Semilinear elliptic problems with asymmetric nonlinearities

Francisco Odair de Paiva and Adilson E. Presoto
Departamento de Matemática – Universidade Federal de São Carlos
13565-905 - São Carlos, SP, Brazil
Email address: odair@dm.ufscar.br and presoto@dm.ufscar.br

Abstract
In this paper we are concerned on the semilinear elliptic problem
\[
\begin{aligned}
-\Delta u &= -\lambda |u|^{q-2} u + au + b(u^+)^{p-1} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \Omega \subseteq \mathbb{R}^N \) is a bounded domain with regular boundary \( \partial \Omega \), \( 1 < q < 2 < p \leq 2^* \). If \( a \) is between two eigenvalues, we get the existence of three nontrivial solutions for the problem above.

Key words. positive solution, indefinite sublinear nonlinearity, concave-convex nonlinearity, critical growth.

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1 Introduction
We consider the semilinear elliptic problem
\[
(P) \quad \begin{cases}
-\Delta u = -\lambda |u|^{q-2} u + au + b(u^+)^{p-1} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \subseteq \mathbb{R}^N \) is a bounded domain with regular boundary \( \partial \Omega \), \( N \geq 3 \), \( 1 < q < 2 < p \leq 2^* \), \( a \in \mathbb{R} \), \( b > 0 \), \( \lambda \) is a positive parameter and \( u^+ = \max \{u, 0\} \).

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†Corresponding author
The weak solutions of the problem \((P)\) correspond to critical points of the \(C^1\) functional \(I_\lambda\), defined on \(H_0^1 := H_0^1(\Omega)\) by

\[
I_\lambda(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{\lambda}{q} \int |u|^q - \frac{a}{2} \int u^2 - \frac{b}{p} \int (u^+)^p, \quad u \in H_0^1.
\]

After the appearance of [1], there has been an increasing concern about multiple solutions of semilinear elliptic problem of the type:

\[
-\Delta u = \mu |u|^{q-2}u + g(u) \quad \text{in} \quad \Omega.
\]

When \(g\) is asymmetric and asymptotically linear this problem was considered in [8, 10, 13, 20]. Here asymmetric means that \(g\) satisfies an Ambrosetti-Prodi type condition (i.e. \(g_- := \lim_{t \to -\infty} g(t)/t < \lambda_k < g_+ := \lim_{t \to +\infty} g(t)/t\)). When \(g\) is asymmetric and superlinear at \(+\infty\), \(g_+ = \infty\), this problem was approached in [8, 13, 17]. In [8] a Neumann problem was considered and in [17] the authors studied a problem involving the \(p\)-Laplace operator. In [13], one was assumed that \(g(t)/t\) crosses an eigenvalue of the Laplacian when the \(t\) varies from 0 to \(-\infty\) (i.e. \(g'(0) < \lambda_k < g_-\)). Similar hypotheses also appears in [20]. Assumptions involving the first eigenvalue, as \(g'(0), g_- \leq \lambda_1\), were considered in [8, 10, 17]. It is known that crossing eigenvalues, in particular the first one, is related to existence and multiplicity for such problems. Notice that the nonlinearity \(g(t) = at + b(t^+)^{p-1}\), with \(a > \lambda_1\), is not included in the cases count on the previous works. Moreover, similar problems with \(\mu = 0\) were studied in [16] for Dirichlet problems, and in [2, 19] for Neumann problems.

Our problem is also closely related to the class of superlinear Ambrosetti-Prodi problem:

\[
-\Delta u = au + (u^+)p + f(x) \quad \text{in} \quad \Omega,
\]

with \(f \in L^2\). For instance, this problem have a solution if \(\|f\|_{L^2}\) is small enough (see [12]). Further results and references for the above problem can be found in [5, 6, 11, 18, 21, 22].

For the critical case, our main motivation to \((P)\) is the Brezis-Nirenberg pioneering work [4], where the following critical problem was considered

\[
\begin{align*}
-\Delta u &= au + |u|^{2^*-2}u \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \(a < \lambda_1\). They noticed that the problem had a breaking of compactness at the value \(\frac{SN/2}{N}\), so that they constructed minimax levels for the energy functional associated below this value. Such ideas have been permeating many later works as well as ours. One of them, it was the Capozzi, Fortunato and Palmieri work [7]. They basically studied the problem above with \(a\) between two eigenvalues. They
showed that the problem above has a nontrivial solution for all $a > 0$ when $N \geq 5$ and for $a$ different from eigenvalues when $N = 4$.

We are denoting by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ the eigenvalues of $(-\Delta, H^1_0(\Omega))$ and by $\varphi_j$ the corresponding eigenfunctions. The $H^1_0(\Omega)$ norm and $L^p(\Omega)$ norm are represented by $\| \cdot \|$ and $| \cdot |_p$ and we denote these spaces by $H^1_0$ and $L^p$, for simplicity, respectively.

In the sequel, we set up precisely the results obtained

**Theorem 1.** Let $N \geq 3$ and $\lambda_k < a < \lambda_{k+1}$. If $2 < p < 2^*$, then, for $\lambda$ small enough, $(P)$ has at least three nontrivial solutions.

**Theorem 2.** Let $N \geq 4$ and $\lambda_k < a < \lambda_{k+1}$. If $p = 2^*$, for $\lambda$ small enough, $(P)$ has at least three nontrivial solutions.

The major arguments of the proofs of our theorems are based on variational methods. As it is well-known, we have to show some geometric conditions and prove a compactness condition. Provided us with these tools, we obtain a negative and a positive solution and the third one comes from linking theorem. In order to do that, we follow some tricks used in [6, 14]. In the next section, we show the (PS) condition for the energy functional. In the third section, we present the proofs of theorems above.

## 2 The (PS) condition

We begin by showing the (PS) condition for $I_\lambda$.

**Lemma 1.** Let $\lambda_1 < a$, $2 < p \leq 2^*$ and $\lambda > 0$. Then every (PS) sequence of $I_\lambda$ is bounded.

**Proof.** Let $(u_n)$ be a (PS) sequence for $I_\lambda$, i.e., it satisfies

\[
\left| \frac{1}{2} \int |\nabla u_n| + \frac{\lambda}{q} \int |u_n|^q - \frac{\lambda}{2} \int u_n^2 - \frac{b}{p} \int (u_n^+)^p \right| \leq C, \tag{4}
\]

\[
\left| \int \nabla u_n \nabla h + \lambda \int |u_n|^{q-2} u_n h - a \int u_n h - b \int (u_n^+)^{p-1} h \right| \leq \epsilon_n \|h\|, \quad \forall h \in H^1_0, \tag{5}
\]

where $\epsilon_n \to 0$ as $n \to \infty$. By (4) and (5) we have

\[
C + \epsilon_n \|u_n\| \geq \left| I_\lambda(u_n) - \frac{1}{2} \langle I'_\lambda(u_n), u_n \rangle \right| = \left| \left( \frac{\lambda}{q} - \frac{\lambda}{2} \right) \int |u_n|^q + \left( \frac{b}{2} - \frac{b}{p} \right) \int (u_n^+)^p \right| \geq \left( \frac{b}{2} - \frac{b}{p} \right) \int (u_n^+)^p. \]

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Since \( p > 2 \) we get

\[
\int (u_n^+)^p \leq C + \epsilon_n \|u_n\|.
\]

We also have by (5),

\[
\langle I'_\lambda(u_n), u_n^- \rangle = \left\| u_n \right\|^q + \lambda |u_n^-|^q - a |u_n^-|^2 \leq C + \epsilon_n \|u_n^-\|,
\]

with \( u^- = \max \{-u, 0\} \). It follows from (4), (6) and (7) that

\[
\frac{1}{2} \left\| u_n^+ \right\|^2 \leq \left( \frac{\lambda}{2} - \frac{\lambda}{q} \right) \int |u_n|^q + \frac{a}{2} \int (u_n^+)^2 + \frac{b}{p} \int (u_n^+)^p + \frac{1}{2} \langle I'_\lambda(u_n), u_n^- \rangle + C
\]

\[
\leq C \int (u_n^+)^p + \epsilon_n \|u_n^-\| + C \leq \epsilon_n \|u_n\| + \epsilon_n \|u_n^-\| + C.
\]

Suppose by contradiction that \( \|u_n\| \to \infty \). We first show that \((u_n^+)^2 \) is bounded in \( H^1_0 \), so that assume also that \( \|u_n^+\| \to \infty \). By (8), \((u_n^-)^2 \) is also unbounded. Let \( v_n = u_n/\|u_n\| \). Since \((v_n)\) is bounded in \( H^1_0 \), there exists \( v \in H^1_0 \) such that

\[
v_n \rightharpoonup v \quad \text{in} \quad H^1_0, \quad v_n \to v \quad \text{in} \quad L^r, \quad \forall \ 1 \leq r < 2^* \quad \text{and} \quad v_n \to v \quad \text{a.e. in} \ \Omega.
\]

Again by (8) there exists \( \delta > 0 \) satisfying

\[
\|v_n^-\| \geq \delta \|u_n^+\|^2
\]

whenever \( n \) is large. Since

\[
v_n^+ = \frac{u_n^+}{\|u_n\|} = \frac{u_n^+}{\left( \|u_n^+\|^2 + \|u_n^-\|^2 \right)^{1/2}} \leq \frac{u_n^+}{\left( \|u_n^+\|^2 + \delta^2 \|u_n^+\|^4 \right)^{1/2}},
\]

we deduce that \( v \leq 0 \). Moreover, by

\[
v_n^- = \frac{u_n^-}{\|u_n\|} = \frac{u_n^-}{\left( \|u_n^+\|^2 + \|u_n^-\|^2 \right)^{1/2}} = \frac{u_n^-}{\|u_n^+\|} \cdot \left( \|u_n^+\|^2 + \|u_n^-\|^2 \right)^{1/2},
\]

and (9), we have \( \|v_n^-\| \to 1 \). Thus, by (7),

\[
-\lambda \frac{|u_n^-|^q}{\|u_n\|^2} + a \frac{|u_n^-|^2}{\|u_n^-\|^2} \to 1.
\]

We also note that by (9) and \( \|v_n^-\| \to 1 \) in \( H^1_0 \),

\[
\frac{u_n^-}{\|u_n\|} - \frac{u_n^-}{\|u_n^-\|} = \frac{u_n^-}{\|u_n\|} \left( \frac{\|u_n\|}{\|u_n^-\|} - 1 \right) \to 0 \quad \text{in} \quad H^1_0.
\]
Hence we may exchange $\|u_n^-\|$ for $\|u_n\|$ in (10). Recalling that $q < 2$, we obtain $|v_n|_2 \to 1/\sqrt{a}$, then $v \neq 0$. We then take $h = \varphi_1$ in (5) to obtain
\[
\int \nabla v_n \nabla \varphi_1 + \frac{\lambda}{\|u_n\|} \int |u_n|^{q-2} u_n \varphi_1 - a \int v_n \varphi_1 - \frac{b}{\|u_n\|} \int (u_n^+)^{p-1} \varphi_1 \to 0,
\]
that is
\[
(\lambda_1 - a) \int v \varphi_1 = 0,
\]
which is a contradiction, because $v \leq 0$, $v \neq 0$ and $\lambda_1 < a$, so that $(u_n^+)$ is bounded.

Finally, suppose that $\|u_n\| \to \infty$ and $\|u_n^+\| \leq C$ for all $n \in \mathbb{N}$. Since $p \leq 2^*$,
\[
\frac{1}{\|u_n\|} \int_\Omega (u_n^+)^p \to 0.
\]
On the other hand, by taking $h = v_n$ in (5) we obtain
\[
a|v_n|_2^2 \to 1,
\]
so that $v_n \to v$ in $L^2$ with $v \neq 0$. Then by (5) we get
\[
\int_\Omega \nabla v \nabla h - a \int v h = 0 \quad \text{for all } h \in H_0^1,
\]
with $v \neq 0$ and $v \leq 0$, which is a contradiction, because $a$ is not the first eigenvalue. Therefore, we conclude that $(u_n)$ must be bounded in $H_0^1$. \(\square\)

In the subcritical case, $1 \leq p < 2^*$, it is well-known that the lemma above implies that $I_\lambda$ satisfies the (PS) condition at every level.

**Lemma 2.** Let $\lambda_1 < a$ and $p = 2^*$. For every $\lambda > 0$, $I_\lambda$ satisfies the (PS) condition at level $c$ with $c < \frac{b^{2-N} N S_0}{N}$. 

**Proof.** Let $(u_n) \subset H_0^1$ be a sequence satisfying
\[
I_\lambda(u_n) \to c \quad \text{and} \quad |\langle I'_\lambda(u_n), h \rangle| \leq \epsilon_n \|h\|, \forall h \in H_0^1,
\]
with $\epsilon_n \to 0$ as $n \to \infty$. By Lemma 1 we have that $(u_n)$ is bounded. Hence, by passing to a subsequence, we may suppose that
\[
(11) \quad u_n \to u \quad \text{in } H_0^1, \quad u_n \to u \quad \text{in } L^2, \quad u_n \to u \quad \text{in } L^q, \quad u_n \to u \quad \text{a.e. in } \Omega.
\]
Since \((u_n^+)\) is bounded in \(H^1_0\), from Gagliardo-Nirenberg Inequality it follows that \((u_n^+)\) is also bounded in \(L^{2^*}\). By passing to a subsequence again, we have \(u_n^+ \rightharpoonup u^+\) in \(L^{2^*}\). Thus, \(u\) solves

\[
\begin{aligned}
-\Delta u &= -\lambda |u|^{q-2}u + au + b(u^+)^{2^*-1} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Note that by (13) we obtain

\[
I_\lambda(u) = \left( \frac{\lambda}{q} - \frac{\lambda}{2} \right) |u|^q_q + \left( \frac{b}{2^*} - \frac{b}{2^*} \right) |u^+|_{2^*}^{2^*} \geq 0.
\]

We denote \(v_n = u_n - u\). By the Brezis-Lieb’s Lemma

\[
\lim_{n \to \infty} (|u_n^+|^p_p - |v_n^+|^p_p) = |u^+|^p_p \quad \text{for all } 1 \leq p \leq 2^*.
\]

Moreover, by (12) we have \(v_n \to 0\) in \(L^q\) and \(L^2\), so that

\[
\lim_{n \to \infty} [I_\lambda(u) + I_\lambda(v_n)] = \lim_{n \to \infty} \left[ \|u\|^2 - \int_\Omega \nabla u_n \nabla u + \frac{1}{2} \|u_n\|^2 + \frac{\lambda}{q} (|u|^q_q + |v_n|^q_q) \right.
\]

\[
- \frac{a}{2} (|u_n|^2 + |v_n|^2) - \frac{b}{2^*} (|u^+|^{2^*} + |v_n^+|^{2^*}) \left. \right] = \lim_{n \to \infty} \left[ \frac{1}{2} \|u_n\|^2 + \frac{\lambda}{q} |u_n|^q_q - \frac{a}{2} |u_n|^2 - \frac{b}{2^*} |u_n^+|^{2^*} \right] = c.
\]

On the other hand, by (11) and again Brezis-Lieb’s Lemma,

\[
\lim_{n \to \infty} \left[ \|v_n\|^2 + \lambda |v_n|^q_q - a |v_n|^2 - b |v_n^+|^{2^*} \right] = \lim_{n \to \infty} \left[ \langle I'_\lambda(u_n), u_n \rangle - 2 \int_\Omega \nabla u_n \nabla u + 2 \|u\|^2 - \langle I'_\lambda(u), u \rangle \right] = 0.
\]

Since \(v_n \to 0\) in \(L^q\) and \(L^2\), we may suppose that

\[
\|v_n\|^2 \to d \quad \text{and} \quad b |v_n^+|_{2^*}^{2^*} \to d.
\]

By Sobolev’s Inequality, \(\|v_n\|^2 \geq S |v_n^+|_{2^*}^{2^*}\), consequently, \(d \geq S (d/b)^{2/2^*}\). If \(d = 0\) the proof is concluded. Otherwise, \(d \geq S^{N/2} b^{2-N/2}\). Then by (14), (15) and (16) we conclude

\[
\frac{S^{N/2} b^{2-N/2}}{N} \leq \left( \frac{1}{2} - \frac{1}{2^*} \right) d \leq c < \frac{b^{2-N} S^{N/2}}{N},
\]

which is a contradiction. \(\square\)
3 Proof of Theorems

3.1 Existence of the nonnegative solution

Consider the functional $I^+_\lambda : H^1_0 \rightarrow \mathbb{R}$ given by

\begin{equation}
I^+_\lambda (u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\lambda}{q} \int_{\Omega} (u^+)^q - \frac{1}{2} \int_{\Omega} (u^+)^2 - \frac{1}{p} \int_{\Omega} (u^+)^p, \ u \in H^1_0.
\end{equation}

It follows that $I^+_\lambda \in C^1$ and the critical points $u_+$ of $I^+_\lambda$ satisfy $u_+ \geq 0$ and so are critical points of $I_\lambda$ as well, actually, $(I^+_\lambda)'(u_+)[(u_+)^-] = -\int_{\Omega} |\nabla (u_+)^-|^2 = 0$.

We will show that $I^+_\lambda$ satisfies the assumptions of the mountain pass theorem. In a similar argument to proofs of Lemmas 1 and 2, we show the (PS) condition for $I^+_\lambda$.

**Lemma 3.** Let $2 < p \leq 2^*$. For all $\lambda > 0$, $I^+_\lambda$ satisfies the (PS) condition at level $c$ with $c < \frac{\lambda^2-N}{N} \frac{S}{N}$.

**Lemma 4.** The trivial solution $u \equiv 0$ is a local minimizer for $I^+_\lambda$, for all $\lambda > 0$.

**Proof.** It suffices to show that $0$ is a local minimizer of $I^+_\lambda$ in the $C^1$ topology (see [3]). Then, for $u \in C^1_0(\Omega)$ we have

\[
I^+_\lambda (u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\lambda}{q} \int_{\Omega} (u^+)^q - \frac{a}{2} \int_{\Omega} (u^+)^2 - \frac{b}{p} \int_{\Omega} (u^+)^p \\
\geq \frac{\lambda}{q} \int_{\Omega} |u|^q - \frac{a}{2} \int_{\Omega} u^2 - \frac{b}{p} \int_{\Omega} |u|^p \\
\geq \left( \frac{\lambda}{q} - \frac{a}{2} |u|_{C^0}^{2-q} - \frac{b}{p} |u|_{C^0}^{p-q} \right) \int_{\Omega} |u|^q \geq 0
\]

whenever $\frac{a}{2} |u|_{C^0}^{2-q} + \frac{b}{p} |u|_{C^0}^{p-q} \leq \frac{\lambda}{q}$. \hfill \Box

**Lemma 5.** There exists $t_0 > 0$ such that $I^+_\lambda(t_0 \varphi_1) \leq 0$, for all $\lambda$ in a bounded set.

**Proof.** Denoting by $\varphi_1$ the positive eigenfunction associated to $\lambda_1$, we have, for $t > 0$,

\[
I^+_\lambda (t \varphi_1) = \frac{t^2}{2} \int_{\Omega} |\nabla \varphi_1|^2 + \frac{t^q \lambda}{q} \int_{\Omega} \varphi_1^q - \frac{at^2}{2} \int_{\Omega} \varphi_1^2 - \frac{tp}{p} \int_{\Omega} \varphi_1^p \\
= \frac{1}{2} t^2 (\lambda_1 - a) \int_{\Omega} \varphi_1^2 + \frac{t^q \lambda}{q} \int_{\Omega} \varphi_1^q - \frac{tp}{p} \int_{\Omega} \varphi_1^p
\]

and, since $\lambda_k < a < \lambda_{k+1}$ and $q < 2 < p$, there exists a choice of $t_0 > 0$ which proves the lemma. \hfill \Box
Finally, define
\[ c^+_\lambda = \inf_{\gamma \in \Gamma^+} \sup_{t \in [0,1]} I^+_\lambda(\gamma(t)), \]
where
\[ \Gamma^+ = \{ \gamma \in C([0,1], H^1_0) ; \gamma(0) = 0, \gamma(1) = t_0 \varphi_1 \}. \]
On the other hand, by the proof of the previous lemma we obtain
\[ I^+_\lambda(t \varphi_1) \leq t^q \lambda \frac{q}{q} \int_\Omega \varphi_1^q. \]
Then, if \( \lambda \) is small enough, \( c^+_\lambda \) is a critical value of \( I^+_\lambda \).

### 3.2 Existence of the nonpositive solution

In order to get the negative solution, consider the following functional \( I^-_\lambda : H^1_0 \to \mathbb{R} \) given by
\[
I^-_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda \frac{q}{q} \int_\Omega (u^-)^q - \frac{a}{2} \int_\Omega (u^-)^2.
\]
Again, \( I^-_\lambda \in C^1 \) and the critical points \( u^- \) of \( I^-_\lambda \) satisfy \( u^- \leq 0 \) and so are critical points of \( I^+_\lambda \) as well. We will apply once again the mountain pass theorem to obtain a critical point of \( I^-_\lambda \).

**Lemma 6.** The trivial solution \( u \equiv 0 \) is a local minimizer for \( I^-_\lambda \), for all \( \lambda > 0 \).

**Proof.** It suffices to show that 0 is a local minimizer of \( I^-_\lambda \) in the \( C^1 \) topology. Then, for \( u \in C^1_0(\Omega) \) we have
\[
I^-_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda \frac{q}{q} \int_\Omega (u^-)^q - \frac{a}{2} \int_\Omega (u^-)^2 \\
\geq \left( \lambda - \frac{a}{2} \right) \int_\Omega (u^-)^q \geq \left( \lambda - \frac{a}{2} \frac{2^{-q}}{C_0} \right) \int_\Omega (u^-)^q \geq 0,
\]
whenever \( \frac{a}{2} |u|^{2-q} \leq \lambda \).

**Lemma 7.** There exists \( t_0 > 0 \) such that \( I^-_\lambda(-t_0 \varphi_1) \leq 0 \), for all \( \lambda \) in a limited set.

**Proof.** We have, for \( t > 0 \),
\[
I^-_\lambda(-t \varphi_1) = \frac{t^2}{2} \int_\Omega |\nabla \varphi_1|^2 dx + \frac{t^q \lambda}{q} \int_\Omega \varphi_1^q dx - \frac{at^2}{2} \int_\Omega \varphi_1^2 dx \\
= \frac{1}{2} t^2 (\lambda_1 - a) \int_\Omega \varphi_1^2 dx + \frac{t^q \lambda}{q} \int_\Omega \varphi_1^q dx
\]
and, since \( \lambda_k < a < \lambda_{k+1} \) and \( q < 2 \), there exists a choice of \( t_0 > 0 \) which proves the lemma.
As in the nonnegative solution case, we obtain a critical value
\[ c^-_\lambda = \inf_{\gamma \in \Gamma^-} \sup_{t \in [0,1]} I^-(\gamma(t)), \]
where
\[ \Gamma^- = \{ \gamma \in C([0,1]); \gamma(0) = 0, \gamma(1) = -t_0 \phi_1 \}. \]
In view of the proof of Lemma 7, we get the estimate
\[ c^-_\lambda \leq \max_{s \in [0,1]} I^-(st_0 \phi_1) \leq \frac{t_0^q \lambda}{q} \int_\Omega \phi_1^q. \]
Then, if \( \lambda \) is small enough, \( c^-_\lambda < b^{2-N} S \frac{N}{N} \), consequently, by the Mountain Pass Theorem, \( c^-_\lambda \) is a critical value of \( I^-_\lambda \).

### 3.3 Existence of the third solution

Denote \( V_k = \langle \phi_1, \cdots, \phi_k \rangle \) and \( W_k = V_k^\perp \). Consider the functions introduced in [14],
\[ \zeta_m = \begin{cases} 
0 & \text{if } x \in B_{1/m}, \\
m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m}, \\
1 & \text{if } x \in \Omega \setminus B_{2/m},
\end{cases} \]
where we may suppose without loss of generality that \( 0 \in \Omega \). Let \( \varphi^m_i = \zeta_m \varphi_i \),
\[ V^m_k = \langle \zeta_m \varphi_1, \cdots, \zeta_m \varphi_k \rangle \]
and \( W^m_k = (V^m_k)^\perp \). For each \( m \in \mathbb{N} \), take a positive cut-off function \( \eta \in C_c^\infty(B_{1/m}) \) and define
\[ \varphi^m_{k+1} = \eta \varphi_{k+1}. \]
It follows from definitions above that
\[ \text{supp} u \cap \text{supp} \varphi^m_{k+1} = \emptyset \]
whenever \( u \in V^m_k \). We use the following lemma from [14].

**Lemma 8.** As \( m \to \infty \) we have
\[ \varphi^m_i \to \varphi_i \text{ in } H^1_0, \text{ and } \max_{\{u \in V^m_k; \int u^2 = 1\}} \|u\|^2 \leq \lambda_k + c_k m^{2-N}. \]
An easy consequence of this result is the following decomposition of \( H^1_0 \),

**Corollary 1.** For \( m \) large enough
\[ V^m_k \oplus W_k = H^1_0. \]
Lemma 9. There exist $\alpha > 0$ and $\rho > 0$ such that
\[
I_\lambda(u) \geq \alpha
\]
whenever $u \in W_k$ and $\|u\| = \rho$.

Proof. Note that, if $u \in W_k$ we have
\[
I_\lambda(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{\lambda}{q} \int |u|^q - \frac{a}{2} \int |u|^2 - \frac{b}{p} \int (u^+)^p
\]
\[
\geq \left( \frac{1}{2} - \frac{a}{2\lambda_{k+1}} \right) \|u\|^2 - \frac{b}{q} |u|^p
\]
\[
\geq \|u\|^2 (A - B\|u\|^{p-2}),
\]
with $A, B > 0$. Then it suffices to take $\rho < (A/B)^{1/(p-2)}$. \hfill \Box

Lemma 10. Given $\lambda_0 > 0$, there exist $m_0 \in \mathbb{N}$ and $R > \rho$ such that
\[
I_\lambda(u) \leq \frac{\lambda}{q} \int |u|^q,
\]
whenever $u \in \partial Q_m$, where $Q_m = (B_R \cap V_k^m) \oplus [0, R\varphi_{k+1}^m]$, $m \geq m_0$ and $\lambda \leq \lambda_0$. Henceforth $\partial$ means the boundary relative to subspace $V_k^m$.

Proof. Let $m$ large enough and $a_k < a$ be such that
\[
\lambda_k + c_k m^{2-N} \leq a_k < a.
\]

Initially, let $u \in V_k^m$, by Lemma 8 and (20) we have
\[
I_\lambda(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{\lambda}{q} \int |u|^q - \frac{a}{2} \int |u|^2 - \frac{b}{p} \int (u^+)^p
\]
\[
\leq \left( \frac{1}{2} - \frac{a}{2a_k} \right) \int |\nabla u|^2 + \frac{\lambda}{q} \int |u|^q - \frac{b}{p} \int (u^+)^p
\]
\[
\leq \frac{\lambda}{q} \int |u|^q.
\]

Also,
\[
I_\lambda(r\varphi_{k+1}^m) = \frac{r^2}{2} \int |\nabla \varphi_{k+1}^m|^2 + \frac{\lambda r^q}{q} \int |\varphi_{k+1}^m|^q - \frac{ar^2}{2} \int |\varphi_{k+1}^m|^2 - \frac{r^{bp}}{p} \int (\varphi_{k+1}^m)^p
\]
\[
\leq \frac{r^2}{2} \int |\nabla \varphi_{k+1}^m|^2 + \frac{\lambda_0 r^q}{q} \int |\varphi_{k+1}^m|^q - \frac{r^{bp}}{p} \int (\varphi_{k+1}^m)^p.
\]

Since $\varphi_{k+1}^m \to \varphi_{k+1}$ in $H_0^1$ as $m \to \infty$ and $\varphi_{k+1}$ changes of signal, there exist $m_0 \in \mathbb{N}$ and $R > 0$ such that
\[
I_\lambda(R\varphi_{k+1}^m) \leq 0 \quad \text{for all } m \geq m_0.
\]

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Then, by (19), (21) and (23) we obtain
\[(24) \quad I_\lambda(u) \leq \frac{\lambda}{q} \int |u|^q,
\]
whenever \( u \in V_k^m \cup (V_k \oplus R\varphi_{k+1}^m) \). By (22), there exists \( \beta > 0 \) satisfying
\[(25) \quad I_\lambda(r\varphi_{k+1}^m) \leq \beta
\]
for all \( m \geq m_0 \) and \( r \geq 0 \). Since \( a > \lambda_k \), we may take \( R > 0 \) such that
\[(26) \quad I_\lambda(u) \leq \left( \frac{1}{2} - \frac{a}{2\lambda_k} \right) \|u\|^2 + \frac{\lambda}{q} \int |u| \leq -\beta + \frac{\lambda}{q} \int |u|^q.
\]
Thus, by (19), (25) and (26) we get
\[(27) \quad I_\lambda(u + r\varphi_{k+1}^m) = I_\lambda(u) + I_\lambda(r\varphi_{k+1}^m) \leq \frac{\lambda}{q} \int |u|^q
\]
for all \( m \geq m_0 \) and \( u \in \partial(B_R \cap V_k^m) \). Therefore, by (24) and (27) we conclude the proof. \( \square \)

**Conclusion of Theorem 1: subcritical case**

Let \( \alpha \) given by the Lemma 9. Take \( \lambda \) enough small in order that
\[\frac{\lambda}{q} \int |u|^q \leq \mu < \alpha
\]
for all \( u \in \partial Q_m \). Then by Lemma 10 we have
\[I_\lambda(u) \leq \mu < \alpha
\]
whenever \( u \in \partial Q_m \) and \( m \geq m_0 \). Applying the Linking Theorem, \( I_\lambda \) possesses a critical point \( u \) at level \( c \), where
\[c_\lambda = \inf_{\Gamma} \max_{u \in Q_m} I_\lambda(h(u))
\]
and
\[\Gamma = \{ h \in \mathcal{C}(\overline{Q_m}, H_0^1); \ h = Id \text{ on } \partial Q_m \}.
\]
Finally, since \( c_\lambda \geq \alpha, I_\lambda(u) \geq \alpha > 0 \) and \( c_\lambda^+ \to 0 \) as \( \lambda \to 0 \). Therefore, if \( \lambda \) is small enough \( c_\lambda^- < \alpha \leq c_\lambda \), and, consequently, \( u \) may be neither of the critical points found above for \( I_\lambda^+ \) and \( I_\lambda^- \). \( \square \)
Conclusion of Theorem 2: critical case

For the critical case, we consider the family of functions taken from [4]

\[ u_\epsilon(x) = \left[ N(N - 2)\epsilon^2 \right]^{N/2} \epsilon^2 \left[ x^2 + |x|^2 \right]^{N/2}, \quad \epsilon > 0. \]

We recall that \( u_\epsilon \) satisfies \( \|u_\epsilon\|^2 = \|u_\epsilon\|_{2^*}^2 = S^{N/2} \) for all \( \epsilon > 0 \). Let \( u_\epsilon^m = \eta u_\epsilon \), where \( \eta \) is given as above, and \( Q_\epsilon^m = (B_R \cap V_{k}^m) \oplus [0, Ru_\epsilon^m] \). Replacing \( u_\epsilon^m \) by \( \varphi_{k+1}^m \) in Lemma 10, we obtain

\[ I_\lambda(u) \leq \lambda \int_\Omega |u|^q \quad \text{for all } u \in \partial Q_\epsilon^m \]

whenever \( m \) is large. Therefore, to conclude the proof of Theorem 2, it remains to show that

\[ \sup_{u \in Q_\epsilon^m} I_\lambda(u) < \frac{b^{2-N} S^{N/2}}{N} \]

for \( \epsilon \) and \( \lambda \) small enough.

**Lemma 11.** There exist \( m_0 > 0 \), \( \lambda_0 > 0 \) and \( \epsilon_0 > 0 \) such that

\[ \sup_{u \in Q_\epsilon^m} I_\lambda(u) < \frac{b^{2-N} S^{N/2}}{N} \]

for all \( m \geq m_0 \), \( \lambda < \lambda_0 \) and \( \epsilon < \epsilon_0 \).

**Proof.** We write

\[ I_\lambda(u) = J(u) + \frac{\lambda}{q} |u|^q, \quad \text{where} \quad J(u) \equiv \frac{1}{2} \|u\|^2 - \frac{a}{2} |u|^2 - \frac{b}{2} |u^+|^2. \]

We note that it is sufficient to prove that there exist \( m_0 > 0 \) and \( \epsilon_0 > 0 \) such that

\[ \sup_{u \in Q_\epsilon^m} J(u) < \frac{b^{2-N} S^{N/2}}{N} \]

for all \( m \geq m_0 \) and \( \epsilon < \epsilon_0 \). Let \( u = v + tu_\epsilon^m \in Q_\epsilon^m \). We first observe that

\[ \max_{t \geq 0} J(tu_\epsilon^m) = \frac{b^{2-N}}{N} \left( \frac{\|u_\epsilon^m\|^2 - a |u_\epsilon^m|^2}{|u_\epsilon^m|_{2^*}^2} \right)^{N/2}. \]

Fixe \( m_0 > 0 \) such that \( \lambda_k + \epsilon_0 m_0^{2-N} \leq \sigma < a \). For \( m \geq m_0 \), we have

\[ J(v) = \frac{1}{2} \|v\|^2 - \frac{a}{2} |v|^2 - \frac{1}{2} |v^+|^2 \leq \frac{1}{2} \|v\|^2 - \frac{a}{2} |v|^2 \leq \frac{\sigma}{2} |v|^2 - \frac{a}{2} |v|^2. \]
Hence

\[ J(u) = J(v) + J(tu^m) \leq J(tu^m). \]

Therefore, it remains to prove that

\[ \frac{\|u^m\|^2 - a|u^m|^2}{|u^m|^2} < S \]

whenever \( \epsilon \) is small. But this follows from identities

\[
\frac{\|u^m\|^2 - a|u^m|^2}{|u^m|^2} = \begin{cases} 
S - ade^2 |\ln \epsilon| + O(\epsilon^{N-2}) & \text{if } N = 4, \\
S - ade^2 + O(\epsilon^{N-2}) & \text{if } N \geq 5,
\end{cases}
\]

for their details see [23, Page 52]. \( \square \)

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References


