Minicourse

Boundary Value Problems for Second Order Elliptic Operators in Lipschitz Domains

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Abstract:

Many boundary value problems in mathematical physics can be phrased as boundary value problems for elliptic differential operators in non-smooth domains. The scope of the course is to present an up-to-date, rigorous, and to a large extent self-contained, treatment of some of the most basic partial differential equations (PDE) of mathematical physics in Lipschitz, via the modern tools of Harmonic Analysis. Examples include the Laplace equation, the Lamé system of elastostatics, the Stokes system of hydrostatics and the Maxwell system of electromagnetism. A common approach to handle such problems is by reducing them to a system of integral equations involving singular integral operators of single and double layer type.

The proof by R. Coifman, A. McIntosh and Y. Meyer in 1982 of the L^2 boundedness of the Cauchy operator on Lipschitz curves, a major problem which was open since the time A. P. Calderón and A. Zygmund have inaugurated the modern era of Singular Integral Operators in the 50's, has been quoted in subsequent American Mathematical Society reports as one of the most striking developments in mathematics in the 20th century. This in turn has opened the door to applying the method of layer potentials for the treating second and higher order elliptic boundary value problems in Lipschitz domains. We will review some of the tremendous advances made in the last two decades in employing the classical method of layer potentials in the treatment of the boundary value problems associated with the aforementioned PDEs.

The course is appropriate for any student who has finished the graduate analysis sequence.

Lecture I. Lipschitz domains and Calderón-Zygmund theory for SIO's

In this lecture we shall discuss the following topics: • Lipschitz domains in the graph case and the bounded case • surface measure and outward unit normal vector • non-tangential approach regions • non-tangential maximal functions • function spaces on Lipschitz boundaries • principal value and weakly SIO's on Lipschitz boundaries • general set up: Dirichlet and Neumann BVP's for the Laplacian • the maximal operator • Calderón-Zygmund operators • the Coifman-McIntosh-Meyer Theorem (statement).

Lecture II. Layer potential operators and functional analytical framework

In this second lecture the plan is to discuss: • double layers (definition, properties, connections with the Dirichlet problem for the Laplacian) • single layer (definition, properties, connections with the Neumann problem for the Laplacian) • semi-Fredholm and Fredholm operators • compact, Hilbert-Schmidt operators • index • Fredholm theory • invariance of index to perturbations • interpolation (the real and the complex methods).

Lecture III. Boundary behavior of layer potentials and Rellich type identities

In the third lecture we shall cover • boundedness of the maximal operator • existence of non-tangential limits to the boundary for L^p functions, $1 < p < \infty$ • the boundedness of the singular version of the double layer • jump relations for single and double layers and the gradient of the single layer • applications to the Dirichlet and Neumann problems for the Laplacian • Relllich type identities (statement and proof of the standard version) • Rellich with λ • spectral theory for K^* • applications to the invertibility of $\pm\frac{1}{2}I + K$ and $\pm\frac{1}{2}I + K^*$.

Lecture IV. L^p BVP problems for the Laplacian on $\mathcal{C}^1$ and Lipschitz domains

In the fourth lecture the plan is to cover the following topics: • the layer potential approach in the L^2 case • solving boundary integral equations • proving uniqueness • the L^p case for $p \geq 2$ in two dimensions • the case of BVP with data from Hölder spaces.

Lecture V. Elliptic systems and Higher Order Elliptic BVPs

In the fifth lecture we shall discuss: • definitions of second and higher elliptic systems, fundamental solution • Legendre-Hadamard condition and strict positive definiteness • conormal derivative • layer potentials, definitions, connections with the case of a single equation • the systems of elastostatics and hydrostatics.