Spectral and dynamical properties of sparse one-dimensional continuous Schrödinger and Dirac operators

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Abstract

Spectral and dynamical properties of some one-dimensional continuous Schrödinger and Dirac operators with a class of sparse potentials (which take nonzero values only at some sparse and suitably randomly distributed positions) are studied. By adapting and extending, to the continuous setting, some of the techniques developed for the corresponding discrete operator cases, the Hausdorff dimension of their spectral measures and lower dynamical bounds for transport exponents are determined. Furthermore, it is found that the condition for the spectral Hausdorff dimension to be positive is the same for the existence of singular continuous spectrum.

1 Introduction

Despite the numerous works and notorious advances in the understanding of spectral and dynamical properties of discrete (tight–binding) Schrödinger and Dirac operators with sparse potentials (see, for instance, [1, 2, 5, 11, 13, 16, 22, 29, 33]), there is a lack of examples of continuous operators for which it is possible to extend these results. We propose in this work some examples of sparse continuous Schrödinger and Dirac operators for which we can actually extend the powerful tools developed in some of the mentioned works, and so determine a number of their spectral and dynamical properties.

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Sparse operators have been used in great profusion in the last years thanks to the possibility of rather detailed spectral and dynamical analysis (there are, in some cases, exact results for the Hausdorff dimension of the spectral measure; see [1, 2, 33]). We understand by sparse operators those with zero potential except at the neighborhood of specific points whose distances are rapidly increasing. We will give a precise definition of the concept further in the text (see [19] for a brief discussion and a collection of results).

We deal essentially with two one-dimensional models: the Schrödinger continuous operator

$$(H_S\psi)(x) = -(\Delta\psi)(x) + (V\psi)(x) = -\psi''(x) + V(x)\psi(x) \; , \tag{1.1}$$

acting in $\mathrm{L}^2(\mathbb{R}_+)$, where $V(x)$ is the potential given by a real bounded function, and which satisfies the boundary condition

$$\psi(0)\cos\phi - \psi'(0)\sin\phi = 0\, , \tag{1.2}$$

with $\phi \in [0, \pi]$; the Dirac continuous operator

$$(H_D\Psi)(x) = \begin{pmatrix} V_1(x) + mc^2 & -cd/dx \\ cd/dx & V_2(x) - mc^2 \end{pmatrix} \Psi(x) \; , \tag{1.3}$$

acting in $\mathrm{L}^2(\mathbb{R}_0^+, \mathbb{C}^2)$, where $c > 0$ represents the speed of light, m the inertial mass of the particle (which can take the value $m = 0$, an important difference with respect to nonrelativistic models), $V_1(x), V_2(x)$ some real bounded functions; we assume that (1.3) satisfies the boundary condition

$$\psi_2(0)\cos\phi - \psi_1(0)\sin\phi = 0\, , \tag{1.4}$$

with $\phi \in [0, \pi]$, $\psi_1(x)$ and $\psi_2(x)$ the components of the "spinorial" wave function $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ which are associated, respectively, with positive and negative energy values (see, for instance, [30]). Since V, V_1 and V_2 are bounded potentials, H_S and H_D are self-adjoint operators, with the domain given by the domain of the free cases (i.e, null potentials).

In two previous works, some lower bounds for the dynamics generated by the discrete counterpart of H_D have been obtained [24], as well as several of its spectral properties [1]. In the present work, we extend our analysis to the operators defined by (1.1) and (1.3), subject to randomly sparse perturbations composed of infinitely many compact 'bumps' whose consecutive distances are rapidly growing.. So, this article is to be considered a natural continuation of such works [1, 24].

It is possible, in principle, to deal with bumps of distinct sizes, that grow, diminish or remain constant. Some randomness in the distribution of the position of the bumps, following an idea in [33] in the discrete case, will play a decisive role in the determination of the exact Hausdorff dimension of the spectral measure and of the lower bounds of the dynamical exponents. We consider a set $\{a_j\}_{j\geq 1}$ of a rapidly increasing sequence of real numbers, so that $V(x) = 0$ if $x \notin [a_j, a_j + 1]$, $j \in \mathbb{N}$, and nonzero elsewhere; see ahead for precise statements.

By considering some sparse potentials, in [1] several techniques and ideas from [13, 22, 33] (see also [2, 3]), in the context of discrete Schrödinger operators, were applied to a discrete counterpart of the Dirac operator (1.3), and a transition between purely point and singular continuous spectrum was also found, as well as the determination of the Hausdorff dimension.

To the best of our knowledge, there are no similar results to the important continuos case, neither for Schrödinger nor for Dirac continuous operators (see [17, 28] for other examples of sparse continuous Schrödinger operators). Thus, it is the aim of this work to apply and extend to the operators (1.1) and (1.3), with suitably chosen potentials, the main tools and results we have just mentioned; this will give us a precise estimate of the norm of transfer matrices, a powerful tool in the determination of some dynamical lower bounds of the transport exponents (according to [11, 24]; see Section 4 for details).

We underline that it is precisely such choice of the potentials, as well as the proposed parametrization of the eigenfunctions to (2.8) and (3.40), that will permit us to apply our strategy. More specifically, we deal with potentials of the form

$$V(x) = \sum_{j=1}^{\infty} v\chi_{[a_j^\omega, a_j^\omega+1]}(x) \ , \tag{1.5}$$

where $0 \neq v \in \mathbb{R}$, $\chi_I(x)$ is the characteristic function of the interval I, and $\mathcal{A} = (a_j^\omega)_{j\geq 1}$ is a random sequence of real numbers of the form $a_j^\omega = a_j + \omega_j$, such that the sequence (a_j) satisfies

$$a_j - a_{j-1} \geq 1 \ , \qquad\qquad j = 2, 3, \ldots$$

and

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = \beta > 1 \ ;$$

$\omega = (\omega_1, \omega_2, \ldots)$ represents a sequence of independent random variables defined on a probability space $(\Xi, \mathcal{B}, \nu)$, so that there is some $\eta > 0$ with ω_j uniformly distributed over the finite set $\{0, 1, 2, \ldots, j^\eta\}$, for all j.

In order to simplify our analysis, we restrict the separation between barriers by the identity

$$a_j - a_{j-1} = \beta^j \ , \qquad\qquad j = 2, 3, \ldots \tag{1.6}$$

with $a_1 + 1 = \beta > 1$. Condition (1.6) guarantees that each "bump" is placed at an interval of unitary size (this is just another convenience which could be removed) without superpositions for any $\eta > 0$, since the distance between (average) consecutive nonzero potential positions grows exponentially with an additional power-law randomness.

Another feature here is that, in the Dirac case, the values of v may be distinct for the potentials $V_1(x)$ and $V_2(x)$; physically, this means that the particle and the antiparticle are subject to different fields, or react differently to the same field. Now we fix some notation.

Definition 1.1 By $H_S(v, \phi)$ we denote the continuous Schrödinger operator (1.1) acting in $\mathrm{L}^2(\mathbb{R}_+)$, with potential $V(x)$ satisfying (1.5)–(1.6), subject to the phase boundary condition (1.2) at $x = 0$.

Definition 1.2 By $H_D(v_1, v_2, \phi)$ we denote the continuous Dirac operator (1.3) acting in $\mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2)$, with potentials $V_1(x), V_2(x)$ satisfying (1.5)–(1.6) (with $v = v_i \in \mathbb{R}$, $i = 1, 2$ in (1.5)), subject to the phase boundary condition (1.4) at $x = 0$.

By taking into account some cited works and the results exposed in the following sections, we would like to point the following major differences between the considered sparse continuous and discrete operators (in both Schrödinger and Dirac settings):

- the existence of several transition points between singular continuous and dense purely point spectra in the continuous operator cases, but only one transition point in the discrete ones;
- the presence of "critical energies" (i.e., values of energy where the norm of the transfer matrices are bounded, leading to a ballistic transport) for continuous operators, in contrast to the absence of this phenomenon for discrete operators.

The paper is organized as follows. In Section 2, we describe some spectral properties of $H_S(v, \phi)$, such as its essential spectrum, the definition of the spectral measure and its classification according to Hausdorff measures. In Section 3, we repeat the analysis developed to the Schrödinger operator in Section 2, to the continuous Dirac operator $H_D(v_1, v_2, \phi)$. In Section 4, we present some lower bounds to the transport exponents, obtained combining results of the previous sections with the techniques developed in [11].

2 Spectral properties of $H_S(v, \phi)$

In this section we discuss spectral properties of the Schrödinger operator $H_S(v, \phi)$. In order to accomplish our task, it will be important to adapt results of [13] to the continuous scenario.

2.1 Essential spectrum

We begin with a characterization of the essential spectrum of $H_S(v,\phi)$ through a theorem due to Klaus, which we reproduce in Theorem 2.3, and whose proof reduces to a direct extension of Theorem 3.13 in [6].

Theorem 2.3 *Let $H_S(v,\phi)$ be the Schrödinger operator in Definition 1.1 and let*

$$H'_S(0,\phi) = H_S(0,\phi) + v\chi_{[0,1]} \ , \tag{2.7}$$

where $(\chi_{[0,1]}\psi)(x) = \chi_{[0,1]}(x)\psi(x)$ for any $\psi \in \mathrm{L}^2(\mathbb{R}_+)$. Then,

$$\sigma_{\mathrm{ess}}(H_S(v,\phi)) = \sigma(H'_S(0,\phi)) \ .$$

The eigenvalues of $H'_S(0,\phi)$ do not belong to the essential spectrum of $H_S(v,\phi)$; hence, they cannot belong to the continuous spectrum of $H_S(v,\phi)$, neither be eigenvalues of infinite multiplicity (since we have a unidimensional problem).

To determine the essential spectrum of $H'_S(0,\phi)$, we need the following

Proposition 2.4 *Let $H'_S(0,\phi)$ be the operator defined by* (2.7). *Then, the spectrum of $H'_S(0,\phi)$ is absolutely continuous, with $\sigma_{\mathrm{ess}}(H'_S(0,\phi)) = \mathbb{R}_+$.*

Given the necessity of some tools which will be presented only in the next subsections, we have moved the proof of Proposition 2.4 to Appendix A.

It follows by Theorem 2.3 and Proposition 2.4, that the spectrum of $H_S(v,\phi)$ is the union of the interval $\mathbb{R}_+ = \sigma_{\mathrm{ess}}(H_S(0,\phi)))$ with the possible addition of a finite number of isolated points (note that if $v \geq 0$, these points are necessarily contained in $\mathbb{R}_+$).

2.2 Transfer matrix and Prüfer-type variables

In order to determine the spectral nature of the operator $H_S(v,\phi)$, we study the exact asymptotic behavior of the solutions to the Schrödinger eigenvalue equation

$$H_S(v,\phi)\psi(x) = E\psi(x) \ , \tag{2.8}$$

with $E \in \mathbb{R}$. This is an important step in our approach; it is here that the concepts of transfer matrix and Prüfer variables play a fundamental role. What follows is an adaptation of the material presented in Sections 3 and 4 in [22] to our continuous setting.

Let $u_D(x,E)$ and $u_N(x,E)$ be the solutions to (2.8) with initial conditions

$$\begin{aligned} u_D(0) &= 0\,, \qquad u'_D(0) = 1\,, \\ u_N(0) &= 1\,, \qquad u'_N(0) = 0\,, \end{aligned} \tag{2.9}$$

which satisfy the Dirichlet and Neumann boundary conditions, respectively.

For arbitrary $x, y \in \mathbb{R}_+$ and $E \in \mathbb{C}$, the transfer matrix is the unique 2×2-matrix $T(x, y; E)$ such that

$$T(x,y;E)\begin{pmatrix} \psi(y) \\ \psi'(y) \end{pmatrix} = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} , \tag{2.10}$$

for every solution ψ of (2.8). In fact, we can represent $T(x, y; E)$ in terms of the solutions to (2.8) subject to the initial conditions (2.9):

$$T(x,y;E) = \begin{pmatrix} u_N(x) & u_D(x) \\ u'_N(x) & u'_D(x) \end{pmatrix} .$$

Simon and Last have shown in [20] that is possible to determine the minimal supports (see [12] for a definition) of the spectral measure from the asymptotic behavior of the norm of the matrix $T(x, 0; E)$, which is directly related to the asymptotic behavior of the solutions to (2.8). This is not, in general, an easy task. We will, nevertheless, make use of our very special potential $V(x)$, combined with the sparsity condition (1.6), to decompose $T(x, 0; E)$ into the product of two types of matrices: a "perturbed" matrix

$$T_v(E) := T(x, x-1; E) = \begin{cases} \begin{pmatrix} \cos\alpha & \dfrac{\sin\alpha}{\alpha} \\ -\alpha\sin\alpha & \cos\alpha \end{pmatrix}, & E > V \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & E = V \\ \begin{pmatrix} \cosh\alpha & \dfrac{\sinh\alpha}{\alpha} \\ \alpha\sinh\alpha & \cosh\alpha \end{pmatrix}, & 0 \le E < V \end{cases} , \tag{2.11}$$

with $\alpha := \sqrt{|E-V|}$, which occurs for $x - 1 = a_j^\omega \in \mathcal{A}$, $j \in \mathbb{N}$, and the so-called "free" matrix

$$T_0(x-y;E) := T(x,y;E) = \begin{pmatrix} \cos k(x-y) & \dfrac{\sin k(x-y)}{k} \\ -k\sin k(x-y) & \cos k(x-y) \end{pmatrix} ,$$

occurring elsewhere, with $k := \sqrt{E}$.

Thus, we write

$$\begin{aligned} T(x,0;E) &= T(x, a_N^\omega + 1; E) T(a_N^\omega + 1, a_N^\omega; E) T(a_N^\omega, a_{N-1}^\omega + 1; E) \cdots \\ &\cdots\ T(a_1^\omega + 1, a_1^\omega; E) T(a_1^\omega, 0; E) \\ &= T_0(x - a_N^\omega - 1; E) T_v(E) T_0(a_N^\omega - a_{N-1}^\omega - 1; E) \cdots \\ &\cdots\ T_v(E) T_0(a_1^\omega; E) , \end{aligned} \tag{2.12}$$

where $a^\omega_N + 1 \le x < a^\omega_{N+1}$ for some $N \in \mathbb{N}$.

Let $E = k^2$, with $k \in \mathbb{R}$, be a parametrization of the continuous part of the essential spectrum of $H_S(v, \phi)$ (see Theorem 2.3). Now note that, for such energies, the free matrix $T_0(x - y; E)$ is similar to a purely clockwise rotation $R((x - y)k)$, that is,

$$UT_0(x-y;E)U^{-1} = \begin{pmatrix} \cos(x-y)k & \sin(x-y)k \\ -\sin(x-y)k & \cos(x-y)k \end{pmatrix} =: R((x-y)k) \ , \tag{2.13}$$

with

$$U := \sqrt{\frac{1+k^2}{2k}} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} .$$

Since the product of rotation matrices is also a rotation, we obtain

$$UT(x,0;E)U^{-1} = R((x-a^\omega_N)k)P(E)R((a^\omega_N - a^\omega_{N-1} - 1)k) \cdots P(E)R((a^\omega_1)k) \tag{2.14}$$

as the conjugation of (2.12) by U^{-1}, for every $x \in \mathbb{R}_+$ and $\omega_j \in \{0, 1, 2, \ldots, j^\eta\}$, $j \ge 1$; $P(E)$ is defined by

$$P(E) = \begin{cases} \begin{pmatrix} \cos\alpha & \dfrac{k}{\alpha}\sin\alpha \\ -\dfrac{\alpha}{k}\sin\alpha & \cos\alpha \end{pmatrix}, & E > V \\ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, & E = V \\ \begin{pmatrix} \cosh\alpha & \dfrac{k}{\alpha}\sinh\alpha \\ \dfrac{\alpha}{k}\sinh\alpha & \cosh\alpha \end{pmatrix}, & 0 \le E < V \end{cases} . \tag{2.15}$$

The next step is crucial in our analysis. Given the sparse structure of the potential and the relation (2.14), some results of [22] inspired us to consider the following change of variables: given the vectors

$$\mathbf{v}_n = (R_{n-1}\cos\theta^\omega_n, R_{n-1}\sin\theta^\omega_n) , \qquad \tilde{\mathbf{v}}_n = \left(R_n\cos\tilde\theta^\omega_n, R_n\sin\tilde\theta^\omega_n\right) , \tag{2.16}$$

the Prüfer-type variables $(R_n, \theta^\omega_n)_{n\ge0}$ satisfy a recurrence relation induced by

$$\mathbf{v}_n = R((a^\omega_n - a^\omega_{n-1} - 1)k)\tilde{\mathbf{v}}_{n-1} \tag{2.17}$$

and

$$\tilde{\mathbf{v}}_n = P(E)\mathbf{v}_n , \tag{2.18}$$

with $\mathbf{v}_1 = R(a_1^\omega k)\tilde{\mathbf{v}}_0$,

$$\tilde{\mathbf{v}}_0(\theta_0) = R_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} = U \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \sqrt{\frac{1+k^2}{2k}} \begin{pmatrix} k\cos\phi \\ \sin\phi \end{pmatrix} ,$$

$$R_0^2 = \frac{1+k^2}{2k}(k^2\cos^2\phi + \sin^2\phi) .$$

Thus, if $\boldsymbol{\psi}(x) = (\psi(x), \psi'(x))$ represents a solution to (3.40) satisfying the initial conditions $\mathbf{u}(0) = (\cos\phi, \sin\phi)$, then

$$\tilde{\mathbf{v}}_n = U\boldsymbol{\psi}((a_n+1)x) .$$

Remark 2.5 The variables defined in (2.16) are a slight variation of the definition of the *EFGP transform*, introduced in [14], for the solutions to the discrete Schrödinger equation. Note that we do not make use of the continuous version of these variables; this is, in fact, an advantage of the choice of potential made here. It is also worth noting that this definition of Prüfer variables, proposed by Marchetti *et. al.* [22], is based on the fact that the radius remains constant after the interaction with the free transfer matrix, which only effect is to displace the angle by a factor k. In sparse models like the one considered here, the great majority of the interactions produces exactly this kind of effect.

These tools, adapted from the discrete operator setting, will give us conditions to determine the exact asymptotic behavior of the solutions to (2.8), as we will see.

By equivalence of norms, the growth of $T(x,0;E)$ may be controlled by the particular norm

$$\|UT(x,0;E)U^{-1}\mathbf{v}(0)\|^2 = \|UT(a_N^\omega+1,0;E)U^{-1}\mathbf{v}(0)\|^2 = R_N^2 ,$$

where the equality holds for any normalized vector $\mathbf{v}(0) = (\cos\theta_0, \sin\theta_0)$ and for each $x \in \mathbb{R}_+$ such that $a_N^\omega + 1 \le x < a_{N+1}^\omega$. Thus, from equations (2.15), (2.17) and (2.18), R_N^2 can be written as

$$(R_N)^2 = (R_0)^2 \prod_{n=1}^{N} \left(\frac{R_n}{R_{n-1}}\right)^2 = (R_0)^2 \left(\exp\left\{\frac{1}{N}\sum_{n=1}^{N} \ln f(\theta_n^\omega, k)\right\}\right)^N , \tag{2.19}$$

with

$$f(\theta^\omega, k) := a(k) + b(k)\cos 2\theta^\omega + c(k)\sin 2\theta^\omega , \tag{2.20}$$

where $a(k) = 1 + \frac{(k^2-\alpha^2)^2}{2\alpha^2k^2}\sin^2\alpha$ if $E > v$, $1 + \frac{(k^2-\alpha^2)^2}{2\alpha^2k^2}\sinh^2\alpha$ if $0 \le E < v$, $1 + k^2/2$ if $E = v$; $b(k) = \frac{(\alpha^4-k^4)}{2\alpha^2k^2}\sin^2\alpha$ if $E > v$, $\frac{(\alpha^4-k^4)}{2\alpha^2k^2}\sinh^2\alpha$ if $0 \le E < v$, $-k^2/2$ if $E = v$; $c(k) = \frac{(k^2-\alpha^2)}{\alpha k}\sin\alpha\cos\alpha$ if $E > v$, $\frac{(k^2+\alpha^2)}{\alpha k}\sinh\alpha\cosh\alpha$ if $0 \le E < v$, k if $E = v$.

The Prüfer angles $(\theta_n^\omega)_{n\geq 1}$ are obtained recursively by

$$\theta_n^\omega = \tan^{-1}\left(\frac{A + B\tan\theta_{n-1}^\omega}{C + D\tan\theta_{n-1}^\omega}\right) - (\beta^n + \omega_n - \omega_{n-1})k \tag{2.21}$$

for $n > 1$, with θ_1^ω given by $\theta_1^\omega = \theta_0 - (a_1 + \omega_1)k$; here, $A(k) = \frac{\alpha}{k}\tanh\alpha$ if $0 \leq E < v$, $\frac{\alpha}{k}\tan\alpha$ if $E > v$, 0 if $E = v$, $B(k) = C(k) = 1$ for every $E \in \mathbb{R}_+$ and $D(k) = \frac{k}{\alpha}\tanh\alpha$ if $0 \leq E < v$, $\frac{k}{\alpha}\tan\alpha$ if $E > v$, k if $E = v$.

Hence, the determination of the exact asymptotic behavior of the sequence $(R_n(\theta_0))_{n\geq 1}$ involves an estimate of the Birkhoff-like sum

$$\frac{1}{N}\sum_{n=1}^{N}\ln f(v, \theta_n^\omega) \tag{2.22}$$

for N large, which, on the other hand, depends on the distribution properties of the sequence $(\theta_n^\omega)_{n\geq 1}$ of the Prüfer angles. This is exactly the same problem present in [1, 3, 22]. By using the ergodic theorem (see Theorem 1.1 in [15]), we may substitute, on the asymptotic limit $N \to \infty$, the average (2.22) by the integral

$$\frac{1}{\pi}\int_0^\pi \ln f(v, \theta) d\theta \ ,$$

in case the sequence $(\theta_n^\omega)_{n\geq 1}$ of the Prüfer angles is uniformly distributed modulo π (shortly, $\mathrm{u.\,d.\,mod}\,\pi$), and $\ln f(v, \theta)$, with $f(v, \theta)$ given by (2.20), is a periodic Riemann integrable function of period π. Recall that a sequence is $\mathrm{u.\,d.\,mod}\,\pi$ if it is equally distributed, in fractional portions, over half-open subintervals of $[0, \pi)$; see Chapter 1 in [15] for a detailed discussion.

Using again the arguments presented in [1, 22] (see, in particular, Section 4 in [22]), one proves the

Lemma 2.6 *The function $h(\theta) := \ln f(v, \theta)$ is a periodic Riemann integrable function of period π, which average is given by*

$$\frac{1}{\pi}\int_0^\pi h(\theta) d\theta = \ln r(v, E) \ ,$$

were

$$r(v, E) = \begin{cases} 1 + \dfrac{v^2}{4E(v-E)}\sinh^2\sqrt{v - E}, & 0 \leq E < v \\ 1 + \dfrac{v}{4}, & E = v \\ 1 + \dfrac{v^2}{4E(E-v)}\sin^2\sqrt{E - v}, & E > v \end{cases} \ . \tag{2.23}$$

Under the hypothesis of uniform distribution modulo π of the sequence $(\theta_n^\omega)_{n\geq 1}$ of Prüfer angles, Lemma 2.6 and Theorem 1.1 in [15] provide a precise estimate for the asymptotic limit of (2.22). Through a direct adaptation of Lemma 3.4 in [1], one proves

Lemma 2.7 *Let $(R_n(\theta_0))_{n\geq 1}$ be the sequence of the Prüfer radii which satisfy the initial conditions $\boldsymbol{\psi}(0) = (\cos\vartheta, \sin\vartheta)$. Suppose there is a set $A \subset \mathbb{R}_+$ of null Lebesgue measure so that the sequence $(\theta_n^\omega)_{n\geq 1}$ of the Prüfer angles is* u. d. mod π *for $k \in \mathbb{R}_+ \backslash A$. Then,*

$$C_N^{-1} r^N \leq (R_N(\theta_0))^2 \leq C_N r^N \,,$$

where C_N is a real number such that $C_N > 1$ and $\lim_{N\to\infty} C_N^{1/N} = (R_0)^2$, with r given by (2.23)*.*

The issue regarding the uniform distribution of the sequence $(\theta_n^\omega)_{n\geq 1}$ is solved by

Theorem 2.8 *The sequence of Prüfer angles $(\theta_n^\omega)_{n\geq 1}$, defined by* (2.21)*, is* u. d. mod π *for all $k \in \mathbb{R}_+ \backslash \mathbb{Q}$ and all $\omega \in \Xi$, apart from a set with null ν measure.*

Proof. The proof is exactly the same of Theorem 3.2 in [3]. □

Remark 2.9 It is our choice of potential and change of variables that give us the dynamical system (2.21), much more tractable than an analogous one obtained from the continuous *EFGP* transform defined by Kiselev *et. al.* (see Section 1 in [14]), which is defined by a transcendental recurrence relation.

2.3 Spectral measure and subordinacy

As is well known, associated with any self-adjoint operator, there exists a monotonically increasing spectral function $\rho(E)$, such that its spectrum corresponds to the complement of the set of points $\lambda \in \mathbb{R}$ where $\rho(E)$ is constant in a neighborhood of λ. Directly related to this spectral function is the so called Weyl-Titchmarsh coefficient, denoted by $m(z)$, defined and analytic in $\mathbb{C}\backslash\sigma$ (σ represents the spectrum of the operator), and Herglotz, which means that $m(z)$ has positive imaginary part in the upper half plane (the set of complex numbers z such that $\Im z > 0$).

In fact, in our setting $m(z)$ can be introduced in such way that

$$\chi(x,z) = -u_2(x,z) + m(z)u_1(x,z) \in \mathrm{L}^2(\mathbb{R}_+, \mathbb{C}), \tag{2.24}$$

where $u_1(x,z)$ and $u_2(x,z)$ are the solutions to (2.8) (with z replacing E) which satisfy the boundary conditions

$$\begin{aligned} u_1(0) &= \sin\phi\,, \qquad & u_1'(0) &= \cos\phi\,, \\ u_2(0) &= \cos\phi\,, \qquad & u_2'(0) &= -\sin\phi\,. \end{aligned} \tag{2.25}$$

It is possible to show that the spectral function $\rho(E)$ and $m(z)$ are related by the formula (see Chapter 9 in [4])

$$\rho(\lambda_2) - \rho(\lambda_1) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im m(E + i\varepsilon) dE \ ,$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$ which are points of continuity of $\rho(E)$. Thus, the boundary behavior of $m(z)$ in the vicinity of the real line can be used to determine the spectral types of the related operator (via de la Vallé-Poussin and Lebesgue-Radon-Nikodym theorems; see section 2 of [12]).

The main contribution obtained by Gilbert and Pearson theory of subordinacy [12] is precisely the connection between this boundary behavior of $m(z)$ and the asymptotic behavior of the solutions to (2.8). However, we will adapt the results of Jitomirskaya and Last [13] (only detailed for discrete Schrödinger operators), based on an extension of the theory of subordinacy, in order to classify the singular continuous spectrum according to the local Hausdorff dimension of the spectral measure $\rho(E)$.

Now we recall some useful definitions. A good description is found in [18]; for a more general approach and applications besides spectral theory, see [10, 25]. Given a Borel set $S \subset \mathbb{R}$ and $\alpha \in [0, 1]$, consider the number

$$Q_{\alpha,\delta}(S) = \inf \left\{ \sum_{\nu=1}^{\infty} |b_\nu|^\alpha : |b_\nu| < \delta; S \subset \bigcup_{\nu=1}^{\infty} b_\nu \right\} \ , \tag{2.26}$$

with the infimum taken over all covers by intervals of size at most δ. The limit

$$h^\alpha(S) = \lim_{\delta \downarrow 0} Q_{\alpha,\delta}(S) \ , \tag{2.27}$$

is called *α-dimensional Hausdorff (outer) measure*. For $\beta < \alpha < \gamma$,

$$\delta^{\alpha-\gamma} Q_{\gamma,\delta}(S) \le Q_{\alpha,\delta}(S) \le \delta^{\alpha-\beta} Q_{\beta,\delta}(S) \ ,$$

holds for any $\delta > 0$ and $S \subset \mathbb{R}$. So, if $h^\alpha(S) < \infty$, then $h^\gamma(S) = 0$ for $\gamma > \alpha$; if $h^\alpha(S) > 0$, then $h^\beta(S) = \infty$ for $\beta < \alpha$. Thus, for every Borel set S, there is a unique α_S such that $h^\alpha(S) = 0$ if $\alpha > \alpha_S$ and $h^\alpha(S) = \infty$ if $\alpha_S < \alpha$. The number α_S is called the *Hausdorff dimension* of the set S.

We also recall the notions of continuity and singularity of a measure with respect to the Hausdorff measure. Given $\alpha \in [0, 1]$, a measure μ is called α-continuous if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$; it is called α-singular if it is supported on some set S with $h^\alpha(S) = 0$.

Another useful concept is the so called *exact dimension* of a measure, taken from [26].

Definition 2.10 A Borel measure μ in $\mathbb{R}$ is said to be of exact dimension α, for $\alpha \in [0,1]$, if two requirements hold: (1) for every $\beta \in [0,1]$ with $\beta < \alpha$ and S a set of dimension β, $\mu(S) = 0$ (which means that $\mu(S)$ gives zero weight to any set S with $h^\alpha(S) = 0$); (2) there is a set S_0 of dimension α which supports μ in the sense that $\mu(\mathbb{R}\backslash S_0) = 0$.

Remark 2.11 There is an equivalent formulation of Definition 2.10: a measure μ is said to have exact dimension α if, for every $\epsilon > 0$, it is simultaneously $(\alpha - \epsilon)$-continuous and $(\alpha + \epsilon)$-singular. This is the definition of exact dimension used in this work.

Definition 2.12 A solution ψ to (2.8) is said to be subordinate if

$$\lim_{l\to\infty} \frac{\|\psi\|_l}{\|\Phi\|_l} = 0$$

holds for any linearly independent solution Φ to (2.8), where $\|\cdot\|_l$ denotes the $\mathrm{L}^2(\mathbb{R}_+)$-norm at the length $l \in \mathbb{R}$, i.e.,

$$\|\psi\|_l^2 := \int_0^l |\psi(x)|^2 dx \ .$$

Following [13], for any given $\epsilon > 0$, introduce the length $l(\epsilon) \in (0,\infty)$ by the equality

$$\|u_1\|_{l(\epsilon)} \, \|u_2\|_{l(\epsilon)} = \frac{1}{2\epsilon} \tag{2.28}$$

(see equation (1.12) in [13]), where u_1 and u_2 are the solutions to (2.8) which satisfy the initial conditions (2.25).

Recall that the Wronskian of two functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{C}$ is given by $W[\varphi,\psi](x) = (\varphi(x)\overline{\psi}'(x) - \varphi'(x)\overline{\psi}(x))$. It follows by Green's identity (see Chapter 9 in [4]) that

$$\begin{aligned}\int_0^N \Big(\overline{\psi}(x)(H_S(v,\phi)\varphi)(x) \quad & -\overline{(H_S(v,\phi)\psi)}(x)\phi(x)\Big)\, dx = \\ & = W[\varphi,\psi](N) - W[\varphi,\psi](0) = 0,\end{aligned}$$

i.e, the Wronskian of the solutions $\{\varphi,\psi\}$ to (2.8) is constant. We observe that we are in the limit-point case and so there is just one (normalized) solution in $\mathrm{L}^2(\mathbb{R}_+)$; this implies that the left-hand side of (2.28) is a monotone increasing function of l, which vanishes at $l = 0$ and diverges as $l \to \infty$. On the other hand, the right-hand side of (2.28) is a monotone decreasing function of ϵ which diverges as $\epsilon \to 0$. It is then concluded that the function $l(\epsilon)$ is a well-defined monotone increasing and continuous function of ϵ which diverges as $\epsilon \to 0$.

What follows are versions of the Jitomirskaya-Last inequalities (Theorem 1.1 in [13]) for continuous Schrödinger operators.

Theorem 2.13 *Let H_S be the Schrödinger operator* (1.1) *with the boundary condition* (1.2)*. Then, given $\epsilon > 0$, one has*

$$\frac{5-\sqrt{24}}{m(E+i\epsilon)} \leq \frac{\|u_1\|_{l(\epsilon)}}{\|u_2\|_{l(\epsilon)}} \leq \frac{5+\sqrt{24}}{m(E+i\epsilon)} \ .$$

Proof. The proof is a direct application of the arguments presented in Section 3 in [13], together with the variation of parameters formula (see Chapter 3 in [4])

$$\begin{aligned} \chi(x,z) &= -u_2(x,E) + m(z)u_1(x,E) + i\epsilon u_2(x,E)\int_0^x u_1(t,E)\chi(t,z)dt \\ &- i\epsilon u_1(x,E)\int_0^x u_2(t,E)\chi(t,z)dt \end{aligned}$$

and the well-known identity

$$\Im m(z) = \epsilon \int_0^\infty |\chi(x,z)|^2 dx$$

(see Chapter 9 in [4] for a proof), where $\chi(x,z)$ represents the unique (up to multiple constants) $\mathrm{L}^2(\mathbb{R}_+)$ solution to (2.8). □

Theorem 1.2 in [13] and its corollaries also hold true here, as direct consequences of Theorem 2.13: if ρ is the spectral measure of H_S, then, with $b = \alpha/(2-\alpha)$,

$$D_\rho^\alpha(E) := \limsup_{\epsilon \downarrow 0} \frac{\rho\left((E-\epsilon, E+\epsilon)\right)}{(2\epsilon)^\alpha} = \infty \tag{2.29}$$

if, and only if,

$$\liminf_{l\to\infty} \frac{\|u_1\|_l}{\|u_2\|_l^b} = 0 \ , \tag{2.30}$$

$D_\rho^\alpha(E)$ the α-*upper derivative* of ρ at E (see [18, 25] for detailed discussions of this definition).

All the remarks made in [18] with respect to the generalized eigenfunction u_1 are equally valid, which combined with Theorem 1.2 in [13] and the constancy of the Wronskian lead to some results regarding continuity properties of the spectral measure with respect to Hausdorff measures.

Before we proceed, we need the following

Lemma 2.14 *Let u_1 and u_2 be the solutions to* (2.8) *that satisfy the boundary conditions* (2.25)*. Then, there exists a real number $c > 0$, which depends only on E and v, such that*

$$\|u_1\|_{l+3}\|u_2\|_{l+3} \geq c\, l \ .$$

Proof. It follows from the Wronskian constancy that

$$\begin{aligned} W[u_2, \overline{u_1}](x) &= \big(u_2(x)u_1'(x) - u_2'(x)u_1(x)\big) = W[u_2, \overline{u_1}](0) \\ &= \cos^2\phi + \sin^2\phi = 1 \ , \end{aligned}$$

for every $x \in \mathbb{R}_+$. Thus,

$$\begin{aligned} l &= \int_0^l W[u_2, \overline{u_1}](x)dx \leq \left| \left\langle \begin{pmatrix} u_2 \\ u_2' \end{pmatrix}, \begin{pmatrix} u_1' \\ -u_1 \end{pmatrix} \right\rangle_l \right| \leq \left\| \begin{pmatrix} u_2 \\ u_2' \end{pmatrix} \right\|_l \left\| \begin{pmatrix} u_1' \\ -u_1 \end{pmatrix} \right\|_l \\ &= \left(\int_0^l |u_2(x)|^2 dx + \int_0^l |u_2'(x)|^2 dx \right) \left(\int_0^l |u_1(x)|^2 dx + \int_0^l |u_1'(x)|^2 dx \right) \ , \end{aligned}$$

by an application of the Cauchy-Schwarz inequality. Now we use Lemma 2.4 in [7]: since

$$\sup_{t \in \mathbb{R}_+} \int_t^{t+1} |V(x) - E| \, dx = |v - E| < \infty \ , \tag{2.31}$$

we have by the referred lemma that

$$\|u_1\|_{l+3} \|u_2\|_{l+3} \geq c\, l \ ,$$

where c is a constant that only depends on E. This concludes the proof of the lemma. $\square$

The proofs of Corollary 2.15*(a)* and *(b)* follow the same lines of the proofs of Corollaries 4.4 and 4.5 in [13], respectively.

Corollary 2.15 *(a) Suppose that for some $\alpha \in [0, 1)$ and every E in some Borel set F, every solution ψ to the Schrödinger equation* (2.8) *obeys*

$$\limsup_{l \to \infty} \frac{\|\psi\|_l^2}{l^{2-\alpha}} < \infty \ .$$

Then, the restriction $\rho(F \cap \cdot)$ is α-continuous.

(b) Suppose that

$$\liminf_{l \to \infty} \frac{\|u_1(E)\|_l^2}{l^\alpha} = 0$$

is satisfied for every E in some Borel set F. Then the restriction $\rho(F \cap \cdot)$ is α-singular.

We will, nevertheless, rewrite item *(a)* in Corollary 2.15 in terms of the one-dimensional 2×2 transfer matrices $T(x, 0; E)$, following the strategy proposed in [2] (Corollary 3.7).

Corollary 2.16 *Suppose that for some $\alpha \in [0, 1)$ and every E in some Borel set $A \subset \mathbb{R}$,*

$$\limsup_{l \to \infty} \frac{1}{l^{2-\alpha}} \int_0^l \|T(x, 0; E)\|^2 < \infty \ , \tag{2.32}$$

with $\|\cdot\|$ some matrix norm. Then the restriction $\rho(A \cap \cdot)$ is α-continuous.

Proof. By choosing $\theta_1 = \arctan((\cot\phi)/k)$ and $\theta_2 = -\arctan((\tan\phi)/k)$, it follows by Theorem 2.1 in [14] that there exists a constant C_1 such that

$$\|T(x,0;E)\| \geq C_1 \max\{R_n(\theta_1), R_n(\theta_2)\} \ ,$$

for all $a_n^\omega \leq x < a_{n+1}^\omega$, where $R_n(\theta)$ is the n-th Prüfer radius starting from the initial condition $\mathbf{v}_\theta = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$; explicitly $C_1 = \max(k, 1/k)$. Since

$$R_n^2(\theta_{1(2)}) = \frac{1+k^2}{2k}\left(k^2\left|u_{1(2)}(x)\right|^2 + \left|u'_{1(2)}(x)\right|^2\right) \ ,$$

we obtain the inequality

$$\frac{1+k^2}{2k}\min(1,k^2)\left[\left|u_{1(2)}(x)\right|^2 + \left|u'_{1(2)}(x)\right|^2\right] \leq R_n^2(\theta_{1(2)}) \ .$$

The last step in this proof is given by Lemma 2.4 in [7]: from (2.31), we have by the referred lemma and the considerations above that

$$\begin{aligned}
\int_0^l \|T(x,0;E)\|^2\,dx &\leq C_2\int_0^l \max_{i=1,2}\left\{|u_i(x,E)|^2 + \left|u_i'(x,E)\right|^2\right\}dx \\
&\leq C_2\sum_{j=0}^{l-1}\int_j^{j+1}\max_{i=1,2}\left\{|u_i(x,E)|^2 + \left|u_i'(x,E)\right|^2\right\}dx \\
&\leq (1+2|v-E|)C_2\max\left\{\|u_1(E)\|^2_{l+3}\,, \|u_2(E)\|^2_{l+3}\right\} \qquad (2.33)
\end{aligned}$$

where $C_2 = (1+k^2)/2$. Hypothesis (2.32), together with (2.33), imply Corollary 2.16. □

Remark 2.17 As remarked in [13], Corollaries 4.4 and 4.5 of [13] does not, in general, hold for continuous operators on $L^2(\mathbb{R}_+)$. It is, nevertheless, the structure of the potential (1.5), which satisfies (2.31), that garantees the applicability of Corollary 2.16 to our model.

2.4 Hausdorff dimension and spectral transition

This subsection is devoted to the determination of the Hausdorff dimension of the spectral measure of $H_S(v,\phi)$ and its spectral types. The next result is a direct adaptation of of Proposition 3.9 in [2] and Proposition 4.3 in [1] to the continuous Schrödinger operator $H_S(v,\phi)$.

Proposition 2.18 *Let $\mathcal{A} = (a_n)_{n\geq 1}$ be given by* (1.6)*, $E\in\mathbb{R}$ and assume that the sequence $(\theta_n^\omega)_{n\geq 1}$ of Prüfer angles* (2.21) *is* u. d. mod π *for every $\theta_0 \in [0,\pi)$, almost every $k\in\mathbb{R}_+$*

(w.r.t. Lebesgue measure) and almost every $\omega \in \Xi$. Then, there is a generalized eigenfunction ψ such that

$$C_n^{-1} r^{n/2} \leq R_n(\theta_0) \leq C_n r^{n/2}$$

holds with r given by (2.23) and $C_n^{1/n} \searrow R_0(\theta_0)$ as $n \to \infty$. In addition, there exists a subordinate solution ϕ (with α^*-phase boundary condition) for energy E such that, for all sufficiently large n, the Prüfer radius associated with ϕ satisfies

$$|R_n(\alpha^*)| \leq \tilde{C}_n r^{-n/2}$$

with $\tilde{C}_n^{1/n} \searrow R_0(\alpha^*)$ as $n \to \infty$.

Now we state and prove one of our main results. Recall that $H_S(v, \phi)$ was introduced in Definition 1.1, and the sparsity parameter satisfies $\beta > 1$.

Theorem 2.19 *Let ρ be the spectral measure of $H_S(v, \phi)$. Given a closed interval of energies $L \subset \mathbb{R}_+$, for almost every $\phi \in [0, \pi]$, and almost every $\omega \in \Xi$, the spectral measure ρ restricted to L has Hausdorff dimension*

$$h_\rho(E) = \max\left\{0, 1 - \frac{\ln r}{\ln \beta}\right\} , \tag{2.34}$$

with $r = r(v, E)$ given by (2.23).

Proof. Despite the proof being similar to the arguments present in the proofs of Theorem 3.11 in [2] and Theorem 1.4 in [1], we expose it for the readers' sake.

Theorem 2.8 implies that the sequence $(\theta_n^\omega)_{n\geq 0}$ of Prüfer angles is u. d. mod π for every $\theta_0 \in [0, \pi]$, every $E \in \mathrm{L}' := L \setminus (\mathbb{Q}\pi)^2$ and almost every $\omega \in \Xi$. We obtain from Proposition 2.18 the estimates

$$\|T(x, 0; E)\| \leq C_n r^{n/2} \leq C_n' a_n^{\gamma/2} \leq C_n'' x^{\gamma/2} , \tag{2.35}$$

which hold for every $E \in \mathrm{L}'$ and every $a_n^\omega \leq x < a_{n+1}^\omega$, with $\gamma := \ln r / \ln \beta$, $C_n'' > 0$ and $\lim_{n\to\infty} (C_n'')^{1/n} < \infty$.

It follows by the constancy of $\|T(x, 0; E)\|$ on $[a_n + 1, a_{n+1}]$ that

$$\int_0^l \|T(x, 0; E)\|^2 \, dx \leq c \, l^{1+\gamma} \tag{2.36}$$

holds for some $c > 0$ and every $E \in \mathrm{L}'$.

The application of Proposition 2.18 guarantees, for $E \in \mathrm{L}'$, the existence of a subordinate solution ϕ^{sub} such that its sequence of Prüfer radii satisfies the estimate

$$|R_n(\theta_{\alpha^*})| \leq C_n''' \, a_n^{-\gamma/2} ,$$

for some constant $C_n''' > 0$.

Since every solution to (2.8) has constant modulus on the interval $[a_n+1, a_{n+1}]$, we have

$$\left\|\Phi^{\rm sub}\right\|_l^2 \le c'\, l^{1-\gamma}\ , \tag{2.37}$$

for some $c' > 0$.

Now we use the subordinacy theory. Being the restriction of the measure ρ to $\mathbb{R}_+$ supported on the set of those E for which $\Phi^{\rm sub}$ satisfies the boundary condition ϕ (thanks to the fact that ρ has no absolutely continuous part; see Theorem 1 in [12]), we have $u_1 = \Phi^{\rm sub}$ for a.e. $E \in I$ with respect to ρ.

Thus, by (2.36) and (2.37), on the intervals (3.39) one has

$$\limsup_{l\to\infty} \frac{1}{l^{2-\alpha}} \int_0^l \|T(x,0;E)\|^2\, dx < \infty$$

and

$$\liminf_{l\to\infty} \frac{\|u_1(E)\|_l^2}{l^{\alpha'}} = 0\ ,$$

provided $2-\alpha = 1+\gamma+\varepsilon$ and $\alpha' = 1-\gamma+\varepsilon$, respectively, where ε is an arbitrary positive number.

It follows by Corollary 2.16 that the spectral measure ρ is simultaneously $(1-\gamma-\varepsilon)$-continuous and $(1-\gamma+\varepsilon)$-singular. Since ε is arbitrary, we have, by Remark 2.11, that the restriction $\rho(I' \cap \cdot)$ has exact Hausdorff dimension given by (2.34), where $I' \subseteq \mathrm{L}'$.

Finally, from the theory of rank one perturbations (more specifically, Theorem 8.1 in [27]), we know that $\rho\left((\mathbb{Q}\pi)^2\right) = 0$ holds for almost every ϕ; therefore, for almost every ϕ, the restriction $\rho(I \cap \cdot)$ has (2.34) as its Hausdorff dimension. This concludes the proof of the theorem. □

Remark 2.20 Note that we have assumed that the minimal support of the absolutely continuous spectrum is an empty set, a result that can be obtained from Theorem 2.21 and is related to the boundedness of the norm of the transfer matrix $\|T(x,0;E)\|$ (see Theorem 1.1 in [20]). In fact, $\|T(x,0;E)\| \le C < \infty$ is satisfied for every $x \in \mathbb{R}_+$ given that $E = v + m^2\pi^2$, $m \in \mathbb{Z}$; however, since the set of energies where this boundedness occurs is enumerable, it does not belong to the minimal support of the absolutely continuous spectrum.

It is interesting to compare the above results obtained for the operator $H_S(v,\phi)$ and its discrete counterpart (as studied in [22, 33]). Before that, let us make some remarks regarding the spectral types of these kind of operators. Let $H_S^c(v,\phi)$ represent the continuous

Schrödinger operator in Definition 1.1 and denote by $H_S^d(v,\phi)$ the discrete Schrödinger operator

$$(H_S^d\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n\psi_n \ ,$$

acting on $l^2(\mathbb{Z}_+)$, where the sequence (V_n) is defined, in analogy to (1.5), as

$$V_n = \begin{cases} v\,, & n = a_j^\omega \in \mathcal{A}\,, \\ 0\,, & n \notin \mathcal{A}\,, \end{cases} \tag{2.38}$$

and which satisfies the boundary condition

$$\psi_{-1}\cos\phi - \psi_0 \sin\phi = 0\,,$$

with $\phi \in [0,\pi]$; the set $\mathcal{A}$ is defined by (1.6).

Let $\sigma_{\text{ess}}^{c(d)}$ denote the essential spectrum of the continuous (discrete) operator, $\beta > 1$ the sparsity parameter and $r(E)$ the asymptotic behavior of the norm $\|T(x,0;E)\|$ (given by (2.23) for $H_S^c(v,\phi)$ and $1+\frac{V^2}{4(E^2-4)}$ for $H_S^c(v,\phi)$; see [33]). In both cases, we can affirm that

Theorem 2.21 *Write*

$$I_{c(d)} := \left\{ E \in \sigma_{\text{ess}}^{c(d)} \setminus A_{c(d)} : r < \beta \right\}$$

with $A_{c(d)}$ some set of Lebesgue zero measure. Then, for ν-almost every $\omega \in \Xi$:

(a) the spectrum of $H_S^{c(d)}(v,\phi)$ restricted to the set $I_{c(d)} \backslash A_{c(d)}$ is purely singular continuous;

(b) the spectrum of $H_S^{c(d)}(v,\phi)$ is purely point when restricted to $\sigma_{\text{ess}}^{c(d)} \setminus I_{c(d)}$ for almost every $\phi \in [0,\pi]$.

Proof. The proof is a slight variation of a proof given in [3] (Theorem 2.4); it is based on the criteria develop by Last-Simon in [20], with some adaptations for sparse operators; see [3, 22] for details. $\square$

The first conclusion drawn from Theorem 2.21 is the absence of the absolutely continuous spectrum for both operators (see Section 4 in [22] and Remark 2.20).

Theorem 2.21 also shows that there exists a sharp transition between singular continuous and purely point spectrum for $H_S^d(v,\phi)$. Note from (2.34) that the condition for the Hausdorff dimension to be positive is the same for the existence of singular continuous spectrum, i.e., $\beta > r$. In fact, the set of energies for which the Hausdorff dimension is zero coincides with the set where the purely point spectrum is supported.

The above discussion implies, from the expression $r(E) = 1 + \frac{V^2}{4(E^2-4)}$, that the dense purely point spectrum is located at the boundaries of $\sigma(H_S^d)$, while the singular continuous spectrum is located at the center of this interval. Nonetheless, this may not occur for $H_S^c(v,\phi)$. According to (2.23), given its oscillatory behavior for $E > v$, $h_\rho(E)$ may vary from zero to one if $1 + v^2/4E(E-v) > \beta$ and E ranges, for instance, the interval $\left[v + (n+1/2)^2\pi^2,\ \ v + (n+3/2)^2\pi^2\right]$ for some integer n. Hence, in this situation, we may have several, but finite (since inevitably $1+v^2/4E(E-v) < \beta$, for E large enough) transition points, giving intervals of dense pure point spectrum intertwined with intervals of singular continuous spectrum.

Note also that $\lim_{E\to\infty} h_\rho(E) = 1$, i.e., for large values of energy, the effects of the sparse perturbation are attenuated. However, we still have a singular continuous spectrum, a result that suggests the dynamical picture that, despite the fact that a particle with energy E is able to transpose the "barriers" of small height v (when compared to E) with high probability, the effects of the infinite number of barriers sum up to the reflection probability; thus, the particle "weakly recurs" to the origin with probability one (see [23, 32] for discussions of these ideas).

3 The Spectral properties of $H_D(v_1, v_2, \phi)$

In this section we reconsider the results discussed in Section 2, but for the Dirac operator $H_D(v_1, v_2, \phi)$, according to Definition 1.2. Given the continual techniques and results from the previous section, we will omit great part of the details here. Note, however, that some expressions take a more complicated form in the Dirac case.

3.1 Essential spectrum

Our first step is the determination of the essential spectrum of the free operator $H_D(0,0,\phi)$.

Proposition 3.22 *The essential spectrum of the free operator is given by*

$$\sigma_{\mathrm{ess}}(H_D(0,0,\phi)) = \left(-\infty, -mc^2\right] \cup \left[mc^2, +\infty\right). \tag{3.39}$$

Proof. In order to prove this proposition, we determine the exact behavior of the function $m_D(E+i\varepsilon)$ as $\varepsilon \downarrow 0$; $m_D(z)$ is defined in such way that

$$\xi(x,z) = \begin{pmatrix} \xi_1(x,z) \\ \xi_2(x,z) \end{pmatrix} = -\mathbf{u}(x,z) + m_D(z)\mathbf{v}(z)$$

is a $\mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ solution to the Dirac eigenvalue equation

$$H_D(v_1, v_2, \phi)\Psi = z\Psi \tag{3.40}$$

for some fixed $z \in \mathbb{C}$, where $\mathbf{u}(x,z)$ and $\mathbf{v}(x,z)$ are also solutions satisfying the boundary conditions

$$\begin{aligned} u_1(0) &= \cos\phi\,, \qquad & u_2(0) &= \sin\phi\,, \\ v_1(0) &= \sin\phi\,, \qquad & v_2(0) &= -\cos\phi\,. \end{aligned} \tag{3.41}$$

After some manipulations, it follows by (1.3) that

$$\psi_j''(x) + \frac{E^2 - m^2c^4}{c^2}\psi_j = 0\,,$$

where $\psi_j, j = 1, 2$, are the components of the spinor Ψ.

Since $m_D(z)$ is uniquely defined imposing that $\xi = -\mathbf{u} + m_D(z)\mathbf{v}$ is in $\mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2)$, it is found that

$$m_D(z) = i\,q(z)\;,$$

with $q(z) = \sqrt{z^2 - m^2c^4}/c$. Now, put

$$\Im\, m_D(E) = \lim_{\varepsilon\downarrow 0} \Im\, m_D(z)\;, \quad z = E + i\varepsilon,$$

and let $L(\rho)$ be the set of all $E \in \mathbb{R}$ for which this limit exists. It is known (see Appendix B in [31]) that the minimal supports $\mathcal{M}$, $\mathcal{M}_{\mathrm{ac}}$ and $\mathcal{M}_{\mathrm{s}}$ of ρ, the absolutely continuous part ρ_{ac} and the singular part ρ_{s} of ρ, with respect to the Lebesgue measure in $\mathbb{R}$, are given by $E \in L(\rho)$ such that $0 < \Im\, m_D(E) \le \infty$, $0 < \Im\, m_D(E) < \infty$ and $\Im\, m_D(E) = \infty$, respectively.

Since

$$\lim_{\varepsilon\downarrow 0} \Im\, m_D(E + i\varepsilon) = \begin{cases} \dfrac{\sqrt{E^2 - m^2c^4}}{c} & \text{if } E^2 > m^2c^4, \\ 0 & \text{otherwise}\;, \end{cases}$$

it follows by the above criteria that the essential spectrum of $H_D(v_1, v_2, \phi)$ satisfies (3.39). $\square$

The next result is obtained again by an adaptation of Theorem 3.13 in [6].

Theorem 3.23 *Let $H_D(v_1, v_2, \phi)$ be the Dirac operator in Definition 1.2 and let*

$$H_D'(0, \phi) = H_D(0, 0, \phi) + \chi_{[0,1]} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}\,, \tag{3.42}$$

where $(\chi_{[0,1]}\psi_i)(x) = \chi_{[0,1]}(x)\psi_i(x)$, $\psi_i(x), i = 1, 2$ representing the components of the spinor $\Psi \in \mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2)$. Then,

$$\sigma_{\mathrm{ess}}(H_D(v_1, v_2, \phi)) = \sigma_{\mathrm{ess}}(H_D'(0, 0, \phi))\;.$$

The essential spectrum of $H_D'(0, 0, \phi)$ is, on the other hand, determined by the following

Proposition 3.24 *Let $H_D'(0, 0, \phi)$ be the operator defined by (3.42). Then, its spectrum is absolutely continuous, with $\sigma_{\mathrm{ess}}(H_D'(0, 0, \phi)) = \big(-\infty, -mc^2\big] \cup \big[mc^2, +\infty\big)$.*

Proof. The proof of the proposition follows the same steps of the proof of Proposition 2.4; the details are left for the avid reader. □

By Theorem 3.23, and Proposition 3.24, the essential spectrum of $H_D(v_1, v_2, \phi)$ is the union of the intervals defined in (3.39) with the possible addition of a finite number of isolated points (note that if $|v_i| > mc$, $i = 1, 2$, these points are necessarily contained in the essential support of $H_D(0, 0, \phi)$).

3.2 Transfer matrix and Prüfer-type variables

Let $\mathbf{u}^D(x, E)$ and $\mathbf{u}^N(x, E)$ be the solutions to (3.40) with

$$\begin{array}{ll} u_1^D(0) = 0\,, & u_2^D(0) = 1\,, \\ u_1^N(0) = 1\,, & u_2^N(0) = 0\,, \end{array} \tag{3.43}$$

which satisfy the Dirichlet and Neumann boundary conditions, respectively.

For arbitrary $x, y \in \mathbb{R}_+$ and $E \in \mathbb{C}$, the transfer matrix is the unique 2×2-matrix $T_D(x, y; E)$ such that

$$T_D(x, y; E) \begin{pmatrix} \psi_1(y) \\ \psi_2(y) \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{3.44}$$

for every solution Ψ to (3.40). Actually, it is possible to represent $T_D(x, y; E)$ in terms of the solutions to (3.40) subject to the boundary conditions (3.43), that is,

$$T_D(x, y; E) = \begin{pmatrix} u_1^D(x) & u_1^N(x) \\ u_2^D(x) & u_2^N(x) \end{pmatrix}$$

Remark 3.25 Definition (3.44) is one among other possible definitions of a transfer matrix related to the solutions to (3.40). The convenience of this choice will be made clear throughout the text.

Thanks to the structure of the potential, we can write $T_D(x, 0; E)$ as

$$\begin{aligned} T_D(x, 0; E) &= T_F(x - a_N - 1; E) T_p(E) T_F(a_N - a_{N-1} - 1; E) \cdots \\ &\cdots \; T_p(E) T_F(a_1; E)\,, \end{aligned}$$

where $a_N^\omega + 1 \le x < a_{N+1}^\omega$ for some $N \in \mathbb{N}$; we write the "perturbed" matrix $T_p(E)$, occurring for $x-1=a_j^\omega \in \mathcal{A}$, $j\in\mathbb{N}$, as

$$T_p(E)=\begin{cases} \begin{pmatrix} \cos\gamma & \eta\sin\gamma \\ -\dfrac{\sin\gamma}{\eta} & \cos\gamma \end{pmatrix}, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D \\ \begin{pmatrix} 1 & \frac{2mc^2+v_1-v_2}{c} \\ 0 & 1 \end{pmatrix}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ \begin{pmatrix} 1 & 0 \\ \frac{2mc^2+v_1-v_2}{c} & 1 \end{pmatrix}, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1= 2mc^2 \\ \begin{pmatrix} \cosh\gamma & \eta\sinh\gamma \\ \dfrac{\sinh\gamma}{\eta} & \cosh\gamma \end{pmatrix}, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D \end{cases}$$

with $\gamma:=\frac{\sqrt{|(E-mc^2-v_1)(E+mc^2-v_2)|}}{c}$, $\eta:=\sqrt{\frac{E+mc^2-v_2}{E-mc^2-v_1}}$,

$$\mathcal{A}_1:=\{E\in\mathbb{R}: E>mc^2+v_1\},\qquad \mathcal{A}_2:=\{E\in\mathbb{R}: E>v_2-mc^2\},$$

$$\mathcal{B}_1:=\{E\in\mathbb{R}: E<mc^2+v_1\},\qquad \mathcal{B}_2:=\{E\in\mathbb{R}: E<v_2-mc^2\},$$

and σ_D given by (3.39). The matrix

$$T_F(x-y;E):=\begin{pmatrix} \cos\kappa(x-y) & \zeta\sin\kappa(x-y) \\ -\dfrac{\sin\kappa(x-y)}{\zeta} & \cos\kappa(x-y) \end{pmatrix}$$

represents the "free" transfer matrix, with $\kappa:=\sqrt{E^2-m^2c^4}/c$, and $\zeta:=\sqrt{\frac{E+mc^2}{E-mc^2}}$.

Let $E=\pm\sqrt{m^2c^4+\kappa^2c^2}$, with $\kappa\in\mathbb{R}$, be a parametrization of the continuous part of the essential spectrum of $H_D(v_1,v_2,\phi)$ (see Theorem 3.23). As in the Schrödinger operator case, the free matrix $T_F(x-y;E)$ is similar to a purely clockwise rotation $R((x-y)\kappa)$, that is, if one considers the non-singular 2×2 matrix

$$U_D:=\sqrt{\frac{1+\zeta^2}{2}}\begin{pmatrix} \dfrac{1}{\zeta} & 0 \\ 0 & 1 \end{pmatrix}$$

one gets $U_D T_F(x-y;E)U_D^{-1}=R((x-y)\kappa)$ (see Equation (2.13)). Thus, relation (2.14) follows in this case with

$$P_D(E)=\begin{cases}\begin{pmatrix}\cos\gamma & \dfrac{\eta}{\zeta}\sin\gamma\\ -\dfrac{\zeta}{\eta}\sin\gamma & \cos\gamma\end{pmatrix}, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D\\ \begin{pmatrix}1 & \frac{2mc^2+v_1-v_2}{c\zeta}\\ 0 & 1\end{pmatrix}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2\\ \begin{pmatrix}1 & 0\\ \frac{(2mc^2+v_1-v_2)}{c}\zeta & 1\end{pmatrix}, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1=2mc^2\\ \begin{pmatrix}\cosh\gamma & \dfrac{\eta}{\zeta}\sinh\gamma\\ \dfrac{\zeta}{\eta}\sinh\gamma & \cosh\gamma\end{pmatrix}, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D\end{cases}$$

Given the sparse nature of the potential, we again adopt the Prüfer type variables given by (2.16) to parametrize the solutions to the Dirac equation (3.40). In particular, these variables satisfy the recurrence relation induced by (2.17) and (2.18), with κ and $P_D(E)$ replacing k and $P(E)$, respectively, where

$$\tilde{\mathbf{v}}_0(\theta_0)=R_0\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}=U_D\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}=\sqrt{\frac{1+\zeta^2}{2}}\begin{pmatrix}\frac{\cos\phi}{\zeta}\\ \sin\phi\end{pmatrix},$$

and

$$R_0^2=\frac{1+\zeta^2}{2\zeta}(\cos^2\phi+\zeta^2\sin^2\phi)\ .$$

If $\mathbf{\Psi}(x)$ represents a solution to (3.40) satisfying the initial conditions $\mathbf{\Psi}(0)=\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$, then

$$\tilde{\mathbf{v}}_n=U_D\mathbf{\Psi}((a_n+1)x)\ .$$

By following the same steps of the previous Schrödinger case, we can express the n-th

Prüfer radius as the Birkhoff sum given by (2.19) (with κ replacing k); but now,

$$a(\kappa) := \begin{cases} 1+\frac{(\zeta^2-\eta^2)^2}{2\zeta^2\eta^2}\sin^2\gamma, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D \\ 1+\frac{(2mc^2+v_1-v_2)^2}{2c^2\zeta^2}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ 1+\frac{(2mc^2+v_1-v_2)^2}{2c^2}\zeta^2, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ 1\,, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1=2mc^2 \\ 1+\frac{(\zeta^2+\eta^2)^2}{2\zeta^2\eta^2}\sinh^2\gamma, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D \end{cases} \tag{3.45}$$

$$b(\kappa) := \begin{cases} \frac{\zeta^4-\eta^4}{2\zeta^2\eta^2}\sin^2\gamma, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D \\ -\frac{(2mc^2+v_1-v_2)^2}{2c^2\zeta^2}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ \frac{(2mc^2+v_1-v_2)^2}{2c^2}\zeta^2, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ 0\,, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1=2mc^2 \\ \frac{\zeta^4-\eta^4}{2\zeta^2\eta^2}\sinh^2\gamma, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D \end{cases} \tag{3.46}$$

and

$$c(\kappa) := \begin{cases} \frac{\eta^2-\zeta^2}{\zeta\eta}\sin\gamma\cos\gamma, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D \\ \frac{2mc^2+v_1-v_2}{c\zeta}, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ \frac{2mc^2+v_1-v_2}{c}\zeta, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ 0\,, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1=2mc^2 \\ \frac{\zeta^2+\eta^2}{\zeta\eta}\sinh\gamma\cosh\gamma, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D \end{cases} \tag{3.47}$$

The Prüfer angles are obtained recursively by (2.21) (k replaced by κ), with $B=C=1$ if $E\in\sigma_D$,

$$A := \begin{cases} -\frac{\zeta}{\eta}\tan\gamma, & E\in[(\mathcal{A}_1\cap\mathcal{A}_2)\cup(\mathcal{B}_1\cap\mathcal{B}_2)]\cap\sigma_D \\ 0, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ \frac{2mc^2+v_1-v_2}{c}\zeta, & E\in\{-mc^2+v_2\}\cap\sigma_D, v_2-v_1\neq 2mc^2 \\ 0, & E\in\{mc^2+v_1\}\cap\sigma_D, v_2-v_1=2mc^2 \\ \dfrac{\zeta}{\eta}\tanh\gamma, & E\in[(\mathcal{B}_1\cap\mathcal{A}_2)\cup(\mathcal{A}_1\cap\mathcal{B}_2)]\cap\sigma_D \end{cases}, \tag{3.48}$$

and

$$D := \begin{cases} \frac{\eta}{\zeta}\tan\gamma, & E \in [(\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}_2)] \cap \sigma_D \\ \frac{2mc^2+v_1-v_2}{c\zeta}, & E \in \{mc^2+v_1\} \cap \sigma_D, v_2 - v_1 \neq 2mc^2 \\ 0, & E \in \{-mc^2+v_2\} \cap \sigma_D, v_2 - v_1 \neq 2mc^2 \\ 0, & E \in \{mc^2+v_1\} \cap \sigma_D, v_2 - v_1 = 2mc^2 \\ \frac{\eta}{\zeta}\tanh\gamma, & E \in [(\mathcal{B}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{B}_2)] \cap \sigma_D \end{cases} . \tag{3.49}$$

Since Theorem 2.8, with its proper adaptations, is applicable to this situation, we may use the ergodic theorem and obtain a Dirac version of Lemma 2.7, that is,

Lemma 3.26 *Let* $(R_n(\theta_0))_{n\geq 1}$ *be the sequence of the Prüfer radii satisfying* (2.19)*, with* a, b, c *given by* (3.45)-(3.47)*, and* $(\theta_n^\omega)_{n\geq 1}$ *the sequence of Prüfer angles satisfying* (2.21)*, with* A, D *given by* (3.48)*,* (3.49)*,* $B = C = 1$*. Then,*

$$C_n^{-1} r_D^n \leq (R_n(\theta_0))^2 \leq C_n r_D^n \ ,$$

for some real number $C_n > 1$ *and* $\lim_{n\to\infty} C_n^{1/n} = (R_0)^2$*, and*

$$r_D(v_1, v_2, E) = \frac{1+a}{2} \ . \tag{3.50}$$

3.3 Spectral measure and subordinacy

Our goal in this subsection is to determine the spectral properties of $H_D(v_1, v_2, \phi)$. For this, we will calculate the limit of $\Im m_D(E + i\varepsilon)$ (see Subsection 2.3 for details) as $\varepsilon \searrow 0$, since, once again,

$$\rho_D(\lambda_2) - \rho_D(\lambda_1) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im m_D(E + i\varepsilon) dE \ , \tag{3.51}$$

where $\rho_D(E)$ represents the spectral function of $H_D(v_1, v_2, \phi)$ (see Chapters 1 and 2 in [21] for a definition of $\rho_D(E)$ and a proof of (3.51)).

The next step is to introduce the concept of subordinate solution to (3.40) and adapt the ideas from [13].

Definition 3.27 A solution Ψ to (3.40) is said to be subordinate if

$$\lim_{l\to\infty} \frac{\|\Psi\|_l}{\|\Phi\|_l} = 0$$

holds for any linearly independent solution Φ to (3.40), where $\|\cdot\|_l$ denotes the $\mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2)$-norm at the length $l \in \mathbb{R}$, i.e.,

$$\|\Psi\|_l^2 := \int_0^l \left(|\psi_1(x)|^2 + |\psi_2(x)|^2\right) dx \ .$$

Introduce, in analogy to (2.28) and equation (4.3) in [1], for any given $\epsilon > 0$, the length $l(\epsilon) \in (0, \infty)$ by the equality

$$\|\mathbf{u}\|_{l(\epsilon)} \|\mathbf{v}\|_{l(\epsilon)} = \frac{c}{2\epsilon} ,$$

where $\mathbf{u}$ and $\mathbf{v}$ are the solutions to (3.40) which satisfy the boundary conditions (3.41).

The Wronskian of two spinors $\Psi, \Phi : \mathbb{R}_+ \to \mathbb{C}$ is defined as $W[\Phi, \Psi](x) = c(\varphi_1(x)\overline{\psi}_2(x) - \varphi_2(x)\overline{\psi}_1(x))$. We have, from Green's identity (see Chapter 1 in [21]), that

$$\int_0^N \left((\overline{\Psi}(x))^t (H_D(v_1, v_2, \phi)\Phi)(x) - (\overline{(H_D(v_1, v_2, \phi)\Psi)}(x))^t \Phi(x) \right) dx$$

$$= W[\Phi, \Psi](N) - W[\Phi, \Psi](0) = 0,$$

i.e, the Wronskian of the solutions $\{\Phi, \Psi\}$ to (3.40) is constant. By the same arguments presented in Subsection 2.3, we conclude that the $l(\epsilon)$ is a well-defined monotone decreasing and continuous function of ϵ, which diverges as $\epsilon \to 0$.

Combining the variation of parameters formula

$$\begin{aligned} \xi(x, z) &= -\mathbf{u}^N(x, E) + m_D(z)\mathbf{u}^D(x, E) - \frac{i\epsilon}{c}\mathbf{u}^N(x, E) \int_0^x (\mathbf{u}^D(t, E))^t \xi(t, z) dt \\ &+ \frac{i\epsilon}{c}\mathbf{u}^D(x, E) \int_0^x (\mathbf{u}^N(t, E))^t \xi(t, z) dt \end{aligned}$$

(see Chapter 3 in [4] and Lemma 4.4 in [1]) with the identity

$$\Im m_D(z) = \epsilon \int_0^\infty |\xi(x, z)|^2 dx ,$$

we obtain the Jitomirskaya-Last inequalities stated in Theorem 2.13. As a direct consequence, Theorem 1.2 in [13] (see equations (2.29), (2.30)) and its corollaries also hold true here; in particular, we have the following analogues of Corollaries 2.15 and 2.16:

Corollary 3.28 *(a) Suppose that for some $\alpha \in [0, 1)$ and every E in some Borel set $A \subset \mathbb{R}$,*

$$\limsup_{l \to \infty} \frac{1}{l^{2-\alpha}} \int_0^l \|T_D(x, 0; E)\|^2 dx < \infty . \tag{3.52}$$

Then the restriction $\rho(A \cap \cdot)$ is α-continuous.

(b) Suppose that

$$\liminf_{l \to \infty} \frac{\|\mathbf{u}(E)\|_l^2}{l^\alpha} = 0$$

holds for every E in some Borel set F. Then the restriction $\rho(F \cap \cdot)$ is α-singular.

Proof. We will only present some details of the proof of *(a)*; item *(b)* can be proven directly by combining the ideas discussed in [1, 13]. By choosing $\theta_1 = \arctan(\zeta \cot \phi)$ and $\theta_2 = -\arctan(\zeta \tan \phi)$, it follows by Theorem 2.1 in [14] that there exists a constant C such that

$$\|T_D(x,0;E)\| \geq C \max\left\{R_n(\theta_1), R_n(\theta_2)\right\} \ ,$$

for all $a_n^\omega \leq x < a_{n+1}^\omega$, where $R_n(\theta)$ is the n-th Prüfer radius starting from the initial condition $\Psi = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$; explicitly $C = \max(\zeta, 1/\zeta)$. Since

$$R_n^2(\theta_{1(2)}) = \frac{1+\zeta^2}{2} \left(\frac{\left|u_1^{1(2)}(x)\right|^2}{\zeta^2} + \left|u_1^{1(2)}(x)\right|^2 \right) \ ,$$

we obtain the inequality

$$\frac{1+\zeta^2}{2} \min(1, 1/\zeta^2) \left[\left|u_1^{1(2)}(x)\right|^2 + \left|u_1^{1(2)}(x)\right|^2 \right] \leq R_n^2(\theta_{1(2)}) \ .$$

Thus, from the considerations above,

$$\int_0^l \|T_D(x,0;E)\|^2 \, dx \geq D \int_0^l \max\left\{ \|\mathbf{u}_1(E)\|_l^2 \, , \|\mathbf{u}_2(E)\|_l^2 \right\} \ , \tag{3.53}$$

where $D = \frac{1+\zeta^2}{2\zeta}$. Hypothesis (3.52), together with (3.53), imply item *(a)* of Corollary 3.28. □

3.4 Hausdorff dimension and spectral transition

Since Proposition 2.18 can be readily adapted to our Dirac operators, we can state the following

Theorem 3.29 *Let ρ_D be the spectral measure of $H_D(v_1, v_2, \phi)$. Given a closed interval of energies*

$$L \subset I = \left(-\infty, -mc^2\right] \cup \left[mc^2, +\infty\right) \ , \tag{3.54}$$

for almost every $\phi \in [0, \pi]$ and almost every $\omega \in \Xi$, the spectral measure ρ_D restricted to L has Hausdorff dimension given by (2.34)*, with ρ_D in substitution to ρ and $r = r(v_1, v_2, E)$ as in* (3.50)*.*

Proof. The proof of Theorem 3.29 has the same structure of the proof of Theorem 2.19, with some minor adjustments. □

We may compare the spectral properties of $H_D(v, v, \phi)$ with its discrete counterpart, studied in [1]. Let $H_D^c(v, \phi)$ represent the continuous Dirac operator in Definition 1.1 (with

$v_1 = v_2$) and define $H_D^d(v,\phi)$ as the discrete Dirac operator (see [8, 9])

$$(H_D^d(v,\phi)\Psi)_n = \begin{pmatrix} (mc^2+V_n)\psi_{1,n} + c(\psi_{2,n}-\psi_{2,n-1}) \\ c(\psi_{1,n+1}-\psi_{1,n}) + (-mc^2+V_n)\psi_{2,n} \end{pmatrix} \tag{3.55}$$

acting on $\Psi \in l^2(\mathbb{Z}_+,\mathbb{C}^2)$, with the sequence (V_n) defined by (2.38) and which satisfies the boundary condition

$$\psi_{2,-1}\cos\phi - \psi_{1,0}\sin\phi = 0\,,$$

$\phi \in [0,\pi]$. We have opted to compare the operator $H_D^d(v,\phi)$ with $H_D^c(v,\phi)$, since [1] gives us an analysis of the spectral properties of $H_D^d(v,\phi)$.

Let Σ_{ess} denote the essential spectrum of both operators, $\beta > 1$ the sparsity parameter and $r(E)$ the asymptotic behavior of the norm $\|T_D(x,0;E)\|$, which is given by (3.50) for $H_D^c(v,\phi)$ and

$$r(E) = 1 + \frac{1}{(m^2c^4+4c^2-E^2)}\frac{v^2}{c^2}\left[\frac{(E^2-m^2c^4)^2+4m^2c^6}{(E^2-m^2c^4)} - 4vE + 2v^2\right]$$

for $H_D^d(v,\phi)$ (see [1]). In both cases, we have

Theorem 3.30 *Write*

$$J_{c(d)} := \left\{E \in \Sigma_{\text{ess}}^{c(d)} \setminus B : r < \beta\right\}$$

with B some set of Lebesgue zero measure. Then, for ν-almost every $\omega \in \Xi$:

(a) the spectrum of $H_D^{c(d)}(v,\phi)$ restricted to the set $J_{c(d)}\setminus B$ is purely singular continuous;

(b) the spectrum of $H_D^{c(d)}(v,\phi)$ is purely point when restricted to $\Sigma_{\text{ess}} \setminus J_{c(d)}$ for almost every $\phi \in [0,\pi]$.

Proof. For the operator $H_D^d(v,\phi)$, see Theorem 1.5 in [1]. For $H_D^c(v,\phi)$, the proof follows the same steps, regarding some minor details. □

As a first remark, a direct consequence of Theorem 3.30 is the absence of absolutely continuous spectrum in both operators (the considerations in Remark 2.20 also applies to $H_D^c(v,\phi)$).

A second point is the location of the dense point and the singular continuous spectra. For $H_D^d(v,\phi)$, we see from the expression of $r(E)$ that the point part is located at the boundaries of Σ, while the singular continuous spectrum is located at the center of this interval; this may not occur for $H_D^c(v,\phi)$, since, given its oscillatory behavior for $E > v$, $h_{\rho_D}(E)$ may vary from zero to one if $1+(m^2c^4v^2)/(E^2-m^2c^4)[(E-v)^2-m^2c^4] > \beta$ and if E ranges in intervals such as $\left[v+\sqrt{(n+1/2)^2\pi^2c^2+m^2c^4}, v+\sqrt{(n+3/2)^2\pi^2c^2+m^2c^4}\right]$ for some integer n. Hence, in this situation, we may have several transition points, providing intervals of dense purely point spectrum intertwined with intervals of singular continuous spectrum.

Note also that in the $\lim_{E\to\pm\infty} h_{\rho_D}(E) = 1$ (i.e., for large absolute values of energy), the effects of the sparse perturbation are attenuated. This also happens with $H_S^c(v,\phi)$, as previously discussed.

Excluding the existence of spectrum for negative values of energy (representing the possible states of antiparticles), surprisingly there are no major spectral differences between the relativistic and non-relativistic operators, as one might expect.

4 Lower bounds of the transport exponents

In this section we employ some results of [11] to get some lower bounds of transport exponents of the continuous operators $H_S(v,\phi)$ and $H_D(v_1,v_2,\phi)$, and compare the results with those related to their discrete counterparts.

We begin by recalling the definition of the averaged moments of order $p > 0$ of the position operator $(\langle X\rangle\varphi) = \langle x\rangle\varphi(x)$, $\langle x\rangle = \sqrt{1+x^2}$ ($\varphi \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}_+)$ in the Schrödinger case, whereas $\varphi \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}_+,\mathbb{C}^2)$ in the Dirac case)

$$\mathbb{M}(p,f,T) = \frac{2}{T}\int_0^\infty e^{-2t/T}\|\langle X\rangle^{p/2}e^{-itH}f(H)\psi_0\|_{\mathcal{H}}^2 dt\ ,$$

associated with the initial state ψ_0 localized at the origin and with energy "localized" in a compact interval $J = [a,b]$, $a < b$, at time T through some positive $f \in \mathrm{C}_0^\infty(J)$.

The presence of transport will be probed by lower bounds for the lower growth exponent, defined by

$$\beta^-(p,f) := \liminf_{T\to\infty}\frac{\ln\mathbb{M}(p,f,T)}{p\ln T}\ ;$$

in order to obtain transport rates nearby a given energy level, we follow [11, 24] and introduce the local lower transport exponent as

$$\beta^-(p,E) := \inf_{J\ni E}\ \sup\{\beta^-(p,f) : 0 \le f \in \mathrm{C}_0^\infty(J)\}\ .$$

Introduce the measurable function $\gamma(E) : \mathbb{R} \to \mathbb{R}_+$ by

$$\gamma(E) = \limsup_{x\to\infty}\frac{\ln\|T(x,0;E)\|}{\ln x}\ .$$

Given a Borel set $S \subset \mathbb{R}$ with $|S| > 0$ ($|\cdot|$ denotes the Lebesgue measure) and $g : S \to \mathbb{R}$ a measurable function, define g^S as the unique real number such that, simultaneously: a) $g(E) \ge g^S$ for almost every E with respect to the Lebesgue measure; b) for all $r > 0$, there exists $S \subset S_r$, $|S_r| > 0$, such that for all $E \in S_r$, one has $g(E) \le g^S + r$.

Since the norms of the transfer matrices related to the solutions to Schrödinger and Dirac equations (2.8) and (3.40), respectively, are polynomially bounded, we can use Theorem 2.2 in [11] and the continuous counterpart of Theorem 2 in [24] to obtain

Theorem 4.1 *Consider $H_S(v,\phi)$ and $H_D(v_1,v_2,\phi)$ with*

$$\psi_0 = \begin{cases} \chi_{[0,1]} & \text{if} \quad \psi_0 \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}_+) \\ \begin{pmatrix} \chi_{[0,1]} \\ 0 \end{pmatrix} & \text{if} \quad \psi_0 \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}_+, \mathbb{C}^2) \end{cases} . \tag{4.1}$$

Then, for almost every $\omega \in \Xi$ and every sparsity parameter $\beta > 1$, the following properties hold true:

1. *For any $0 \le f \in \mathrm{C}_0^\infty(J)$, with $J \subset (0,+\infty)$, and any $s > 0$, there exists a finite constant $C_S(p,J,s) > 0$ such that, for all sufficiently large T,*

   $$\mathbb{M}_S(p,f,T) \ge C_S(p,J,s) T^{p-2\gamma^J-s} ,$$

   for $p > 2\gamma^J + s$ with $\gamma^J = \inf_{E\in J}\{\ln r(v,E)\}/(2\ln\beta)$, $r(v,E)$ given by (2.23)*. As a consequence, for any $E \in \mathbb{R}_+$,*

   $$\beta_S^-(p,E) \ge 1 - \frac{\ln r(v,E)}{p \ln \beta} .$$

2. *For any $0 \le f \in \mathrm{C}_0^\infty(J)$, and $J \subset (-\infty, -mc^2) \cup (mc^2, +\infty)$, for any $s > 0$, there exists a finite constant $C_D(p,J,s) > 0$ such that, for all sufficiently large T,*

   $$\mathbb{M}_D(p,f,T) \ge C_D(p,J,s) T^{p-2\gamma^J-s} ,$$

   for $p > 2\gamma^J + s$ with $\gamma^J = \inf_{E\in J}\{\ln r_D(v_1,v_2,E)\}/(2\ln\beta)$, $r_D(v_1,v_2,E)$ given by (3.50)*. As a consequence, for any $E \in (-\infty,-mc^2) \cup (mc^2,+\infty)$,*

   $$\beta_D^-(p,E) \ge 1 - \frac{\ln r_D(v_1,v_2,E)}{p\ln\beta} .$$

Proof. Theorem 4.1 is a direct consequence of Theorem 2.2 in [11], Theorem 2 in [24], and

$$\gamma(E) = \limsup_{x\to\infty} \frac{\ln \|T(x,0;E)\|}{\ln x} = \frac{\ln r(E)}{2\ln\beta} , \tag{4.2}$$

where $r(E)$ satisfies (2.23) for the Schrödinger operator, (3.50) for the Dirac operator; note that $\gamma(E)$ is a continuous function in both cases. □

Remark 4.2 The adaptation of Theorem 2 in [24] to the continuous Dirac operator H_D is straightforward and, therefore, will be omitted.

A first conclusion taken from Theorem 4.1 is the asymptotic ballistic transport at large values of energy in both cases, since we obtain $\lim_{E\to\infty}\gamma(E) = 0$ ($\lim_{E\to\infty} r(E) = 1$) for the Schrödinger operator $H_S(v,\phi)$, and $\lim_{E\to\pm\infty}\gamma(E) = 0$ ($\lim_{E\to\pm\infty} r_D(E) = 1$) for $H_D(v_1,v_2,\phi)$, independently of the sparsity parameter $\beta > 1$. Since the Hausdorff dimension of the spectral measures of both operators, as proved in the last section, converges to 1 in this asymptotic limit, the general inequality

$$\beta^-(p,f) \geq h_\rho \tag{4.3}$$

(see [5] for a discussion of this problem) is, in fact, sharp and is independent of p.

Another important feature of these operators is the existence of "critical energies"; by critical energy we refer to a specific value of energy for which the solutions to (2.8) and (3.40) are bounded. Note that this definition is slightly different from the one given in [7].

If we denote the set of critical energies of $H_S(v,\phi)$ by $\mathcal{E}_S$ and the set of critical energies of $H_D(v_1,v_2,\phi)$ by $\mathcal{E}_D$, we see from (2.35) that $\mathcal{E}_S$ and $\mathcal{E}_D$ coincide, respectively, with the sets of energies where $r(v,E)$ and $r_D(v_1,v_2,E)$ are equal to one; thus, according to (2.23) and (3.50),

$$\begin{aligned}
\mathcal{E}_S &= \{E \in \mathbb{R}_+ : E = v + (n+1/2)^2\pi^2, n \in \mathbb{Z}\},\\
\mathcal{E}_D &= \{E \in (-\infty,-mc^2] \cup [mc^2,+\infty) : \gamma(E) = (n+1/2)\pi, n \in \mathbb{Z}\},
\end{aligned}$$

$\gamma(E) = \sqrt{|(E-mc^2-v_1)(E+mc^2-v_2)|}/c$.

Let E_S^c and E_D^c denote, respectively, points of $\mathcal{E}_S$ and $\mathcal{E}_D$. By Theorem 4.1, we obtain $\beta_S^-(p,E_S^c) \geq 1$, $\beta_D^-(p,E_D^c) \geq 1$, and consequently, existence of ballistic transport, although the spectrum is singular continuous at these points. A similar phenomenon was observed in [7] for the Bernoulli-Anderson model: singular spectra and super-diffusive transport.

This is perhaps the major difference between the discrete and continuous operators, and the reason is simple: the absence of critical energies in the discrete cases, due to the nature of the solutions to the discrete versions of (2.8) and (3.40).

At the critical points, the inequality expressed by (4.3) turns into an equality to one; thus, at least at these points, the Hausdorff and packing dimensions of the spectral measures coincide (see Section 1 in [5] for some statements regarding this issue).

Remark 4.3 We could have adapted the results in [7] in order to obtain a different lower bound of the dynamical exponents $\beta^-_{S(D)}(p,E)$. In fact, it follows by an adapted version of its Corollary 2.1 that

$$\beta^-_{S(D)}(p,E) \geq \frac{1-(1+2\gamma_{S(D)}(E))/p}{1+\gamma_{S(D)}},$$

with $\gamma_{S(D)}(E)$ satisfying (4.2). Despite the rather crude bound given above, the method developed in [7] is particularly efficient when the eigenfunctions are polynomially bounded only at single points of the spectrum: this suffices to guarantee nontrivial dynamical lower bounds, in contrast to Theorem 4.1, which demands a set of positive Lebesgue measure where the eigenfunctions are bounded.

A Proof of Proposition 2.4

Proof. We only present is a proof for the operator $H'_S(0,0)$, since the general case follows from the well-known stability of the absolutely continuous spectrum with respect to rank-one perturbations.

In order to prove Proposition 2.4, we will establish the exact boundary behavior of the Weyl-Titchmarsh $m(E+i\varepsilon)$ function, defined by (2.24), as $\varepsilon \downarrow 0$. We obtain from (2.24) and (2.9) that

$$m(E+i0) = \frac{\chi'(0,E)}{\chi(0,E)} \; ; \tag{A.1}$$

since, for $x \geq 1$, the potential is null, the $\mathrm{L}^2(\mathbb{R}_+,\mathbb{C})$ solution $\chi(x,E)$ to (2.8) is uniquely defined (up to a multiplicative constant $c \neq 0$) as

$$\chi(x,E) = \begin{cases} ce^{ikx}, & E \in \mathbb{R}_+ \\ ce^{-kx}, & E \in (-\infty,0) \end{cases} , \tag{A.2}$$

where $k^2 := |E|$.

Thus, it follows from the definition of transfer matrix (2.10) that

$$\begin{pmatrix} \chi(0,z) \\ \chi'(0,z) \end{pmatrix} = T^{-1}(1,0;E) \begin{pmatrix} \chi(1,z) \\ \chi'(1,z) \end{pmatrix} ,$$

which, combined with (2.2), (A.1), (A.2) and the considerations present in the proof of Proposition 3.39, lead to

$$\Im m(E) \;=\; \begin{cases} \dfrac{k\alpha^2}{\alpha^2\cos^2\alpha + k^2\sin^2\alpha}, & E > v \\ \dfrac{k}{1+k^2}, & E = v \\ \dfrac{k\alpha^2}{\alpha^2\cosh^2\alpha + k^2\sinh^2\alpha}, & E > v \end{cases}$$

if $E \geq 0$, and $\Im m(E) = 0$ if $E < 0$. This concludes the proof of the Proposition. $\square$

Remark A.4 It is worth noting that differently from the analogous discrete operator, $H'_S(0,\phi)$ cannot be regarded as a rank-one, or more generally as a compact perturbation of the free operator $H_S(0,\phi)$.

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