GENERIC QUASILOCALIZED AND QUASIBALLISTIC DISCRETE SCHRÖDINGER OPERATORS

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Abstract. We derive sufficient conditions for the presence of generic sets of discrete Schrödinger operators on $l^2(\mathbb{Z}^d)$, $d \geq 1$, with both quasilocalized and quasiballistic dynamics, and apply them to three operator spaces, that is, with uniformly bounded, analytic quasiperiodic and unbounded potentials. It is concluded, for these spaces, that the dynamics is typically (from the topological viewpoint) nontrivial, whereas quantum intermittency is exceptional.

1. Introduction

We study aspects of the dynamics of discrete Schrödinger operators acting on the Hilbert spaces $l^2(\mathbb{Z}^d)$, $d \geq 1$. If T is an instance of such self-adjoint operators, its dynamics is given by the one-parameter unitary group $\mathbb{R} \ni t \mapsto e^{-itT}$, and our emphasis is on the very localized initial condition δ_0. Thus, $e^{-itT}\delta_0$ is the unique solution to the Schrödinger equation $\partial_t \psi = -iT\psi$, with $\psi(0) = \delta_0$.

More specifically, our interest is in operators $T = H^v$ with action

$$(H^v\psi)_n = \sum_{|j|=1} \psi_{n+j} + v_n\psi_n \,, \tag{1.1}$$

where the potential $v = (v_n)$ is a real multisequence. We have denoted the norm $|n| = \sum_{p=1}^{d} |n_p|$.

It is known that the spectral properties of T have important influence on the long time asymptotics of $e^{-itT}\delta_0$, and it is common to probe the localization versus transport behavior through the lower $\beta^-(q)$ and upper $\beta^+(q)$ power-law exponents of the (time) average $\langle M_T^q \rangle(t)$ of the q-moment $M_T^q(t)$ of the position operator, for $q > 0$ (we recall relevant definitions in Section 2). The larger $\beta^\pm$, the faster the transport; localization is associated with null exponents.

Under suitable conditions (see Section 2 for detailed statements), it is known that $0 \leq \beta^-(q) \leq \beta^+(q) \leq 1$, and we say that the dynamics is

- *localized*, or T has *dynamical localization*, if the function $t \mapsto M_T^q(t)$ is bounded for each $q > 0$;
- *essentially localized* if $\beta^+(q) = 0$ for all q;
- *quasilocalized* if $\beta^-(q) = 0$ for all q;
- *quasiballistic* if $\beta^+(q) = 1$ for all q;
- *ballistic* if $\beta^-(q) = 1$ for all q.

There is a vast literature concerning the dynamics of such operators, with studies and techniques that go beyond the pure spectral properties, with different proposals

2010 *Mathematics Subject Classification.* 81Q10 (35J10 28A80).

of generalized dimensions and direct inspections of the transfer matrices; see the review [5] and references therein. It is worth mentioning that "pathological" examples with pure point spectra and quasiballistic dynamics have been found (see [4, 6]).

The possibility of nonconstant functions $q \mapsto \beta^{\pm}(q)$ (see (2.1)) has been referred, in the physical literature, as *quantum intermittency* (see [9]). Theoretical and numerical simulations (see [8, 11]) indicate that quantum intermittency is a common phenomenon. However, we are going to show here that three classes of potentials (v_n), i.e.,

1. $|v_n| \le a$ for all n (with the topology of pointwise convergence),
2. $v_n = u(\nu n)$, $\nu \in \mathbb{R}^d/\mathbb{Z}^d$, and u a real analytic function on the d-dimensional torus (with metric $\sum_{1\le p\le d} |\sin(\nu_p - \nu_p')/2|$),
3. arbitrary multisequences (v_n) (with pointwise convergence),

give rise to spaces of operators with generic (i.e., a set that contains a dense G_δ subset) quasilocalized and quasiballistic dynamics, so that generically we have nontrivial dynamics (i.e., $0 = \beta^- < \beta^+ = 1$) but with constant lower and upper exponents; hence, for such systems, intermittency is the exception from the topological viewpoint (see Section 4).

Let us discuss an application of our results. In [6], it was proved that given a Schrödinger operator in $l^2(\mathbb{Z})$ (or in $l^2(\mathbb{N})$, satisfying in this case a phase boundary condition at $n = 0$), if there exist a positive γ and $N_0 \in \mathbb{N}$ such that $\|A(E,N)\| \le C(E)N^\gamma$ for every $E \in S$ (with $S \subset \mathbb{R}$ some set of positive Lebesgue measure) and $N \ge N_0$, then, for every $f \in \mathrm{C}^\infty_{0,+}(\mathbb{R})$ (compactly supported positive smooth functions) such that $f = 1$ on S, one has

$$\beta^-_{T;f}(q) \ge 1 - \frac{2\gamma}{q};$$

that is, if the norms of the transfer matrices

$$A(E,N) := \prod_{n=0}^{N} \begin{pmatrix} E - v_n & -1 \\ 1 & 0 \end{pmatrix}$$

do not grow faster than polynomially in N on a set of energies E of positive Lebesgue measure, then there is a nontrivial lower bound of the dynamics. Note that the result is independent of the nature of the spectrum and of the Hausdorff (or packing) dimension of the spectral measure μ_{δ_0}. See also [2] for related results, although with different techniques.

Examples of classes of one-dimensional bounded operators with a nontrivial lower bound of $\beta^-(q)$ are random operators with decaying potential and sparse operators (see [6] for definitions and details), as well as some substitution, Sturmian, hierarchical and moderately sparse potentials (as the prime model; see [2]). According to Theorem 4.2 ahead, this phenomenon can only occur for Schrödinger operators in a meager set (i.e., of Baire first category) of the metric spaces X_a of operators (1.1) with potentials satisfying $|v_n| \le a$, for all $n \in \mathbb{Z}^d$ (X_a is endowed with the topology of pointwise convergence).

Precise statements of the above mentioned results are described in the next sections. After presenting some necessary details of dynamical quantities in Section 2, an abstract result showing that suitable subsets of self-adjoint operators are G_δ is discussed in Section 3; this abstract result is the base of our applications to discrete Schrödinger operators in Section 4. It is worth emphasizing that some results in these sections can be adapted to continuous Schrödinger operators, that is, operators defined in $\mathrm{L}^2(\mathbb{R}^d)$.

Given a self-adjoint operator T acting on the Hilbert space $l^2(\mathbb{Z}^d)$, its spectrum will be denoted by $\sigma(T)$, and for a vector ξ, the corresponding spectral measure will be indicated by μ_ξ^T, whereas $P^T(I)$ will denote its spectral projection to the interval I. For a nonempty open interval $I \subset \mathbb{R}$, let $\mathrm{C}^\infty_{0,+}(I)$ be the set of smooth functions compactly supported on I and taking positive values.

2. Some dynamical quantities and results

In this section, we collect dynamical quantities often used to characterize the (quantum) dynamics generated by a self-adjoint operator T acting on the Hilbert space $\mathcal{H} = l^2(\mathbb{Z}^d)$, with $d \geq 1$, and add remarks suitable for the next sections. Let I be, unless explicitly stated, a nonempty open interval in $\mathbb{R}$. Here we consider wave packets represented by the initial state $\psi = f(T)\delta_0$, usually for some $f \in \mathrm{C}^\infty_{0,+}(I)$, or even by $\psi = \delta_0$ in some situations.

2.1. Moments and escaping probabilities.

First, we discuss the power-law growth of the time-average q-moments $\langle M^q_{T;f}\rangle(t)$ of the position operator $\mathbf{x}$, which are of physical interest since they are used to characterize the spreading of initially localized quantum states. Let $\{\delta_n\}$, $\delta_n(k) = \delta_{n_1 k_1} \cdots \delta_{n_d k_d}$, be the canonical basis of $\mathcal{H}$.

Such growth is described by the lower and upper *transport exponents* (see [5] and references therein), given respectively, for each $q > 0$, by

$$\beta^-_{T;f}(q) := \liminf_{t\to\infty} \frac{\ln\langle M^q_{T;f}\rangle(t)}{q \ln t}, \quad \beta^+_{T;f}(q) := \limsup_{t\to\infty} \frac{\ln\langle M^q_{T;f}\rangle(t)}{q \ln t}, \tag{2.1}$$

with the q-moment given by

$$M^q_{T;f}(t) := \left\| |\mathbf{x}|^{\frac{q}{2}} e^{-itT} f(T)\delta_0 \right\|^2 = \sum_{n\in\mathbb{Z}^d} |n|^q \left|\langle e^{-itT} f(T)\delta_0, \delta_n\rangle\right|^2,$$

and its average defined as

$$\langle M^q_{T;f}\rangle(t) := \frac{2}{t}\int_0^\infty e^{-2u/t} M^q_{T;f}(u)\, \mathrm{d}u.$$

The following result, extracted from [7], guarantees that $\langle M^q_{T;f}\rangle(t)$ is well defined.

Proposition 2.1. *Let T be a self-adjoint operator in $\mathcal{H}$, $f \in \mathrm{C}^\infty_{0,+}(\mathbb{R})$ and $\psi = f(T)\delta_0$. Then, $e^{-itT}\psi \in \operatorname{dom}|\mathbf{x}|^q$ for every $q, t > 0$, and*

1. *$\beta^\pm_{T;f}(q)$ are nondecreasing functions of q;*
2. *$\beta^\pm_{T;f}(q) \in [0,1]$, for all $q > 0$.*

Observe that, in case of bounded operators T, one may take $\psi = \delta_0$ or any other vector in $\mathcal{H}$ (without the need of the function f), to guarantee that $e^{-itT}\psi \in \operatorname{dom}|\mathbf{x}|^q$ for every $q, t > 0$. In this particular setting, the function f can still be useful to restrict the spectrum to some particular interval.

Definition 2.2. Let T be a self-adjoint operator in $\mathcal{H}$, I a nonempty open interval of $\mathbb{R}$, $f \in \mathrm{C}^\infty_{0,+}(I)$ and $\psi = f(T)\delta_0$. We say that:

1. T has *dynamical localization* on I if $\sup_t M^q_{T;f}(t) < \infty$, for every $q > 0$ and every $f \in \mathrm{C}^\infty_{0,+}(I)$.

(2) T is *essentially localized* (resp. *quasilocalized*) on I if $\beta^+_{T;f}(q) = 0$ (resp. $\beta^-_{T;f}(q) = 0$) for every $q > 0$ and every $f \in \mathrm{C}^\infty_{0,+}(I)$.
(3) T is *quasiballistic* (resp. *ballistic*) on I if $\beta^+_{T;f}(q) = 1$ (resp., $\beta^-_{T;f}(q) = 1$) for every $q > 0$ and every $f \in \mathrm{C}^\infty_{0,+}(I)$.

If $I = \mathbb{R}$ we just say that T is quasilocalized, and so on.

Remark 2.3. In the case $\psi = \delta_0$ (remember that this is a possible definition of ψ for every bounded operator), we say that T is *quasilocalized* if $\beta^-_{T;h}(q) = 0$ for every $q > 0$, with the constant function $h(x) = 1$, and so on.

Other quantities used to describe the quantum dynamics of such systems are *escaping probabilities*. Define the time-average outside (of the ball of radius R) probability of the initial state $\psi = f(T)\delta_0$ by

$$P_{T;f}(R,t) := \sum_{|n|\geq R} \frac{2}{t}\int_0^\infty e^{-2u/t}|\langle e^{-iuT}\psi, \delta_n\rangle|^2\,\mathrm{d}u.$$

Note that $P_{T;f}(R,t)$ can also be defined in case $f = \chi$ is the characteristic function of an open interval; we will use this possibility ahead.

By setting $P^\alpha_{T;f}(t) := P_{T;f}(t^\alpha - 1, t)$ for $\alpha \geq 0$, we follow [6] and introduce its growth exponents as

$$S^-_{T;f}(\alpha) = -\liminf_{t\to\infty}\frac{\ln P^\alpha_{T;f}(t)}{\ln t} \tag{2.2}$$

and

$$S^+_{T;f}(\alpha) = -\limsup_{t\to\infty}\frac{\ln P^\alpha_{T;f}(t)}{\ln t}. \tag{2.3}$$

If $P^\alpha_{T;f}(t) = 0$ for some $\alpha > 0$ and every $t \geq t_0 > 0$, we set $S^\pm_{T;f}(\alpha) = \infty$. Thus, for every α, $0 \leq S^+_{T;f}(\alpha) \leq S^-_{T;f}(\alpha) \leq \infty$.

If $S^\pm_{T;f}(\alpha) = \infty$, then $P^\alpha_{T;f}(t) = \mathcal{O}(t^{-m})$ for every $m > 0$, so that for large t, only a negligible part of the wave packet escapes from the ball of radius t^α. On the other hand, if $S^\pm_{T;f}(\alpha) = 0$, then $P^\alpha_{T;f}(t) = \mathcal{O}(1)$ and the essential part of the wave packet travels faster than t^α. Thus, it is natural to introduce the *dynamical exponents*

$$\alpha^\pm_{\mathrm{l}}(T,f) := \sup\{\alpha \geq 0 \mid S^\pm_{T;f}(\alpha) = 0\}, \quad \alpha^\pm_{\mathrm{u}}(T,f) := \sup\{\alpha \geq 0 \mid S^\pm_{T;f}(\alpha) < \infty\}. \tag{2.4}$$

Since $S^\pm_{T;f}(\alpha)$ are nondecreasing functions, $0 \leq \alpha^\pm_{\mathrm{l}}(T,f) \leq \alpha^\pm_{\mathrm{u}}(T,f) \leq \infty$, and $\alpha^\pm_{\mathrm{l}}(T,f)$ represent the lower (-) and upper (+) rates of propagation of the slowest part of the wave packet, whereas $\alpha^\pm_{\mathrm{u}}(T,f)$ quantify the rates of propagation of the fastest part of the wave packet.

The above dynamical and transport exponents are not completely independent, and now we recall some relations taken from [6]. Set $\beta^\pm_{T;f}(\infty) := \lim_{q\to\infty}\beta^\pm_{T;f}(q)$ and $\beta^\pm_{T;f}(0) := \lim_{q\to 0}\beta^\pm_{T;f}(q)$.

Theorem 2.4. *Let T, I, f and ψ be as in Definition 2.2. Then, for every $q > 0$, one has*

$$\alpha^-_{\mathrm{u}}(T,f) = \beta^-_{T;f}(\infty) \geq \beta^-_{T;f}(q) \geq \beta^-_{T;f}(0) = \alpha^-_{\mathrm{l}}(T,f) \geq \dim^+_{\mathrm{H}}(\mu^T_\psi)\,,$$
$$\alpha^+_{\mathrm{u}}(T,f) = \beta^+_{T;f}(\infty) \geq \beta^+_{T;f}(q) \geq \beta^+_{T;f}(0) = \alpha^+_{\mathrm{l}}(T,f) \geq \dim^+_{\mathrm{P}}(\mu^T_\psi)\,,$$

with $\dim_{\mathrm{K}}^+(\mu_\psi^T) := \inf\{\dim_{\mathrm{K}}(E) \mid \mu_\psi^T(E) = \mu_\psi^T(\mathbb{R}),\ E$ a Borel subset of $\mathbb{R}\}$ *representing the upper* K *dimension of* μ_ψ^T *(*K *stands for either packing (*P*) or Hausdorff (*H*)).*

In Theorem 2.4 and ahead, μ_ψ^T denotes the spectral measure of the operator T with respect to ψ. In the next statement, we set lim-L := lim sup if L = +, lim-L := lim inf if L = −.

Lemma 2.5. *Let T, I and f be as above, and $\alpha, \gamma > 0$. Then, for every $0 < \varepsilon \leq \gamma$, one has*

(2.5)
$$\{f \mid \operatorname*{lim\text{-}L}_{t\to\infty} t^\gamma P^\alpha_{T;f}(t) < \infty\} \subset \{f \mid S^{\mathrm{L}}_{T;f}(\alpha) \geq \gamma\} \subset \{f \mid \operatorname*{lim\text{-}L}_{t\to\infty} t^{\gamma-\varepsilon} P^\alpha_{T;f}(t) < \infty\}.$$

Proof. If, for a fixed $\alpha > 0$, $P^\alpha_{T;f}(t) = 0$ for every $t > t_0 > 0$, then $S^{\mathrm{L}}_{T;f}(\alpha) = \infty$, and the inclusions in (2.5) are trivially satisfied. Otherwise, the result follows directly from definitions (2.2) and (2.3). □

Lemma 2.6. *Let T and I be as above, and suppose that $\alpha_{\mathrm{u}}^+(T,\chi) \leq \alpha$ for some $\alpha \geq 0$, where χ represents the characteristic function of I. Then, for every $f \in \mathrm{C}^\infty_{0,+}(I)$, $\alpha_{\mathrm{u}}^+(T,f) \leq \alpha$.*

Proof. Pick an arbitrary $f \in \mathrm{C}^\infty_{0,+}(I)$ and define $C^{f,I} := \sup\{(f(x))^2 \mid x \in I\}$. Since $0 < C^{f,I} < \infty$, one has, for every $R, t > 0$,

$$P_{T;f}(R,t) = \sum_{|n|\geq R} \frac{2}{t}\int_0^\infty e^{-2u/t} |\langle e^{-iuT} f(T)\delta_0, \delta_n\rangle|^2 \,\mathrm{d}u \leq C^{f,I} P_{T;\chi}(R,t).$$

Now, fix $\beta > 0$; it follows from (2.3) that

(2.6)
$$S^+_{T;f}(\beta) = -\limsup_{t\to\infty} \frac{\ln P^\beta_{T;f}(t)}{\ln t} \geq -\limsup_{t\to\infty}\left(\frac{\ln P^\beta_{T;\chi}(t)}{\ln t} + \frac{\ln C^{f,I}}{\ln t}\right) = S^+_{T;\chi}(\beta).$$

Since, by hypothesis, $\alpha_{\mathrm{u}}^+(T,\chi) \leq \alpha$, this implies, for every $\beta > \alpha$, that $S^+_{T;\chi}(\beta) = \infty$. Hence, by (2.6), $S^+_{T;f}(\beta) = \infty$ for every $\beta > \alpha$, and the result follows. □

2.2. **Localization.** Here, we discuss how the presence of a complete set of specially localized eigenfunctions of an operator T affects its dynamical exponents. We begin with

Definition 2.7. Let T be a self-adjoint operator in $\mathcal{H} = l^2(\mathbb{Z}^d)$, I some subset of $\mathbb{R}$ (not necessarily an open interval) and $P^T(I)$ the spectral projection for the operator T onto the set I . We say that T has SULE [4] on I (or just has SULE, in case of $I \supset \sigma(T)$) if:

(i) T has a complete set $\{\varphi_k\}_{k\geq 1}$, in $P^T(I)\mathcal{H}$, of orthonormal eigenfunctions (say, $T\varphi_k = \lambda_k\varphi_k$, with $\lambda_k \in I$);
(ii) there are $r > 0$ and $n_k \in \mathbb{Z}^d$ (the "localization centers") such that for every $\delta > 0$, there is a C_δ so that, for every $n \in \mathbb{Z}^d$, $k \geq 1$,
$$|\varphi_k(n)| \leq C_\delta\, e^{\delta|n_k| - r|n-n_k|}.$$

The next result is a direct consequence of the following observations:

- If T has SULE on I, then it has dynamical localization on I.

- If T has dynamical localization on I, it immediately follows that T is essentially localized on I (see Definition 2.2). In particular, T is quasilocalized on I and $\alpha_{\mathrm{u}}^{+}(T,f)=0$ for every $f\in \mathrm{C}_{0,+}^{\infty}(I)$.

Proposition 2.8. *Let T be a self-adjoint operator in $\mathcal{H}$, I a nonempty open interval of $\mathbb{R}$, and suppose that T has SULE on I. Then, $\alpha_{\mathrm{u}}^{+}(T,f)=0$ for every $f\in \mathrm{C}_{0,+}^{\infty}(I)$.*

An important example of operator that has SULE is the so-called one-dimensional Anderson operator T_ω, represented by the action (1.1) ($d=1$) with potential $v(\omega)=(v_n(\omega))_{n\in\mathbb{Z}}$, a compactly supported sequence of independent and identically distributed (real) random variables with probability distribution $g(x)\mathrm{d}\ell(x)$, $g\in \mathrm{L}^\infty(\mathbb{R})$; think of ω as an element of (Ω,τ), where $\Omega=\mathbb{R}^{\mathbb{Z}}$ and τ is the probability measure obtained as the product of such distributions. In fact, it is shown in [4], Section 7, that T_ω has SULE for τ a.e. ω. Thus, a direct consequence of Proposition 2.8 is the following

Corollary 2.9. *Let (Ω,τ) be a probability measure space, and for each $\omega\in\Omega$, let T_ω be an Anderson operator. Then, for τ a.e. ω and every $f\in \mathrm{C}_{0,+}^{\infty}(\mathbb{R})$, $\alpha_{\mathrm{u}}^{+}(T_\omega,f)=0$.*

The next result is important for our applications.

Lemma 2.10. *Let J be a bounded self-adjoint operator acting on $l^2(\mathbb{Z})$ such that $\sigma(J)=[a,b]$ and J has SULE. Consider, in $l^2(\mathbb{Z}^d)$, the Kronecker sum $T=J\otimes \mathbf{1}\otimes\cdots\otimes\mathbf{1}+\ldots+\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes J$, and write $T'=T+\sum_{|j|\le n} v_j\langle\delta_j,\cdot\rangle\delta_j$ (that is, a finite-rank perturbation of T). Then, for every $f\in \mathrm{C}_{0,+}^{\infty}((a\cdot d,b\cdot d))$, one has $\alpha_{\mathrm{u}}^{+}(T',f)=0$.*

Proof. We initially show that T has SULE (on $[a\cdot d,b\cdot d)$). Denote by $\{\varphi_l\}$ the complete set, in $l^2(\mathbb{Z})$, of orthonormal eigenfunctions of J (such that $J\varphi_l=\lambda_l\varphi_l$). Since $\{\Psi_k\}_k$, with $\Psi_k(n):=\varphi_{k_1}(n_1)\otimes\cdots\otimes\varphi_{k_d}(n_d)$, is a complete set, in $P^T([a\cdot d,b\cdot d])l^2(\mathbb{Z}^d)$, of orthonormal eigenfunctions of T (such that $T\Psi_k=\Lambda_k\Psi_k$, with $\Lambda_k=\lambda_{k_1}+\cdots+\lambda_{k_d}\in[a\cdot d,b\cdot d]$), we need to prove that there are $r>0$ and $l_k\in\mathbb{Z}^d$ so that, for each $\delta>0$, there is a C_δ such that

$$|\Psi_k(l)|\le C_\delta\, e^{\delta|l_k|-r|l-l_k|},$$

for all $l\in\mathbb{Z}^d$ and $k\in\mathbb{N}$. But then, since there are, for every $p=1,\ldots,d$, $r^p>0$ and $l_{k_p}\in\mathbb{Z}$ so that, for each $\delta>0$, there is a C_δ^p such that

$$|\varphi_{k_p}(l_p)|\le C_\delta^p\, e^{\delta|l_{k_p}|-r^p|l_p-l_{k_p}|},$$

the result follows by setting $C_\delta:=\prod_{p=1}^d C_\delta^p$, $r:=\min r^p$ and $l_k:=(l_{k_1},\ldots l_{k_p})$.

Now, fix $m\in\mathbb{Z}^d$, $|m|>n$ and $u>0$. By a Duhamel's formula, one obtains

$$\langle\delta_m,e^{-iuT'}\delta_0\rangle=\langle\delta_m,e^{-iuT}\delta_0\rangle-i\int_0^u\sum_{|j|\le n} v_j\langle\delta_m,e^{-isT}\delta_j\rangle\langle\delta_j,e^{-i(u-s)T'}\delta_0\rangle\,\mathrm{d}s.$$

Since T has SULE, it follows from Theorem 7.5 in [4] that there is a $\rho>0$ such that, for every $\delta>0$, there is a finite D_δ so that, for every $q,r\in\mathbb{Z}^d$,

$$\sup_u|\langle\delta_q,e^{-iuT}\delta_r\rangle|\le D_\delta\, e^{\delta|r|-\rho|r-q|}.$$

Inserting the previous result in Duhamel's formula, and using, for every $y \in \mathbb{R}$, that $|\langle \delta_j, e^{-iyT'}\delta_0\rangle| \le 1$, one has

$$\begin{aligned} |\langle \delta_m, e^{-iuT'}\delta_0\rangle| &\le |\langle \delta_m, e^{-iuT}\delta_0\rangle| + \sum_{|j|\le n} |v_j| \int_0^u |\langle \delta_m, e^{-isT}\delta_j\rangle| \, \mathrm{d}s \\ &\le D_\delta e^{-\rho|m|} + D_\delta u \sum_{|j|\le n} |v_j| \, e^{\delta|j|-\rho|j-m|} \\ &\le D_\delta e^{-\rho|m|}(1 + uA\, e^{(\delta+\rho)n}), \end{aligned}$$

where $A := \sum_{|j|\le n} |v_j|$. Now, for every $t \in \mathbb{R}_+$ and $r > 0$, it follows that

$$\begin{aligned} P_{T';\chi}(t) &\le \sum_{|m|\ge t^r-1} \frac{2}{t}\int_0^\infty e^{-2u/t} D_\delta^2 e^{-2\rho|m|} \left(1 + uAe^{(\delta+\rho)n}\right)^2 \mathrm{d}u \\ &= \sum_{|m|\ge t^r-1} D_\delta^2\, e^{-2\rho|m|} \left(1 + Ae^{(\delta+\rho)n}t + \left(A\, e^{(\delta+\rho)n}t\right)^2 / 2\right) \\ &\le E_\delta e^{-2\rho t^r} \left(1 + A\, e^{(\delta+\rho)n}t + \left(A\, e^{(\delta+\rho)n}t\right)^2 / 2\right) \sum_{p=1}^{d} \frac{(d-1)!}{(d-p)!} \frac{t^{r(d-p)}}{(2\rho)^p}, \end{aligned}$$

with E_δ some finite constant, and consequently,

$$S^+_{T';\chi}(r) \ge \limsup_{t\to\infty} \left[2\rho \frac{t^r}{\ln t} - 2 - r(d-1)\right] = \infty.$$

Since $r > 0$ is arbitrary, it follows from (2.4) that $\alpha_{\mathrm{u}}^+(T', \chi) = 0$. An application of Lemma 2.6 concludes the proof. □

3. Quasilocalized and quasiballistic sets

The symbol X will denote a subset of operators (1.1) with a complete metric so that the metric convergence implies strong resolvent convergence of operators. Throughout this section, we fix $-\infty \le a < b \le -\infty$, $f \in \mathrm{C}^\infty_{0,+}((a,b))$, and denote, for each $T \in X$, $\psi = f(T)\delta_0$. To state our main result in this section, i.e., Theorem 3.1, we introduce the sets $U^f := \{T \in X \mid \mathrm{supp}(\mu_\psi^T) \supset (a,b)\}$, $C^f_{0\mathrm{lp}} := \{T \in X \mid \alpha_{\mathrm{u}}^-(T,f) = 0\}$ and $C^f_{1\mathrm{up}} := \{T \in X \mid \alpha_{\mathrm{l}}^+(T,f) = 1\}$.

Theorem 3.1. *If each of the sets $C^f_{0\mathrm{lp}}$, $C^f_{1\mathrm{up}}$ and U^f is dense in X, then $C^f_{\text{QL-QB}} := \{T \in X \mid \mathrm{supp}(\mu_\psi^T) \supset (a,b)$, $\beta^-_{T;f}(q) = 0$ and $\beta^+_{T;f}(q) = 1$ for every $q > 0\}$ is generic in X.*

Our strategy to prove Theorem 3.1 is through Proposition 3.2; then we prove an auxiliary lemma and finally, the proposition itself.

Proposition 3.2. *Each of the sets U^f, $C^f_{0\mathrm{lp}}$ and $C^f_{1\mathrm{up}}$ is a G_δ set in X.*

Proof of Theorem 3.1. By Proposition 3.2, U^f, $C^f_{0\mathrm{lp}}$ and $C^f_{1\mathrm{up}}$ are G_δ sets. By Theorem 2.4, we have the inclusions $C^f_{0\mathrm{lp}} \subset C^f_{\mathrm{QL}} := \{T \in X \mid \beta^-_{T;f}(q) = 0$ for every $q > 0\}$, $C^f_{1\mathrm{up}} \subset C^f_{\mathrm{QB}} := \{T \in X \mid \beta^+_{T;f}(q) = 1$ for every $q > 0\}$. Now, by the hypotheses of the theorem, it follows that $C^f_{\text{QL-QB}} = C^f_{\mathrm{QL}} \cap C^f_{\mathrm{QB}} \cap U^f$ is generic in X. □

Lemma 3.3. *One has*

$$C^f_{0\text{lp}} = \bigcap_{r=1}^{\infty}\bigcap_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{1/r+1/p}_{k-1/l} \tag{3.1}$$

and

$$C^f_{1\text{up}} = \bigcap_{r=2}^{\infty}\bigcap_{p=2}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{l=1}^{\infty}\bigcap_{n=1}^{\infty} Y^{1-1/r-1/p}_{1/k+1/l}(n)\,, \tag{3.2}$$

where, for $\alpha > 0$ and each $c > 0$,

$$\begin{aligned} Q^{\alpha}_{k-1/l} &:= \big\{T \in X \mid \text{for each } m > 0,\ \exists\, t > m \text{ with } t^{k-1/l} P^{\alpha}_{T;f}(t) < c\big\}\,, \\ Y^{\alpha}_{1/k+1/l}(n) &:= \big\{T \in X \mid \text{for each } m > 0,\ \exists\, t > m \text{ with } t^{1/k+1/l} P^{\alpha}_{T;f}(t) > n\big\}\,. \end{aligned}$$

Proof. We will only present a proof of relation (3.1); the proof of (3.2) is similar. If $f \in \mathrm{C}^{\infty}_{0,+}((a,b))$ is such that, for every $T \in X$ and every $t > t_0 > 0$, $P^{\alpha}_{T;f}(t) = 0$ (which is the case when, for every $T \in X$, $\mathrm{supp}(\mu^T_{\psi}) = \emptyset$), then $Q^{\alpha}_{k-1/l} = X = C^f_{0\text{lp}}$. Otherwise, note that, for every $\alpha, \gamma > 0$ and $0 < \varepsilon \le \gamma$, (2.5) is equivalent to

$$\big\{f \mid \lim_{t\to\infty}\text{-L}\, t^{\gamma} P^{\alpha}_{T;f}(t) = 0\big\} \subset \big\{f \mid S^{\mathrm{L}}_{T;f}(\alpha) \ge \gamma\big\} \subset \big\{f \mid \lim_{t\to\infty}\text{-L}\, t^{\gamma-\varepsilon} P^{\alpha}_{T;f}(t) = 0\big\}\ ;$$

thus, we may work with any $c > 0$ and any fixed $f \in \mathrm{C}^{\infty}_{0,+}((a,b))$ in the relation (take $\mathrm{L} = -$)

$$Q^{\alpha}_{\gamma} = \bigcap_{m=0}^{\infty}\bigcup_{t>m} \big\{T \in X \mid t^{\gamma} P^{\alpha}_{T;f}(t) < c\big\} \subset \big\{T \in X \mid S^{-}_{T;f}(\alpha) \ge \gamma\big\} \subset \bigcap_{m=0}^{\infty}\bigcup_{t>m} \big\{T \in X \mid t^{\gamma-\varepsilon} P^{\alpha}_{T;f}(t) < c\big\} = Q^{\alpha}_{\gamma-\varepsilon}\,.$$

Now, replacing γ by $\gamma - \varepsilon$ and taking $\gamma = k$, $k \ge 1$, $\varepsilon = 1/l$, $l \ge 2$, one obtains the inclusions

$$\bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{\alpha}_{k-1/l} \subset \bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} \{T \in X \mid S^{-}_{T;f}(\alpha) \ge k - 1/l\} \subset \bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{\alpha}_{k-2/l}\,,$$

and consequently,

$$\{T \in X \mid S^{-}_{T;f}(\alpha) = \infty\} = \bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{\alpha}_{k-1/l}. \tag{3.3}$$

Finally, one has, from (2.4) and (3.3), the inclusions

$$\bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{\alpha}_{k-1/l} \subset \big\{T \in X \mid \alpha^{-}_{\mathrm{u}}(T,f) < \alpha + \delta\big\} \subset \bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{\alpha+\delta}_{k-1/l}\,,$$

valid for every $\delta > 0$. Thus, replacing α by $\alpha + \delta$ and taking $\alpha = 1/r$, $\delta = 1/p$, $p, r \ge 1$, one gets, since $\bigcap_{m\ge 1} C^f_{(1/m)\text{lp}} = \{T \in X \mid \alpha^{-}_{\mathrm{u}}(T,f) \le 1/m$ for every $m \in \mathbb{N}\} = C^f_{0\text{lp}}$,

$$\bigcap_{r=1}^{\infty}\bigcap_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{1/r+1/p}_{k-1/l} \subset C^f_{0\text{lp}} \subset \bigcap_{r=1}^{\infty}\bigcap_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{l=2}^{\infty} Q^{1/r+2/p}_{k-1/l}\,,$$

and we are done. □

Proof of Proposition 3.2. First, we discuss the sets $C^f_{0\mathrm{lp}}$ and $C^f_{1\mathrm{up}}$. If, for every $T \in X$, there is a $t_0 > 0$ such that, for every $t > t_0$, $P_{T;f}(0,t) = 0$, then $\alpha^-_{\mathrm{u}}(T,f) = \alpha^+_{\mathrm{l}}(T,f) = 0$ (since in this case, $S^{\pm}_{T;f}(\alpha) = \infty$ for every $\alpha > 0$), and consequently, $C^f_{0\mathrm{lp}} = X$ and $C^f_{1\mathrm{up}} = \emptyset$ are G_δ sets in X.

Otherwise, for each fixed $(a,b) \subset \mathbb{R}$, $f \in \mathrm{C}^\infty_{0,+}((a,b))$ and $t > 0$, the mapping $X \ni T \mapsto \frac{2}{t}\int_0^\infty e^{-2x/t}|\langle e^{-ixT} f(T)\delta_0, \delta_n\rangle|^2 \,\mathrm{d}x$ is continuous, since $e^{-ix\cdot} f(\cdot) : \mathbb{R} \to \mathbb{C}$ is bounded and continuous for every $x \in \mathbb{R}$; see Proposition 10.1.9 in [3]. Now, it follows from this remark, the estimate

$$\frac{2}{t}\int_0^\infty e^{-2x/t}|\langle e^{-ixT} f(T)\delta_0, \delta_n\rangle|^2 \,\mathrm{d}x \le |\langle f(T)\delta_0, \delta_n\rangle|^2,$$

valid for every $n \in \mathbb{Z}^d$, and Weierstrass M-test, that the mapping $X \ni T \mapsto P_{T;f}(t^\alpha - 1, t)$ is also continuous, for every $\alpha > 0$; this implies, for every $c > 0$, that each of the sets

$$C^f_{0\mathrm{lp}} = \bigcap_{r\ge1}\bigcap_{p\ge1}\bigcap_{k\ge1}\bigcap_{l\ge2}\bigcap_{m\ge0}\bigcup_{t>m}\left\{T \in X \mid t^{k-1/l} P_{T;f}(t^{1/r+1/p} - 1, t) < c\right\}$$

$$C^f_{1\mathrm{up}} = \bigcap_{r\ge2}\bigcap_{p\ge2}\bigcap_{k\ge1}\bigcap_{l\ge1}\bigcap_{n\ge0}\bigcap_{m\ge0}\bigcup_{t>m}\left\{T \in X \mid t^{1/k+1/l} P_{T;f}(t^{1-1/r-1/p} - 1, t) > n\right\}$$

is a G_δ set in X.

Now we discuss U^f. We follow the steps of the proof of Theorem 1.3 in [10]. Let λ_n be a countable dense set in $[a,b]$. Then,

$$\{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset [a,b]\} = \bigcap_n \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \ni \lambda_n\},$$

so we need only to consider the cases where we replace $[a,b]$ by $\{\lambda\}$. Since

$$\{T \in X \mid \mathrm{supp}(\mu^T_\psi) \not\ni \lambda\} = \bigcup_{m=1}^\infty \left\{T \in X \mid \mathrm{supp}(\mu^T_\psi) \cap (\lambda - 1/m, \lambda + 1/m) = \emptyset\right\},$$

the continuity of the mapping $X \ni T \mapsto \mu^T_\psi \in \mathcal{M}_+(I)$ (since strong resolvent convergence results in vague convergence of the spectral measures; $\mathcal{M}_+(I)$ represents the set of positive finite Borel measures on the open interval I, endowed with the vague topology) says that $\{T \in X \mid \mathrm{supp}(\mu^T_\psi) \not\ni \lambda\}$ is an F_σ. Thus, its complement is a G_δ. We conclude the proof by observing that $\{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)\} = \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset [a,b]\}$ (since the support of a measure is closed). □

4. Applications

As mentioned in the Introduction, in this section we discuss some applications of our results to classes of discrete Schrödinger operators as defined in (1.1) and acting on $l^2(\mathbb{Z}^d)$. The symbol X (as well as with suitable indices) denotes a subset of such operators with a complete metric so that the metric convergence implies strong resolvent convergence of operators. Note that it is enough that convergence in the metric implies pointwise convergence of potentials (see Lemma 4.4 for the unbounded case).

Remark 4.1. An important feature of these applications is that the vector δ_0 belongs to the set of cyclic vectors of every $T \in X$, so the results stated in Theorem 3.1 can be strengthened. First, note the inclusion $\sigma(T) \supset \text{supp}(\mu^T_{\delta_0})$, which makes $U := \{T \mid \sigma(T) \supset I\}$ a G_δ set for each open interval I.

Suppose that U is dense and let $T \in U$; if $\mu^T_{\delta_0}$ is absolutely continuous (a.c.) with respect to the one-dimensional Lebesgue measure, then μ^T_ψ is a.c. for every vector $\psi = f(T)\delta_0$, with $f \in \mathrm{C}^\infty_{0,+}(I)$. Thus, if the set $\{T \in X \mid \sigma(T) \supset I$ and $\mu^T_{\delta_0}$ is a.c$\}$ is dense, then $\{T \in X \mid \sigma(T) \supset I$ and T is quasiballistic on $I\}$ is generic (since $\dim^+_{\mathrm{P}}(\mu^T_\psi) = 1$ if μ^T_ψ is a.c., it follows, by Theorem 2.4, that the inclusions $\{T \in X \mid \mu^T_\psi$ is a.c., $\psi = f(T)\delta_0\} \subset C^f_{1\mathrm{up}} \subset C^f_{\mathrm{QB}}$ are valid for every $f \in \mathrm{C}^\infty_{0,+}(I)$).

Now, if the set $\{T \in X \mid \sigma(T) \supset I$ and $\alpha^-_{\mathrm{u}}(T,\chi) = 0\}$ (with $\chi(\cdot) := \chi_I(\cdot)$) is also dense, then Lemma 2.6 may be used to obtain generic sets of operators with quasilocalized dynamics on I.

4.1. **Bounded potentials.** Pick $a > 0$ and let X_a be the set of Schrödinger operators (1.1) on $l^2(\mathbb{Z}^d)$ with real multisequences (v_n) satisfying $|v_n| \le a$, for all $n \in \mathbb{Z}^d$; endow X_a with the topology of pointwise convergence. We have the following application of our previous results.

Theorem 4.2. *Fix $a > 0$. Then, the set $\{T \in X_a \mid \sigma(T) \supset [-a-2d, a+2d]$, T is quasilocalized and quasiballistic on $(-2d, 2d)\}$ is generic in X_a.*

Proof. By following an idea presented in Theorem 4.1 in [10], let ζ be the product of the normalized Lebesgue measures $(2a/d)^{-1}\ell_n$, $n \in \mathbb{Z}^d$, restricted to $[-a/d, a/d]$, so that $\text{supp}\,\zeta = [-a/d, a/d]^{\mathbb{Z}}$. Let $D = \{J \in X^{a/d}_1 \mid \text{supp}\,\mu^J_{\delta_0} = [-2-a/d, 2+a/d]$, J has SULE$\}$, where $X^{a/d}_1$ represents the set of one-dimensional Schrödinger operators with potentials (v_n) satisfying $|v_n| \le a/d$, for all $n \in \mathbb{Z}$, also endowed with the topology of the pointwise convergence. Then, by Corollary 2.9, $\zeta(X^{a/d}_1 \setminus D) = 0$, which implies, taking into account the support of ζ, that D is dense in $X^{a/d}_1$.

Hence, pick an arbitrary $J \in D$, defined by the sequence $\{w_m\}$, and set the multisequence $v^0 = (v^0_n)$, $v^0_n = w_{n_1} + \ldots + w_{n_d}$, so that the corresponding operator, T_0, can be written as the Kronecker sum $J \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \ldots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes J$. Clearly, one has $\sigma(T_0) = [-2d-a, 2d+a]$.

Now, let $T = H^v \in X_a$ be defined by $v = (v_n)$ and consider the sequence of operators $T_j \in X_a$ with $v^j = (v^j_n)$, where

$$v^j_n = \begin{cases} v_n, & |n| \le j \\ v^0_n, & |n| > j \end{cases} .$$

It is clear that each T_j is a finite-rank perturbation of T_0. Thus,

- $\sigma_{\mathrm{ess}}(T_j) = \sigma_{\mathrm{ess}}(T_0) = [-2d-a, 2d+a]$, by Weyl's criterion (Corollary 11.3.7 in [3]);
- for every $f \in \mathrm{C}^\infty_{0,+}((-2d-a, 2d+a))$, $\alpha^-_{\mathrm{u}}(T_j, f) = 0$, by Lemma 2.10.

Since $v^j \to v$ pointwise, $T_j \to T$ in the strong resolvent sense. Therefore, it follows from the remarks above that, for every $f \in \mathrm{C}^\infty_{0,+}((-2d-a, 2d+a))$, the set $\{T \in X_\phi \mid \sigma(T) \supset [-2d-a, 2d+a]$, $\alpha^-_{\mathrm{u}}(T, f) = 0\}$ is dense in X_a.

Finally, given an arbitrary $T \in X_a$, consider the sequence of operators $T_j \in X_a$ with potentials $v^j = (v^j_n)$, where

$$v^j_n = \begin{cases} v_n, & |n| \le j \\ 0, & |n| > j \end{cases}.$$

Since each T_j is a finite rank perturbation of H^0, the free operator, each of the spectral measures $\mu^j_{\delta_0}$ of T_j has an absolutely continuous part that is supported on $[-2d, 2d]$. Hence, since $[-2d, 2d] \subset \sigma(T)$ for every T in a dense set, one has, by Remark 4.1, that the set $C^f_{1\mathrm{up}}$ is also dense in X_a, for every $f \in \mathrm{C}^\infty_{0,+}((-2d, 2d))$.

A use of Theorem 3.1 concludes the proof. □

4.2. **Analytic quasiperiodic potentials.** Here we consider quasiperiodic Schrödinger operators (1.1) acting on $l^2(\mathbb{Z})$, with potentials generated by a nonconstant real analytic function $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$, that is, with $v_n = \lambda u(\nu n + \theta)$, $n \in \mathbb{Z}$, and fixed $\lambda \in \mathbb{R}$, $\nu, \theta \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$; we denote such operator by $H^u_{\lambda,\nu,\theta}$.

For each $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$, $\theta \in \mathbb{T}^d$ and $0 \ne \lambda \in \mathbb{R}$, we consider the space of self-adjoint operators

$$X^u_{\lambda,\theta} = \{H^u_{\lambda,\nu,\theta} \mid \nu \in \mathbb{T}^d\},$$

endowed with the topology induced by the metric

$$d(H^u_{\lambda,\nu,\theta}, H^u_{\lambda,\nu',\theta}) = \sum_{p=1}^{d} \left| \sin\left(\frac{\nu_p - \nu'_p}{2}\right) \right|.$$

Theorem 4.3. *Fix a nonconstant $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$. Then, there is $\lambda_0(u) > 0$ so that, for every $\lambda > \lambda_0(u)$ and every $\theta \in \mathbb{T}^d$, the set of operators $H^u_{\lambda,\nu,\theta}$ such that $\beta^-_{H^u_{\lambda,\nu,\theta};h}(q) = 0$ and $\beta^+_{H^u_{\lambda,\nu,\theta};h}(q) = 1$, for every $q > 0$, is generic in $X^u_{\lambda,\theta}$, where $h : \mathbb{R} \mapsto \mathbb{R}$ is the constant function $h(x) = 1$.*

Proof. Since the spectrum of $H^u_{\lambda,\nu,\theta}$ is purely absolutely continuous for every $\lambda \in \mathbb{R}$, $\theta \in \mathbb{T}^d$ and $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$ when $\nu \in \mathbb{Q}^d/\mathbb{Z}^d$, the metric and the density of the rational numbers in $\mathbb{R}$ imply the density of the set $C_{\mathrm{ac}} := \{T \in X^u_{\lambda,\theta} \mid \sigma(T)$ is purely absolutely continuous$\}$ in $X^u_{\lambda,\theta}$. Thus, for each $\theta \in \mathbb{T}^d$ and $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$, it follows that $C^h_{1\mathrm{up}}$ is dense in $X^u_{\lambda,\theta}$.

On other hand, it is known that given an arbitrary nonconstant $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$, there is a positive $\lambda_0(u)$ such that, for every $\lambda > \lambda_0(u)$ and every $\theta \in \mathbb{T}^d$, $H^u_{\lambda,\nu,\theta}$ satisfies dynamical localization (and consequently, SULE) for all ν outside a set of zero Lebesgue measure (depending on θ; this is a Corollary in the Introduction in [1]).

Thus, one obtains, for each fixed nonconstant $u \in \mathrm{C}^\omega(\mathbb{T}^d, \mathbb{R})$, $\lambda > \lambda_0(u)$ and $\theta \in \mathbb{T}^d$, that the set $C^h_{0\mathrm{lp}}$ is dense in $X^u_{\lambda,\theta}$. An application of Theorem 3.1 concludes the proof. □

4.3. **Unbounded discrete Schrödinger operators.** Now we consider unbounded Schrödinger operators defined as follows: take the space of real multisequences $\tilde{X} := \{v = (v_n) \mid n \in \mathbb{Z}^d, v_n \in \mathbb{R}\}$ with some enumeration $(v_k)_{k\in\mathbb{N}}$, and the metric

$$\tilde{d}(u, v) := \sum_{k=0}^{\infty} 2^{-k} \frac{|u_k - v_k|}{1 + |u_k - v_k|}, \quad u, v \in \tilde{X}.$$

Convergence in $(\tilde{X}, \tilde{d})$ implies pointwise convergence, and this is a complete metric space.

Then, for each $v \in \tilde{X}$, we associate the self-adjoint discrete Schrödinger operator H^v, with action (1.1) and appropriate domain. By denoting

$$X := \{H^v \mid v \in \tilde{X}\},$$

the pair (X, d), with $d(H^u, H^v) := \tilde{d}(u, v)$, is a complete metric space, and from now on we naturally identify elements $v \in \tilde{X}$ with $H^v \in X$.

Lemma 4.4. *Convergence in d implies strong resolvent convergence in X.*

Proof. Let $v^j \to v$ in $\tilde{X}$ as $j \to \infty$. It is enough to check the convergence of resolvents $R_i(H^{v^j})\xi \to R_i(H^v)\xi$ for vectors ξ in a dense subset A of $l^2(\mathbb{Z}^d)$.

Since the subspace $l_0(\mathbb{Z}^d)$ of multisequences with only a finite number of nonzero entries is a common core of all $H^v \in X$, the set $A := (H^v - i\mathbf{1})l_0(\mathbb{Z}^d)$ is dense in $l^2(\mathbb{Z}^d)$ for any $v \in \tilde{X}$. Thus, for $A \ni \xi = (H^v - i\mathbf{1})\eta$, $\eta \in l_0(\mathbb{Z}^d)$, the second resolvent identity implies that

$$\begin{aligned} \left\| R_i(H^{v^j})\xi - R_i(H^v)\xi \right\| &= \left\| R_i(H^{v^j})(v^j - v)R_i(H^v)\xi \right\| \\ &\le \left\| (v^j - v)R_i(H^v)\xi \right\| = \left\| (v^j - v)\eta \right\| \stackrel{j\to\infty}{\longrightarrow} 0\,, \end{aligned}$$

and we are done. □

Next, our application in this setting.

Theorem 4.5. *The set $\{T \in X \mid \sigma(T) = \mathbb{R}$, T is quasilocalized and quasiballistic on $(-2d, 2d)\}$ is generic in X.*

Proof. By Lemma 4.4, pointwise convergence of sequences in $\tilde{X}$ leads to strong resolvent convergence of operators in X. For each integer $n > 0$, the proof follows exactly the same steps of the proof of Theorem 4.2 with $\{T \in X \mid \sigma(T) \supset [-2d - n, 2d + n]$, T is quasilocalized and quasiballistic on $(-2d, 2d)\}$ being generic in X; the result follows then for $\sigma(T) = \mathbb{R}$. □

Acknowledgments

CRdeO was partially supported by CNPq (a Brazilian government agency).

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