0$ denotes the location of the nth barrier and $h_n \geq 0$ denotes its respective height (see Section 3). We shall obtain for the first class (i.e., $0 \leq h_n \leq a$ for all $n \geq 1$ and some $a > 0$) $\beta_m^-(p, E) \geq 1 - \frac{2\gamma_m(E)}{p}$, for second class (i.e., $h_n \to \infty$) $\beta_m^+(p, E) \geq 1 - \frac{\nu}{p}$, for some $\nu > 0$, and for third class (i.e., $h_n \to 0$) we have a *ballistic dynamics*, that is, $\beta_m^-(p, E) = 1$.

Since a modification of the boundary condition just corresponds to a rank-one perturbation, it has no impact on the growth of the norms of transfer matrices so, from now on, we will assume Dirichlet boundary conditions at the position $n = 0$, and only the lower component of the "spinor" will be assigned a boundary value at $n = 0$ (see Section 2).

The organization of the paper is as follows. In Section 2 we present an abstract result about quantum transport (Theorem 1) and its consequences (Theorems 2 and 3) for the Dirac model (1), whose proofs appear in Section 5. Section 3 is devoted to our main applications, that is, dynamical lower bounds for discrete Dirac operators with sparse potentials. In Section 4 we collect some preliminary results and spectral bounds that will be used in the proofs of Theorems 1, 2 and 3.

2. Dynamical Lower Bounds

In this section we present general results about quantum transport for the Dirac operators $\mathbf{D}(m, c)$ defined by (1). Write the transfer matrices [11] $\Phi_m(E, n, k)$ between sites k and n, for $n \geq k \geq 1$, as

$$\Phi_m(E, n, k) = \begin{pmatrix} u_D^+(E, n) & u_N^+(E, n) \\ u_D^-(E, n-1) & u_N^-(E, n-1) \end{pmatrix},$$

where $u_D = \begin{pmatrix} u_D^+ \\ u_D^- \end{pmatrix}$ and $u_N = \begin{pmatrix} u_N^+ \\ u_N^- \end{pmatrix}$ denote the solutions to the "eigenvalue" equation $\mathbf{D}(m,c)u = Eu$, $E \in \mathbb{R}$, satisfying, respectively,

$$\begin{pmatrix} u_D^+(E,k) \\ u_D^-(E,k-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_N^+(E,k) \\ u_N^-(E,k-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The subscripts D and N stand for Dirichlet and Neumann, respectively, in particular at position $k = 1$ they define the corresponding boundary conditions. It follows that if $u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$ is a general solution of $\mathbf{D}(m,c)u = Eu$, then

$$\begin{pmatrix} u^+(E,n) \\ u^-(E,n-1) \end{pmatrix} = \Phi_m(E,n,k) \begin{pmatrix} u^+(E,k) \\ u^-(E,k-1) \end{pmatrix}.$$

Hence the transfer matrix $\Phi_m(E,n,k)$ can be written as

$$\Phi_m(E,n,k) = \begin{cases} T_m(E,V(n-1)) \cdots T_m(E,V(k)) & n > k \geq 1, \\ \\ I_2 & n = k, \end{cases}$$

with

$$T_m(E,V(j)) = \begin{pmatrix} 1 + \dfrac{m^2c^4 - (E - V(j))^2}{c^2} & \dfrac{mc^2 + E - V(j)}{c} \\ \\ \dfrac{mc^2 - E + V(j)}{c} & 1 \end{pmatrix}.$$

We denote by $\delta_n^\pm$ the elements of the canonical position basis of $\ell^2(\mathbb{N}, \mathbb{C}^2)$, $n \geq 1$, for which all entries are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ except the nth one, which is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the superscript indices $+$ and $-$, respectively.

Introduce the measurable function $\gamma_m : \mathbb{R} \longrightarrow [0, +\infty]$ by

$$\gamma_m(E) := \limsup_{n \to +\infty} \frac{\log \|\Phi_m(E,n,1)\|}{\log n} \;. \tag{6}$$

Moreover, we shall denote by ℓ the Lebesgue measure, by $[a]$ the integer part of the real number a, by $\sigma(\mathbf{D}(m,c))$ the spectrum of the operator $\mathbf{D}(m,c)$ and by μ_m (resp. $\mu_{m,f}$) the spectral measure for $\mathbf{D}(m,c)$ associated to the initial state δ_1^+ (resp. $f(\mathbf{D}(m,c))\delta_1^+$). Introduce the local spectral moments [13]

$$K_{\mu_{m,f}}(q,\epsilon) := \frac{1}{\epsilon} \int_{\mathbb{R}} (\mu_{m,f}(x-\epsilon, x+\epsilon))^q \, dx \tag{7}$$

defined for $q > 0$ and $\epsilon > 0$. Now we are in position to state an abstract result about quantum transport for Dirac operators.

Theorem 1. *Let* $\mathbf{D}(m,c)$ *be the operator defined by* (1) *with* V *satisfying condition* (2). *Let* $S \subset [-L,L]$ *with* $\ell(S)>0$ *and* $0 \le f \in \mathrm{C}_0^\infty(\mathbb{R})$ *with* $f=1$ *on* S. *Then:*
(**i**) *For any* $q \in (0,1)$ *and* $\tau>0$, *there exist constants* $C_1 = C_1(q,m,c) > 0$ *and* $C_2 = C_2(q,m,c,f,\tau,a,b,L)>0$ *such that for all*

$$0 < \epsilon < \min\left\{1, c\sqrt{3+m^2c^2} - c\sqrt{1+m^2c^2}\right\}$$

one has

$$K_{\mu_{m,f}}(q,\epsilon) \ge C_1 \epsilon^{q-1} \int_S \frac{dE}{\|\Phi_m(E,N,1)\|^{2q}} - C_2\epsilon \ ,$$

where $N = \left[\epsilon^{-(1+\tau)}\right]$.
(**ii**) *For any* $p>0$ *and* $\tau>0$, *there exist constants* $C_p(m,c)>0$ *and* $C_3 = C_3(p,m,c,f,\tau,a,b,L)>0$ *such that for* $T>0$ *large enough,*

$$M_m(p,f,T) \ge C_p(m,c) T^p \left(\frac{1}{\log T} \int_S \frac{dE}{\|\Phi_m(E,N,1)\|^{2/(p+1)}} \right)^{p+1} - C_3 \ ,$$

where $N = \left[T^{1+\tau}\right]$.

Remark 1. *Theorem 1 can be adapted to Dirac operators* $\mathbf{D}(m,c)$ *acting in* $\ell^2(\mathbb{Z},\mathbb{C}^2)$. *One just needs to replace* $\|\Phi_m(E,N,1)\|$ *in (i) and (ii) by* $\min\{\|\Phi_m(E,N,1)\|, \|\Phi_m(E,-N,1)\|\}$.

Given a Borel set $S \subset \mathbb{R}$, with $\ell(S)>0$, and $g: S \to \mathbb{R}$ a measurable function, denote by g^S the unique real number so that $g(E) \ge g^S$ for ℓ-a.e. E, and for all $r>0$, there exists $S_r \subset S$, $\ell(S_r)>0$, such that for all $E \in S_r$, one has $g(E) \le g^S + r$.

Theorem 2. *Let* $\mathbf{D}(m,c)$ *be as in Theorem 1 and* γ_m *given by* (6).
(**i**) *Suppose there exists a bounded Borel set* $S \subset [-L,L]$ *with* $\ell(S)>0$, *such that* $\gamma_m^S < \infty$. *Then, for all* $0 \le f \in \mathrm{C}_0^\infty(\mathbb{R})$, *with* $f=1$ *on* S, *one has for all* $p>0$,

$$\beta_m^-(p,f) \ge 1 - \frac{2\gamma_m^S}{p}.$$

(**ii**) *If* $\overline{\gamma}_m(E) := \sup_{\delta>0} \gamma_m^{(E-\delta,E+\delta)} < \infty$ *then for all* $p>0$,

$$\beta_m^-(p,E) \ge 1 - \frac{2\overline{\gamma}_m(E)}{p}.$$

Remark 2. *In the particular case where* $\gamma_m(E)=0$ *for* ℓ-a.e. $E \in I$, *where* I *is some open interval, Theorem 2 asserts that* $\beta_m^-(p,E)=1$ *on* I. *One may see this as a relativistic dynamical version of results for Schrödinger operators saying that if the transfer matrices are bounded, then the spectrum on* I *has an absolutely continuous component* [20], *which implies ballistic motion* $\beta^-(p,E) = 1$ [14].

Now, in order to state a version of Theorem 2 for upper transport exponents, it is convenient to introduce the following notation: for a given increasing sequence $(n_i)_{i\geq 1}$ such that $\lim_{i\to\infty} n_i = +\infty$, consider the measurable function $\gamma_{m,(n_i)} : \mathbb{R} \longrightarrow [0,+\infty]$ given by

$$\gamma_{m,(n_i)}(E) := \limsup_{i\to+\infty} \frac{\log \|\Phi_m(E,n_i,1)\|}{\log n_i} \ . \tag{8}$$

Theorem 3. *Let* $\mathbf{D}(m,c)$ *be as in Theorem 1 and* $\gamma_{m,(n_i)}$ *given by* (8).
(**i**) *Suppose there exists a bounded Borel set* $S \subset [-L,L]$ *with* $\ell(S) > 0$*, such that* $\gamma^S_{m,(n_i)} < \infty$*. Then, for all* $0 \leq f \in \mathrm{C}_0^\infty(\mathbb{R})$*, with* $f = 1$ *on* S*, one has for all* $p > 0$*,*

$$\beta^+_m(p,f) \geq 1 - \frac{2\gamma^S_{m,(n_i)}}{p}.$$

(**ii**) *If* $\overline{\gamma}_{m,(n_i)}(E) := \sup_{\delta>0} \gamma^{(E-\delta,E+\delta)}_{m,(n_i)} < \infty$ *then for all* $p > 0$*,*

$$\beta^+_m(p,E) \geq 1 - \frac{2\overline{\gamma}_{m,(n_i)}(E)}{p}.$$

The proofs of the above theorems rely on preliminary results discussed in Section 4, and will be completed in Section 5.

3. Applications to Sparse Potentials

This section is devoted to applications of Theorems 2 and 3 to Dirac operators $\mathbf{D}(m,c)$, given by (1), with different classes of sparse potentials of general form

$$V = \sum_{n=1}^{\infty} h_n \delta_{x_n}, \tag{9}$$

that is, $V(x_n) = h_n$ and $V(k) = 0$ if $k \neq x_n$ for all $n \geq 1$, with $x_n > 0$, $h_n \geq 0$ denoting the location of the nth barrier and its height, respectively.

We study quantum transport for three classes of sparse potentials of the type (9), and different conditions on the location of the barriers x_n will be imposed according to how their heights behave:

- **(Bounded)** the heights of the barriers h_n are bounded, i.e., $0 \leq h_n \leq a$, for all $n \geq 1$ and for some $a > 0$;
- **(Diverging)** the heights of the barriers h_n grow to infinity, that is, $h_n \to \infty$ as $n \to \infty$;
- **(Vanishing)** the heights of the barriers h_n go to zero, that is, $h_n \to 0$ as $n \to \infty$.

Denoting by $\mathcal{I}_0(m,c) := \mathrm{int}[\sigma(\mathbf{D}_0(m,c))]$ the interior of the spectrum $\sigma(\mathbf{D}_0(m,c))$, it is known [11] that

$$\mathcal{I}_0(m,c) = \left(-c\sqrt{4+m^2c^2}, -mc^2\right) \cup \left(mc^2, c\sqrt{4+m^2c^2}\right).$$

3.1. **Bounded barriers.** For the first class of sparse potentials it is supposed that there exist numbers a, α so that $a > 0$ and $\alpha \in (0,1)$ so that $0 \le h_n \le a$ and $x_n \ge \alpha^{-n}$, for all $n \ge 1$.

Theorem 4. *Let $(x_n)_{n\ge 1}$ and $(h_n)_{n\ge 1}$ be as above and the operator $\mathbf{D}(m,c)$ given by (1) with potential (9). Then for any $E \in \mathcal{I}_0(m,c)$, there exists a constant $C = C(E,m,c) > 0$ such that for all $p > 0$ one has*

$$\beta_m^-(p,E) \ge 1 - \frac{2\gamma_m(E)}{p},$$

with

$$\gamma_m(E) = \frac{\log\left[C\left(2 + \frac{mc^2 + c\sqrt{4+m^2c^2}+a}{c}\right)^2\right]}{\log(1/\alpha)}.$$

Remark 3. *Note that the lower the mass $m \ge 0$ the larger the lower bound of the dynamical exponents β_m^- and so the faster the quantum transport. With respect to such lower bounds, the mass plays a role similar to the maximum potential intensity a.*

Proof. For each $E \in \mathcal{I}_0(m,c)$, there exists a constant $C = C(E,m,c) > 0$ such that, for any $k \ge 0$, one has

$$\left\|T_m(E,0)^k\right\| = \left\| \begin{pmatrix} 1 + \dfrac{m^2c^4 - E^2}{c^2} & \dfrac{mc^2 + E}{c} \\ \dfrac{mc^2 - E}{c} & 1 \end{pmatrix}^k \right\| \le C. \tag{10}$$

The constant C diverges to $+\infty$ for $E \to \pm mc^2$ or $E \to \pm c\sqrt{4 + m^2c^2}$, but it is continuous in E and thus remains uniformly bounded on any compact subset of $\mathcal{I}_0(m,c)$. The sparseness of the potential implies that, for any $E \in \mathcal{I}_0(m,c)$,

$$\begin{aligned} \left\|\Phi_m(E,N,1)\right\| &\le \mathrm{C}^{n+1} \prod_{j=1}^{n} \left(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_j}{c}\right)^2 \\ &\le C\left[C\left(2 + \frac{mc^2 + c\sqrt{4+m^2c^2}+a}{c}\right)^2\right]^n \\ &\le CN^{\gamma_m(E)}, \end{aligned} \tag{11}$$

if $x_n \le N < x_{n+1}$, with $\gamma_m(E)$ as in the statement of the theorem.

It follows from (11), Theorem 2 and the continuity of $\gamma_m(E)$ in the set $\mathcal{I}_0(m,c)$, that for all $p > 0$,

$$\beta_m^-(p,E) \ge 1 - \frac{2\gamma_m(E)}{p},$$

and the proof is complete. □

3.2. **Diverging barriers.** We now consider the second class of sparse potentials and assume that the heights of the barriers $(h_n)_{n\geq 1}$ constitute of a sequence with $\lim_{n\to\infty} h_n = \infty$. By applying Theorem 3 we will get the following result.

Theorem 5. *Let $(h_n)_{n\geq 1}$ be as above and $\nu > 0$. For each $n \geq 1$ pick x_n so that*

$$x_n \geq \prod_{j=1}^{n} \Big(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_j}{c}\Big)^{4/\nu},$$

and consider the operator $\mathbf{D}(m,c)$ defined by (1) with potential (9). Then, for any $p > \nu$ and $E \in \mathcal{I}_0(m,c)$ one has

$$\beta_m^+(p,E) \geq 1 - \frac{\nu}{p}.$$

Remark 4. *In this case, the lower bounds of the dynamical exponents do not depend on m, however, the lower the mass the lower the required degree of sparseness as ruled by the sequence (x_n).*

Proof. Take $f \in \mathrm{C}_0^\infty(\mathcal{I}_0(m,c))$ and consider the quantity

$$C(f,m,c) := \sup_{E\in \mathrm{supp} f} C(E,m,c),$$

where $C(E,m,c)$ is given by (10) above. Fix $r > 0$. Since $h_n \to \infty$, it follows that

$$\Big(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_n}{c}\Big)^2 \geq C(f,m,c)^{1/r}$$

for any n larger than some n_r. Following the argument described above, we have uniformly in $E \in \mathrm{supp} f$,

$$\begin{aligned}
&\big\|\Phi_m(E, x_{n+1}-1, 1)\big\| \leq \\
\leq\;& C(f,m,c)^{n+1} \prod_{j=1}^{n} \Big(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_j}{c}\Big)^2 \\
\leq\;& C(f,m,c)^{n_r+1} C(f,m,c)^{n-n_r} \prod_{j=1}^{n} \Big(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_j}{c}\Big)^2 \\
\leq\;& C(f,m,c)^{n_r+1} \Big[\prod_{j=1}^{n} \Big(2 + \frac{mc^2 + c\sqrt{4+m^2c^2} + h_j}{c}\Big)^2\Big]^{1+r} \\
\leq\;& C(f,m,c)^{n_r+1} \left(x_{n+1}\right)^{\nu(1+r)/2}.
\end{aligned}$$

Therefore, for all $r > 0$,

$$\gamma_{m,(x_n)}(E) = \limsup_{n\to+\infty} \frac{\log \|\Phi_m(E, x_{n+1}-1, 1)\|}{\log(x_{n+1}-1)} \leq \nu(1+r)/2.$$

It then follows by Theorem 3 that, for any $E \in \mathcal{I}_0(m,c)$ and $p > \nu$, one has

$$\beta_m^+(p,E) \geq 1 - \frac{\nu}{p},$$

which completes the proof. □

3.3. **Vanishing barriers.** Finally, we consider the third class of sparse potentials to which we shall again apply Theorem 2. Now assume that

$$\limsup_{n\to\infty} \frac{x_n}{x_{n+1}} < 1 \tag{12}$$

and $\lim_{n\to\infty} h_n = 0$.

Theorem 6. *Let (x_n) and (h_n) be as above and consider the Dirac operator $\mathbf{D}(m,c)$ given by (1) with potential (9). Then, for all $p > 0$ and for any $E \in \mathcal{I}_0(m,c)$, one has*

$$\beta_m^-(p,E) = 1.$$

Remark 5. *Under the sparseness condition (12), the vanishing of the heights results in ballistic transport independently of the mass.*

Proof. The idea is to look at the sequence $h_n \to 0$ as a perturbation of the potential $V = 0$ and employ the general perturbation method developed in [17] (see also [8]).

Consider the eigenvalue equation for the free Dirac operator $\mathbf{D}_0(m,c)$,

$$(\mathbf{D}_0(m,c)\Phi)(n) = E\Phi(n)\ , \quad \Phi(n) = \begin{pmatrix} \phi^+(n) \\ \phi^-(n) \end{pmatrix}, \tag{13}$$

and the eigenvalue equation for the Dirac operator $\mathbf{D}(m,c)$,

$$(\mathbf{D}(m,c)\Psi)(n) = E\Psi(n)\ , \quad \Psi(n) = \begin{pmatrix} \psi^+(n) \\ \psi^-(n) \end{pmatrix}. \tag{14}$$

Let $\Psi_D = \begin{pmatrix} \psi_D^+ \\ \psi_D^- \end{pmatrix}$ and $\Psi_N = \begin{pmatrix} \psi_N^+ \\ \psi_N^- \end{pmatrix}$ be the solutions to (14) with initial conditions

$$\begin{pmatrix} \psi_D^+(1) \\ \psi_D^-(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_N^+(1) \\ \psi_N^-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the transfer matrix for $\mathbf{D}(m,c)$ is given by

$$\Phi_m(E,n,1) = \begin{pmatrix} \psi_D^+(n) & \psi_N^+(n) \\ \psi_D^-(n-1) & \psi_N^-(n-1) \end{pmatrix}. \tag{15}$$

Fix a complex solution $\Phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix}$ to (13). For example, we could set $\Phi = \Phi_D + i\Phi_N$, where Φ_D and Φ_N solve (13) and have the same initial conditions as Ψ_D and Ψ_N. For each energy $E \in \mathcal{I}_0(m,c)$ there exists a constant $C = C(E,m,c) > 0$ such that

$$\|\Phi(n)\| \leq C\ , \quad \forall n \geq 1. \tag{16}$$

Let $\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ be one of the basic solutions Ψ_D, Ψ_N to (14). Define $\rho(n) \in \mathbb{C}$ by

$$\begin{aligned} \begin{pmatrix} \psi^+(n) \\ \psi^-(n-1) \end{pmatrix} &= \\ &= \frac{1}{2i}\left[\rho(n)\begin{pmatrix} \phi^+(n) \\ \phi^-(n-1) \end{pmatrix} - \overline{\rho(n)}\begin{pmatrix} \overline{\phi^+(n)} \\ \overline{\phi^-(n-1)} \end{pmatrix}\right] \\ &= \operatorname{Im}\left[\rho(n)\begin{pmatrix} \phi^+(n) \\ \phi^-(n-1) \end{pmatrix}\right]. \end{aligned} \tag{17}$$

Write $\phi^+(n)$, $\phi^-(n)$ and $\rho(n)$ in polar coordinates,

$$\phi^+(n) = |\phi^+(n)|e^{i\gamma^+(n)}\ ,\ \phi^-(n) = |\phi^-(n)|e^{i\gamma^-(n)}\ ,\ \rho(n) = |\rho(n)|e^{i\eta(n)}$$

and define

$$\theta^+(n) = \eta(n) + \gamma^+(n)\ ,\ \theta^-(n) = \eta(n+1) + \gamma^-(n).$$

Denote by $W[\Phi, \Psi](n)$ the Wronskian of Φ and Ψ in the position n. By (17), we have that for every $n \geq 2$,

$$\begin{aligned} W[\overline{\Phi}, \Psi](n-1) &= \overline{\phi^+(n)}\psi^-(n-1) - \overline{\phi^-(n-1)}\psi^+(n) \\ &= \rho(n)\omega, \qquad \omega \neq 0. \end{aligned} \tag{18}$$

By using (13), (14), (17) and (18), one has

$$\begin{aligned} &\rho(n+1) - \rho(n) = \\ =\ & \frac{1}{\omega}\left[\overline{\phi^+(n+1)}\psi^-(n) - \overline{\phi^-(n)}\psi^+(n+1)\right] - \\ & -\frac{1}{\omega}\left[\overline{\phi^+(n)}\psi^-(n-1) - \overline{\phi^-(n-1)}\psi^+(n)\right] \\ =\ & \frac{1}{\omega}\left[\left(\overline{\phi^+(n+1)} - \overline{\phi^+(n)}\right)\psi^-(n) - \left(\psi^-(n-1) - \psi^-(n)\right)\overline{\phi^+(n)}\right] - \\ & -\frac{1}{\omega}\left[\left(\psi^+(n+1) - \psi^+(n)\right)\overline{\phi^-(n)} - \left(\overline{\phi^-(n-1)} - \overline{\phi^-(n)}\right)\psi^+(n)\right] \\ =\ & \frac{1}{\omega c}\left[\left(E + mc^2\right)\overline{\phi^-(n)}\psi^-(n) - \left(E - V(n) - mc^2\right)\psi^+(n)\overline{\phi^+(n)}\right] - \\ & -\frac{1}{\omega c}\left[\left(E - V(n) + mc^2\right)\psi^-(n)\overline{\phi^-(n)} - \left(E - mc^2\right)\overline{\phi^+(n)}\psi^+(n)\right] \\ =\ & \frac{1}{\omega c}V(n)\left(\psi^+(n)\overline{\phi^+(n)} + \psi^-(n)\overline{\phi^-(n)}\right) \\ =\ & \frac{1}{\omega c}V(n)\rho(n)\left|\phi^+(n)\right|^2\sin(\theta^+(n))e^{-i\theta^+(n)} + \\ & +\frac{1}{\omega c}V(n)\rho(n+1)\left|\phi^-(n)\right|^2\sin(\theta^-(n))e^{-i\theta^-(n)}, \end{aligned}$$

which implies

$$|\rho(n+1)| \leq \frac{\left(1+\frac{1}{|\omega|c}|V(n)|\,|\phi^{+}(n)|^{2}\right)}{\left(1-\frac{1}{|\omega|c}|V(n)|\,|\phi^{-}(n)|^{2}\right)}|\rho(n)|. \tag{19}$$

By (12), there are $\gamma > 1$ and $n_0 \in \mathbb{N}$ such that $\frac{x_n}{x_{n-1}} > \gamma$ and $x_n > \gamma^n$, for all $n > n_0$. Since $V(x_n) = h_n \to 0$, it follows by (16) and (19) that, for each $\epsilon > 0$, there exists a constant $C_0 > 0$ such that

$$|\rho(x_{n+1})| \leq C_0\ \frac{(1+\epsilon)^n}{(1-\epsilon)^n}\ ,$$

for n sufficiently large and for energies $E \in \mathcal{I}_0(m,c)$. This implies that

$$\frac{(\log\gamma)\ \log|\rho(x_{n+1})|}{\log(x_{n+1})} \ \leq\ \frac{\log C_0}{n} + \log\left(\frac{1+\epsilon}{1-\epsilon}\right),$$

for all $\epsilon > 0$ and n sufficiently large. Hence, by taking $\epsilon \to 0$ we obtain

$$\limsup_{n\to\infty} \frac{\log|\rho(x_{n+1})|}{\log(x_{n+1})} = 0. \tag{20}$$

On the other hand, it follows by (15), (16) and (17), that

$$\begin{aligned} &\|\Phi_m(E,n,1)\| \leq \\ \leq\ &\sqrt{2}\ \max\left\{\left\|\begin{pmatrix} \psi_D^{+}(n) \\ \psi_D^{-}(n-1)\end{pmatrix}\right\|, \left\|\begin{pmatrix} \psi_N^{+}(n) \\ \psi_N^{-}(n-1)\end{pmatrix}\right\|\right\} \\ \leq\ &\sqrt{2}\ C\ |\rho(n)|, \end{aligned} \tag{21}$$

for all $n \geq 1$ and $E \in \mathcal{I}_0(m,c)$.

Therefore, from (20) and (21) we conclude that

$$\gamma_m(E) = \limsup_{n\to+\infty} \frac{\log\|\Phi_m(E,n,1)\|}{\log n} = 0,$$

and by Theorem 2 we obtain

$$\beta_m^{-}(p,E) = 1,$$

for all $p > 0$ and $E \in \mathcal{I}_0(m,c)$. □

4. Preliminaries

In this section we collect some results that will be used in the proofs of Theorems 1, 2 and 3.

For the proof of Theorem 1(ii) we will use the following lemma, which provides a lower bound for the moments $M_m(p,f,T)$ in terms of local spectral moments (7).

Lemma 1. *Let* $\mathbf{D}(m,c)$ *be the operator defined by* (1). *Then for all* $p>0$ *and* $0 \le f \in \mathrm{C}_0^\infty(\mathbb{R})$, *there exists a constant* $C(m,p)>0$ *such that for all* $T>0$,

$$M_m(p,f,T) \ge \left(\frac{C(m,p)}{\log T} \; K_{\mu_{m,f}}(q,T^{-1}) \right)^{1/q} \quad , \quad q = \frac{1}{1+p} \ ,$$

where $K_{\mu_{m,f}}(q,T^{-1})$ *is defined by* (7).

Lemma 1 can be easily adapted, to the Dirac operator $\mathbf{D}(m,c)$ (details omitted), from corresponding results in references [1, 2], and it directly implies:

Lemma 2. *For all* $p>0$, $q=(1+p)^{-1}$ *and functions* f *with compact support, one has*

$$\beta_m^-(p,f) \ge \liminf_{\epsilon\to 0} \frac{\log K_{\mu_{m,f}}(q,\epsilon)}{(q-1)\log\epsilon} \quad , \quad \beta_m^+(p,f) \ge \limsup_{\epsilon\to 0} \frac{\log K_{\mu_{m,f}}(q,\epsilon)}{(q-1)\log\epsilon} \ .$$

The next result is a version of the well-known Combes-Thomas estimate in the Schrödinger setting for the Dirac model $\mathbf{D}(m,c)$.

Proposition 1. *Let* $\mathbf{D}(m,c)$ *be the operator defined by* (1). *If* $z \notin \sigma(\mathbf{D}(m,c))$, *then there exist constants* $\eta=\eta(m,c)>0$ *and* $a=a(m,c)>0$ *such that*

$$\left| \left\langle \delta_j^\pm , (\mathbf{D}(m,c)-z\mathrm{I})^{-1}\, \delta_k^\pm \right\rangle \right| \le \frac{2}{\eta}\; e^{-a|j-k|} \ ,$$

where $\{\delta_n^\pm\}$ *is the canonic basis of* $\ell^2(\mathbb{N},\mathbb{C}^2)$.

Proof. For $\alpha\in\mathbb{C}$, let U_α be the diagonal operator $U_\alpha \delta_j^\pm := e^{\alpha j}\delta_j^\pm$ with

$$\mathrm{dom}\ U_\alpha = \left\{ \Psi \in \ell^2(\mathbb{N},\mathbb{C}^2) : (e^{\alpha n}\Psi(n))_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{C}^2) \right\}.$$

If $\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = (\Psi(n))_{n\in\mathbb{N}} \in \mathrm{dom}\ U_\alpha$, one has

$$(U_{-\alpha}\mathbf{D}(m,c)U_\alpha)\,\Psi = \mathbf{D}(m,c)\Psi + Q(\alpha)\Psi \ , \tag{22}$$

where

$$Q(\alpha) = \begin{pmatrix} c(e^{-\alpha}-1)\ \psi^-(n-1) \\ c(e^{\alpha}-1)\ \psi^+(n+1) \end{pmatrix}.$$

It follows by (22) that $\mathrm{dom}\,(U_{-\alpha}\mathbf{D}(m,c)U_\alpha) = \mathrm{dom}\,(\mathbf{D}(m,c))$, $\forall\alpha\in\mathbb{C}$ and we have $\|Q(\alpha)\| \le 2c\left(e^{|\alpha|}-1\right)$. Thus, $Q(\alpha)$ is bounded and does not depend on the potential V.

Let $\eta = \mathrm{dist}\,(z,\sigma(\mathbf{D}(m,c)))>0$. By imposing that

$$2c\left(e^{|\alpha|}-1\right) < \frac{\eta}{2} \ , \tag{23}$$

one has $\|Q(\alpha)\| < \frac{\eta}{2}$. Note that

$$\mathbf{D}(m,c) - z\mathrm{I} + Q(\alpha) = (\mathbf{D}(m,c)-z\mathrm{I})\left[I + (\mathbf{D}(m,c)-z\mathrm{I})^{-1}Q(\alpha)\right],$$

which implies

$$(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1}=\left(\sum_{n=0}^{\infty}(\mathbf{D}(m,c)-zI)^{-n}\,Q(\alpha)^n\right)(\mathbf{D}(m,c)-z\mathrm{I})^{-1}.$$

As $z\notin\sigma\left(\mathbf{D}(m,c)\right)$, then $\left\|(\mathbf{D}(m,c)-z\mathrm{I})^{-1}\right\|\le\frac{1}{\eta}$. It follows from the above equality that

$$\left\|(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1}\right\|\le\frac{1}{1-1/2}\left\|(\mathbf{D}(m,c)-z\mathrm{I})^{-1}\right\|\le\frac{2}{\eta}\;. \tag{24}$$

Relation (22) implies, for α satisfying (23),

$$U_{-\alpha}(\mathbf{D}(m,c)-z\mathrm{I})^{-1}U_{\alpha}=(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1},$$

and consequently,

$$\left\langle\delta_j^{\pm},(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1}\,\delta_k^{\pm}\right\rangle=e^{-\alpha j}e^{\alpha k}\left\langle\delta_j^{\pm},(\mathbf{D}(m,c)-zI)^{-1}\,\delta_k^{\pm}\right\rangle.$$

Therefore, by choosing α real ($\alpha\ge0$ if $j-k<0$ and $\alpha<0$ if $j-k\ge0$) satisfying (23) and by using (24), we obtain

$$\begin{aligned}\left|\left\langle\delta_j^{\pm},\Big(\mathbf{D}(m,c)\right.\right.&-\left.\left.zI\Big)^{-1}\delta_k^{\pm}\right\rangle\right|=\\&=e^{-|\alpha||j-k|}\left|\left\langle\delta_j^{\pm},(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1}\,\delta_k^{\pm}\right\rangle\right|\\&\le e^{-|\alpha||j-k|}\|\delta_j^{\pm}\|\left\|(\mathbf{D}(m,c)-z\mathrm{I}+Q(\alpha))^{-1}\right\|\|\delta_k^{\pm}\|\\&\le\frac{2}{\eta}\,e^{-|\alpha||j-k|}\;.\end{aligned}$$

□

By Proposition 1, the Helffer-Sjöstrand formula [15] (in case of the Dirac operator $\mathbf{D}(m,c)$) and analogous arguments utilized in the Appendix A2 of [13], we obtain a version of the approximation lemma for $\mathbf{D}(m,c)$, that is,

Lemma 3. *Let $\mathbf{D}_1(m,c)=\mathbf{D}_0(m,c)+V_1\mathrm{I}_2$ and $\mathbf{D}_2(m,c)=\mathbf{D}_0(m,c)+V_2\mathrm{I}_2$ be Dirac operators defined by (1), such that $V_1(n)=V_2(n)$ for all $n\le N$, for some $N>1$. We shall assume the polynomial bound*

$$|V_1(n)-V_2(n)|\le a(1+n^2)^{b/2}\;,\quad\forall n\in\mathbb{N},$$

with positive constants a,b. Let $M>0$ and $\tau>0$ be given. Then, if I is a compact interval, there exists a constant $\tilde{C}=\tilde{C}(m,c,I,M,\tau,a,b)>0$ such that for any $\epsilon>N^{-\frac{1}{1+\tau}}$ and $x\in I$,

$$\mu_m^{(1)}(x-\epsilon,x+\epsilon)\ge\mu_m^{(2)}\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)-\tilde{C}\epsilon^M,$$

where $\mu_m^{(i)},i=1,2,$ denotes the spectral measure of $\mathbf{D}_i(m,c)$ associated to the vector δ_1^+.

For each pair $E_0 \in \mathbb{R}$ and $N > 1$, introduce the operator $\mathbf{D}(m,c)^{(E_0,N)}$ on the space $\ell^2(\mathbb{N},\mathbb{C}^2)$ by

$$\mathbf{D}(m,c)^{(E_0,N)} := \mathbf{D}_0(m,c) + V\chi_{[1,N]}\mathrm{I}_2 + E_0(1-\chi_{[1,N]})\mathrm{I}_2 \ .$$

Write $\mu_m^{(E_0,N)}$ for the spectral measure of $\mathbf{D}(m,c)^{(E_0,N)}$ associated to δ_1^+, and $R_m^{(E_0,N)}(z) = \left(\mathbf{D}(m,c)^{(E_0,N)} - zI\right)^{-1}$ for the corresponding resolvent. Note that $\mu_m^{(E_0,N)}$ is absolutely continuous with respect to Lebesgue measure ℓ, since $\mathbf{D}(m,c)^{(E_0,N)}$ is a finite rank perturbation of a purely absolutely continuous operator. Denote by $\Omega_1 = c\sqrt{3+m^2c^2}$ and $\Omega_2 = c\sqrt{1+m^2c^2}$; we need the following technical result.

Lemma 4. *Let $\mathbf{D}(m,c)$ be the Dirac operator defined by* (1). *There exists a finite constant $C_1 = C_1(m,c) > 0$ such that for any $E_0 \in \mathbb{R}$,*

$$E \in [E_0-\Omega_1, E_0-\Omega_2] \cup [E_0+\Omega_2, E_0+\Omega_1]$$

and $N > 1$,

$$\frac{d\mu_m^{(E_0,N)}}{dx}(E) \ \geq \ \frac{C_1}{\|\Phi_m(E,N,1)\|^2} \ .$$

Proof. Let E_0 and E be as in the statement of the lemma. We will make use of Stone's formula [9] $(-\infty < a < b < \infty)$

$$\begin{aligned} &\frac{1}{2}\left[\chi_m^{(E_0,N)}\left([a,b]\right) + \chi_m^{(E_0,N)}\left((a,b)\right)\right] = \\ &= \lim_{\eta\to 0}\frac{1}{2\pi i}\int_a^b \left(R_m^{(E_0,N)}(E+i\eta) - R_m^{(E_0,N)}(E-i\eta)\right)dE, \end{aligned}$$

where $\chi_m^{(E_0,N)}(I)$ is the spectral projection of $\mathbf{D}(m,c)^{(E_0,N)}$ onto the interval I. Using that $\mu_m^{(E_0,N)}(I) = \left\langle \delta_1^+, \chi_m^{(E_0,N)}(I)\delta_1^+\right\rangle$, follows by Stone's formula and Radon-Nikodym Theorem that

$$\begin{aligned} \frac{d\mu_m^{(E_0,N)}}{dx}(E) &= \frac{1}{\pi}\lim_{\eta\to 0}\operatorname{Im}\left\langle \delta_1^+, R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\rangle \\ &= \frac{1}{\pi}\lim_{\eta\to 0}\eta\left\|R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\|^2. \end{aligned} \tag{25}$$

Let $\varphi_m = R_m^{(E_0,N)}(E+i\eta)\delta_1^+, \ \ \eta > 0$. Thus,

$$\varphi_m(N) = \begin{pmatrix} \varphi_m^+(N) \\ \varphi_m^-(N) \end{pmatrix} = \begin{pmatrix} \left\langle \delta_N^+, \varphi_m\right\rangle \\ \left\langle \delta_N^-, \varphi_m\right\rangle \end{pmatrix}.$$

We have the following estimate from below:

$$\left\|R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\|^2 \ \geq \ \frac{1}{2}\sum_{n=N}^{+\infty}\left(\left|\varphi_m^+(n)\right|^2 + \left|\varphi_m^-(n-1)\right|^2\right). \tag{26}$$

Using the transfer matrices representation and the fact that $\Phi_m(E+i\eta, n, N)$ is a 2×2 matrix with determinant 1, one has for any $n \geq N$,

$$\begin{aligned}(27)\qquad \left|\varphi_m^+(N)\right|^2 &+ \left|\varphi_m^-(N-1)\right|^2 = \\ &= \left\|\Phi_m^{-1}(E+i\eta,n,N)\begin{pmatrix}\varphi_m^+(n)\\ \varphi_m^-(n-1)\end{pmatrix}\right\|^2 \\ &\leq \left\|\Phi_m(E+i\eta,n,N)\right\|^2\left(\left|\varphi_m^+(n)\right|^2+\left|\varphi_m^-(n-1)\right|^2\right),\end{aligned}$$

where Φ_m is the transfer matrix corresponding to the operator $\mathbf{D}(m,c)^{(E_0,N)}$. Note that $\Phi_m(E+i\eta,n,N) = \Phi_m^0(E-E_0+i\eta,n,N)$ for $n \geq N$, where Φ_m^0 is the transfer matrix for the free Dirac operator $\mathbf{D}_0(m,c)$, that is,

$$\Phi_m^0(z,n,k) = \left(A_m^0(z)\right)^{n-k} \quad , \quad A_m^0(z) = \begin{pmatrix} 1+\dfrac{m^2c^4-z^2}{c^2} & \dfrac{mc^2+z}{c} \\ \dfrac{mc^2-z}{c} & 1 \end{pmatrix}.$$

Note that for any real number

$$\omega \in [-\Omega_1,-\Omega_2]\cup[\Omega_2,\Omega_1] \tag{28}$$

one has $\Phi_m^0(\omega,n,k) = \left(A_m^0(\omega)\right)^{n-k}$ with $A_m^0(\omega)$ elliptic, so that

$$\left\|\Phi_m^0(\omega,n,k)\right\| = C_2\ ,$$

for all $n,k\in\mathbb{N}$, where $C_2 = C_2(m,c) > 0$ is a finite constant. It follows, by Lemma 5 of [18], that for any ω as in (28), $|\eta|\leq 1$ and $n\geq k\geq 1$,

$$\left\|\Phi_m^0(\omega+i\eta,n,k)\right\| \leq C_2\left[1+\frac{|\eta|}{c}\left(\frac{|\eta|}{c}+C_3\right)C_2\right]^{n-k}, \tag{29}$$

for some $0 < C_3 < \infty$. As a consequence, by inserting (27) and (29) into (26) gives

$$\begin{aligned}(30)\ \ \eta\left\|R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\|^2 &\geq \\ \geq\ & \frac{\eta}{2(C_2)^2}\left(\left|\varphi_m^+(N)\right|^2+\left|\varphi_m^-(N-1)\right|^2\right)\sum_{n=N}^{+\infty}\left[1+\frac{\eta}{c}\left(\frac{\eta}{c}+C_3\right)C_2\right]^{-2(n-N)} \\ \geq\ & C_4\left(\left|\varphi_m^+(N)\right|^2+\left|\varphi_m^-(N-1)\right|^2\right),\end{aligned}$$

for some $0 < C_4 < \infty$.

Analogous to the Schrödinger case, we obtain

$$\begin{aligned}\varphi_m(N) &:= \begin{pmatrix}\left\langle \delta_N^+, R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\rangle \\ \left\langle \delta_N^-, R_m^{(E_0,N)}(E+i\eta)\delta_1^+\right\rangle\end{pmatrix} \\ &= u_N(E+i\eta,N)+F_m(E+i\eta)u_D(E+i\eta,N),\end{aligned}$$

where $F_m(z)$ is the Weyl function of operator $\mathbf{D}(m,c)^{(E_0,N)}$ (the Borel transform of its spectral measure) and u_D, u_N are the two linearly independent solutions used in the definition of the transfer matrices (see Section 2). One observes that the vectors

$$\begin{pmatrix} u_D^+(E,n) \\ u_D^-(E,n-1) \end{pmatrix}, \quad \begin{pmatrix} u_N^+(E,n) \\ u_N^-(E,n-1) \end{pmatrix},$$

for $n \le N$, are the same for both $\mathbf{D}(m,c)^{(E_0,N)}$ and $\mathbf{D}(m,c)$, since the potentials coincide on $[1,N]$. Since $\mu_m^{(E_0,N)}$ is absolutely continuous and $u_D(E,n), u_N(E,n)$ are both real, we have

$$\lim_{\eta\to 0} \left|\varphi_m^+(N)\right|^2 \ge \left(u_N^+(E,N) + \mathrm{Re}(F_m(E+i0))u_D^+(E,N)\right)^2 \tag{31}$$

and

$$\lim_{\eta\to 0} \left|\varphi_m^-(N)\right|^2 \ge \left(u_N^-(E,N) + \mathrm{Re}(F_m(E+i0))u_D^-(E,N)\right)^2, \tag{32}$$

with $F_m(E+i0) < \infty$. It follows by (25), (30), (31) and (32) that

$$\begin{aligned} &\frac{d\mu_m^{(E_0,N)}}{dx}(E) \ge \\ \ge\ & C_5 \left(u_N^+(E,N) + \mathrm{Re}(F_m(E+i0))u_D^+(E,N)\right)^2 + \\ &+ C_5 \left(u_N^-(E,N-1) + \mathrm{Re}(F_m(E+i0))u_D^-(E,N-1)\right)^2, \end{aligned} \tag{33}$$

for some $0 < C_5 < \infty$.

Now consider the polynomial function $g(t) = (a+tb)^2 + (c+td)^2$. One readily checks that $t_0 = -(ab+cd)/(b^2+d^2)$ is the point of minimum of g. Thus, $g(t) \ge g(t_0) = (bc-da)^2/(b^2+d^2)$, for all $t \in \mathbb{R}$. By using this in (33) and the fact that the Wronskian of u_0 and u_N is equal to one, we finally get

$$\frac{d\mu_m^{(E_0,N)}}{dx}(E) \ \ge\ \frac{C_5}{(u_0^+(E,N))^2 + (u_0^-(E,N-1))^2} \ \ge\ \frac{C_5}{\|\Phi_m(E,N,1)\|^2},$$

and the proof is complete. □

The next result converts upper bounds on the norms of transfer matrices into lower bounds on the spectral measures.

Proposition 2. *Let* $\mathbf{D}(m,c)$ *be the operator defined by* (1), *with potential* V *satisfying* (2), *and* I *a compact interval. There exist a constant* $C_1 = C_1(m,c) > 0$ *and, for each* $M > 0$ *and* $\tau > 0$, *a constant* $C_2 = C_2(m,c,I,M,\tau,a,b) > 0$ *such that for all* $\epsilon > 0$ *with* $\epsilon < \min\{1, \Omega_1 - \Omega_2\}$ *and all* $\lambda \in \mathbb{R}$ *with* $\lambda_m^\pm = \lambda \pm \frac{1}{2}(\Omega_1+\Omega_2) \in I$, *one has*

$$\mu_m(\lambda_m^\pm - \epsilon, \lambda_m^\pm + \epsilon) \ge C_1 \int_{\lambda_m^\pm - \frac{\epsilon}{2}}^{\lambda_m^\pm + \frac{\epsilon}{2}} \frac{dE}{\left\|\Phi_m(E,N,1)\right\|^2} \ -\ C_2\epsilon^M \ ,$$

where $N = [\epsilon^{-(1+\tau)}]$.

Proof. For $\lambda \in \mathbb{R}$ with $\lambda_m^\pm \in I$ and $N > 1$ given, we shall apply Lemma 3 with $\mathbf{D}_1(m,c) = \mathbf{D}(m,c)$ and $\mathbf{D}_2(m,c) = \mathbf{D}(m,c)^{(\lambda_m^\pm, N)}$. Note that $V_1(n) = V_2(n)$ for all $n \in [1, N]$.

Since I is compact and the potential V is polynomially bounded, we have that $|V_1(n) - V_2(n)| \leq a(1+n^2)^{b/2}$, for all $n \in \mathbb{N}$, with constants $a, b > 0$ uniform in $\lambda_m^\pm \in I$ and N.

Let $M > 0$ and $\tau > 0$. It follows, by Lemmas 3 and 4, that for all $\epsilon > 0$ with $\epsilon < \min\{1, \Omega_1 - \Omega_2\}$ and for all $\lambda \in \mathbb{R}$ with $\lambda_m^\pm \in I$,

$$\begin{aligned} \mu_m(\lambda_m^\pm - \epsilon, \lambda_m^\pm + \epsilon) &\geq \mu_m^{(\lambda_m^\pm, N)}\left(\lambda_m^\pm - \frac{\epsilon}{2}, \lambda_m^\pm + \frac{\epsilon}{2}\right) - C_2\epsilon^M \\ &= \int_{\lambda_m^\pm - \frac{\epsilon}{2}}^{\lambda_m^\pm + \frac{\epsilon}{2}} \frac{d\mu_m^{(\lambda_m^\pm, N)}}{dx}(E) \ - \ C_2\epsilon^M \\ &\geq \int_{\lambda_m^\pm - \frac{\epsilon}{2}}^{\lambda_m^\pm + \frac{\epsilon}{2}} \frac{C_1 \, dE}{\left\|\Phi_m(E, N, 1)\right\|^2} \ - \ C_2\epsilon^M \ , \end{aligned}$$

where $N = [\epsilon^{-(1+\tau)}]$ and $0 < C_1, C_2 < \infty$. □

5. Proofs of Abstract Lower Bounds

In this section the proofs of Theorems 1, 2 and 3 will be presented. We first prove Theorem 1; it will be a combination of Proposition 2 and Lemma 1.

Proof. **(Theorem 1)**
(i) Let $0 \leq f \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp} f \subset [-L, L]$ and $f = 1$ on S. Since f is uniformly continuous, there exists $\eta > 0$ with $\eta < \min\{1, \Omega_1 - \Omega_2\}$ such that if $|x - y| \leq 2\eta$ then $|f(x) - f(y)| \leq \frac{1}{2}$ (Ω_1, Ω_2 were introduced just before Lemma 4). Define the set

$$J = \{x \in \mathbb{R} : d(x, S) \leq \eta\}\,.$$

For all $\epsilon < \eta$ and $x \in J$, one verifies that

$$\mu_{m,f}(x - \epsilon, x + \epsilon) = \int_{x-\epsilon}^{x+\epsilon} f(y) d\mu_m(y) \geq \frac{1}{2}\ \mu_m(x - \epsilon, x + \epsilon). \tag{34}$$

Furthermore, for any given $M > 0$ and $\tau > 0$ we get, by Proposition 2, that for all $\epsilon > 0$ with $\epsilon < \min\{1, \Omega_1 - \Omega_2\}$ and all $x \in \mathbb{R}$ with $x_m^\pm = x \pm \frac{1}{2}(\Omega_1 + \Omega_2) \in I = [-L - 1, L + 1]$,

$$\mu_m(x_m^\pm - \epsilon, x_m^\pm + \epsilon) \ \geq \ C_1 \int_{x_m^\pm - \frac{\epsilon}{2}}^{x_m^\pm + \frac{\epsilon}{2}} \frac{dE}{\left\|\Phi_m(E, N, 1)\right\|^2} \ - \ C_2\epsilon^M,$$

where $N = [\epsilon^{-(1+\tau)}]$.

By using that $\mu_m(x_m^\pm - \epsilon, x_m^\pm + \epsilon) \geq 0$ and the Jensen's inequality, one has for all $q \in (0,1)$,

$$
\begin{aligned}
(35) \qquad & \big(\mu_m(x_m^\pm - \epsilon, x_m^\pm + \epsilon)\big)^q \geq \\
\geq \quad & \left(\int_{x_m^\pm - \frac{\epsilon}{2}}^{x_m^\pm + \frac{\epsilon}{2}} \frac{\epsilon\; C_1}{\big\|\Phi_m(E,N,1)\big\|^2}\, \frac{dE}{\epsilon}\right)^q - C_2^q \epsilon^{qM} \\
\geq \quad & C_1^q\; \epsilon^{q-1} \int_{x_m^\pm - \frac{\epsilon}{2}}^{x_m^\pm + \frac{\epsilon}{2}} \frac{dE}{\big\|\Phi_m(E,N,1)\big\|^{2q}} - C_2^q \epsilon^{qM} \; .
\end{aligned}
$$

Now, by combining (34) and (35), one gets, for all $\epsilon < \eta$,

$$
\begin{aligned}
K_{\mu_{m,f}}(q,\epsilon) \;&=\; \frac{1}{\epsilon}\int_{\mathbb{R}} (\mu_{m,f}(x-\epsilon, x+\epsilon))^q \, dx \\
&\geq\; \frac{1}{2^{q+1}\epsilon}\int_J \big(\mu_m(x_m^+ - \epsilon, x_m^+ + \epsilon)\big)^q \, dx_m^+ \; + \\
&\quad +\; \frac{1}{2^{q+1}\epsilon}\int_J \big(\mu_m(x_m^- - \epsilon, x_m^- + \epsilon)\big)^q \, dx_m^- \\
&\geq\; \frac{C_1^q \epsilon^{q-2}}{2^{q+1}} \int_J \int_{x_m^+ - \frac{\epsilon}{2}}^{x_m^+ + \frac{\epsilon}{2}} \frac{dE}{\big\|\Phi_m(E,N,1)\big\|^{2q}}\; dx_m^+ \; + \\
&\quad +\; \frac{C_1^q \epsilon^{q-2}}{2^{q+1}} \int_J \int_{x_m^- - \frac{\epsilon}{2}}^{x_m^- + \frac{\epsilon}{2}} \frac{dE}{\big\|\Phi_m(E,N,1)\big\|^{2q}}\; dx_m^- \; - \tilde{C}_2 \frac{\epsilon}{2^q} \; ,
\end{aligned}
$$

where $M = 2/q$. Let $E \in S$. If $|x_m^\pm - E| \leq \epsilon/2$ and $\epsilon < \eta$, then $x_m^\pm \in J$. Note that $\ell\left(\{x_m^\pm : |x_m^\pm - E| \leq \epsilon/2\} \cap J\right) \geq \epsilon/2$. Therefore, upon integrating in the variable E over the set S (and applying Fubini's Theorem), one obtains

$$
K_{\mu_{m,f}}(q,\epsilon) \geq C_q \epsilon^{q-1} \int_S \frac{dE}{\|\Phi_m(E,N,1)\|^{2q}} \; - \tilde{C}_2 \frac{\epsilon}{2^q} \; ,
$$

where $N = \left[\epsilon^{-(1+\tau)}\right]$. The bound in Theorem 1(i) follows with the constants $C_1 = C_q$ and $C_2 = \frac{\tilde{C}_2}{2^q}$.

(**ii**) By Lemma 1, it follows that for all $p > 0$, $0 \leq f \in \mathrm{C}_0^\infty(\mathbb{R})$, and $T > 0$,

$$
(36) \qquad M_m(p,f,T) \geq \left(\frac{C(m,p)}{\log T}\; K_{\mu_{m,f}}(q, T^{-1})\right)^{1/q} \quad , \quad q = \frac{1}{1+p} \; .
$$

Insert the bound obtained in (i) into (36) with $\epsilon = T^{-1}$; hence, for all $p > 0$ and $T > 0$ large enough,

$$
M_m(p,f,T)^q \geq \frac{C(m,p)}{\log T} C_1 T^{1-q} \int_S \frac{dE}{\|\Phi_m(E,N,1)\|^{2q}} - C_2(m,p,f),
$$

which implies

$$
M_m(p,f,T) \geq C_p(m,c) T^p \left(\frac{1}{\log T}\int_S \frac{dE}{\|\Phi_m(E,N,1)\|^{2/(p+1)}}\right)^{p+1} - C_3 \; ,
$$

with $N = \left[T^{1+\tau}\right]$. □

Proof. **(Theorem 2)**
(**i**) Let S and f be as in the statement of the theorem. By hypothesis, we have that $\gamma_m^S < \infty$. Then, for all $r > 0$, there exists $S_r \subset S$ with $\ell(S_r) > 0$, such that

$$\gamma_m(E) \le \gamma_m^S + r, \quad \forall E \in S_r. \tag{37}$$

Define, for each $E \in S_r$, the measurable function

$$h_m(E) = \sup_{n>1} \frac{\|\Phi_m(E,n,1)\|}{n^{\gamma_m^S + r}}.$$

Hence, we have

$$\|\Phi_m(E,N,1)\| \le h_m(E) N^{\gamma_m^S + r}, \quad \forall E \in S_r \ , \ \forall N > 1. \tag{38}$$

Using the definition (6) of $\gamma_m(E)$ in (37), together with the hypothesis $\gamma_m^S < \infty$, it follows that $0 < h_m(E) < \infty$ for all $E \in S_r$. Since S_r is a bounded set and $f = 1$ on S_r, by applying Theorem 1 with $S = S_r$, $\tau = r$, and using (38), it is found that for all $q \in (0,1)$ and for all $\epsilon > 0$ with $\epsilon < \min\{1, \Omega_1 - \Omega_2\}$,

$$K_{\mu_{m,f}}(q,\epsilon) \ge C_1 \epsilon^{q-1} N^{-2q(\gamma_m^S + r)} \int_{S_r} h_m(E)^{-2q} dE - C_2 \epsilon \ , \tag{39}$$

with $N = \left[\epsilon^{-(1+r)}\right]$. Since $0 < h_m(E) < \infty$ for all $E \in S_r$, we have

$$\int_{S_r} h_m(E)^{-2q} dE \ge \left(\sup_{E \in S_r} h_m(E) \right)^{-2q} \ell(S_r) = C(q,m,S,r)$$

with $C(q,m,S,r)$ a positive constant depending on q,m,S,r. It follows by (39) that for all $q \in (0,1)$, $0 < \epsilon < \min\{1, \Omega_1 - \Omega_2\}$ and for any $r > 0$,

$$K_{\mu_{m,f}}(q,\epsilon) \ge C(q,m,c,S,r) \left(\frac{1}{\epsilon}\right)^{[p - 2(1+r)(\gamma_m^S + r)]/(p+1)} - C_2 \epsilon \ , \tag{40}$$

with $q = \frac{1}{1+p}$ and $C(q,m,c,S,r) = C_1 C(q,m,S,r)$ a positive constant depending on q,m,c,S,r. By Lemma 2 and (40), it follows that for all $p > 0$ and any $r > 0$,

$$\beta_m^-(p,f) \ge \liminf_{\epsilon \to 0} \frac{\log K_{\mu_{m,f}}(q,\epsilon)}{(q-1)\log \epsilon} \ge 1 - \frac{2(1+r)(\gamma_m^S + r)}{p}.$$

Therefore, for all $p > 0$,

$$\beta_m^-(p,f) \ge 1 - \frac{2\gamma_m^S}{p}.$$

(**ii**) Choose $E \in \mathbb{R}$ such that $\overline{\gamma}_m(E) < \infty$. For any bounded open interval I such that $E \in I$, it is found that $\gamma_m^I < \infty$; then (i) and the definition of the transport exponents implies that $\beta_m^-(p,f) \ge 1 - \frac{2\gamma_m^I}{p}$ for all $f \in \mathrm{C}_0^\infty(I)$.

Since this is true for all such intervals I, we get, by taking into account equation (4), that $\beta_m^-(p,E) \geq 1 - \frac{2\overline{\gamma}_m(E)}{p}$. □

Proof. **(Theorem 3)**

The proof is quite similar to the proof of Theorem 2, but with the subsequences N_i and $T_i = N_i^{(1+\tau)^{-1}}$, so that one can only infer conclusions about upper limits. □

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References

[1] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev, Fractal dimensions and the phenomenon of intermittency in quantum dynamics, Duke Math. J. 110 (2001) 161–193.
[2] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev, Generalized fractal dimensions: equivalence and basic properties, J. Math. Pure et Appl. 80 (2001) 977–1012.
[3] Y. P. Bliokh, V. Freilikher, S. Savel'ev, et al., Transport and localization in periodic and disordered graphene superlattices, Phys. Rev. B 79 (2009) 075123.
[4] S. L. Carvalho, D. H. U. Marchetti, W. F. Wreszinski, Sparse block-Jacobi matrices with arbitrarily accurate Hausdorff dimension, J. Math. Anal. Appl. 368 (2010) 218–234.
[5] S. L. Carvalho, C. R. de Oliveira, R. A. Prado, Sparse one-dimensional discrete Dirac operators II: spectral transitions and Hausdorff dimension, J. Math. Phys. 52 (2011) 072105.
[6] T. O. Carvalho, C. R. de Oliveira, Critical energies in random palindrome models, J. Math. Phys. 44 (2003) 945–961.
[7] D. Damanik, D. Lenz, G. Stolz, Lower transport bounds for one-dimensional continuum Schrödinger operators, Math. Ann. 336 (2006) 361–389.
[8] D. Damanik, A. Sütő, S. Tcheremchantsev, Power-law bounds on transfer matrices and quantum dynamics in one dimension II, J. Funct. Anal. 216 (2004) 362–387.
[9] C. R. de Oliveira, Intermediate Spectral Theory and Quantum Dynamics, Progress in Mathematical Physics, vol. 54, Birkhäuser, Boston, 2008.
[10] C. R. de Oliveira, R. A. Prado, Dynamical delocalization for the 1D Bernoulli discrete Dirac operator, J. Phys. A: Math. Gen. 38 (2005) L115–L119.
[11] C. R. de Oliveira, R. A. Prado, Spectral and localization properties for the one-dimensional Bernoulli discrete Dirac operator, J. Math. Phys. 46 (2005) 072105 17 pp.
[12] C. R. de Oliveira, R. A. Prado, Quantum Hamiltonians with quasi-ballistic dynamics and point spectrum, J. Differential Equations 235 (2007) 85–100.
[13] F. Germinet, A. Kiselev, S. Tcheremchantsev, Transfer matrices and transport for 1D Schrödinger operators, Ann. Inst. Fourier 54 (2004) 787–830.
[14] I. Guarneri, Spectral properties of quantum diffusion on discrete lattices, Europhys. Lett. 10 (1989) 95–100.
[15] B. Helffer, J. Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, in Schrödinger Operators, Lectures Notes in Physics, 345, 118–197, Springer, Berlin, 1989.

[16] S. Jitomirskaya, H. Schulz-Baldes, G. Stolz, Delocalization in random polymer models, Commun. Math. Phys. 233 (2003) 27–48.
[17] A. Kiselev, C. Remling, B. Simon, Effective perturbation methods for one-dimensional Schrödinger operators, J. Differential Equations 151 (1999) 290–312.
[18] R. A. Prado, C. R. de Oliveira, Dynamical lower bounds for 1D Dirac operators, Math. Z. 259 (2008) 45–60.
[19] V. S. Rabinovich, S. Roch, Essential spectra and exponential estimates of eigenfunctions of lattice operators of quantum mechanics, J. Phys. A: Math. Theor. 42 (2009) 385207.
[20] B. Simon, Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators, Proc. AMS. 124 (1996) 3361–3369.

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