On the spectrum and weakly effective operator for Dirichlet Laplacian in thin deformed tubes

César R. de Oliveira* and Alessandra A. Verri
Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil

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Abstract

We study the Laplacian in deformed thin (bounded or unbounded) tubes in $\mathbb{R}^3$, i.e., tubular regions along a curve $r(s)$ whose cross sections are multiplied by an appropriate deformation function $h(s) > 0$. One the main requirements on $h(s)$ is that it has a single point of global maximum. We find the asymptotic behaviors of the eigenvalues and weakly effective operators as the diameters of the tubes tend to zero. It is shown that such behaviors are not influenced by some geometric features of the tube, such as curvature, torsion and twisting, and so a huge amount of different deformed tubes are asymptotically described by the same weakly effective operator.

Keywords: spectrum; thin tubes; Laplacian; dimensional reduction.

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1 Introduction

The Laplacian in tubular domains has been studied in various situations [4, 8, 10, 18]. A common tubular region Ω is as follows: let $I \subseteq \mathbb{R}$ be an interval of $\mathbb{R}$, $r : I \to \mathbb{R}^3$ a curve in $\mathbb{R}^3$, parametrized by its arc length s, and $k(s)$ and $\tau(s)$ denote its curvature and torsion at the point $s \in I$, respectively. Let S be an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. Move the region S along $r(s)$ and at each point s allow the region to rotate by an angle $\alpha(s)$ (see details in Section 2). A problem of interest is the description of the spectral properties of the Laplacian in such tubes and weakly effective operators (see the definition just after Theorem 1.1) when the region Ω is

*Corresponding author. Email: oliveira@pq.cnpq.br. Telephone: +55 16 8135 0039. Fax: +55 16 3351 8218.

"squeezed" to the curve $r(s)$, that is, one considers the sequence of tubes Ω_ε generated by the cross section εS and analyze the limit $\varepsilon \to 0$.

Let $-\Delta_\varepsilon$ be the Dirichlet Laplacian in Ω_ε. For bounded tubes, i.e., when I is a bounded interval of $\mathbb{R}$, the spectrum of $-\Delta_\varepsilon$ is purely discrete because in this case its resolvent is compact. In [4] it was analyzed the convergence of the eigenvalues $\{\lambda_i^\varepsilon : i \in \mathbb{N}\}$ as $\varepsilon \to 0$ and shown that

$$\lambda_i^\varepsilon = \frac{\lambda_0}{\varepsilon^2} + \mu_i^\varepsilon, \qquad \mu_i^\varepsilon \to \mu_i,$$

where λ_0 is the first, i.e., the lowest, eigenvalue of the Laplacian in the Sobolev space $\mathcal{H}_0^1(S)$, and μ_i are the eigenvalues of the one-dimensional operator

$$w(s) \;\mapsto\; -w''(s) + \left[C(S)(\tau(s) + \alpha'(s)) - \frac{k(s)^2}{4}\right] w(s), \tag{1}$$

acting in $\mathrm{L}^2(I)$. Here $C(S)$ is a nonnegative number depending only on the transverse region S [4]. Note that this effective operator explicitly depends on the geometric shape of the reference curve $r(s)$ (and so of the tube).

An interesting problem is to know if there exists a similar result about convergence of eigenvalues for unbounded tubes. For such tubes, in [18] it is shown that if $(\tau + \alpha')(s) = 0$ and $k(s) \neq 0$, then the discrete spectrum is nonempty, whereas if $(\tau + \alpha')(s) \neq 0$ and $k(s) = 0$, then the discrete spectrum is empty. In [8], by using Γ-convergence in case of unbounded tubes, a strong resolvent convergence was proven and the same action (1) for the respective effective operator (now acting in $\mathrm{L}^2(\mathbb{R})$) was found as $\varepsilon \to 0$.

The Dirichlet Laplacian in strips of $\mathbb{R}^2$ has been studied in many works [2, 14, 13, 19]. For the case of the constraints of planar motion to curves there are results about the limit operator in [9, 1], and the effective potential is written in terms of the curvature. The main novelty, when we pass from planar domain to tubes in $\mathbb{R}^3$, as considered in [4, 8, 18, 15], is the additional presence of torsion and twisting (i.e., a nonzero $\tau(s) + \alpha'(s)$) in the effective potential, since the case of untwisted tubes has also been previously studied (see, for instance, [10, 6, 7, 11, 12, 17]).

In [14, 13] the authors consider a family of deformed strips

$$\left\{(s, y) \in \mathbb{R}^2 : s \in J,\, 0 < y < \varepsilon h(s)\right\},$$

where $J = [-a, b]$, $0 < a, b \leq \infty$, and $h(s) > 0$ is a continuous function satisfying:

(i) $h(s)$ is a C^1 function in $J \setminus \{0\}$ and $\|h'/h\|_\infty < \infty$;

(ii) near the origin h behaves as

$$h(s) = M - s^2 + O(|s|^3), \qquad M > 0, \tag{2}$$

and $s = 0$ is a single point of global maximum for h;

(iii) in case $I=\mathbb{R}$ it is assumed that $\limsup_{|s|\to\infty} h(s) < M$.

In what follows we assume that h satisfies the above conditions.

It was shown [14, 13] that, for ε small enough, the discrete spectrum of the Laplacian is always nonempty and the eigenvalues $\lambda_j(\varepsilon)$ have the following behavior

$$\mu_j = \lim_{\varepsilon\to 0} \varepsilon \left(\lambda_j(\varepsilon) - \frac{\pi^2}{\varepsilon^2 M^2}\right),$$

where μ_j are the eigenvalues of the operator in $\mathrm{L}^2(\mathbb{R})$ (it acts on a subspace of $\mathrm{L}^2(\mathbb{R})$, independently if the interval I is bounded or not) given by

$$(Tw)(s) = -w''(s) + 2\frac{\pi^2}{M^3}\, s^2\, w(s),$$

so that we say that T is a *weakly effective operator* (WEO) in such situation.

In this work we show that these results hold in a more general setting. We consider a sequence of tubes Ω_ε in the space $\mathbb{R}^3$, as presented at the beginning of this introduction, but we deform them by multiplying their cross sections by the above function $h(s)$. Here the tubes may be bounded or not. Then we analyze the asymptotic behavior of eigenvalues and the weakly effective operators in the limit $\varepsilon\to 0$. The situation here differs from [14, 13], since besides the different dimensions (we consider regions in 3$-$dimensional space), the reference curves defining our tubes are allowed to have nontrivial curvatures and torsion. These tubes, which we shall call *deformed tubes*, will generically be denoted by Λ_ε (see details in Section 2).

Our main goal is to study how curvature and torsion of the reference curve, together with the deforming function h, influence the WEO and eigenvalues as $\varepsilon\to 0$. To this end, we introduce some notation right now. Recall that λ_0 is the lowest eigenvalue of the negative Laplacian with Dirichlet conditions in the region S, and let u_0 be the corresponding (positive) normalized eigenfunction, that is,

$$-\Delta u_0 = \lambda_0 u_0, \quad u_0 \in \mathcal{H}_0^1(S), \quad \int_S u_0(y)^2 dy = 1. \tag{3}$$

Furthermore, denote by $\mathcal{L}$ the subspace of $\mathrm{L}^2(I\times S)$ generated by functions $w(s)u_0(y)$ with $w\in \mathrm{L}^2(I)$.

We study three distinct cases. First, the tubes are bounded since the interval I is of the form $I=[-a,b]$ with $0<a,b<\infty$, and we consider the Dirichlet condition at the boundary $\partial\Lambda_\varepsilon$. In the second case, the tubes are bounded but the Dirichlet condition at the vertical part of the $\partial\Lambda_\varepsilon$, that is, $\{(-a)\times S \cup b\times S\}$, is replaced by Neumann. In the third case we consider $I=\mathbb{R}$ with Dirichlet condition at $\partial\Lambda_\varepsilon$.

If the tubes are not deformed, according to the results of [4, 8], the effective operator (1) presents an additional potential

$$C(S)\left(\tau+\alpha'\right)(s) - k^2(s)/4$$

derived from geometric features of the tube. Hence, here there is a kind of competition between geometric properties of the tube and the behavior at its single maximum of the deformation function h. Roughly speaking, it is expected that the behavior of h at the single maximum will control the limit $\varepsilon \to 0$, since the geometric effects gives a contribution of order zero, whereas the single maximum of the deformation function h gives a contribution of order $1/\varepsilon$. However, this requires a proof which turns out to be far from trivial, and so for the three cases mentioned in the previous paragraph, we prove the following result:

Theorem 1.1. Let I denote either $\mathbb{R}$ or a bounded interval $[-a, b]$ as above; in case $I = \mathbb{R}$ assume that $\lim_{|s|\to\infty} k(s) = 0$. If $l_j(\varepsilon)$ denote the eigenvalues of the Dirichlet $-\Delta_\varepsilon$ in the deformed tube Λ_ε, then, the limits

$$\mu_j = \lim_{\varepsilon\to 0} \varepsilon \left(l_j(\varepsilon) - \frac{\lambda_0}{\varepsilon^2 M^2} \right) \tag{4}$$

exist, where $\mu_j = (2j+1)(2\lambda_0/M^3)^{1/2}$ are the eigenvalues of the self-adjoint operator T, acting in $\mathrm{L}^2(\mathbb{R})$, given by

$$(Tu)(s) = -u''(s) + 2\frac{\lambda_0}{M^3}\, s^2\, u(s). \tag{5}$$

Due to the conclusions of Theorem 1.1, T is a WEO for $-\Delta_\varepsilon$ as $\varepsilon \to 0$. Note that T has purely discrete spectrum since the potential

$$V(s) = 2\frac{\lambda_0}{M^3}\, s^2 \to \infty, \qquad |s| \to \infty;$$

in this case it is the harmonic oscillator potential (but see (7) below). Therefore, for deformed tubes as above, the weakly effective operators T do not depend on some geometric features of the tube, although the curvature of the reference curve must vanish at infinity. The additional potential $V(s)$ is related to the behavior of $h(s)$ near it maximum (at the origin). Hence, the eigenvalues of the Laplacian in quite different deformed tubes are described by the same WEO as $\varepsilon \to 0$!

In Section 2 we present a detailed construction of the deformed tubes Λ_ε. Our study and technique are focused on analyzing the sequence of quadratic forms

$$F_\varepsilon(\psi) = \int_{\Lambda_\varepsilon} \left(|\nabla\psi|^2 - \frac{\lambda_0}{\varepsilon^2 M^2}|\psi|^2 \right) dx, \qquad \mathrm{dom}\, F_\varepsilon = \mathcal{H}_0^1(\Lambda_\varepsilon). \tag{6}$$

In Section 3 it will become clear why we subtract terms of the form $\lambda_0/(\varepsilon^2 M^2)|\psi|^2$ from the quadratic forms; we think this is in fact a natural choice. In Section 3 we also perform a change of variables so that the integration region and the corresponding domains in (6) remain fixed. In Section 4, we show that our analysis can be restricted to a specific subspace; we will see that this subspace can be identified with the Sobolev space $\mathcal{H}_0^1(I)$, and we call this fact a *reduction of dimension.* Finally, in Sections 5, 6 and 7, we discuss details of the three cases previously mentioned.

We remark that although we rely on [14, 13], the generalization to our setting is not immediate and different techniques are added to those of the original works. Furthermore, as an alternative to (2), all results can be easily adapted to cover more general deformation functions $h(s)$, as considered in [14, 13], so that near the unique global maximum at the origin they behave as

$$h(s) = \begin{cases} M - c_+ s^m + O(s^{m+1}), & \text{if} \quad s > 0 \\ M - c_- |s|^m + O(|s|^{m+1}), & \text{if} \quad s < 0 \end{cases}, \tag{7}$$

for some positive numbers $M, m, c_\pm$. For the sake of simplicity, in Equation (2) we have particularized to $m = 2$ and $c_+ = c_- = 1$.

An interesting problem would be if the maximum of h would be reached at an interval of values of the parameter s instead of a single point (see [3] for results in this direction in case of bounded domains, as well as [5, 16]); we are currently working on a related problem.

2 Geometry of the tubes

Let $I = [-a, b]$, with either $0 < a, b < \infty$ or $a = b = \infty$, be an interval of $\mathbb{R}$, $r : I \subseteq \mathbb{R} \to \mathbb{R}^3$ a simple C^2 curve in $\mathbb{R}^3$ parametrized by its arc length parameter s and, as in the previous section, $k(s)$ is its curvature. The vectors

$$T(s) = r'(s), \qquad N(s) = \frac{1}{k(s)} T'(s), \qquad B(s) = T(s) \times N(s),$$

denote, respectively, the tangent, normal and binormal vectors of the curve. We assume that Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\tau(s)$ is the torsion of the curve $r(s)$.

Let S be an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. The set

$$\Omega = \left\{ x \in \mathbb{R}^3 : x = r(s) + y_1 N(s) + y_2 B(s),\ s \in I, y = (y_1, y_2) \in S \right\}$$

is obtained by translating the region S along the curve r. At each point $r(s)$ we allow a rotation of the region S by an angle $\alpha(s)$ with respect to $\alpha(0) = 0$, so that the new region is given by

$$\Omega^\alpha = \left\{ x \in \mathbb{R}^3 : x = r(s) + y_1 N_\alpha(s) + y_2 B_\alpha(s),\ s \in I, (y_1, y_2) \in S \right\},$$

where

$$N_\alpha(s) := \cos\alpha(s)N(s) + \sin\alpha(s)B(s),$$
$$B_\alpha(s) := -\sin\alpha(s)N(s) + \cos\alpha(s)B(s).$$

Next, for each $0 < \varepsilon < 1$, we "squeeze" the cross sections of the above region, that is, we consider

$$\Omega_\varepsilon^\alpha = \left\{x \in \mathbb{R}^3 : x = r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s),\ s \in I, (y_1, y_2) \in S\right\}.$$

Note that $\Omega_\varepsilon^\alpha$ approaches the curve $r(s)$ as $\varepsilon \to 0$.

Finally, we consider the function $h(s)$ defined in the Introduction, so that each region $\Omega_\varepsilon^\alpha$ is properly deformed, and the result is

$$\Lambda_\varepsilon^\alpha := \left\{x \in \mathbb{R}^3 : x = r(s) + \varepsilon h(s) y_1 N_\alpha(s) + \varepsilon y_2 h(s) B_\alpha(s),\ s \in I, (y_1, y_2) \in S\right\}.$$

From now on we will omit the symbol α in most notations and write $dx = ds dy_1 dy_2$ and $dy = dy_1 dy_2$.

In this work we study the behavior of a free quantum particle that moves in Λ_ε, and initially with Dirichlet boundary condition at the boundary $\partial\Lambda_\varepsilon$. Thus, we initially consider the family of quadratic forms

$$b_\varepsilon(\psi) := \int_{\Lambda_\varepsilon} |\nabla\psi|^2 dx, \qquad \operatorname{dom} b_\varepsilon = \mathcal{H}_0^1(\Lambda_\varepsilon), \tag{8}$$

which is associated with the Dirichlet Laplacian operator $-\Delta_\varepsilon$ in Λ_ε. The symbol $\nabla = (\partial_s, \nabla_y)$, $\nabla_y = (\partial_{y_1}, \partial_{y_2})$, denotes the gradient in the coordinates (s, y_1, y_2) in $\mathbb{R}^3$.

3 Quadratic forms

As usual in this kind of problems, in this section we perform a change of variables so that the integration region in (8), and consequently the domains, become independent of $\varepsilon > 0$. Then, for the singular limit $\varepsilon \to 0$, customary "regularizations" will be employed.

Consider the mapping

$$\begin{array}{rccl} f_\varepsilon : & I \times S & \to & \Lambda_\varepsilon \\ & (s, y_1, y_2) & \mapsto & r(s) + \varepsilon\, h(s)\,(y_1 N_\alpha(s) + y_2 B_\alpha(s)), \end{array}$$

and suppose the boundedness $\|k\|_\infty, \|\tau\|_\infty, \|\alpha'\|_\infty < \infty$. These conditions are to guarantee that f_ε will be a diffeomorphism. With this change of variables we work with a fixed region for all $\varepsilon > 0$; more precisely, the domain of the quadratic form (8) turns out to be $\mathcal{H}_0^1(I \times S)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon^\alpha$ which is induced by f_ε, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j\rangle = G_{ji}, \qquad 1 \le i, j \le 3,$$

where

$$e_1 = \frac{\partial f_\varepsilon}{\partial s}, \qquad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \qquad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$\begin{aligned} J &= \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ &= \begin{pmatrix} \beta_\varepsilon & -\varepsilon h(\tau+\alpha')\langle z_\alpha^\perp, y\rangle + \varepsilon h'\langle z_\alpha, y\rangle & \varepsilon h(\tau+\alpha')\langle z_\alpha, y\rangle + \varepsilon h'\langle z_\alpha^\perp, y\rangle \\ 0 & \varepsilon h\cos\alpha & \varepsilon h \sin\alpha \\ 0 & -\varepsilon h \sin\alpha & \varepsilon h\cos\alpha \end{pmatrix}, \end{aligned}$$

where

$$\beta_\varepsilon(s,y) = 1 - \varepsilon h(s)k(s)\langle z_\alpha, y\rangle, \quad z_\alpha := (\cos\alpha, -\sin\alpha), \quad z_\alpha^\perp := (\sin\alpha, \cos\alpha).$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} \frac{1}{\beta_\varepsilon} & \frac{1}{\beta_\varepsilon}\left[(\tau+\alpha')y_2 - \frac{h'}{h}y_1\right] & \frac{1}{\beta_\varepsilon}\left[-(\tau+\alpha')y_1 - \frac{h'}{h}y_2\right] \\ 0 & \frac{\cos\alpha}{\varepsilon h} & \frac{-\sin\alpha}{\varepsilon h} \\ 0 & \frac{\sin\alpha}{\varepsilon h} & \frac{\cos\alpha}{\varepsilon h} \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2 h^2(s)\beta_\varepsilon(s,y)$. Since k and h are bounded functions, for ε small enough β_ε does not vanish in $I \times S$. Thus, $\beta_\varepsilon > 0$ and f_ε is a local diffeomorphism. By requiring that f_ε is injective (that is, the tube is not self-intersecting), a global diffeomorphism is obtained.

Introducing the notation

$$||\psi||_G^2 := \int_{I\times S} |\psi(s,y)|^2 \varepsilon^2 h^2(s)\beta_\varepsilon(s,y)\, ds dy,$$

we obtain a sequence of quadratic forms

$$\tilde{b}_\varepsilon(\psi) := \left|\left|J^{-1}\nabla\psi\right|\right|_G, \qquad \operatorname{dom}\tilde{b}_\varepsilon = \mathcal{H}_0^1(I\times S, G).$$

More precisely, the above change of coordinates was obtained by a unitary transformation

$$\begin{array}{rccl} U_\varepsilon: & \mathrm{L}^2(\Lambda_\varepsilon) & \to & \mathrm{L}^2(I\times S, G) \\ & \phi & \mapsto & \phi\circ f_\varepsilon \end{array}.$$

However, we still denote $U_\varepsilon\psi$ by ψ.

Recall that λ_0 is the lowest eigenvalue of the negative Laplacian with Dirichlet boundary conditions in the cross section region S, and $u_0 \geq 0$ (see Equation (3)) the corresponding eigenfunction of this restricted problem. This eigenfunction u_0 is directly related to transverse oscillations in Λ_ε. Due to this fact, in [4, 8] the authors have remove the diverging energy

λ_0/ε^2 from their quadratic forms. In our case, as the boundary of the tubes were multiplied by $h(s)$, we subtract the terms of the form $\lambda_0/(\varepsilon M)^2$, i.e., since $0 < h(s) \le M$, for all $s \in I$, we eliminate the possible "least transverse energy."

Therefore, we turn to the study of the sequence of quadratic forms

$$\tilde{g}_\varepsilon(\psi) := \left(\left|\left|J^{-1}\nabla\psi\right|\right|_G^2 - \frac{\lambda_0}{\varepsilon^2 M^2}\|\psi\|_G^2 + c\|\psi\|_G^2 \right),$$

where c is a positive constant to be chosen later on. After the norms are written out, we obtain

$$\begin{aligned}\tilde{g}_\varepsilon(\psi) &= \varepsilon^2 \int_{I\times S} \Big(\frac{1}{\beta_\varepsilon^2(s,y)} \left| \psi' + \nabla_y\psi \cdot Ry(\tau+\alpha')(s) - \nabla_y\psi \cdot y\,\frac{h'(s)}{h(s)} \right|^2 \\ &+ \frac{|\nabla_y\psi|^2}{\varepsilon^2 h(s)^2} - \frac{\lambda_0}{\varepsilon^2 M^2}|\psi|^2 + c|\psi|^2 \Big) h(s)^2 \beta_\varepsilon(s,y)\, ds dy\,.\end{aligned}$$

Note that $\operatorname{dom}\tilde{g}_\varepsilon = \mathcal{H}_0^1(I\times S)$ is a subspace of $\mathrm{L}^2(I\times S, h(s)^2\beta_\varepsilon(s,y))$. We observe that the factor $|\nabla_y\psi|^2/(\varepsilon h(s))^2$ is directly related to transverse oscillations of the particle. This term diverges as $\varepsilon \to 0$, but we control this fact by subtracting $\lambda_0/(\varepsilon M)^2|\psi|^2$ from the quadratic form (a renormalization).

It will be convenient to work in the space $\mathrm{L}^2(I\times S, \beta_\varepsilon(s,y))$; so we consider the isometry

$$\begin{array}{ccc} \mathrm{L}^2(I\times S,\beta_\varepsilon) & \to & \mathrm{L}^2(I\times S, h(s)^2\beta_\varepsilon) \\ v & \mapsto & vh^{-1} \end{array}.$$

This change of variables and the division by the global factor ε^2 (a common singular factor due to the "change of dimension" as $\varepsilon \to 0$) leads to

$$\begin{aligned}\hat{g}_\varepsilon(v) &:= \int_{I\times S} \Big(\frac{1}{\beta_\varepsilon(s,y)} \left| v' - v\,\frac{h'(s)}{h(s)} + \nabla_y v \cdot Ry(\tau+\alpha')(s) - \nabla_y v \cdot y\,\frac{h'(s)}{h(s)} \right|^2 \\ &+ \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 h(s)^2}|\nabla_y v|^2 - \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 M^2}|v|^2 + c\beta_\varepsilon(s,y)|v|^2 \Big)\, ds dy,\end{aligned}$$

with $\operatorname{dom}\hat{g}_\varepsilon = \mathcal{H}_0^1(I\times S)$, again as a subspace of $\mathrm{L}^2(I\times S, \beta_\varepsilon(s,y))$. However, this latter space can be identified with $\mathrm{L}^2(I\times S)$, for all $\varepsilon > 0$, since $\beta_\varepsilon(s,y)$ converges uniformly to 1 as $\varepsilon \to 0$. Hence we introduce the form

$$\begin{aligned}g_\varepsilon(v) &:= \int_{I\times S} \Big(\left| v' - v\,\frac{h'(s)}{h(s)} + \nabla_y v \cdot Ry(\tau+\alpha')(s) - \nabla_y v \cdot y\,\frac{h'(s)}{h(s)} \right|^2 \\ &+ \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 h(s)^2}|\nabla_y v|^2 - \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 M^2}|v|^2 + c|v|^2 \Big)\, ds dy,\end{aligned}$$

with $\operatorname{dom} g_\varepsilon = \mathcal{H}_0^1(I\times S)$.

Let $\hat{G}_\varepsilon$ and G_ε be the self-adjoint operators associated with the quadratic forms $\hat{g}_\varepsilon$ and g_ε, respectively.

Theorem 3.1. For ε small enough, there exists $C > 0$ so that

$$\left\| \hat{G}_\varepsilon^{-1} - G_\varepsilon^{-1} \right\| \leq C\varepsilon.$$

This theorem follows basically from the fact that $\beta_\varepsilon(s, y) \to 1$ uniformly as $\varepsilon \to 0$. Its proof is presented in the Appendix.

Due to the above changes of variables and Theorem 3.1, we may consider the sequence of quadratic forms g_ε in what follows.

4 Reduction of dimension

Recall that $u_0(y)$ is the positive and normalized eigenfunction corresponding to the first eigenvalue λ_0 of the Laplacian in $\mathcal{H}_0^1(S)$. After the orthogonal decomposition $\mathrm{L}^2(\mathbb{R} \times S) = \mathcal{L} \oplus \mathcal{L}^\perp$, for $\psi \in \mathrm{L}^2(\mathbb{R} \times S)$, we can write

$$\psi(s, y) = w(s)u_0(y) + \eta(s, y),$$

with $w \in \mathrm{L}^2(I)$ and $\eta \in \mathcal{L}^\perp$. We observe that $\eta \in \mathcal{L}^\perp$ implies

$$\int_S u_0(y)\eta(s, y)dy = 0, \qquad \text{a.e.}[s].$$

Note that $wu_0 \in \mathcal{H}_0^1(I \times S)$ if $w \in \mathcal{H}_0^1(I)$. For $\psi \in \mathcal{H}_0^1(\mathbb{R} \times S)$, write $\psi = wu_0 + \eta$ with $w \in \mathcal{H}_0^1(I)$ and $\eta \in \mathcal{H}_0^1(\mathbb{R} \times S) \cap \mathcal{L}^\perp$.

First we study the quadratic form g_ε restricted to the subspace $\mathcal{H}_0^1(I \times S) \cap \mathcal{L}$. For $w \in \mathcal{H}_0^1(I)$, some calculations show that

$$g_\varepsilon(wu_0) = \int_I \left[|w'|^2 + \vartheta(s)|w|^2 + \zeta_\varepsilon(s, y) \left(\frac{\lambda_0}{\varepsilon^2 h^2(s)} - \frac{\lambda_0}{\varepsilon^2 M^2} \right) |w|^2 + c|w|^2 \right] ds,$$

where

$$\vartheta(s) = C_1(S)(\tau(s)+\alpha'(s))^2 + (C_2(S)-1) \left(\frac{h'(s)}{h(s)} \right)^2 - 2C_3(S)(\tau(s)+\alpha'(s)) \frac{h'(s)}{h(s)}$$

and

$$\zeta_\varepsilon(s, y) = 1 - \varepsilon\, k(s)h(s) \left\langle z_\alpha(s), F(S) \right\rangle.$$

The constants $C_1(S)$, $C_2(S)$ and $C_3(S)$ that appear in the definition of ϑ depend only on the region S and are explicitly given by

$$C_1(S) = \int_S |\langle \nabla_y u_0, Ry \rangle|^2 dy, \qquad C_2(S) = \int_S |\langle \nabla_y u_0, y \rangle|^2 dy,$$

and

$$C_3(S) = \int_S \langle \nabla_y u_0, Ry \rangle \langle \nabla_y u_0, y \rangle dy.$$

The vector $F(S) = (F_1(S), F_2(S))$ in the definition of ζ_ε also depends only on the region S, and its components are given by

$$F_1(S) = \int_S y_1 |u_0|^2 dy \qquad \text{and} \qquad F_2(S) = \int_S y_2 |u_0|^2 dy.$$

Under such restrictions, the quadratic form b_ε in $\mathcal{H}_0^1(I)$ can be written in terms of the form $t_\varepsilon = t_{\varepsilon,c}$ given by

$$t_\varepsilon(w) := g_\varepsilon(w u_0) = \int_I \left(|w'|^2 + W_\varepsilon(s)|w|^2 \right) ds, \tag{9}$$

with

$$W_\varepsilon(s) := \vartheta(s) + c + \zeta_\varepsilon(s,y) \left(\frac{\lambda_0}{\varepsilon^2 h^2(s)} - \frac{\lambda_0}{\varepsilon^2 M^2} \right). \tag{10}$$

We choose the constant c so that $c > \|v\|_\infty + (1/M^2)\|k(s)^2/4\|_\infty$.

Since $k(s)$ and $h(s)$ are bounded functions, there exist $\varepsilon_1 > 0$ and $\delta > 0$ so that, for all $s \in I$,

$$1 - \varepsilon\, k(s)h(s)\langle z_\alpha(s), F(S)\rangle > \delta \qquad \text{and} \qquad 1 - \varepsilon\, k(s)h(s)\langle z_\alpha(s), y\rangle > \delta,$$

for all $\varepsilon < \varepsilon_1$. In what follows, we tacitly assume that $\varepsilon < \varepsilon_1$.

The self-adjoint operator associated with t_ε in $\mathrm{L}^2(I)$ is

$$(T_{\varepsilon,c} w)(s) := -w''(s) + W_\varepsilon(s) w(s), \qquad \operatorname{dom} T_{\varepsilon,c} = \mathcal{H}^2(I) \cap \mathcal{H}_0^1(I).$$

From now on we denote by $-\Delta_{\varepsilon,c}$ the operator $-\Delta_\varepsilon + c\mathbf{1}$ and write $T_\varepsilon = T_{\varepsilon,c} - c\mathbf{1}$. Next we discuss how the resolvent operator $\left(-\Delta_{\varepsilon,c} - \lambda_0/\varepsilon^2 M^2 \mathbf{1}\right)^{-1}$ can be approximated by $T_{\varepsilon,c}^{-1} \oplus 0$, where 0 is the null operator on the subspace $\mathcal{L}^\perp$. Such result gives a quantitative indication of how $-\Delta_\varepsilon$ is approximated by T_ε.

Lemma 4.1. Suppose that I is a bounded interval. Then, there exists $C_6 > 0$ so that

$$t_\varepsilon(w) \geq C_6^{-1}\, \varepsilon^{-1} \int_I |w|^2 ds, \qquad \forall w \in \mathcal{H}_0^1(I),\ 0 < \varepsilon < \varepsilon_1.$$

By noting that

$$\frac{\varepsilon^2 W_\varepsilon(s)}{s^2} \geq \frac{\lambda_0}{s^2} \delta \left(\frac{1}{h(s)^2} - \frac{1}{M^2} \right),$$

the proof of Lemma 4.1 is similar to the proof of Lemma 2.1 in [14], and it will not be reproduced here.

By following [4], for each $\xi \in \mathbb{R}^2$, we consider the following perturbed problem

$$-\text{div}[(1-(\xi \cdot y))\nabla_y u] = \lambda(1-(\xi\cdot y))u, \qquad u \in \mathcal{H}_0^1(S).$$

By taking $\xi = \varepsilon h(s)k(s)z_\alpha$, for ε small enough, the perturbed operator is positive and with compact resolvent. Denote by $\lambda(\xi) > 0$ its first eigenvalue, i.e.,

$$\lambda(\xi) = \inf_{\{u\in\mathcal{H}_0^1(S):\, u\neq 0\}} \frac{\int_S (1-(\xi\cdot y))|\nabla_y u|^2 dy}{\int_S (1-(\xi\cdot y))|u|^2 dy}.$$

Thus, for $v \in \mathcal{H}_0^1(\mathbb{R}\times S)$,

$$\frac{1}{\varepsilon^2}\int_S \beta_\varepsilon(s,y)(|\nabla_y v|^2 - \lambda_0|v|^2)dy \geq \gamma_\varepsilon(s)\int_S \beta_\varepsilon(s,y)|v|^2 dy \qquad \text{a.e.}[s], \quad (11)$$

where

$$\gamma_\varepsilon(s) := \frac{\lambda(\varepsilon h(s)k(s)z_\alpha(s)) - \lambda_0}{\varepsilon^2}.$$

Using the fact that $h(s)$ and $k(s)$ are bounded functions, it is possible to prove that $\gamma_\varepsilon(s)$ converges uniformly as $\varepsilon \to 0$ to a bounded function (see Proposition 4.1 in [4]). This will be used in the proof of Lemma 4.2.

Lemma 4.2. Let I denote either $\mathbb{R}$ or a bounded interval. Then, for $\eta \in \mathcal{H}_0^1(I\times S)\cap\mathcal{L}^\perp$, there exists $C_7 \in \mathbb{R}$ so that, for ε small enough,

$$g_\varepsilon(\eta) \geq \frac{C_7}{\varepsilon^2 M^2}\|\eta\|^2.$$

Proof: Let λ_1 be the second eigenvalue of the Laplacian in $\mathcal{H}_0^1(S)$, and pick $\eta \in \mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}^\perp$.

Since $h(s) \leq M$, for all $s \in I$, we have

$$\int_S \beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2 h(s)^2} - \lambda_1\frac{|\eta|^2}{\varepsilon^2M^2}\right)dy \geq \int_S \beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2M^2} - \lambda_1\frac{|\eta|^2}{\varepsilon^2M^2}\right)dy.$$

By (11), it follows that

$$\int_S \beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2M^2} - \lambda_1\frac{|\eta|^2}{\varepsilon^2M^2}\right)dy \geq \frac{\gamma_\varepsilon(s)}{M^2}\int_S \beta_\varepsilon(s,y)|\eta|^2 dy.$$

Since $\gamma_\varepsilon(s)$ converges uniformly as $\varepsilon \to 0$, there exists $C_8 \in \mathbb{R}$ so that, for ε small enough,

$$\frac{\gamma_\varepsilon(s)}{M^2} \geq C_8, \qquad \forall s \in I.$$

Thus,

$$\frac{\gamma_\varepsilon(s)}{M^2}\int_S \beta_\varepsilon(s,y)|\eta|^2 dy \geq C_8\int_S \beta_\varepsilon(s,y)|\eta|^2 dy,$$

and so

$$\int_{I\times S}\beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2h(s)^2}-\lambda_1\frac{|\eta|^2}{\varepsilon^2M^2}\right)dyds\geq C_8\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dyds.$$

Adding and subtracting the term $\frac{\lambda_0}{\varepsilon^2M^2}\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dyds$ on the left hand side of the above inequality, we obtain

$$\begin{aligned}\int_{I\times S}&\beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2h(s)^2}-\lambda_0\frac{|\eta|^2}{\varepsilon^2M^2}\right)dyds\\ &\geq C_8\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dy+\frac{(\lambda_1-\lambda_0)}{\varepsilon^2M^2}\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dyds.\end{aligned}$$

Now, for ε small enough, there exists C_9 so that

$$\begin{aligned}g_\varepsilon(\eta)&\geq\int_{I\times S}\beta_\varepsilon(s,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2h(s)^2}-\lambda_0\frac{|\eta|^2}{\varepsilon^2M^2}\right)dyds+c\int_{I\times S}|\eta|^2dyds\\ &\geq C_8\delta\int_{I\times S}|\eta|^2dyds+\frac{(\lambda_1-\lambda_0)}{\varepsilon^2M^2}\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dyds+c\int_{I\times S}|\eta|^2dyds\\ &\geq\frac{C_9}{\varepsilon^2M^2}\int_{I\times S}\beta_\varepsilon(s,y)|\eta|^2dyds\\ &\geq\frac{C_9}{\varepsilon^2M^2}\delta\int_{I\times S}|\eta|^2dyds.\end{aligned}$$

Finally, it is enough to take $C_7=C_9\,\delta$ to complete the proof of the lemma. ■

Now we are ready to state and prove the main result of this section; it will rest on results presented in Section 3 of [13], combined with the previous lemmas.

Theorem 4.3. Let I denote either $\mathbb{R}$ or a bounded interval. Then there exists $C_{10}>0$ so that, for ε small enough,

$$\left\|\left(-\Delta_{\varepsilon,c}-\frac{\lambda_0}{\varepsilon^2M^2}\mathbf{1}\right)^{-1}-\left(T_{\varepsilon,c}^{-1}\oplus 0\right)\right\|\leq C_{10}\,\varepsilon^{3/2},$$

where 0 denotes the null operator on the subspace $\mathcal{L}^\perp$.

Proof: For $\psi\in\mathcal{H}_0^1(\mathbb{R}\times S)$ write

$$\psi(s,y)=w(s)u_0(y)+\eta(s,y),$$

with $w\in\mathcal{H}_0^1(I)$ and $\eta\in\mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}^\perp$. Thus, the quadratic form $g_\varepsilon(\psi)$ can be rewritten as

$$g_\varepsilon(\psi)=t_\varepsilon(w)+g_\varepsilon(\eta)+2m_\varepsilon(wu_0,\eta),$$

where $t_\varepsilon(w) = g_\varepsilon(wu_0)$ (see (9)) and

$$\begin{aligned} m_\varepsilon(wu_0,\eta) &= \int_{I\times S} dyds\Big[\left(w'u_0 - wu_0\,\frac{h'}{h} + w\nabla_y u_0\cdot Ry(\tau+\alpha') - w\nabla_y u_0\cdot y\,\frac{h'}{h}\right) \\ &\times \left(\eta' - \eta\,\frac{h'}{h} + \nabla_y\eta\cdot Ry(\tau+\alpha') - \nabla_y\eta\cdot y\,\frac{h'}{h}\right)\Big] \\ &- \int_{I\times S} dyds\; k(s)h(s)\langle z_\alpha, y\rangle w\left(\frac{\nabla_y u_0\nabla_y\eta}{\varepsilon h^2} - \lambda_0\frac{u_0\eta}{\varepsilon M^2}\right). \end{aligned}$$

We are going to show that $t_\varepsilon(w)$, $g_\varepsilon(\eta)$ and $m_\varepsilon(wu_0,\eta)$ satisfy the conditions (3.2), (3.3), (3.4) and (3.5) in Section 3 of [13], and so the theorem will follow. Conditions (3.2), (3.3) and (3.4) are obtained by applying Lemmas 4.1 and 4.2 above. We need only to verify condition (3.5), i.e., that there exists a function $q(\varepsilon)$ so that for each $\psi\in\mathcal{H}_0^1(I\times S)$

$$|m_\varepsilon(wu_0,\eta)|^2 \le q(\varepsilon)^2\, t_\varepsilon(w)\, g_\varepsilon(\eta), \qquad q(\varepsilon)\to 0 \qquad (\varepsilon\to 0). \tag{12}$$

We write

$$m_\varepsilon(wu_0,\eta) = m_\varepsilon^1(wu_0,\eta) - m_\varepsilon^2(wu_0,\eta) + m_\varepsilon^3(wu_0,\eta) - m_\varepsilon^4(wu_0,\eta) - m_\varepsilon^5(wu_0,\eta),$$

where

$$\begin{aligned} m_\varepsilon^1(wu_0,\eta) &:= \int_{\mathbb{R}\times S} w'u_0\left(\eta' - \eta\,\frac{h'}{h} + \nabla_y\eta\cdot Ry(\tau+\alpha') - \nabla_y\eta\cdot y\,\frac{h'}{h}\right) dsdy\,, \\ m_\varepsilon^2(wu_0,\eta) &:= \int_{\mathbb{R}\times S} wu_0\,\frac{h'}{h}\left(\eta' - \eta\,\frac{h'}{h} + \nabla_y\eta\cdot Ry(\tau+\alpha') - \nabla_y\eta\cdot y\,\frac{h'}{h}\right) dsdy\,, \\ m_\varepsilon^3(wu_0,\eta) &:= \int_{\mathbb{R}\times S} w\nabla_y u_0\cdot Ry(\tau+\alpha')\left(\eta' - \eta\,\frac{h'}{h} + \nabla_y\eta\cdot Ry(\tau+\alpha') - \nabla_y\eta\cdot y\,\frac{h'}{h}\right) dyds\,, \\ m_\varepsilon^4(wu_0,\eta) &:= \int_{\mathbb{R}\times S} w\nabla_y u_0\cdot y\,\frac{h'}{h}\left(\eta' - \eta\,\frac{h'}{h} + \nabla_y\eta\cdot Ry(\tau+\alpha') - \nabla_y\eta\cdot y\,\frac{h'}{h}\right) dsdy\,, \\ m_\varepsilon^5(wu_0,\eta) &:= \int_{I\times S} \frac{k(s)h(s)\langle z_\alpha, y\rangle}{\varepsilon} w\left(\frac{\nabla_y u_0\nabla_y\eta}{h^2} - \lambda_0\frac{u_0\eta}{M^2}\right) dyds. \end{aligned}$$

Now we are going to estimate each of the above terms. Let

$$H_1 := \left|\left|\frac{h'}{h}\right|\right|_\infty, \qquad H_2 := \left|\left|\tau+\alpha'\right|\right|_\infty,$$

and recall that

$$C_1(S) = \int_S |\langle\nabla_y u_0, Ry\rangle|^2 dy \qquad \text{and} \qquad C_2(S) = \int_S |\langle\nabla_y u_0, y\rangle|^2 dy.$$

By Green identities and some calculations, we get

$$\int_S u_0\langle\nabla_y\eta, Ry\rangle dy = -\int_S \langle\nabla_y u_0, Ry\rangle\eta dy,$$

$$\int_S u_0 \langle \nabla_y \eta, y\rangle dy = -\int_S \langle \nabla_y u_0, y\rangle \eta dy.$$

Hence:

- $$\begin{aligned} |m^1_\varepsilon(wu_0,\eta)| &\leq H_2^{1/2} C_1(S) \left(\int_I |w'|^2 ds\right)^{1/2} \left(\int_{I\times S} |\eta|^2 dyds\right)^{1/2} \\ &+ H_1^{1/2} C_2(S) \left(\int_I |w'|^2 ds\right)^{1/2} \left(\int_{I\times S} |\eta|^2 dyds\right)^{1/2} \\ &\leq C_{11}\, \varepsilon\, t_\varepsilon(w)^{1/2} g_\varepsilon(\eta)^{1/2}; \end{aligned}$$

- $$\begin{aligned} |m^2_\varepsilon(wu_0,\eta)| &\leq \left(\int_{I\times S} |w|^2 |u_0|^2 \left(\frac{h'}{h}\right)^2 dyds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq H_1 \left(\int_I |w|^2 ds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq \frac{H_1}{C_6^{1/2}}\, \varepsilon^{1/2}\, t_\varepsilon(w)^{1/2} g_\varepsilon(\eta)^{1/2}; \end{aligned}$$

- $$\begin{aligned} |m^3_\varepsilon(wu_0,\eta)| &\leq \left(\int_{I\times S} |w|^2 |\langle \nabla_y u_0, Ry\rangle|^2 (\tau+\alpha')^2 dyds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &= \left(\int_I |w|^2 C_1(S)(\tau+\alpha')^2 ds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq C_1(S)^{1/2} H_2 \left(\int_I |w|^2 ds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq C_1(S)^{1/2} \frac{H_2}{C_6^{1/2}}\, \varepsilon^{1/2}\, t_\varepsilon(w)^{1/2} g_\varepsilon(\eta)^{1/2}; \end{aligned}$$

- $$\begin{aligned} |m^4_\varepsilon(wu_0,\eta)| &\leq \left(\int_{I\times S} |w|^2 |\langle \nabla_y u_0, y\rangle|^2 \left(\frac{h'}{h}\right)^2 dyds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &= \left(\int_I |w|^2 C_2(S) \left(\frac{h'}{h}\right)^2 ds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq C_2(S)^{1/2} H_1 \left(\int_I |w|^2 ds\right)^{1/2} g_\varepsilon(\eta)^{1/2} \\ &\leq C_2(S)^{1/2} \frac{H_1}{C_6^{1/2}}\, \varepsilon^{1/2}\, t_\varepsilon(w)^{1/2} g_\varepsilon(\eta)^{1/2}. \end{aligned}$$

Additional calculations show that

$$\int_S y_1 \langle \nabla_y u_0, \nabla_y \eta \rangle = -\int_S f_1(y) \eta dy ds,$$

$$\int_S y_2 \langle \nabla_y u_0 \nabla_y \eta \rangle = -\int_S f_2(y) \eta dy ds,$$

where

$$f_1(y) := \left(\frac{\partial u_0}{\partial y_1} + y_1 \frac{\partial^2 u_0}{\partial y_1^2} + y_1 \frac{\partial^2 u_0}{\partial y_2^2} \right),$$

$$f_2(y) := \left(\frac{\partial u_0}{\partial y_2} + y_2 \frac{\partial^2 u_0}{\partial y_2^2} + y_2 \frac{\partial^2 u_0}{\partial y_1^2} \right).$$

Thus, there exist C_{12} and C_{13} so that

$$\begin{aligned} |m_\varepsilon^5(wu_0, \eta)| &\leq \left| \int_{I\times S} k(s) \cos \alpha(s) y_1 w \frac{\nabla_y u_0 \nabla_y \eta}{\varepsilon h} dy ds \right| \\ &+ \left| \int_{I\times S} k(s) h(s) \cos \alpha(s) y_1 \lambda_0 w \frac{u_0 \eta}{\varepsilon M^2} dy ds \right| \\ &+ \left| \int_{I\times S} k(s) \sin \alpha(s) y_2 w \frac{\nabla_y u_0 \nabla_y \eta}{\varepsilon h} dy ds \right| \\ &+ \left| \int_{I\times S} k(s) h(s) \sin \alpha(s) y_2 \lambda_0 w \frac{u_0 \eta}{\varepsilon M^2} dy ds \right| \\ &\leq \frac{C_{12}}{\varepsilon} \left(\int_{\mathbb{R}} |w|^2 ds \right)^{1/2} \left(\int_{\mathbb{R}\times S} |\eta|^2 dy ds \right)^{1/2} \\ &\leq C_{13}\, \varepsilon^{1/2}\, t_\varepsilon(w)^{1/2} g_\varepsilon(\eta)^{1/2}. \end{aligned}$$

By the above estimates it follows that there exists $C_{14} > 0$ so that

$$|m_\varepsilon(wu_0, \eta)|^2 \leq C_{14}\, \varepsilon\, t_\varepsilon(w)\, g_\varepsilon(\eta),$$

and so (12) is proven. By applying Proposition 3.1 of [13], it is found that there exists C_{10} so that, for ε small enough,

$$\left\| \left(-\Delta_{\varepsilon,c} - \frac{\lambda_0}{\varepsilon^2 M^2} \mathbf{1} \right)^{-1} - \left(T_{\varepsilon,c}^{-1} \oplus 0 \right) \right\| \leq C_{10}\, \varepsilon^{3/2}.$$

The proof of the theorem is complete. ∎

5 Bounded interval and Dirichlet condition

In this section we suppose that $I = [-a, b]$ is a bounded interval and the condition at the boundary $\partial\Lambda_\varepsilon$ is Dirichlet. Since I is bounded, the spectrum of $-\Delta_{\varepsilon,c}$ in Λ_ε is purely discrete and we denote its eigenvalues by $l_j^c(\varepsilon)$.

The main result in this section, that is, Theorem 5.1, is a version of Theorem 1.1 in this context.

Theorem 5.1. The limits

$$\mu_j = \lim_{\varepsilon \to 0} \varepsilon \left(l_j^c(\varepsilon) - \frac{\lambda_0}{\varepsilon^2 M^2} \right) \tag{13}$$

exist, where μ_j are the eigenvalues of a self-adjoint operator T (see Equation (5)) acting in $\mathrm{L}^2(\mathbb{R})$.

To prove this theorem we need some previous results; we will follow [14]. Introduce the family of segments

$$I_\varepsilon = (-a\varepsilon^{-1/2}, b\varepsilon^{-1/2}), \qquad \varepsilon > 0,$$

and the family of unitary operators $J_\varepsilon : \mathrm{L}^2(I) \to \mathrm{L}^2(I_\varepsilon)$ generated by the dilation $s \mapsto s\varepsilon^{1/2}$, that is,

$$(J_\varepsilon \psi)(s) = \varepsilon^{1/4}\, \psi(\varepsilon^{1/2} s),$$

and identify $\mathrm{L}^2(I_\varepsilon)$ with the subspace

$$\left\{ u \in \mathrm{L}^2(\mathbb{R}) : u(s) = 0 \;\text{ a.e. in }\; \mathbb{R} \backslash I_\varepsilon \right\}.$$

Set

$$\hat{T}_{\varepsilon,c} := \varepsilon J_\varepsilon T_{\varepsilon,c} J_\varepsilon^{-1}, \tag{14}$$

which is a self-adjoint operator acting in $\mathrm{L}^2(I_\varepsilon)$.

Theorem 5.2. In case $I = [-a, b]$ is a bounded interval, one has

$$\left|\left| \hat{T}_{\varepsilon,c}^{-1} \oplus 0 - T^{-1} \right|\right| \to 0, \qquad \text{as} \qquad \varepsilon \to 0,$$

where 0 is the null operator on the subspace $\mathrm{L}^2(\mathbb{R} \backslash I_\varepsilon)$.

We have $\varepsilon J_\varepsilon W_\varepsilon(s) J_\varepsilon^{-1} = \varepsilon W_\varepsilon(\varepsilon^{1/2} s)$, and a direct calculation shows that

$$\varepsilon J_\varepsilon W_\varepsilon(s) J_\varepsilon^{-1} = \zeta_\varepsilon(\varepsilon^{1/2} s, y)\lambda_0 \left[M^{-3} s^2 + \rho(\varepsilon^{1/2} s) s^3 \varepsilon^{1/2} \right] + \varepsilon\, \vartheta(\varepsilon^{1/2} s) + \varepsilon c,$$

with $\rho \in \mathrm{L}^\infty(I)$. Since $\zeta_\varepsilon(\varepsilon^{1/2} s, y) \to 1$ uniformly as $\varepsilon \to 0$, the proof of Theorem 5.2 is similar to the proof of Theorem 1.3 in [14], and so it will not be repeated here.

Proof of Theorem 5.1: Let $l_j(T_{\varepsilon,c})$, $l_j(\hat{T}_{\varepsilon,c})$ denote the eigenvalues of $T_{\varepsilon,c}$ and $\hat{T}_{\varepsilon,c}$ respectively. Let $\psi_{j,\varepsilon}^c$ denote the eigenfunction associated with eigenvalue $l_j^c(\varepsilon)$ of $-\Delta_{\varepsilon,c}$. Thus, there exist functions $w_{\varepsilon,c} \in \mathrm{L}^2(I)$ and $U \in \mathcal{L}$ so that $\psi_{j,\varepsilon}^c = w_{j,\varepsilon}^c u_0 + U$. Since $\mathcal{L}$ is invariant under $(-\Delta_{\varepsilon,c} - \lambda_0/\varepsilon^2 M^2 \mathbf{1})$, it follows that $w_{j,\varepsilon}^c u_0$ is the eigenfunction associated with the eigenvalue

$l_j(T_{\varepsilon,c})$. Observe also that the nonzero eigenvalues of $T_{\varepsilon,c}^{-1} \oplus 0$ are exactly the eigenvalues of $T_{\varepsilon,c}^{-1}$. Hence, by Theorem 4.3, we have

$$\begin{aligned}\left| \left(l_j^c(\varepsilon) - \frac{\lambda_0}{\varepsilon^2 M^2} \right)^{-1} - l_j^{-1}(T_{\varepsilon,c}) \right| &\leq \left\| \left(-\Delta_{\varepsilon,c} - \frac{\lambda_0}{\varepsilon^2 M^2} \mathbf{1} \right)^{-1} - T_{\varepsilon,c}^{-1} \oplus 0 \right\| \\ &\leq C_{10}\, \varepsilon^{3/2}.\end{aligned}$$

Thus,

$$\left| \frac{1}{\varepsilon} \left(l_j^c(\varepsilon) - \frac{\lambda_0}{\varepsilon^2 M^2} \right)^{-1} - \frac{1}{\varepsilon l_j^{-1}(T_{\varepsilon,c})} \right| \leq C_{10}\, \varepsilon^{1/2}.$$

Since $l_j(\hat{T}_{\varepsilon,c}) = \varepsilon l_j(T_{\varepsilon,c})$, by Theorem 5.2, we find

$$\varepsilon l_j(T_{\varepsilon,c}) \to \mu_j, \quad \varepsilon \to 0,$$

and (13) follows. ■

6 The Neumann case

Here we again consider that $I = [-a, b]$ is a bounded interval, but the Dirichlet condition at the vertical part of the boundary $\partial(I \times S)$, that is, $\{(-a) \times S \cup b \times S\}$, is replaced by Neumann condition. Our point is that the conclusions of Theorem 5.1 also hold true in this case. Although in our case the curvature and torsion can be nontrivial, the proof in this case are similar to the proof of Theorem 5.1 above (and taking into account [13]); for this reason, details will not be presented.

7 The case $I = \mathbb{R}$ and Dirichlet condition

In this section we study the case $I = \mathbb{R}$. First we give sufficient conditions for a nonempty discrete spectrum of the Dirichlet Laplacian, and then discuss the WEO and eigenvalue approximations.

7.1 The discrete spectrum

Now the spectrum of the Laplacian $-\Delta_{\varepsilon,c}$ in Λ_ε is not necessarily discrete, but in this section we will see that the essential spectrum $\sigma_{\text{ess}}(-\Delta_{\varepsilon,c})$ depends on the behavior of the curvature at infinity; it will then follow that if $k(s) \to 0$ as $|s| \to \infty$, then the discrete spectrum of $-\Delta_{\varepsilon,c}$ is nonempty for ε small enough.

Denote $\nu(\varepsilon) := \inf \sigma_{\text{ess}}(-\Delta_{\varepsilon,c})$ and let $l_j^c(\varepsilon)$ be the eigenvalues of $-\Delta_{\varepsilon,c}$ (recall the Dirichlet boundary condition).

Theorem 7.1. If $I = \mathbb{R}$ and the curvature satisfies

$$\lim_{|s|\to\infty} k(s) = 0, \tag{15}$$

then $\nu(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof: Let $N := \limsup_{|s|\to\infty} h(s) < M$ and $\hat{I} = [-a, a]$ and define

$$\Omega_{a,\varepsilon} = \left\{(s,y) : s \in \hat{I}\right\} \qquad \text{and} \qquad \Omega'_{a,\varepsilon} = \left\{(s,y) : s \notin \hat{I}\right\}.$$

Let $-\Delta^c_{a,\varepsilon,D}$, $-\Delta'^c_{a,\varepsilon,D}$ be the Dirichlet Laplacian in $\Omega_{a,\varepsilon}$ and $\Omega'_{a,\varepsilon}$ respectively. Similarly, let $-\Delta^c_{a,\varepsilon,DN}$, $-\Delta'^c_{a,\varepsilon,DN}$ be the above Laplacian operators but with Neumann condition at the vertical part of the boundaries of $\Omega_{\alpha,\varepsilon}$ and $\Omega'_{\alpha,\varepsilon}$, respectively. Note that

$$-\Delta^c_{a,\varepsilon,DN} + \left(-\Delta'^c_{a,\varepsilon,DN}\right) < -\Delta_{\varepsilon,c} < -\Delta^c_{a,\varepsilon,D} + \left(-\Delta'^c_{a,\varepsilon,D}\right). \tag{16}$$

Therefore $\inf \sigma_{\text{ess}}(-\Delta_{\varepsilon,c}) \geq \inf \sigma_{\text{ess}}(-\Delta'^c_{a,\varepsilon,DN})$.

Let $q'_{a,\varepsilon,DN}$ the quadratic form associated with the operator $-\Delta'^c_{a,\varepsilon,DN}$. Write $K_\varepsilon = \sup_{(s,y)\in\mathbb{R}\times S} \beta_\varepsilon(s,y)$; we have

$$\begin{aligned}
q'_{a,\varepsilon,DN}(\psi) &\geq \left(\inf_{(s,y)\in\mathbb{R}\times S} \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 h(s)}\right) \int_{(\mathbb{R}\setminus\hat{I})\times S} |\nabla_y \psi|^2 dy ds \\
&\geq \lambda_0 \left(\inf_{(s,y)\in\mathbb{R}\times S} \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 h(s)}\right) \int_{(\mathbb{R}\setminus\hat{I})\times S} |\psi|^2 dy ds \\
&\geq \lambda_0 \left(\inf_{(s,y)\in\mathbb{R}\times S} \frac{\beta_\varepsilon(s,y)}{\varepsilon^2 h(s)}\right) \frac{1}{K_\varepsilon} \int_{(\mathbb{R}\setminus\hat{I})\times S} \beta_\varepsilon(s,y) |\psi|^2 dy ds,
\end{aligned}$$

for all $\psi \in \operatorname{dom} q'_{a,\varepsilon,DN}$. Since k satisfies (15), it follows that the essential spectrum of $-\Delta'^c_{a,\varepsilon,DN}$ is estimated from below by λ_0 times a function that converges to $\frac{1}{\varepsilon^2 N}$ as $a \to \infty$. Since the essential spectrum is a closed subset, it follows that $\nu(\varepsilon) \geq \frac{\lambda_0}{\varepsilon^2 N^2}$ and consequently $\nu(\varepsilon) \to \infty$ as $\varepsilon \to 0$. ■

We conclude that, under condition (15), for ε small enough the discrete spectrum of $-\Delta_{\varepsilon,c}$ is nonempty. We again stress that we have got another property that does not depend on important geometric features of the tube. Also the WEO T (see also Subsection 7.2), which weakly describes the asymptotic behaviors of the eigenvalues of $-\Delta_{\varepsilon,c}$ in the sense of (13), is not influenced by such geometric features.

7.2 Weakly effective operator

The goal of this section is to show that Theorems 4.3, 5.1 and 5.2 have a similar counterpart in case $I = \mathbb{R}$. In [13] these theorems are proven for two dimensional strips, and here we argue that those proofs can be adapted to our three dimensional setting. The proof of Lemma 7.2 will be postponed to the end of this subsection.

Lemma 7.2. There exists $C_6 > 0$ so that, for ε small enough,

$$t_\varepsilon(w) \geq C_6^{-1}\,\varepsilon^{-1} \int_{\mathbb{R}} |w|^2 ds, \qquad \forall w \in \mathcal{H}_0^1(\mathbb{R}).$$

The proof of the next theorem is similar to the proof of Theorem 4.3; it is enough to take into account Lemma 4.2, and then Lemma 7.2 instead of Lemma 4.1. Recall that $\mathcal{L}$ is the subspace generated by functions $w(s)u_0(y)$ with $w \in \mathrm{L}^2(\mathbb{R})$

Theorem 7.3. Let $I = \mathbb{R}$. Then, there exists $C_{10} > 0$ so that, for ε small enough,

$$\left|\left|\left(-\Delta_{\varepsilon,c} - \frac{\lambda_0}{\varepsilon^2 M^2}\mathbf{1}\right)^{-1} - \left(T_{\varepsilon,c}^{-1} \oplus 0\right)\right|\right| \leq C_{10}\,\varepsilon^{3/2},$$

where 0 denotes the null operator on the subspace $\mathcal{L}^\perp$.

As in the previous section, consider the self-adjoint operators

$$\hat{T}_{\varepsilon,c} := \varepsilon J_\varepsilon T_{\varepsilon,c} J_\varepsilon^{-1},$$

where $J_\varepsilon : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ is the previously discussed unitary operator generated by the dilation $s \mapsto s\varepsilon^{1/2}$.

Theorem 7.4. For $\varepsilon \to 0$ one has

$$\left|\left|\hat{T}_{\varepsilon,c}^{-1} - T^{-1}\right|\right| \to 0,$$

where T is the operator (5).

As in the Section 5, we have that $\varepsilon J_\varepsilon W_\varepsilon(s) J_\varepsilon^{-1}$ equals

$$\lambda_0\,\zeta_\varepsilon(\varepsilon^{1/2}s, y)\left[M^{-3}s^2 + \rho(\varepsilon^{1/2}s)s^3\varepsilon^{1/2}\right] + \varepsilon\,\vartheta(\varepsilon^{1/2}s) + \varepsilon c.$$

Again, since $\zeta_\varepsilon(\varepsilon^{1/2}s, y) \to 1$ uniformly as $\varepsilon \to 0$, the proof of Theorem 7.4 is similar to the proof of Theorem 1.3 in [14], and details will be skipped.

Proof of Lemma 7.2: Theorem 7.4 guarantees that

$$\varepsilon^{-1}\,||T_{\varepsilon,c}^{-1}|| \to ||T^{-1}|| \qquad (\varepsilon \to 0),$$

and so there exists $C_6 > 0$ so that

$$||T_{\varepsilon,c}^{-1}|| \leq C_6\,\varepsilon.$$

The proof is complete. ■

A Proof of Theorem 3.1

It will be shown that, for ε small enough, there exists $C_5 > 0$ so that

$$\left\| \hat{G}_\varepsilon^{-1} - G_\varepsilon^{-1} \right\| \leq C_5\, \varepsilon.$$

Remember that $c > \|v\|_\infty + (1/M^2)\|k(s)^2/4\|_\infty$, thus, there exists a number $d > 0$ so that $c = \|v\|_\infty + (1/M^2)\|k(s)^2/4\|_\infty + d$.

Since $\zeta_\varepsilon \to 1$ uniformly as $\varepsilon \to 0$, there exist $\varepsilon_1 > 0$ and numbers $\sigma_1, \sigma_2 > 0$ so that $\sigma_1 \leq \beta_\varepsilon \leq \sigma_2$, for all $\varepsilon < \varepsilon_1$. Thus,

$$\hat{g}_\varepsilon(\psi) \geq \sigma_1 d \|\psi\|^2 \qquad \text{and} \qquad g_\varepsilon(\psi) \geq d\|\psi\|^2,$$

for all $\varepsilon < \varepsilon_1$. Consequently,

$$\|\hat{G}_\varepsilon^{-1}\| \leq \frac{1}{\sigma_1 d} \qquad \text{and} \qquad \|G_\varepsilon^{-1}\| \leq \frac{1}{d},$$

for all $\varepsilon < \varepsilon_1$.

Since $k, h \in \mathrm{L}^\infty(\mathbb{R})$, $y \in S$ and S is a bounded region, there exist $\varepsilon_0 > 0$ ($\varepsilon_0 < \varepsilon_1$) and $C_1, C_2 > 0$ so that

$$\left| \left(\frac{1}{\zeta_\varepsilon} - 1 \right) \right| = \left| \frac{\varepsilon\, k(s)h(s)(y \cdot z_\alpha(s))}{\zeta_\varepsilon} \right| \leq C_1\, \varepsilon,$$

and

$$c\,|(\zeta_\varepsilon - 1)| \leq C_2\, \varepsilon,$$

for all $\varepsilon < \varepsilon_0$. Under such conditions we have

$$\begin{aligned}
& |\hat{g}_\varepsilon(\psi) - g_\varepsilon(\psi)| \\
\leq\ & \int_{I\times S} \left| \left(\frac{1}{\zeta_\varepsilon} - 1 \right) \right| \left| \psi' - \psi\, \frac{h'}{h} + (\nabla_y \psi \cdot Ry)(\tau + \alpha') - (\nabla_y \psi \cdot y) \frac{h'}{h} \right|^2 dyds \\
+\ & \int_{I\times S} c|(\zeta_\varepsilon - 1)||\psi|^2\, dsdy \\
\leq\ & C_1 \varepsilon \int_{I\times S} \left| \psi' - \psi\, \frac{h'}{h} + (\nabla_y \psi \cdot Ry)(\tau + \alpha') - (\nabla_y \psi \cdot y) \frac{h'}{h} \right|^2 + C_2 \varepsilon \int_{I\times S} |\psi|^2 dyds \\
\leq\ & C_3 \varepsilon g_\varepsilon(\psi)
\end{aligned}$$

for some $C_3 > 0$. Hence,

$$(1 - C_3\varepsilon) g_\varepsilon(\psi) \leq \hat{g}_\varepsilon(\psi) \leq (1 + C_3\varepsilon) g_\varepsilon(\psi),$$

for all $\varepsilon < \varepsilon_0$. The first inequality implies that it is possible to find $\varepsilon_0' > 0$ ($\varepsilon_0' < \varepsilon_0$) and a constant $C_4 > 0$ so that

$$g_\varepsilon(\psi) \leq C_4\, \hat{g}_\varepsilon(\psi),$$

for all $\varepsilon < \varepsilon_0'$.

By Schwarz's Inequality for bilinear forms, we have

$$|\hat{g}_\varepsilon(\psi_1, \psi_2)| \le [\hat{g}_\varepsilon(\psi_1)]^{1/2} [\hat{g}_\varepsilon(\psi_2)]^{1/2},$$

$$|g_\varepsilon(\psi_1, \psi_2)| \le [g_\varepsilon(\psi_1)]^{1/2} [g_\varepsilon(\psi_2)]^{1/2},$$

for all $\psi_1, \psi_2 \in \mathcal{H}_0^1(\mathbb{R} \times S)$. Thus, by using the above estimates, for each pair $\psi_1, \psi_2 \in \mathcal{H}_0^1(\mathbb{R} \times S)$ we have

$$\begin{aligned}
\left|\left\langle \hat{G}_\varepsilon^{\,1/2}\psi_1, \hat{G}_\varepsilon^{\,1/2}\psi_2\right\rangle\right. &- \left.\left\langle G_\varepsilon^{1/2}\psi_1, G_\varepsilon^{1/2}\psi_2\right\rangle\right| \\
&= \left|\hat{g}_\varepsilon(\psi_1, \psi_2) - g_\varepsilon(\psi_1, \psi_2)\right| \\
&\le C_3\, \varepsilon\, [g_\varepsilon(\psi_1)]^{1/2} [g_\varepsilon(\psi_2)]^{1/2} \\
&\le C_3 \sqrt{C_4}\, \varepsilon\, [g_\varepsilon(\psi_1)]^{1/2} [\hat{g}_\varepsilon(\psi_2)]^{1/2}.
\end{aligned}$$

By picking $\psi_1 = G_\varepsilon^{-1} f$, $\psi_2 = \hat{G}_\varepsilon^{-1} g$, where $f, g \in \mathrm{L}^2(\mathbb{R} \times S)$ are arbitrary, we obtain

$$\begin{aligned}
\left|\left\langle \hat{G}_\varepsilon^{\,-1} f, g\right\rangle\right. &- \left.\left\langle G_\varepsilon^{-1} f, g\right\rangle\right| \\
&\le C_3 \sqrt{C_4}\, \varepsilon \left[\left\langle \hat{G}_\varepsilon^{-1} g, g\right\rangle \left\langle G_\varepsilon^{-1} g, g\right\rangle\right]^{1/2} \\
&\le \frac{C_3 \sqrt{C_4}}{d\sqrt{\sigma_1}}\, \varepsilon\, \|f\|\, \|g\|,
\end{aligned}$$

for all $\varepsilon < \varepsilon_0'$. Therefore,

$$\left\|\hat{G}_\varepsilon^{-1} - G_\varepsilon^{-1}\right\| \le C_5\, \varepsilon,$$

for all $\varepsilon < \varepsilon_0'$, with $C_5 = C_3\sqrt{C_4}/(d\sqrt{\sigma_1})$. This completes the proof of the theorem.

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