NONNEGATIVE SOLUTIONS OF ELLIPTIC PROBLEMS WITH SUBLINEAR INDEFINITE NONLINEARITY

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Abstract. We are concerned with existence, nonexistence and multiplicity of nonnegative solutions for the elliptic problem

\[-\Delta u = a(x)u^q + \lambda b(x)u^p \quad \text{in} \quad \Omega \]
\[u = 0 \quad \text{on} \quad \partial \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(\lambda \in \mathbb{R}\), \(0 < q < 1 < p \leq 2^* - 1\) and \(a, b\) are bounded functions, with \(b(x) \geq 0\) and \(a(x)\) changes its sign.

1. Introduction

There has recently been increasing interest in questions about positive solutions of semilinear elliptic problem of the type:

\[-\Delta u = f(x, u, \lambda),\]

\[x \in \Omega \subset \mathbb{R}^N \quad \text{and} \quad \lambda \in \mathbb{R}.\] One interesting situation appears when \(f\) is sublinear at the origin in some open subset \(\Omega' \subset \Omega\), i.e., if the following condition,

\[\lim_{u \to 0^+} \frac{f(x, u, \lambda)}{u} = \infty,\]

holds uniformly for \(x \in \Omega'\) and \(\lambda \in \mathbb{R}\). One direction of research is looking for an interval \(\Lambda \subset \mathbb{R}\), such that \(-\Delta u = f(x, u, \lambda)\) has two solution for \(\lambda \in \Lambda\).

In this paper we deal with the following class of parameterized elliptic problems

\[(Q_\lambda) \quad \begin{cases} -\Delta u = a(x)u^q + \lambda b(x)u^p & \text{in} \quad \Omega \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial \Omega, \end{cases}\]

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where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $0 < q < 1 < p \leq 2^* - 1$ and $a, b$ are bounded functions. Here we will assume that $a(x)$ changes its sign in $\Omega$, so the Maximum Principle is not applicable, thus the solutions can vanish on parts of $\Omega$ (see for instance [11]). By solutions we mean weak solutions in $H^1_0(\Omega)$, i.e. the critical points of the associated $C^1$ functional $F_\lambda : H^1_0(\Omega) \to \mathbb{R}$, given by

\[
F_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{q + 1} \int_\Omega a(x)(u^+)^{q+1} - \frac{\lambda}{p + 1} \int_\Omega b(x)(u^+)^{p+1}.
\]

Our general assumptions concerning the functions $a(x)$ and $b(x)$ are that $a, b \in L^\infty(\Omega)$ and the sets

\[
\begin{align*}
\Omega_a &= \{x \in \Omega : a(x) \geq 0\}, & \Omega_a^+ &= \{x \in \Omega : a(x) > 0\}, \\
\Omega_a^- &= \{x \in \Omega : a(x) < 0\} & \Omega_a^+ &= \{x \in \Omega : b(x) > 0\},
\end{align*}
\]

are nonempty. Moreover, we will make the following assumptions:

\[
\begin{align*}
(a) & \quad \Omega_a^+ \text{ is open, } |\Omega_a^-| > 0 \text{ and } \overline{\Omega_a^+} \cap \overline{\Omega_a^-} = \emptyset; \\
(b) & \quad \text{int}(\Omega_a^+) \neq \emptyset \text{ and } b \geq 0; \\
(c) & \quad \Omega_a^+ \subset \Omega_a^+ \text{ and } \overline{\Omega_a^-} \subset \Omega; \\
(d) & \quad \text{int}(\Omega_a) = \bigcup_{i=1}^k U_i, \quad U_i \text{ connected, and } U_i \cap \Omega_a^+ \neq \emptyset;
\end{align*}
\]

As a consequence of assumption (d), by the Maximum Principle, if $u$ is a solution of $(Q_\lambda)$ such that $u$ is nontrivial in the components of $\Omega_a$, then $u > 0$ in $\text{int}(\Omega_a) \supset \Omega_a^+$. This motivates the following definition:

**Definition 1.** We say that $u \in H^1_0(\Omega)$ is a solution to $(P_\lambda)$ if $u$ solves $(Q_\lambda)$ in the weak sense and $u(x) > 0$ a.e. $x \in \Omega_a^+$.

The aim of this paper is to obtain, assuming the above hypotheses, a global result in the following sense: if

\[
\lambda^* = \sup \{\lambda > 0; \ (P_\lambda) \text{ has a solution}\},
\]

then for all $0 < \lambda < \lambda^*$, $(P_\lambda)$ has at least two nontrivial solutions (in the critical case we have an additional hypothesis). This kind of result was proved in [2] for the case $a \equiv b \equiv 1$. When $a \geq 0$ and $b$ is indefinite, the same result was obtained in [9].

Before presenting our results, we notice that elliptic problems with indefinite nonlinearities have been widely studied recently. For instance, problems where the nonlinearity are composed by a linear part and an indefinite superlinear were considered in [3, 4, 5, 20]. For concave-convex nonlinearities, in addition to [9], we can cite [10], for $p$-Laplacian problems, and [1] for a semilinear equation with Neumann boundary condition (see also [12, 13, 19]). Moreover, we refer to [8, 16, 17, 18] for local results, i.e., when $\lambda$ is in a small neighborhood
of the origin. Notice that in [8] both coefficients, $a$ and $b$, could change sign, but the authors have proved only local results.

The main results of this paper are stated in the theorems below. First, we consider existence results:

**Theorem 1.** Assume $(a)-(d)$ and $0 < q < 1 < p < 2^* - 1$. Then there is $\lambda^* \in (0, \infty)$ such that problem $(P_{\lambda})$ has at least one solution for $0 < \lambda < \lambda^*$, moreover, this solution is a local minimum of $F_{\lambda}$. Furthermore, $(P_{\lambda})$ has no solution for $\lambda > \lambda^*$; and, if $\Omega$ is smooth then $(P_{\lambda^*})$ has at least one solution.

**Remark 1.** Assumption $\Omega^+_a \cap \Omega^-_a = \emptyset$ has been considered by many authors in the study of elliptic problems with indefinite nonlinearities, see for instance [3, 4]. Assumption $(d)$, that appears in [1, 6] in the study of a problem with Neumann boundary condition, will be essential to prove that the solution obtained in Theorem 1 is a local minimum in $H^1_0(\Omega)$.

Now we consider multiplicity in the subcritical case. Let $\lambda^*$ be given in previous theorem.

**Theorem 2.** Assume $(a)-(d)$ and $0 < q < 1 < p < 2^* - 1$. Then problem $(P_{\lambda})$ has at least two solutions for $0 < \lambda < \lambda^*$.

Aiming now to multiplicity in the critical case, we will assume that $N \geq 3$ and, without loss of generality, that $0 \in \Omega$. Moreover, in addition to the above hypotheses, we assume that $b$ satisfies:

(e) for some $\delta > 0$, $M > 0$ and some $\gamma > 2$, one has

$$0 \leq ||b||_{\infty} - b(x) \leq M|x|^{\gamma} \text{ a.e. } x \in B_{\delta}(0).$$

Let $\lambda^*$ be given in Theorem 1, we have:

**Theorem 3.** Assume $(a) - (e)$ and $0 < q < 1 < p = 2^* - 1$. Then for all $0 < \lambda < \lambda^*$ problem $(P_{\lambda})$ has at least two solutions.

**Remark 2.** Assumption $(e)$ appears in [10], but there $\gamma$ depends on the dimension in the following way: $\gamma > 2^*$ when $N \geq 5$, $\gamma \geq 2^*$ when $N = 4$ and $\gamma > 3/5$ when $N = 3$.

Throughout this paper, we will use the following notations: $|| \cdot ||$ to the norm of $H^1_0(\Omega)$, $|| \cdot ||_p$ to the norm of $L^p(\Omega)$ and $C$ to several different positive constants.

2. **Proof of Theorem 1**

Existence for $0 < \lambda < \lambda^*$:
First, we shall prove that problem \((P_\lambda)\) has a supersolution for \(\lambda > 0\) small enough.

**Claim 1.** There is \(\epsilon_0 > 0\) such that for \(0 < \lambda \leq \epsilon_0\), problem \((P_\lambda)\) has a supersolution.

In fact, let \(e\) be a solution of

\[-\Delta u = 1 \quad \text{in} \quad \Omega \]
\[u = 0 \quad \text{on} \quad \partial \Omega.\]

Since \(0 < q < 1 < p\), we can find \(m > 0\) and \(\epsilon_0 > 0\) such that

\[m \geq ||a||_\infty ||me||_\infty^q + \epsilon_0 ||b||_\infty ||me||_\infty^p.\]

It follows that \(me\) is a supersolution to \((P_\lambda)\), since \(0 < \lambda \leq \epsilon_0\), q.e.d.

Let \(\lambda\) be such that \(0 < \lambda \leq \epsilon_0\), where \(\epsilon_0\) is as in Claim 1. Defining \(u := me\), where \(m\) and \(e\) are as in Claim 1, we have that \(u\) is a supersolution for \((Q_\lambda)\). Moreover, \(u = 0\) is a solution, and so a subsolution. Consider the following minimization problem

\[
\inf_M F_\lambda, \quad \text{where} \quad M = \{u \in H^1_0 : u(x) \leq u(x) \leq \bar{u}(x) \text{ a.e. } x \in \Omega\}.
\]

By Theorem 1.2.4 from [21], the above infimum is achieved at \(u_\lambda \in M\) and, in addition, \(u_\lambda\) is a solution of \((Q_\lambda)\). It remains to show that \(u_\lambda\) solves \((P_\lambda)\) (see Definition 1). Suppose, by contradiction, that \(u_\lambda \equiv 0\) a.e. \(x \in \Omega_+^\ast\). Let \(\varphi \in C^1_0(\Omega_+^\ast)\) be nonnegative and nontrivial. Therefore \(u_\lambda + s\varphi \in M\), for sufficiently small \(s > 0\), and

\[
F_\lambda(u_\lambda + s\varphi) = F_\lambda(u_\lambda) + F_\lambda(s\varphi)
= F_\lambda(u_\lambda) + \frac{s^2}{2} ||\varphi||^2
- \frac{s^{q+1}}{q+1} \int_\Omega a(x)\varphi^{q+1} dx - \frac{s^{p+1}}{p+1} \lambda \int_\Omega b(x)\varphi^{p+1} dx.
\]

It follows that \(F_\lambda(u_\lambda + s\varphi) < F_\lambda(u_\lambda)\) if \(s > 0\) is small enough. This contradicts the definition of \(u_\lambda\), and so \(u_\lambda\) is a solution of \((P_\lambda)\).

Now, we can define

\[(2) \quad \Lambda := \{\lambda > 0 : (P_\lambda) \text{ has a solution}\}, \quad \text{and} \quad \lambda^* := \sup \Lambda.
\]

By the previous paragraph, we have \(\Lambda \neq \emptyset\), and so \(\lambda^*\) is well defined.
Now, we will prove existence of a solution for $0 < \lambda < \lambda^*$. Let $\lambda$ be such that $\lambda < \overline{\lambda} < \lambda^*$, with $\overline{\lambda} \in \Lambda$. Let $\pi$ be a solution of $(P_\lambda)$, then

$$-\Delta \pi = a(x)\pi^q + \lambda b(x)\pi^p \geq a(x)\pi^q + \lambda b(x)\pi^p,$$

and so $\pi$ is a supersolution for $(P_\lambda)$. Consider $M = \{u \in H^1_0 : 0 \leq u \leq \pi \}$. Let $u_\lambda \in M$ be such that $F_\lambda(u_\lambda) = \inf_M F_\lambda$. As before, $u_\lambda$ is a solution of $(Q_\lambda)$. Suppose, by contradiction, that $u_\lambda$ does not solve $(P_\lambda)$, i.e. $u_\lambda \equiv 0$ a.e. $x \in \Omega^+_a$. Let $\varphi \in C^1_0(\Omega^+_a)$ be nonnegative and nontrivial such that $\varphi \pi \geq 0$ a.e. $x \in \Omega^+_a$. So we get $u_\lambda + s\varphi \pi \in M$ for $s > 0$ and sufficiently small. Arguing as in Claim 1, we can conclude that $F_\lambda(u_\lambda + s\varphi \pi) < F_\lambda(u_\lambda)$ if $s > 0$ is small enough, which contradicts the definition of $u_\lambda$. Thus $u_\lambda$ is a solution of $(P_\lambda)$. The proof that $u_\lambda$ is a local minimum of $F_\lambda$ is deferred to an appendix. \textit{q.e.d.}

**Proof of $\lambda^* < \infty$:**

First, note that

$$a(x)t^q + b(x)t^p \geq \lambda^{1-q} m(x)t, \quad \text{a.e. } x \in \Omega_a \quad \text{and for all } t \geq 0,$$

where $m(x) = C(p, q)a(x)^{p-1}b(x)^{p-\frac{q}{\sigma}}$ (see [8, Lemma 3.6]).

Let $u$ be a solution of $(P_\lambda)$. Let $B$ be a ball in $\Omega^+_a$. Let $\mu_1$ be the first eigenvalue of $(-\Delta, H^1_0(B))$, with the weight $m(x)$, and $\phi_1$ the associated eigenfunction, i.e. $-\Delta \phi_1 = \mu_1 m(x)\phi_1$ in $B$. We have

$$\int_B \nabla u \nabla \phi_1 dx = \mu_1 \int_B m(x)u\phi_1 dx.$$

On the other hand

$$\int_B \nabla u \nabla \phi_1 dx = \int_B (a(x)u^q + b(x)u^p)\phi_1 dx.$$

It follows that

$$\lambda^{1-q} \int_B m(x)u\phi_1 dx \leq \mu_1 \int_B m(x)u\phi_1 dx,$$

which implies that $\lambda^{1-q} \leq \mu_1$. Thus $\lambda^*$ is finite and, by the definition of $\lambda^*$, it follows that $(P_\lambda)$ has no solution for $\lambda > \lambda^*$. \textit{q.e.d.}

**Existence for $\lambda = \lambda^*$:**

We begin by recalling that, under the assumption $0 < q < p \leq 2^* - 1$, the solutions of $(P_*^\lambda)$ are in $C^0_0(\Omega)$ (see [21, p. 245]).

By the definition of $\lambda^*$, there is a sequence $\lambda_k \in \Lambda$ such that $\lambda_k \nearrow \lambda^*$ and $(P_{\lambda_k})$ has a solution. Let $u_k$ be a solution of $(P_{\lambda_k})$. First, let us
show that $||u_k||$ is bounded. Actually, since $F'(u_k) = 0$ and $F(u_k) \leq 0$ with $F(u_k) \leq 0$ (see the first part of the proof), we have

$$pF(u_k) - F'(u_k) \cdot u_k \leq C||u_k||,$$

that means

$$\left(\frac{p}{2} - 1\right)||u_k||^2 \leq C||u_k||^{q+1} + C.$$

It follows that $||u_k||$ is bounded, since $q < 1$.

Thus we can assume that $u_k \rightharpoonup u_*$ in $H^1_0$. Hence $u$ solves $(Q_{\lambda^*})$ and $F(u) \leq 0$. Moreover, by standard bootstrap, we can assert that $u_k \rightarrow u_*$ in $C^1_0(\Omega)$.

We have to prove that $u_*$ is a solution of $(P_{\lambda^*})$. For this purpose, assume, by contradiction, that $u_* = 0$ in $\Omega^+_\alpha$. Let $\varphi_1 > 0$ be the associated eigenfunction to the eigenvalue $\lambda_1(B_1)$, where $B_1$ is an open ball in $\Omega^+_\alpha$. We have

$$\lambda_1(B_1) \int_{B_1} u_k \varphi_1 dx = \int_{B_1} \nabla u_k \nabla \varphi_1 dx \leq \int_{B_1} \varphi_1 \left(a(x)u_k^q + \lambda^*b(x)u_k^p\right) dx.$$ 

Then

$$\int_{\Omega} c_1 u_k^q \varphi_1 dx \leq \int_{B_1} \varphi_1 \left(\lambda_1(B_1)u_k - \lambda^*b(x)u_k^p\right) dx.$$

It is a contradiction, if $k$ is large enough, provided

$$c_1 u_k(x)^q \geq \lambda_1(B_1) u_k - \lambda^*b(x)u_k^p$$

for a.e. $x \in B_1$, since $u_k \rightarrow u_*$ in $C^1_0(\Omega)$. This concludes the proof of Theorem 1. \hfill \Box

3. Proof of Theorem 2

It follows, by Theorem 1, that for each $\lambda \in (0, \lambda^*)$ the functional $(F_\lambda)$ has a local minimum $u_\lambda$, that satisfies $F_\lambda(u_\lambda) \leq 0$. We will look for a second solution of the form

$$v = u_\lambda + u, \text{ with } u \geq 0.$$
It is equivalent to find a critical point of the following functional, defined in $H_0^1(\Omega)$,

$$I_\lambda(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{q+1} \int a(x)((u^+ + u_\lambda)^{q+1} - (u_\lambda)^{q+1} - (q+1)u_\lambda^q u^+] - \frac{\lambda}{p+1} \int b(x)((u^+ + u_\lambda)^{p+1} - (u_\lambda)^{p+1} - (p+1)u_\lambda^p u^+].$$

Indeed, if $u$ is a critical point of $I_\lambda$, then $u + u_\lambda$ is a critical point of $F_\lambda$. Moreover, using $-u^-$ as a test function, we can conclude that $u \geq 0$. Thus, we have to prove that $I_\lambda$ has a nontrivial critical point. To this end, we will show that $I_\lambda$ satisfies the assumptions of the relaxed mountain pass theorem, see [14, Corollary 5.11].

First, observe that $I_\lambda(0) = F_\lambda(u_\lambda)$. Thus we have to show that:

(i) there is $r > 0$ such that $I_\lambda(u) \geq F_\lambda(u_\lambda)$ for all $u \in H_0^1$ with $||u|| = r$;

(ii) there is $w_1 \in H_0^1$ such that $I(w_1) \leq F_\lambda(u_\lambda)$ and $||w_1|| > r$; and

(iii) $I_\lambda$ satisfies the $(PS)$ condition.

The item (i) is a consequence of $u_\lambda$ being a local minimum of $F_\lambda$. In order to prove item (ii), let $v_1 \in C_0^1(\Omega_0^+)$ be nonnegative, nontrivial and such that $\int_\Omega b(x)v_1^{p+1} > 0$. We have, for large $s$,

$$I_\lambda(sv_1) = \frac{s^2}{2} \int_\Omega |\nabla v_1|^2 - \frac{s^{q+1}}{q+1} \int_\Omega a(x)((v_1 + \frac{u_\lambda}{s})^{q+1} - (\frac{u_\lambda}{s})^{q+1} - (q+1)\frac{u_\lambda^q}{s^q} v_1] - \frac{\lambda s^{p+1}}{p+1} \int_\Omega b(x)((v_1 + \frac{u_\lambda}{s})^{p+1} - (\frac{u_\lambda}{s})^{p+1} + (p+1)\frac{u_\lambda^p}{s^p} v_1] = O(s^{q+1}) - \frac{\lambda s^{p+1}}{p+1} \int_\Omega b(x)(v_1 + \frac{u_\lambda}{s})^{p+1} \to -\infty \text{ as } s \to \infty,$$

and so (ii) follows.

For (iii), notice that $F_\lambda$ satisfies $(PS)$, see for instance [8]. Now, let $u_n$ be a $(PS)$ sequence of $I_\lambda$ at level $c$, it follows that $u_1 + u_n$ is a $(PS)$ sequence for $F_\lambda$, and so has a convergent subsequence. Thus, $I_\lambda$ satisfies the $(PS)$ condition. This concludes the proof of Theorem 2. □
4. Proof of Theorem 3

As in the previous section, we have to prove that the functional $I_\lambda$ has a nontrivial critical point. Again, $I_\lambda$ has a local minimum at the origin and we can find $e \in H^1_0$, with $\|e\|$ large enough, such that $I_\lambda(e) \leq 0$. We are assuming that $p = 2^* - 1$, then $I_\lambda$ fails to satisfies $(PS)$ condition. In order to avoid this difficulty we follow the ideas in [7].

We argue by contradiction, i.e., suppose that 0 is the unique critical point of $I_\lambda$. Consider the mountain pass level
\[ c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)), \]
where $\Gamma = \{ \gamma \in C([0, 1], H^1_0) : \gamma(0) = 0, \gamma(1) \leq 0 \}$. The next two lemmas will be proved below.

**Lemma 1.** The following is true:
\[ c_\lambda < \frac{S^{\frac{2}{p}}}{N \|b\|_{\infty}^{\frac{2}{p}}} \]

**Lemma 2.** $I_\lambda$ satisfies the $(PS)_c$ condition for all $c < \frac{S^{\frac{2}{p}}}{N \|b\|_{\infty}^{\frac{2}{p}}}$.

Now, by the mountain pass theorem, there is $w_n \in H^1_0(\Omega)$ a sequence such that
\[ I'_\lambda(w_n) \to 0 \quad \text{and} \quad I_\lambda(w_n) \to c_\lambda, \]
where $0 \leq c_\lambda < \frac{S^{\frac{2}{p}}}{N \|b\|_{\infty}^{\frac{2}{p}}}$, by Lemma 1. Lemma 2 implies that $w_n \to w_0$ in $H^1_0(\Omega)$. Thus $w_0$ is a critical point of $I_\lambda$, and so, by our assumption, $w_0 = 0$. We have that, by (3),
\[ \frac{1}{p+1} I'_\lambda(w_n) \cdot (u_\lambda + w_n) - I_\lambda(w_n) = \frac{1}{N} \|w_n\|^2 + o(1) \to c_\lambda. \]

If $c_\lambda = 0$, using the version of mountain pass theorem by Ghoussoub & Preiss [15, Theorem 1], we can take $w_n$ satisfying $\|w_n\| \to r$, $r > 0$, what is a contradiction with (4). Thus we should have $c_\lambda > 0$. Since $I'(w_n) \cdot w_n \to 0$, we conclude that
\[ \lim_{n \to \infty} \|w_n\|^2 = \lim_{n \to \infty} \int b(x)(w_n^+)^{2^*} = Nc_\lambda. \]

By definition of $S$, we have
\[ \|w_n\|^2 \geq S \left( \int |w_n|^{2^*} \right)^{\frac{2}{2^*}} \geq \frac{S}{\|b\|_{\infty}^{\frac{2}{p}}} \left( \int b(x)|w_n^+|^{2^*} \right)^{\frac{2}{2^*}}. \]
Passing to the limit in the above inequality, we get
\[ c_{\lambda}N \geq \frac{S}{\|b\|^{\frac{N}{\infty}}} \left( c_{\lambda}N \right)^{\frac{2}{1}}. \]
It follows that \( c_{\lambda} \geq \frac{S^{\frac{N}{2}}}{N\|b\|^{\frac{N}{\infty}}} \), provided \( c > 0 \). This contradicts Lemma 2. Thus Theorem 3 is proved. 

\[ \square \]

**Proof of Lemma 1:** As usual, we will follow the approach from [7]. Define
\[ u_{\epsilon}(x) = C_{\epsilon} \frac{N-2}{N(N-2)} \left( \frac{\epsilon^2 + |x|^2}{N-2} \right)^{\frac{N-2}{2}}, \]
where \( C_{\epsilon} = (N(N-2))^{\frac{N-2}{2}} \), so that \( u_{\epsilon} \) satisfies
\[-\Delta u_{\epsilon} = u_{\epsilon}^{2^*-1} \text{ in } \mathbb{R}^N.\]
Pick a function \( \eta \in C^\infty_0(B_{\rho}(0)) \) such that \( 0 \leq \eta(x) \leq 1 \) and \( \eta(x) = 1 \) for all \( x \in B_{\rho/2}(0) \) (\( \rho \) as in (e)). Then set
\[ u_{\epsilon}(x) = \eta(x) v_{\epsilon}(x). \]
It is easy to see that for \( \epsilon_0 \), sufficiently small, there is \( R > 0 \) such that
\[ I_{\lambda}(R u_{\epsilon}) < 0, \text{ for all } \epsilon \in (0, \epsilon_0). \]
It means that if we put \( \gamma(t) = t R u_{\epsilon}, t \in [0, 1] \), then \( \gamma \in \Gamma \), and hence
\[ c_{\lambda} \leq \max_{t \in [0,1]} I_{\lambda}(t u_{\epsilon}). \]
Thus we need to show that
\[ \max_{t \in [0,1]} I_{\lambda}(t u_{\epsilon}) < \frac{S^{\frac{N}{2}}}{N\|b\|^{\frac{N}{\infty}}} \]
First, we remark some standard estimates,
\[ ||v_{\epsilon}||^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \quad ||v_{\epsilon}||_{2^*}^2 = S^{\frac{N}{2}} + O(\epsilon^{N}), \]
and, for some constants \( K_1, K_2 \) and \( K_3 \),
\[ ||v_{\epsilon}||_2^2 = \begin{cases} K_1 \epsilon^2 + O(\epsilon^{N-2}) & \text{if } N \geq 5, \\ K_2 |\ln \epsilon| + O(\epsilon^2) & \text{if } N = 4, \\ K_3 \epsilon + O(\epsilon^2) & \text{if } N = 3. \end{cases} \]
Moreover,
\[
\int_{\Omega} |u_\epsilon|^{q+1} \, dx \leq \int_{B_\epsilon} \frac{(C_N \epsilon)^{\frac{N-2}{2} (q+1)}}{\epsilon^{(N-2)(q+1)}} + \int_{B_\epsilon \setminus B_\epsilon} \frac{(C_N \epsilon)^{\frac{N-2}{2} (q+1)}}{|x|^{(N-2)(q+1)}} \\
+ C \epsilon^{\frac{(N-2)(1-q)+4}{2}} C \epsilon^{\frac{N}{2} - \frac{q+1}{2}} \int_\epsilon^\rho \rho^{-q(2-N)+1} \, dr
\]
and so
\[
\int_{\Omega} |u_\epsilon|^{q+1} \, dx \leq \left\{ \begin{array}{ll}
C \epsilon^{\frac{N-2}{2} (q+1)}, & \text{if } q \neq \frac{2}{(N-2)} \\
C \epsilon^{\frac{(N-2)(1-q)+4}{2}} C \epsilon^{\frac{N}{2} - \frac{q+1}{2}} \ln \epsilon, & \text{if } q = \frac{2}{(N-2)}.
\end{array} \right.
\]
Thus
\[
(7) \quad \int_{\Omega} |u_\epsilon|^{q+1} \, dx \leq \left\{ \begin{array}{ll}
o(\epsilon^2), & \text{if } N \geq 6 \\
o(\epsilon^{\frac{N-2}{2}}), & \text{if } 3 \leq N \leq 5.
\end{array} \right.
\]
Now,
\[
\int_{B_\rho} b(x) u_\epsilon^{2^*} = ||b||_\infty \int_{B_\rho(0)} u_\epsilon^{2^*} - \int_{B_\rho(0)} (||b||_\infty - b(x)) u_\epsilon^{2^*}.
\]
Using (c) and doing a change of variables, one has
\[
\int_{B_\rho(0)} (||b||_\infty - b(x)) u_\epsilon^{2^*} \leq \int_{B_\rho(0)} |x|^\eta u_\epsilon^{2^*} \\
= \int_{B_{\rho \epsilon}(0)} |\epsilon x|^\eta u_\epsilon^{2^*} (\epsilon x) + O(\epsilon^N) \\
= \epsilon^\eta \int_{B_{\rho \epsilon}(0)} |x|^\eta C_N^{2^*} (1 + |x|)^N + O(\epsilon^N) \\
= O(\epsilon^\eta) + O(\epsilon^N).
\]
Thus
\[
(8) \quad \int b(x) u_\epsilon^{2^*} = ||b||_\infty ||u_\epsilon||_{2^*}^{2^*} + O(\epsilon^\eta) + O(\epsilon^N).
\]
Now, we divide the proof in two cases:

Case \(N \geq 6\): Note that, see Appendix B, we have
\[
a(x) \left[ \frac{(tu_\epsilon + u_\lambda)^q + 1}{q + 1} - u_\lambda^q (tu_\epsilon) \right] \geq -C(tu_\epsilon)^{q+1}
\]
and
\[
b(x) \left[ \frac{(tu_\epsilon + u_\lambda)^p + 1}{p + 1} - u_\lambda^p (tu_\epsilon) \right] \geq b(x) \left[ \frac{(tu_\epsilon)^p + 1}{p + 1} + u_\lambda^{p-1} (tu_\epsilon)^2 \right].
\]
Thus

\[
I_\lambda(tu_\epsilon) \leq \frac{t^2}{2} (\|u_\epsilon\|^2 - C\|u_\epsilon\|_2^2) - \frac{t^{2^*}}{2^*} \int b(x) u_\epsilon^{2^*} + C t^{q+1} \int u_\epsilon^{q + 1}
\]

\[
\leq \frac{1}{N} \left[ \frac{\|u_\epsilon\|^2 - C\|u_\epsilon\|_2^2}{\left( \int b(x) u_\epsilon^{2^*} \right)^{\frac{N}{2^*}}} \right]^{\frac{N}{2^*}} + C t^{q+1} \int u_\epsilon^{q + 1}
\]

Using (5)-(8), one has

\[
I_\lambda(tu_\epsilon) \leq \frac{1}{N} \left[ \frac{S_\epsilon^N - C\epsilon^2 + O(\epsilon^{N-2})}{\left( \|b\|_\infty S_\epsilon^N + O(\epsilon) + O(\epsilon^N) \right)^{\frac{N}{2}}} \right]^{\frac{N}{2}} + o(\epsilon^2)
\]

\[
= \frac{1}{N} \left[ \frac{S_\epsilon^N - C\epsilon^2 + o(\epsilon^2)}{\left( \|b\|_\infty S_\epsilon^N + o(\epsilon^2) \right)^{\frac{N}{2}}} \right]^{\frac{N}{2}} + o(\epsilon^2)
\]

\[
= \frac{S_\epsilon^N}{N\|b\|_\infty^{\frac{N}{2}}} \left[ 1 - C\epsilon^2 + o(\epsilon^2) \right] + o(\epsilon^2)
\]

\[
< \frac{S_\epsilon^N}{N\|b\|_\infty^{\frac{N}{2}}}
\]

for \( \epsilon > 0 \) sufficiently small (above, we used that \( N \geq 6 \) and \( \eta > 2 \)).

**Case 3 \( \leq N \leq 5 \): Using (7) and (12), in Appendix B, we get

\[
I_\lambda(tu_\epsilon) \leq \frac{t^2}{2} \|u_\epsilon\|^2 - \frac{t^{2^*}}{2^*} \int b(x) u_\epsilon^{2^*} - C_0 \frac{t^p}{p} \|u_\epsilon\|^p + o(\epsilon^{N-2}), \quad C_0 > 0.
\]

Now, noting that \( \|u_\epsilon\|_{2^*} = C_1 \epsilon^{rac{N}{2} - 2} + O(\epsilon^{N+2}), \quad C_1 > 0 \), it follows that

\[
I_\lambda(tu_\epsilon) \leq \frac{t^2}{2} S_\epsilon^N - \frac{t^{2^*} \|b\|_\infty \|S_\epsilon^N \| S_\epsilon^N - C_0 \frac{t^p}{p} \epsilon^{rac{N}{2}}}
\]

\[
+ o(\epsilon^{N-2}) + O(\epsilon^{N+2}) + O(\epsilon^N) + O(\epsilon^\eta)
\]

\[
= \frac{t^2}{2} S_\epsilon^N - \frac{t^{2^*} \|b\|_\infty \|S_\epsilon^N \| S_\epsilon^N - C_0 \frac{t^p}{p} \epsilon^{rac{N}{2}}}
\]

\[
+ o(\epsilon^{N-2})
\]

where we used (5), (8) and that \( \eta > (N-2)/2 \). Calling \( t_\epsilon \) the maximum of the right-hand side for \( t \in [0,1] \), then \( t_\epsilon \) satisfy

\[
S_\epsilon^N = t_\epsilon^{2^*-2} \|b\|_\infty S_\epsilon^N + t_\epsilon^{2^*-3} C \epsilon^{rac{N}{2}} + o(\epsilon^\eta).
\]
It follows that
\[ t_\epsilon = \frac{1}{||b||_{\infty}} - C\epsilon^{\frac{N-2}{2}}t_\epsilon^{2^{*}-3} + o(\epsilon^{\frac{N-2}{2}}). \]

Thus
\[
\max_{t \in [0, R]} I_\lambda(tu_\epsilon) \leq \frac{t_\epsilon^2}{2} S_{\frac{N}{2}} - \frac{t_\epsilon^{2^{*}}}{2^*} ||b_{\infty}|| S_{\frac{N}{2}} - C t_\epsilon^{2^{*}-3} \epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}})
\]
\[
= \frac{1}{2} \frac{S_{\frac{N}{2}}}{||b||_{\frac{N-2}{2}}} - \frac{1}{2^*} \frac{S_{\frac{N}{2}}}{||b||_{\frac{N-2}{2}}} - C t_\epsilon^{2^{*}-3} \epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}})
\]
\[
= \frac{1}{N} \frac{S_{\frac{N}{2}}}{||b||_{\frac{N-2}{2}}} - C t_\epsilon^{2^{*}-3} \epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}})
\]
\[
< \frac{1}{N} \frac{S_{\frac{N}{2}}}{||b||_{\frac{N-2}{2}}},
\]
for \( \epsilon \) sufficiently small. This completes the proof of Lemma 1.

Proof of Lemma 2: Let \( w_n \in H^1_0(\Omega) \) be a sequence such that

\[ I'_\lambda(w_n) \to 0 \quad \text{and} \quad I_\lambda(w_n) \to c < \frac{S_{\frac{N}{2}}}{N||b||_{\frac{N-2}{2}}}. \]

Notice that \( w_n \) is bounded, actually, we have

\[ \frac{1}{p+1} I'_\lambda(w_n) \cdot (u_\lambda + w_n) - I_\lambda(w_n) \leq \epsilon_n||u_\lambda + w_n||, \quad \epsilon_n \to 0. \]

In the above expression the terms of power \( p + 1 \) are cancelled, then it can be rewritten as

\[ ||w_n||^2 \leq C(||w_n||^{q+1} + ||w_n|| + 1), \]

which yields the boundedness of \( ||w_n|| \). Passing to a subsequence, \( w_n \to w_0 \) in \( H^1_0(\Omega) \), \( w_n \to w_0 \) in \( L^r(\Omega) \), \( 1 < r < 2^* \). Moreover, \( u_\lambda + w_0 \) is a solution of \( (Q_\lambda) \) and so a critical point of \( F_\lambda \). Thus \( w_0 \) is a critical point of \( I_\lambda \). By assumption, we have \( w_0 = 0 \). Now,

\[ \frac{1}{p+1} I'_\lambda(w_n) \cdot (u_\lambda + w_n) - I_\lambda(w_n) = \frac{1}{N}||w_n||^2 + o(1) \to c. \]

If \( c = 0 \) then \( w_n \to 0 \) in \( H^1_0(\Omega) \) and the proof is finished. We claim that \( c = 0 \) is the unique possibility. Assume, by contradiction, that
\( c \neq 0 \). We can assume that \( ||w_n|| \) converges and since \( I'(w_n) \cdot w_n \to 0 \), we conclude that
\[
\lim_{n \to \infty} ||w_n||^2 = \lim_{n \to \infty} \int b(x)(w_n^+)^{2^*} = Nc.
\]
By definition of \( S \), we have
\[
||w_n||^2 \geq S \left( \int |w_n|^{2^*} \right)^{\frac{2}{2^*}} \geq \frac{S}{||b||_{\infty}^{\frac{2}{2^*}}} \left( \int b(x)|w_n^+|^{2^*} \right)^{\frac{2}{2^*}}.
\]
Passing to the limit in the above inequality, we get
\[
cN \geq \frac{S}{||b||_{\infty}^{\frac{2}{2^*}}} (cN)^{\frac{2}{2^*}}.
\]
It follows that \( c \geq \frac{S}{N||b||_{\infty}^{\frac{2}{2^*}}} \), provided \( c > 0 \). This contradiction completes the proof of Lemma 2.

5. Appendix

5.1. Appendix A.

This appendix has the purpose of proving that the solution \( u_\lambda \), obtained in the first part of the proof of Theorem 1, is a local minimum of \( F_\lambda \) for \( 0 < \lambda < \lambda^* \), where \( \lambda^* \) is defined by (2).

Let us remember that
\[
F_\lambda(u_\lambda) = \inf_M F_\lambda \text{ where } M = \{ u \in H^1_0(\Omega); 0 \leq u \leq \bar{u} \}.
\]
Here \( \bar{u} \) is a solution of \((P_\lambda)\) for some \( \lambda < \bar{\lambda} < \lambda^* \).

**Lemma 3.** We have that \( u_\lambda < \bar{u} \) in \( U = \{ \bar{u} > 0 \} \cap \Omega_b^+ \).

**Proof.** Let \( v = \bar{u} - u_\lambda \geq 0 \) a.e. in \( \Omega \), then
\[
-\Delta v + m(x)v \geq 0, \text{ where } m(x) := a^{-\frac{\|v\|}{\bar{u} - u}}.
\]
Suppose, by contradiction, that \( v(x_0) = 0 \) for some \( x_0 \in U \). We can choose \( r > 0 \) such that the ball \( B_r(x_0) \subset U \). We have that \( m \) is uniformly bounded in \( B_r(x_0) \), so by the Strong Maximum Principle we get \( v = 0 \) in \( B_r(x_0) \). It means that \( u_\lambda = \bar{u} \) in \( B_r(x_0) \), what contradicts the equations satisfied by these functions, since \( \lambda < \bar{\lambda} \) and \( b > 0 \) in \( U \). \( \square \)
Suppose, by contradiction, that $u_\lambda$ is not a local minimum of $F_\lambda$. Then we can choose $u_n \in H_0^1$ with $\|u_n - u_\lambda\| \to 0$ and $F_\lambda(u_n) < F_\lambda(u_\lambda)$. Let

$$v_n = \max\{0, \min\{u_n, \bar{u}\}\}, \quad w_n = (u_n - \bar{u})^+,$$

so that $u_n^+ = v_n + w_n$ and $v_n \in \mathcal{M}$. Define the sets $R_n = \{x \in \Omega : 0 \leq u_n(x) \leq \bar{u}\}$, $S_n = \text{supp}(w_n)$ and $T_n = \text{supp}(u_n^-)$, and the functions

$$h(x, t) = a(x)(t^+)^q + \lambda b(x)(t^+)^p \quad \text{and} \quad H(x, t) = \int_0^t h(x, s)ds.$$

Then, we can rewrite $F_\lambda(u_n^+)$ as

$$F_\lambda(u_n^+) = \int_{S_n} \left(\frac{\nabla u_n^2}{2} - H(x, u_n)\right) + \int_{R_n} \left(\frac{\nabla v_n^2}{2} - H(x, v_n)\right).$$

Notice that

$$\int_{S_n} \left(\frac{\nabla u_n^2}{2} - H(x, u_n)\right) = \int_{S_n} \left(\frac{\nabla (\bar{u} + w_n)^2}{2} - H(x, \bar{u} + w_n)\right),$$

and moreover

$$\int_{R_n} \left(\frac{\nabla v_n^2}{2} - H(x, v_n)\right) = F_\lambda(v_n) - \int_{S_n} \left(\frac{\nabla \bar{u}^2}{2} - H(x, \bar{u})\right).$$

Therefore

$$F_\lambda(u_n) = F_\lambda(u_n^+) + F_\lambda(u_n^-)$$

$$= \int_{S_n} \left(\frac{(|\nabla (\bar{u} + w_n)|^2 - |\nabla \bar{u}|^2)}{2} - (H(x, \bar{u} + w_n) - H(x, \bar{u}))\right)$$

$$+ F_\lambda(v_n) + \int_{T_n} \left(\frac{\nabla u_n^2}{2} - H(x, u_n)\right).$$

By using that

$$\int_\Omega \nabla \bar{u} \nabla w_n dx \geq \int_\Omega h(x, \bar{u})w_n dx,$$

we get

$$F_\lambda(u_n) \geq F_\lambda(v_n) + \frac{1}{2} \int_\Omega |\nabla w_n|^2 dx + \frac{1}{2} \int_\Omega |\nabla u_n^-|^2 dx$$

$$- \int_{S_n} (H(x, \bar{u} + w_n) - H(x, \bar{u}) - h(x, \bar{u})w_n) dx.$$

We can then conclude that

$$\frac{1}{2}||w_n||^2 + \frac{1}{2}||u_n^-||^2 < \int_{S_n} (H(x, \bar{u} + w_n) - H(x, \bar{u}) - h(x, \bar{u})w_n) dx.$$

The proof follows from the next claim.
**Claim.** We have
\[ \int_{S_n} (H(x, \bar{u} + w_n) - H(x, \bar{u}) - h(x, \bar{u})w_n) \, dx \leq o(1)\|w_n\|^2. \]

Assuming the above claim, we have
\[ \frac{1}{2}\|u_n\| + \frac{1}{2}(\|w_n\|^2 - o(1)\|w_n\|^2) < 0, \]
which implies that \( w_n, u_n = 0 \) for \( n \) large enough. It follows that \( u_n = v_n \in M \), for \( n \) large, and so \( F_\lambda(u_n) \geq F_\lambda(u_\lambda) \), what is a contradiction. \( \square \)

**Proof of the Claim.**
First, consider the following splitting for the function \( H_n = H_{1n} + H_{2n} \), where
\[ H_{1n}(x) = \frac{\lambda b(x)}{p+1}[(\bar{u} + w_n)^{p+1} - \bar{u}^{p+1}] - \lambda b(x)\bar{u}^p w_n, \] and
\[ H_{2n}(x) = \frac{a(x)}{q+1}[(\bar{u} + w_n)^{q+1} - \bar{u}^{q+1}] - a(x)\bar{u}^q w_n. \]

**Superlinear term:** Note that, there are \( s(x), \theta(x) \in (0, 1) \) such that
\[ H_{1n}(x) = \lambda b(x)[(\bar{u} + \theta w_n)^p - \bar{u}^p]w_n \]
\[ \leq C(\bar{u} + s\theta w_n)^{p-1}\theta w_n^2. \]

Moreover, \((\bar{u} + s\theta w_n)^{p-1}w_n^2 \leq w_n^{p+1} \) in \( B = \Omega \setminus A \), where \( A = \{u > 0\} \), then
\[ \int_{S_n \setminus A} H_{1n}(x) \, dx \leq C\|w_n\|^{p+1} \leq o(1)\|w_n\|^2. \] On the other hand, \((\bar{u} + s\theta w_n)^{p-1}w_n^2 \leq Cw_n^2 + Cw_n^{p+1} \) in \( A \), then
\[ \int_{S_n \cap A} H_{1n}(x) \, dx \leq \int_{S_n \cap A \cap \Omega^+_b} H_{1n}(x) \, dx \]
\[ \leq C \int_{S_n \cap A \cap \Omega^+_b} w_n^2 dx + C\|w_n\|^{p+1} \]
\[ \leq |S_n \cap A \cap \Omega^+_b| \frac{2}{p+1} \left( \int_{\Omega} w_n^{\frac{2(p+1)}{p-1}} dx \right)^{\frac{p-1}{2}} + C\|w_n\|^{p+1} \]
\[ \leq C|S_n \cap A \cap \Omega^+_b| \frac{2}{p} \|w_n\|^2 + o(1)\|w_n\|^2. \]

Now, we claim that \( |S_n \cap A \cap \Omega^+_b| \to 0 \) as \( n \to \infty \). Actually, given \( \epsilon > 0 \), by Lemma 3, we can choose \( \delta > 0 \) such that \( |A \cap \Omega^+_b \cap \{\bar{u} \leq u_\lambda + \delta\}| < \epsilon. \)
However, we have
\[ S_n \subset \{ \overline{u} \leq u_\lambda + \delta \} \cup \{ u_n > \overline{u} > u_\lambda + \delta \}, \]
and since \( u_n \to u_\lambda \) in \( L^2 \) there is \( n_0 \) such that for all \( n \geq n_0 \),
\[ \epsilon \delta^2 \geq \int_{\Omega} (u_n - u_\lambda)^2 \geq \int_{\{ u_n > u_\lambda + \delta \}} (u_n - u_\lambda)^2 \]
\[ \geq \int_{\{ u_n > u_\lambda + \delta \}} \delta^2 \{ u_n > u_\lambda + \delta \}. \]
Thus \( |S_n \cap A \cap \Omega^+| \leq |A \cap \Omega^+_b \cap \{ \overline{u} \leq u_\lambda + \delta \}| + |\{ u_n > u_\lambda + \delta \}| \leq 2\epsilon. \)
It follows that
\[ \int_{S_n \cap A} H_{1,n}(x) dx \leq o(1) ||w_n||^2. \]

**Sublinear term:** First, observe that \( \Omega^+_a \subset A \) and \( B \subset \Omega \setminus \Omega^+_a \). We have, for \( x \in A \setminus \Omega^+_a \),
\[ H_{2n}(x) = a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \leq 0. \]
Thus
\[ \int_{S_n \cap (A \setminus \Omega^+_a)} H_{2n}(x) dx \leq 0 \leq o(1) ||w_n||^2. \]
In the other hand, note that \( \overline{\Omega}^+_a \subset int(\Omega_a) \) and \( \overline{u} > 0 \) in \( int(\Omega_a) \), and so there is \( \delta > 0 \) such that \( \overline{u}(x) \geq \delta \) in \( \Omega^+_a \). Then, for \( x \in \Omega^+_a \),
\[ H_{2n}(x) = a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \]
\[ = a(x)(\overline{u} + s \theta w_n)^{q-1} \theta w_n^2 \]
\[ \leq C \delta^{q-1} w_n^2 \leq C w_n^2, \]
with \( \theta(x), s(x) \in (0, 1) \). Thus
\[ \int_{S_n \cap \Omega^+_a} H_{2n}(x) dx \leq C \int_{S_n \cap \Omega^+_a} w_n^2 \leq |S_n \cap \Omega^+_a| \delta \|w_n\|^2 \leq o(1) \|w_n\|^2, \]
where the last inequality is a consequence of \( S_n \cap \Omega^+_a \subset S_n \cap A \cap \Omega^+_b \) and \( |S_n \cap A \cap \Omega^+_b| = o(1) \).
Now, for \( x \in B \) we have \( H_{2n}(x) = -\frac{a^-(x)}{q+1} w_n^{q+1} \) since \( B \subset \Omega^-_a \). Thus
\[ (10) \int_B H_{2n}(x) dx = -\int_B \frac{a^-(x)}{q+1} w_n^{q+1} dx \leq 0. \]
The claim follows from the above estimates.
5.2. Appendix B.

Lemma 4. (1) There is a constant $C(p) > 0$ such that

\[
\frac{(r+s)^{p+1} - r^{p+1}}{p+1} - r^p s \geq \frac{s^{p+1}}{p+1} + Cr^p s^2, \quad r, s \geq 0,
\]

(2) for $r, s \geq 0$, we claim that there is a constant $C(q) > 0$ such that

\[
\frac{(r+s)^{q+1} - r^{q+1}}{q+1} - r^q s \leq C(q)s^{q+1}.
\]

Proof. For (1), see [2, pag. 537]. Item (2) is left as an exercise to the reader.

\[
\Box
\]

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References


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