# Steklov-Neumann eigenproblens: a spectral characterization of the Sobolev trace spaces *

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**Abstract**

We will study two classes of the eigenvalues problems for elliptic systems: one of them for the Steklov and another one for Neumann. In both problems we guarantee the existence of an increasing unbounded sequence of eigenvalues. The results were basically justified through of the variational arguments.

**Keywords** Steklov-Neumann eigenvalue, variational methods, and elliptic system.

## 1 Introduction

In this paper we will study two classes of eigenvalue problems for elliptic systems, namely, Steklov and Neumann eigenvalue problem. The first one, so called Steklov eigensystem, is given by

$$\textbf{(I)} \quad \begin{cases} -\triangle U + C(x)U &= 0 \quad \text{in} \quad x \in \Omega, \\ \dfrac{\partial U}{\partial \eta} &= \mu U \quad \text{on} \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary, denoted by $\partial\Omega$, of class $C^{0,1}$, $U = (u, v) \in H^1(\Omega) \times H^1(\Omega)$ and the matrix

$$C(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}$$

verifies the following condition:

**(P)** $C(x)$ is a positive definite matrix on $\mathbb{R}^2$, almost everywhere $x \in \Omega$, with $a, b, c \in L^p(\Omega)$, for $p \geq \frac{N}{2}$, when $N \geq 3$ and $p > 1$, when $N = 2$.

In this problem, we establish the existence of an increasing unbounded sequence of eigenvalues

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \to +\infty.$$

We will call these eigenvalues by the Steklov eigenvalues.

The main difficulty is to lead with the trace operator. In the scalar case, we would like to cite papers by [5, 6, 7, 9, 10, 11] and [14], while for the system case, to the best of our knowledge, we not able to find any reference treating this kind of problem.

The second problem is concerning to the Neumann eigensystem, namely

$$\textbf{(II)} \quad \begin{cases} -\triangle U + C(x)U &= \lambda U \quad \text{in} \quad x \in \Omega, \\ \dfrac{\partial U}{\partial \eta} &= 0 \quad \text{on} \quad x \in \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary, $\partial\Omega$, of class $C^{0,1}$, and $C(x)$ is a matrix verifying **(P)**. In this second problem, we also establish the existence of an increasing unbounded sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty.$$

We shall denominate these eigenvalues by the Neumann eigenvalues.

For the scalar case, we would like to refer, for instance, the works [4, 15, 16, 20], and [1, 2] for the system case.

Recently, in a very interesting paper, Auchmuty in [3] proved an abstract eigenproblem which contain former results either in the scalar case or in the uncoupled system.

## 2 Notations and definitions

To put our results into the context, we have collected in this short section some relevant notations and definitions for our purposes.

From the condition **(P)** together with the hypothesis on $\Omega$, we have that:

(1) the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ is continuous for $1 \leq p \leq p(N)$ and compact for $1 \leq p < p(N)$, where $p(N) = 2N/(N-2)$ if $N \geq 3$ and $p(N) = \infty$ if $N = 2$ (see [12], for more details).

(2) for each $u \in H^1(\Omega)$ the trace of $u$ on $\partial\Omega$, denoted by $\Gamma(u)$, is well-defined and it is a Lebesgue integrable function with respect to Hausdorff $(N-1)$-dimensional measure $\sigma$, in other words, $\int_{\partial\Omega} \Gamma(u) d\sigma$ is finite (see [12]).

(3) the trace mapping $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is compact (see [13]).

(4) if we denote by $H(\Omega) = H^1(\Omega) \times H^1(\Omega)$, then

$$\langle U, V \rangle_C = \int_\Omega \left[ \nabla U \cdot \nabla V + \langle C(x)U, V \rangle \right] dx$$

defines an inner product for $H(\Omega)$, with associated norm $\| \cdot \|_C$.

To conclude this section, we will present some definitions.

**Definition 2.1.** *The pair $(\mu, U) \in \mathbb{R} \times H(\Omega)$ is called a weak solution of **(I)** if*

$$\int_\Omega \left[ \nabla U \cdot \nabla V + \langle C(x)U, V \rangle \right] dx = \mu \int_{\partial\Omega} U \cdot V d\sigma, \ \forall \ V \in H(\Omega).$$

*In this case, when $U \in H(\Omega) \setminus \{0\}$, we say that $U$ is a Steklov eigenfunction associated to the Steklov eigenvalue $\mu$.*

**Definition 2.2.** *The pair $(\lambda, U) \in \mathbb{R} \times H(\Omega)$ is called a weak solution of **(II)** if*

$$\int_\Omega \left[ \nabla U \cdot \nabla V + \langle C(x)U, V \rangle \right] dx = \lambda \int_\Omega U \cdot V dx, \ \forall \ V \in H(\Omega).$$

*In this case, when $U \in H(\Omega) \setminus \{0\}$, we say that $U$ is a Neumann eigenfunction associated to the Neumann eigenvalue $\lambda$.*

**Definition 2.3.** *$U \in H(\Omega)$ is called a weak $H$-solution of*

$$- \triangle U + C(x)U = 0 \quad in \quad \Omega, \tag{1}$$

*when $\langle U, \Theta \rangle_C = 0$, for all $\Theta \in \left[ C_c^1(\Omega) \right]^2$.*

## 3 Main results

In this section, we state the main results of our work. We consider the maximization problem

$$\alpha_1 = \sup_{U \in \mathbb{K}} \beta(U),$$

where

- $\mathbb{K} = \{U \in H(\Omega) : \|U\|_C \leq 1\}$;
- $\beta(U) = \|U\|^2_{2,\partial} = \|u\|^2_{2,\partial} + \|v\|^2_{2,\partial}$, $U = (u,v) \in H(\Omega)$;
- $\|w\|^q_{q,\partial} \doteq \int_{\partial\Omega} |w|^q d\sigma$, for $1 \leq q < \infty$.

Our results for the problem **(I)** are following:

**Theorem 3.1.** *Suppose **(P)**. Then*
*(F-1) there exists $U_1 \in \mathbb{K}$, such that $\|U_1\|_C = 1$ and $\beta(U_1) = \alpha_1$. Therefore, $\alpha_1 > 0$;*
*(F-2) $\mu_1 \doteq \alpha_1^{-1}$ is an eigenvalue of Steklov eigensystem **(I)** with associated eigenfunction $U_1$;*
*(F-3) $\mu_1$ is the least positive eigenvalue of the Steklov eigensystem **(I)**.*

**Theorem 3.2.** *Suppose **(P)**. Then there exists a sequence of weak solutions from **(I)**, namely $((\mu_j, U_j))$, in $\mathbb{R} \times [H(\Omega) \setminus \{0\}]$. Furthermore, if we define, for each $s \in \mathbb{N} = \{1, 2, \ldots, n, \ldots\}$,*

$$\mathbb{K}_0 = \mathbb{K} \text{ and } \mathbb{K}_s = \{U \in \mathbb{K} : \ \langle U, U_k\rangle_{2,\partial} = 0, \ \text{for } 1 \leq k \leq s\},$$

*then, for each $j \in \mathbb{N}$,*

$$\alpha_j \doteq \sup_{U \in \mathbb{K}_{j-1}} \beta(U) = \beta(U_j) > 0 \text{ and } \mu_j = \alpha_j^{-1}.$$

*Here, $\langle U, V\rangle_{2,\partial} = \int_{\partial\Omega} U \cdot V d\sigma$. Moreover, the sequence $(\mu_j, U_j)$, above constructed, satisfies:*
*(S-1) $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_j \leq \cdots$;*
*(S-2) $\langle U_j, U_k\rangle_{2,\partial} \doteq \int_{\partial\Omega} U \cdot V d\sigma = \mu_j^{-1}\delta_{jk}$ and $\|U_j\|_C = 1$, for any $j, k \in \mathbb{N}$;*
*(S-3) $\lim_{j\to+\infty} \mu_j = +\infty$;*
*(S-4) The dimension of the eigenspace associated to $\mu_j$ is finite, for each $j$.*

The following result establishes a relation between the space $H(\Omega)$ and Steklov eigenspace. Denote $H_0(\Omega) = H^1_0(\Omega) \times H^1_0(\Omega)$.

**Theorem 3.3.** *Suppose **(P)**. Let $\mathcal{M}_k = \{V^k_1, V^k_2, \ldots, V^k_{m_k}\} \subset H(\Omega) \cap \left[L^2(\Omega)\right]^2$ be the $C$-orthonormal basis of Steklov egenfunctions associated to $\mu_k$.*
*(D-1) If $U \in \mathcal{M}_l$ and $V \in \mathcal{M}_k$, with $l \neq k$, then $\langle U, V\rangle_C = 0$;*
*(D-2) Define*

$$\mathcal{S} = \{V^1_1, V^1_2, \ldots, V^1_{m_1}, V^2_1, V^2_2, \ldots, V^2_{m_2}, \ldots, V^k_1, V^k_2, \ldots, V^k_{m_k}, \ldots\}.$$

*Then $\mathcal{S}$ is a $C$-orthonormal subset in $(H(\Omega), \|\cdot\|_C)$;*
*(D-3) For each $j \in \mathbb{N}$ define the subspaces*

$$\mathbb{V}_j = span\left[\bigcup_{k=1}^{j} \mathcal{M}_k\right], \ \mathbb{Y}_j = span\left[\overline{\bigcup_{k=j+1}^{\infty} \mathcal{M}_k}\right] \text{ and } \mathbb{X}_j = \mathbb{Y}_j \oplus_C H_0(\Omega),$$

*then*

$$H(\Omega) = \mathbb{V}_j \oplus_C \mathbb{X}_j.$$

The proof of our results were motivated by various papers above cited, mainly that by [5].

Define

$$\lambda_1 = \inf_{U \in \mathbb{L}} \Upsilon_C(U),$$

where

- $\mathbb{L} = \{U \in H(\Omega) : \|U\|_2 = 1\}$;

- $\Upsilon_C(U) = \|U\|_C^2$;
- $\|w\|_s^s = \int_\Omega |w|^s dx$ and $\|U\|_s^s = \|u\|_s^s + \|v\|_s^s$, for $U = (u, v) \in [L^s(\Omega)]^2$ and $w \in L^s(\Omega)$, with $1 \le s < \infty$.

Our results for the problem **(II)** are following:

**Theorem 3.4.** *Suppose **(P)**. Then*
*(N-1) there exists $U_1 \in H(\Omega)$ such that $\Upsilon_C(U_1) = \lambda_1$ and $\lambda_1 > 0$;*
*(N-2) $\lambda_1$ is the least eigenvalue of the Neumann eigensystem **(II)**;*
*(N-3) $\|U\|_C^2 \ge \lambda_1 \|U\|_2^2, \ \forall\ U \in H(\Omega)$.*

**Theorem 3.5.** *Suppose **(P)**. Then there exists a sequence of pairs $((\lambda_k, U_k))$ in $\mathbb{R} \times [H(\Omega) \setminus \{0\}]$ of weak solutions from **(II)**, verifying*

$$\lambda_k \doteq \inf_{U \in \mathbb{L}_{k-1}} \Upsilon_C(U)$$

*where, for $s \in \mathbb{N}$,*

$$\mathbb{L}_0 \doteq \mathbb{L} \ \text{and}\ \mathbb{L}_s \doteq \{U \in \mathbb{L} : \langle U, U_i \rangle_2 = 0, \ \text{for}\ 1 \le i \le s\}.$$

*Moreover, the sequence $((\lambda_k, U_k))$ satisfies the following properties:*
*(SN-1) $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$;*
*(SN-2) $\langle U_k, U_l \rangle_2 \doteq \int_\Omega U_k \cdot U_l dx = \delta_{kl}, \ \forall\ k, l \in \mathbb{N}$;*
*(SN-3) $\lim_{j \to +\infty} \lambda_j = +\infty$;*
*(SN-4) The dimension of the eigenspace associated to $\lambda_j$ is finite, for each $j$.*

Now, we establish a relation between the space $H(\Omega)$ and the Neumann eigenspace.

**Theorem 3.6.** *Suppose **(P)**. Let $\mathcal{B}_k = \{U_1^k, U_2^k, \ldots, U_{\tau_k}^k\} \subset H(\Omega) \cap \left[L^2(\Omega)\right]^2$ be an orthonormal basis for the eigenspace associated to $\lambda_k$.*
*(DN-1) Then the set $\mathcal{N} = \{U_1^1, \ldots, U_{\tau_1}^1, U_1^2, \ldots, U_{\tau_2}^2, \ldots, U_1^k, \ldots, U_{\tau_k}^k, \ldots\}$ is an orthonormal in $\left[L^2(\Omega)\right]^2$;*
*(DN-2) For each $j \in \mathbb{N}$, fixed, consider $\mathbb{F}_j$, the subspace generated by*

$$\mathbb{E}_j = \{U_1^1, \ldots, U_{\tau_1}^1, \ldots, U_1^j, \ldots, U_{\tau_j}^j\},$$

*which has dimension $\tau_1 + \tau_2 + \cdots + \tau_j$. Then*

$$\left[L^2(\Omega)\right]^2 = \mathbb{F}_j \oplus \mathbb{F}_j^\perp,$$

*where*

$$\mathbb{F}_j^\perp = \{U \in \left[L^2(\Omega)\right]^2 : \langle U, V \rangle_2 = 0, \ \forall\ V \in \mathbb{F}_j\}.$$

*Here, $\langle U, V \rangle_2 = \int_\Omega U.V dx$. Also, we obtain the decomposition*

$$H(\Omega) = \mathbb{F}_j \oplus \left[\mathbb{F}_j^\perp \cap H(\Omega)\right].$$

# 4 Preliminary results

In this section we present some auxiliary results which will be needed, in the sequel, for the proof of our results. Firs of all, we consider

- $(H(\Omega), \|\cdot\|_H)$ with $\|U\|_H = \|u\|_{1,2} + \|v\|_{1,2}$ and $U = (u, v) \in H(\Omega)$;

- $\tilde{A}, \tilde{C} : H^1(\Omega) \to \mathbb{R}$, defined by

$$\tilde{A}(u) = \int_\Omega a(x)u^2 dx \text{ and } \tilde{C}(u) = \int_\Omega c(x)u^2 dx, \ \forall\ u \in H^1(\Omega);$$

- $\tilde{B} : (H(\Omega), \|\cdot\|_H) \to \mathbb{R}$, defined by

$$\tilde{B}(u,v) = \int_\Omega b(x)uv dx, \ \forall\ u,v \in H^1(\Omega);$$

- $\tilde{G} : H^1(\Omega) \to \mathbb{R}$, defined by

$$\tilde{G}(u) \doteq \int_\Omega |\nabla u|^2 dx.$$

As a consequence of the Sobolev embeddings of $H^1(\Omega)$ into $L^p(\Omega)$, for $1 \leq p \leq p(N)$, of the Brezis-Lieb Lemma (see [8]) and of the Hölder's inequalities, we conclude that $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are continuous functionals. The proof of the continuity for $\tilde{G}$ is standard (see [19] and [17]).

**Proposition 4.1.** *Suppose **(P)**. Then the functional* $\Lambda_C : (H(\Omega), \|\cdot\|_H) \to \mathbb{R}$, *defined by* $\Lambda_C(U) = \|U\|_C$, *is sequentially weakly continuous.*

**Proof:** It suffices to prove that $\Lambda_C$ is continuous and convex. $\Lambda_C$ is continuous, because, for $U = (u,v) \in H(\Omega)$,

$$\Lambda_C(U)^2 = \|U\|_C^2 = \Upsilon_C(U), \Upsilon_C(U) = \tilde{G}(u) + \tilde{G}(v) + \tilde{A}(u) + 2\tilde{B}(u,v) + \tilde{C}(v)$$

and the functionals $\tilde{G}, \tilde{A}, \tilde{B}$, $\tilde{C}$ are continuous. Note that $\Lambda_C = \sqrt{\cdot} \circ \Upsilon_C$. The proof of the convexity for $\Lambda_C$ follows, recalling that $\|\cdot\|_C$ is a norm in $H(\Omega)$. Therefore, $\Lambda_C$ is sequentially weakly continuous. □

**Remark 4.1.** It is a consequence from Proposition 4.1 that for all sequence $(U_n)$ such that $U_n \rightharpoonup U$ weakly in $(H(\Omega), \|\cdot\|_H)$ satisfies

$$\liminf_{n \to +\infty} \Lambda_C(U_n) \geq \Lambda_C(U).$$

**Proposition 4.2.** *Suppose that **(P)** holds. Then*

*(i) there exists* $\delta > 0$ *such that* $\Upsilon_C(U) = \|U\|_C^2 \geq \delta \|U\|_2^2$, *for all* $U \in H(\Omega)$;

*(ii) the norms* $\|\cdot\|_H$ *and* $\|\cdot\|_C$ *are equivalents in* $H(\Omega)$.

**Proof:** Define $\mathbb{S} = \{U \in H(\Omega) : \|U\|_2 = 1\}$ and $\delta = \inf_{U \in \mathbb{S}} \Upsilon_C(U)$.

**Claim 1:** There exists $\widehat{U} \in \mathbb{S}$ such that $\Upsilon_C(\widehat{U}) = \delta$.

Indeed, by the definition of $\delta$, there exists sequence $(U_m)$ in $\mathbb{S}$ such that

$$\Upsilon_C(U_m) \to \delta \text{ and } \Upsilon_C(U_m) < \delta + 1. \tag{2}$$

Since $U_m \in \mathbb{S}$, from **(P)** and (2), the sequence $(U_m)$ is bounded in $(H(\Omega), \|\cdot\|_H)$, which implies that there exist a subsequence $(U_{m_k})$ of $(U_m)$ and $\widehat{U} \in H(\Omega)$ such that

$$U_{m_k} \rightharpoonup \widehat{U} \text{ weakly in } (H(\Omega), \|\cdot\|_H).$$

Consequently, from the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, we have that $U_{m_k} \to \widehat{U}$, in $([L^2(\Omega)]^2, \|\cdot\|_2)$. Thus $\widehat{U} \in \mathbb{S}$. Furthermore, it follows from Remark 4.1 that

$$\|\widehat{U}\|_C \leq \liminf_{k \to +\infty} \|U_{m_k}\|_C = \delta^{\frac{1}{2}}.$$

Therefore $\delta = \Upsilon_C(\widehat{U})$.

**Claim 2:** $\delta > 0$.

Clearly, $\delta \geq 0$. If $\delta = 0$, then $\|\widehat{U}\|_C^2 = \Upsilon_C(\widehat{U}) = 0$ and thus $\widehat{U} = 0$ in $H(\Omega)$, which contradicts the assumption that $\widehat{U} \in \mathbb{S}$. Therefore $\delta > 0$.

It follows from Claim 1 and 2 that $\Upsilon_C(U) = \|U\|_C^2 \geq \delta \|U\|_2^2,\ \forall\, U \in H(\Omega)$.

The proof of the second part is also immediate, by the continuity of $\Upsilon_C$, there exists a constant $C_2 > 0$ such that

$$\|U\|_C \leq \sqrt{C_2}\|U\|_H,\ \forall\, U \in H(\Omega). \tag{3}$$

Now, from **(P)**, we have $\|U\|_H^2 \leq \|U\|_C^2 + \|U\|_2^2,\ \forall\, U \in H(\Omega)$. Therefore by the first part

$$\|U\|_H \leq \sqrt{\left(1 + \frac{1}{\delta}\right)}\|U\|_C,\ \forall\, U \in H(\Omega). \tag{4}$$

The equivalences desired follow from (3) and (4). □

**Proposition 4.3.** *Suppose* ***(P)****. Then*

*(1)* $\beta$ *and* $\Upsilon_C$ *are elements of* $C^1((H(\Omega), \|\cdot\|_H), \mathbb{R})$ *and, for* $U \in H(\Omega)$,

$$\Upsilon_C^{'}(U)(V) = 2\langle U, V\rangle_C \text{ and } \beta^{'}(U)(V) = 2\langle U, V\rangle_{2,\partial},\ \forall\, V \in H(\Omega). \tag{5}$$

*Moreover,* $\beta$ *is weakly continuous function;*

*(2) for each* $U \in H(\Omega)$ *fixed,* $\Pi_U, G_U : H(\Omega) \to \mathbb{R}$, *defined by*

$$\Pi_U(V) = \langle V, U\rangle_{2,\partial} \text{ and } G_U(V) = \langle V, U\rangle_2,\ \forall\, V \in H(\Omega),$$

*are elements of* $C^1((H(\Omega), \|\cdot\|_H), \mathbb{R})$ *with Fréchet derivatives in* $V \in H(\Omega)$ *given by*

$$\Pi_U^{'}(V)(W) = \langle W, U\rangle_{2,\partial} \text{ and } G_U^{'}(V)(W) = \langle W, U\rangle_2,\ \forall\, W \in H(\Omega). \tag{6}$$

**Proof:** It follows from the Sobolev embedding of $H^1(\Omega)$ into $L^2(\Omega)$, of the continuity of trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$ and of the Hölder's inequalities that $\Upsilon_C, \beta, \Pi_U$ and $G_U$ are elements of $C^1((H(\Omega), \|\cdot\|_H), \mathbb{R})$ and satisfy the equalities (5) and (6). We shall now prove that $\beta$ is weakly continuous function. Let $(U_m)$ and $U \in H(\Omega)$ such that $U_m \rightharpoonup U$ weakly in $(H(\Omega), \|\cdot\|_H)$. By the continuity of trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, we have $U_m \to U$ in $(\left[L^2(\partial\Omega)\right]^2, \|\cdot\|_{2,\partial})$. Thus, there exists $T > 0$ such that $\|U_m\|_{2,\partial} \leq T,\ \forall\, m \in \mathbb{N}$.

Therefore, by

$$|\beta(U_m) - \beta(U)| \leq \|U_m\|_{2,\partial}\|U_m - U\|_{2,\partial} + \|U\|_{2,\partial}\|U_m - U\|_{2,\partial}$$

and $U_m \to U$ in $(\left[L^2(\partial\Omega)\right]^2, \|\cdot\|_{2,\partial})$, we get that $\beta$ is weakly continuous function. □

**Proposition 4.4.** *Define, for* $J \in \mathbb{N}$, *the sets*

$$\mathbb{K} = \{U \in H(\Omega) :\ \Upsilon_C(U) \leq 1\}$$

*and*

$$\mathbb{K}_J = \{U \in \mathbb{K} : \langle U, W_k\rangle_{2,\partial} = 0,\ \text{for } 1 \leq k \leq J\},$$

*where* $W_1, W_2, \ldots, W_J$ *are fixed elements in* $H(\Omega)$. *Then* $\mathbb{K}$ *and* $\mathbb{K}_J$ *are weakly compact in* $(H(\Omega), \|\cdot\|_H)$.

**Proof:** It suffices to prove that the sets are closed, bounded and convex. The set $\mathbb{K}$ is convex, because $\|\cdot\|_C$ define a norm in $H(\Omega)$, while the convexity of $\mathbb{K}_J$ follows from the definition of the inner produt $\langle\cdot,\cdot\rangle_{2,\partial}$ and by the convexity of $\mathbb{K}$. By Proposition 4.2 we conclude that $\mathbb{K}$ is bounded, and the boundedness of the $\mathbb{K}_J$ is immediate, because $\mathbb{K}_J \subset \mathbb{K}$. By definition, $\mathbb{K}$ is closed, while for $\mathbb{K}_J$, notice that

$$\mathbb{K}_J = \mathbb{K} \cap \bigcap_{k=1}^{J} \Pi_k^{-1}(\{0\}),$$

where $\Pi_k : H(\Omega) \to \mathbb{R}$ is a continuous functional, defined by $\Pi_{W_k}(V) = \langle V, W_k\rangle_{2,\partial}$, with $W_k \in H(\Omega)$ fixed $(k = 1, 2, \ldots, J)$ and $V \in H(\Omega)$. Then $\mathbb{K}_J$ is closed. □

# 5 Proof of Theorem 3.1

Some of the arguments from the proof were inspired in that was used in [1, 6, 5, 7]. Firstly, we note that due to Proposition 4.4, $\mathbb{K}$ is weakly compact in $(H(\Omega), \|\cdot\|_H)$ and by the Proposition 4.3, $\beta$ is weakly continuous. This implies that there exists $U_1 \in \mathbb{K}$ such that $\alpha_1 = \beta(U_1)$.

**Claim 1:** $\alpha_1 > 0$.

Clearly we have $\alpha_1 \geq 0$. Suppose that $\alpha_1 = 0$. Then, by the definition of $\alpha_1$, we conclude that $\beta(U) = 0$, for $U \in \mathbb{K}$. Taking $\Psi = (\varphi_1, \varphi_2) \in H(\Omega)$, defined by $\varphi_1(x) = 1$ and $\varphi_2(x) = 0$, for $x \in \Omega$, we obtain

$$\beta(\Psi) = \int_{\partial\Omega} \Psi \cdot \Psi d\sigma = \int_{\partial\Omega} \left[\varphi_1^2 + \varphi_2^2\right] d\sigma = \int_{\partial\Omega} 1 d\sigma = |\partial\Omega|_\sigma > 0.$$

Therefore, putting $\Phi = \frac{\Psi}{\|\Psi\|_C}$, we have $\Phi \in \mathbb{K}$ and $\beta(\Phi) = \frac{\beta(\Psi)}{\|\Psi\|_C^2} > 0$, which is a contradiction. Therefore the Claim 1 is proved.

**Claim 2:** $\|U_1\|_C = 1$.

In fact, notice that since $U_1 \in \mathbb{K}$, we have $\|U_1\|_C^2 = \Upsilon_C(U_1) \leq 1$. Suppose by contradiction that $\|U_1\|_C < 1$. Then there exists $r > 1$ such that $rU_1 \in \mathbb{K}$. Hence,

$$\alpha_1 \geq \beta(rU_1) = r^2\beta(U_1) > \beta(U_1),$$

which is a contradiction to the definition of $\alpha_1$ (notice that $\beta(U_1) \neq 0$).

Now, invoking the Lagrange multiplier Theorem (see [18]), there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\partial\Omega} U_1 \cdot V d\sigma = \lambda \int_\Omega \left[\nabla U_1 \cdot \nabla V + \langle C(x)U_1, V\rangle\right] dx, \ \forall\, V \in H(\Omega). \tag{7}$$

**Claim 3:** $\lambda = \alpha_1$.

Indeed, taking $V = U_1$ in (7) and remind that $\|U_1\|_C = 1$, we obtain

$$\lambda = \langle U_1, U_1\rangle_{2,\partial} = \|U_1\|_{2,\partial}^2 = \alpha_1 \geq 0.$$

Consequently, from (7), Claim 1 and Claim 2, we have that $\mu_1 \doteq \lambda^{-1}$ is a Steklov eigenvalue for the eigensystem **(I)** with associated Steklov eigenfunction $U_1$.

Finally, we will prove that $\mu_1 = \alpha_1^{-1}$ is the least positive eigenvalue of the Steklov eigensystem **(I)**. If not, there exist $\tilde{U} \in H(\Omega) \setminus \{0\}$ and $0 < \tilde{\mu} < \mu_1$ such that

$$\langle \tilde{U}, V\rangle_C = \tilde{\mu}\langle \tilde{U}, V\rangle_{2,\partial}, \ \forall\, V \in H(\Omega).$$

Putting $V = \frac{\tilde{U}}{\|\tilde{U}\|_C^2}$, then $\tilde{\mu}\beta\left(\frac{\tilde{U}}{\|\tilde{U}\|_C}\right) = 1$. So that $\beta\left(\frac{\tilde{U}}{\|\tilde{U}\|_C}\right) = \frac{1}{\tilde{\mu}} > \frac{1}{\mu_1} = \alpha_1$, which is a contracdition, since $\dfrac{\tilde{U}}{\|\tilde{U}\|_C} \in \mathbb{K}$. □

# 6 Proof of Theorem 3.2

The proof of this Theorem were inspired in some arguments used in [1, 5, 6, 7].

We will prove this Theorem using a finite induction argument in $j$. The Theorem 3.1 implies that for $j = 1$ holds. Suppose that the Theorem holds for $1 \leq j \leq J$ and we will verify for $J + 1$.

It follows from the Proposition 4.4 and Proposition 4.3 that there exists $U_{J+1}$ in $\mathbb{K}_J$ such that

$$\alpha_{J+1} = \sup_{U \in \mathbb{K}_J} \beta(U) = \beta(U_{J+1}).$$

**Claim 1:** $\alpha_{J+1} > 0$.

Indeed, by hypothesis of the finite induction Theorem, we have that there exist $U_1, U_2, \ldots, U_J$ in $H(\Omega)$ such that

$$\alpha_l = \sup_{U \in \mathbb{K}_{l-1}} \beta(U) = \beta(U_l) > 0, \text{ for } 1 \leq l \leq J.$$

Moreover, we can write

$$\left[L^2(\partial\Omega)\right]^2 = [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)] \oplus [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]^{\perp}, \tag{8}$$

where $\Sigma(U) \doteq (\Gamma(u), \Gamma(v))$, for $U = (u, v) \in H(\Omega)$ (sometimes we will just use $U$ in place of $\Sigma(U)$). Also, the inclusion operator $i : H^{\frac{1}{2}}(\partial\Omega) \to L^2(\partial\Omega)$ is linear, injective and continuous. Thus, if we define

$$\begin{aligned} H_*^{\frac{1}{2}}(\partial\Omega) &= H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \\ &= \{V \in [L^2(\partial\Omega)]^2 : \exists\, U \in H(\Omega) \text{ with } \Sigma(U) = V\}, \end{aligned}$$

we conclude that the operator $\mathbb{I} = i \times i : H_*^{\frac{1}{2}}(\partial\Omega) \to \left[L^2(\partial\Omega)\right]^2$, given by $\mathbb{I}(U) = U$, for $U \in H_*^{\frac{1}{2}}(\partial\Omega)$ is injective, linear and continuous.

The proof of Claim 1 relies on the following auxiliary result.

**Auxiliary result 1:** There exists $V \in H_*^{\frac{1}{2}}(\partial\Omega) \setminus \{0\}$ such that $\mathbb{I}(V) = V$ is an element of $[\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]^{\perp}$. Suppose that it does not hold and let $V \in H_*^{\frac{1}{2}}(\partial\Omega)$. Then there exists $U \in H(\Omega)$ such that $\Sigma(U) = V \in \left[L^2(\partial\Omega)\right]^2$. Hence, from (8), there exist real numbers $\delta_i$, for $i = 1, 2, \ldots, J$, and $\tilde{U} \in [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]^{\perp}$ such that

$$\Sigma(U) = V = \delta_1 \Sigma(U_1) + \delta_2 \Sigma(U_2) + \cdots + \delta_J \Sigma(U_J) + \tilde{U}. \tag{9}$$

In virtue of the linearity of the operator $\Sigma$, of the characterization of $H_*^{\frac{1}{2}}(\partial\Omega)$ and of the equality (9), we conclude that $\tilde{U} \in H_*^{\frac{1}{2}}(\partial\Omega)$ and $\tilde{U} = \mathbb{I}(\tilde{U})$. Also, $\tilde{U} \in [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]^{\perp}$. But, by assumption, $\tilde{U} = 0$. Consequently, by (9), we get $V \in [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]$. Therefore, since $V$ is an arbitrary element in $H_*^{\frac{1}{2}}(\partial\Omega)$, we conclude that $H_*^{\frac{1}{2}}(\partial\Omega) = [\Sigma(U_1), \Sigma(U_2), \ldots, \Sigma(U_J)]$, which is a contradiction, because the dimension of $H_*^{\frac{1}{2}}(\partial\Omega)$ is infinite. This proves the Auxiliary result 1.

Now, since $V \in H_*^{\frac{1}{2}}(\partial\Omega)$, there exists $W \in H(\Omega)$ such that $\Sigma(W) = V$ and since $V \neq 0$ in $H_*^{\frac{1}{2}}(\Omega)$, we infer that $\beta(W) > 0$. In fact, if $\beta(W) = 0$, then $\|W\|_{2,\partial} = 0$, so that $\Sigma(W) = 0$ in $\left[L^2(\partial\Omega)\right]^2$. Moreover, $\Sigma(W) = V$. Hence, $V = 0$ in $\left[L^2(\partial\Omega)\right]^2$. But, since $V \in H_*^{\frac{1}{2}}(\partial\Omega)$, $\mathbb{I}$ is injective and $\mathbb{I}(V) = 0 = \mathbb{I}(0)$, we find that $V = 0$ in $H_*^{\frac{1}{2}}(\partial\Omega)$, which is an absurd. Also, we have that $W \neq 0$ in $H(\Omega)$. Indeed, if $W = 0$ in $H(\Omega)$, then $V = \Sigma(W) = 0$ in $\left[L^2(\partial\Omega)\right]^2$, so that $V = 0$ in $H_*^{\frac{1}{2}}(\partial\Omega)$, which also is an absurd. Thus, $\|W\|_C \neq 0$.

Now, considering $\tilde{W} = \frac{W}{\|W\|_C} \in H(\Omega)$, we obtain that

$$\beta(\tilde{W}) = \frac{\beta(W)}{\|W\|_C^2} = \frac{\beta(V)}{\|W\|_C^2} \text{ and } \langle \tilde{W}, U_k \rangle_{2,\partial} = 0 \text{ for } 1 \leq k \leq J.$$

It follows from $\|\tilde{W}\|_C = 1$ that $\tilde{W} \in \mathbb{K}_J$. Therefore,

$$\alpha_{J+1} = \sup_{U \in \mathbb{K}_J} \beta(U) \geq \beta(\tilde{V}) = \frac{\beta(V)}{\|\tilde{W}\|_C^2} > 0.$$

and thus the Claim 1 is proved.

As in the case $j = 1$, the following is true:

**Claim 2:** $\|U_{J+1}\|_C = 1$.

In fact, since $U_{J+1} \in \mathbb{K}_J$ we have that $U_{J+1} \in \mathbb{K}$. Then, $\|U_{J+1}\|_C \leq 1$. Suppose that $\|U_{J+1}\|_C < 1$, then there exists $r > 1$ such that $rU_{J+1} \in \mathbb{K}_{J+1}$. Consequently,

$$\beta(rU_{J+1}) = r^2 \beta(U_{J+1}) > \beta(U_{J+1}) = \alpha_{J+1},$$

which is a contradiction (notice that $\beta(U_{J+1}) \neq 0$ by the Claim 1).

Because of the Claim 1, one can define $\mu_{J+1} \doteq \alpha_{J+1}^{-1}$.

**Claim 3:** The pair $(\mu_{J+1}, U_{J+1})$ is a weak solution from the eigensystem **(I)**.

Indeed, because of the Claims 1 and 2, $\beta(U_{J+1}) = \alpha_{J+1}$ is an extremal for $\beta$ constrained to

$$\mathbb{K}_J = \Upsilon_C^{-1}\left(\{\Upsilon_C(U_{J+1})\}\right) \cap \left[\bigcap_{k=1}^{J} \Pi_k^{-1}\left(\{\Pi_k(U_{J+1})\}\right)\right],$$

where $\Pi_k \doteq \Pi_{U_k}$, for $k = 1, 2, \ldots, J$, as in the Proposition 4.3. Also, by the Proposition 4.3, $\Upsilon_C$, $\beta$ and $\Pi_k$ are elements of $C^1(H(\Omega), \mathbb{R})$. Furthermore, by the induction assumption, $det A(U_{J+1}, U_1, U_2, \ldots, U_J) = 2\alpha_1\alpha_2\cdots\alpha_J > 0$ with

$$A(U_{J+1}, U_1, \ldots, U_J) = \begin{pmatrix} \Upsilon_C'(U_{J+1})(U_{J+1}) & \Upsilon_C'(U_{J+1})(U_1) & \cdots & \Upsilon_C'(U_{J+1})(U_J) \\ \Pi_1'(U_{J+1})(U_{J+1}) & \Pi_1'(U_{J+1})(U_1) & \cdots & \Pi_1'(U_{J+1})(U_J) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \Pi_J'(U_{J+1})(U_{J+1}) & \Pi_J'(U_{J+1})(U_1) & \cdots & \Pi_J'(U_{J+1})(U_J) \end{pmatrix},$$

it follows from the Lagrange multipliers Theorem (see [18]) that there exist constants $\lambda, \lambda_k \in \mathbb{R}$, for $k = 1,\ 2, \ldots, J$, such that

$$\beta'(U_{J+1})(V) = \lambda\Upsilon_C'(U_{J+1})(V) + \sum_{k=1}^{J} \lambda_k \Pi_k'(U_{J+1})(V),\ \forall\ V \in H(\Omega). \tag{10}$$

Remembering that, for $U, V \in H(\Omega)$ and $k = 1, 2, \cdots, J$,

$$\Upsilon'(U)(V) = 2\langle U, V\rangle_C,\ \beta'(U)(V) = 2\langle U, V\rangle_{2,\partial} \text{ and } \Pi_k'(U)(V) = \langle V, U_k\rangle_{2,\partial},$$

we have, from (10), that

$$2\langle U_{J+1}, V\rangle_{2,\partial} = 2\lambda\langle U_{J+1}, V\rangle_C + \sum_{k=1}^{J} \lambda_k\langle V, U_k\rangle_{2,\partial},\ \forall\ V \in H(\Omega). \tag{11}$$

Finally, the proof of Claim 3 relies on the following auxiliary result:

**Auxiliary result 2:** $\lambda_k = 0$, for $k \in \{1, 2, \ldots, J\}$.

In fact, considering $V = U_s$, for $s \in \{1, 2, \ldots, J\}$, in (11), we obtain the identity

$$2\langle U_{J+1}, U_s\rangle_{2,\partial} = 2\lambda\langle U_{J+1}, U_s\rangle_C + \sum_{k=1}^{J} \lambda_k\langle U_s, U_k\rangle_{2,\partial}.$$

Thus, by $U_{J+1} \in \mathbb{K}_J$ and by the induction assumption,

$$0 = 2\lambda\mu_s\langle U_{J+1}, U_s\rangle_{2,\partial} + \lambda_s\langle U_s, U_s\rangle_{2,\partial} = 2\lambda_s\mu_s^{-1}\|U_s\|_c^2 = 2\lambda_s\mu_s^{-1}.$$

Since $\mu_s \neq 0$, for $s \in \{1, 2, \ldots, J\}$ (by the induction assumption), we conclude that $\lambda_s = 0$, for $\forall\ s = 1, 2, \ldots, J$, so that Auxiliary result 2 is justified.

Choosing $V = U_{J+1}$ in (11) we obtain that

$$\alpha_{J+1} = \|U_{J+1}\|_{2,\partial}^2 = \langle U_{J+1}, U_{J+1}\rangle_{2,\partial} = \lambda\langle U_{J+1}, U_{J+1}\rangle_C = \lambda\|U_{J+1}\|_C^2 = \lambda.$$

It follows from Auxiliary result 2 that

$$\langle U_{J+1}, V\rangle_{2,\partial} = \lambda\langle U_{J+1}, V\rangle_C,\ \forall\ V \in H(\Omega).$$

Hence, as $\lambda = \alpha_{J+1} > 0$ (Claim 1), we have

$$\alpha_{J+1}^{-1}\langle U_{J+1}, V\rangle_{2,\partial} = \langle U_{J+1}, V\rangle_C,\ \forall\ V \in H(\Omega),$$

in other words, the pair $(\mu_{J+1}, U_{J+1})$ is a weak solution for the eigensystem **(I)**. Consequently, by the Claims 1, 2 and 3, $(\mu_j)$ is a sequence of Steklov eingenvalues for **(I)**.

**Proof of (S-1):** By the above construction, for $l \geq 1$,

$$\mu_l = \alpha_l^{-1} \text{ and } \alpha_l = \sup_{U\in\mathbb{K}_{l-1}} \beta(U).$$

Now, since $\mathbb{K}_{j-1} \subset \mathbb{K}_{j-2}$, for $j \geq 2$ and $\alpha_j \leq \alpha_{j-1}$, we obtain that $\mu_{j-1} \leq \mu_j$, for all $j \geq 2$. Finally, as $\mu_1 > 0$ (Theorem 3.1), we conclude that

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots .$$

**Proof of (S-2):** By the previous construction, since $U_j \in \mathbb{K}_{j-1} \subset \mathbb{K}$, for $j \in \mathbb{N}$,

$$\|U_j\|_C = 1 \text{ and } \langle U_j, U_k\rangle_{2,\partial} = 0,\ \text{for } k < j. \tag{12}$$

If $j < k$, then, from (12), we have that $\langle U_j, U_k\rangle_{2,\partial} = 0$. Thus, considering $j = k$, we obtain $\langle U_j, U_j\rangle_{2,\partial} = \mu_j^{-1}\langle U_j, U_j\rangle_C = \mu_j^{-1}\|U_j\|_C^2 = \mu_j^{-1}$. Therefore the condition (S-2) is valid.

**Proof of (S-3):** Suppose by contradiction that there exists $K \in \mathbb{R}$ such that $\mu_j \leq K$, $\forall\ j \in \mathbb{N}$. Consequently, for $V_j = \frac{U_j}{\|U_j\|_{2,\partial}} \in H(\Omega)$,

$$\|V_j\|_C^2 = \frac{\|U_j\|_C^2}{\|U_j\|_{2,\partial}^2} = \frac{1}{\|U_j\|_{2,\partial}^2} = \mu_j \leq K,\ \forall\ j \in \mathbb{N},$$

that is, $(V_j)$ is a bounded sequence in $(H(\Omega), \|\cdot\|_C)$ and as a consequence from the equivalence of norms $\|\cdot\|_C$ and $\|\cdot\|_H$ in $H(\Omega)$, $(V_j)$ is also a bounded sequence in $(H(\Omega), \|\cdot\|_H)$. Hence, there exist a subsequence $(V_{j_k})$ of $(V_j)$ and $\tilde{V} \in H(\Omega)$ such that $V_{j_k} \rightharpoonup \tilde{V}$ weakly in $(H(\Omega), \|\cdot\|_H)$.

Now, by the continuity of the trace mapping of $H^1(\Omega)$ into $L^2(\partial\Omega)$, we conclude that $V_{j_k} \to \tilde{V}$ in $(\left[L^2(\partial\Omega)\right]^2, \|\cdot\|_{2,\partial})$, which implies that $(V_{j_k})$ is a Cauchy sequence in $\left[L^2(\partial\Omega)\right]^2$. However, for $j_k, j_l$ large enough and $j_k \neq j_l$, we obtain from (S-2) that

$$\|V_{j_k} - V_{j_l}\|_{2,\partial}^2 = \left\| \frac{U_{j_k}}{\|U_{j_k}\|_{2,\partial}} - \frac{U_{j_l}}{\|U_{j_l}\|_{2,\partial}} \right\|_{2,\partial}^2 = \frac{\|U_{j_k}\|_{2,\partial}^2}{\|U_{j_k}\|_{2,\partial}^2} + \frac{\|U_{j_l}\|_{2,\partial}^2}{\|U_{j_l}\|_{2,\partial}^2} = 2,$$

which is an absurd. Therefore $\lim\limits_{j\to+\infty} \mu_j = +\infty$.

**Proof of (S-4):** Assume, by contradiction, that there exists $k \in \mathbb{N}$ such that the dimension of the eigenspace associated to the eigenvalue $\mu_k$ is infinite. Hence, there exists a $C$-orthonormal sequence of Steklov eigenfunctions $(W_j)$ associated to the eigenvalue $\mu_k$. Consequently, for $r, s \in \mathbb{N}$ with $r \neq s$,

$$\langle W_r, W_s\rangle_{2,\partial} = \mu_k^{-1}\langle W_r, W_s\rangle_C = 0 \text{ and } 1 = \|W_r\|_C^2 = \mu_k\|W_r\|_{2,\partial}^2.$$

Following the same argument as in the proof of condition (S-3) for the sequence $(V_j)$, defined by $V_j = \frac{W_j}{\|W_j\|_{2,\partial}}$, we conclude that $(V_j)$ is a Cauchy sequence and it satisfies, for $j_r, j_s$ large enough and $j_r \neq j_s$,

$$\|V_{j_r} - V_{j_s}\|_{2,\partial}^2 = \left\| \frac{W_{j_r}}{\|W_{j_r}\|_{2,\partial}} - \frac{W_{j_s}}{\|W_{j_s}\|_{2,\partial}} \right\|_{2,\partial}^2 = \frac{\|W_{j_r}\|_{2,\partial}^2}{\|W_{j_r}\|_{2,\partial}^2} + \frac{\|W_{j_s}\|_{2,\partial}^2}{\|W_{j_s}\|_{2,\partial}^2} = 2,$$

which is an absurd. Thus (S-4) is proved. □

# 7 Proof of theorem 3.3

The next three lemmas are the main steps to prove the Theorem 3.3.

**Lemma 7.1.** *Suppose **(P)**. Then $U \in H(\Omega)$ is a weak $H$-solution of (1) if and only if $U \in H_0(\Omega)^{\perp}$.*

**Proof:** First, suppose that $U \in H(\Omega)$ is a weak $H$-solution of (1) and let $V \in H_0(\Omega)$, so that $V = (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Since $C_c^1(\Omega)$ is dense in $H_0^1(\Omega)$, with the norm $\|\cdot\|_{1,2}$, defined by

$$\|u\|_{1,2} = \left(\|u\|_2^2 + \||\nabla u|\|_2^2\right)^{\frac{1}{2}},$$

there exist sequences $(\varphi_n)$, $(\psi_n)$ in $C_c^1(\Omega)$ such that

$$\varphi_n \to u \text{ and } \psi_n \to v \text{ in } (H^1(\Omega), \|\cdot\|_{1,2}).$$

Now, noting that $\|U\|_H^2 = \|u\|_{1,2}^2 + \|v\|_{1,2}^2$, we conclude that $\Psi_n \to V$ in $(H(\Omega), \|\cdot\|_H)$, where, for each $n \in \mathbb{N}$, $\Psi_n = (\varphi_n, \psi_n)$. Hence, by the equivalence of the norms $\|\cdot\|_H$ and $\|\cdot\|_C$ in $H(\Omega)$, $\Psi_n \to V$ in $(H(\Omega), \|\cdot\|_C)$. Consequently $\langle U, \Psi_n\rangle_C \to \langle U, V\rangle_C$. Thus, in virtue of $U$ be a weak $H$-solution of (1) and $\Psi_n \in \left[C_c^1(\Omega)\right]^2$, $\langle U, \Psi_n\rangle_C = 0$, for all $n \in \mathbb{N}$. It follows that $\langle U, V\rangle_C = 0$. Therefore $U \in H_0(\Omega)^{\perp}$, because $V \in H_0(\Omega)$ is arbitrary.

Conversely, if $U \in H_0(\Omega)^{\perp}$, then $U \in H(\Omega)$ and $\langle U, V\rangle_C = 0$, for all $V \in H_0(\Omega)$. In particular, by $\left[C_c^1(\Omega)\right]^2 \subset H_0(\Omega)$, we have that $\langle U, \Theta\rangle_C = 0$, for all $\Theta \in \left[C_c^1(\Omega)\right]^2$, that is, $U$ is a weak $H$-solution of (1). □

**Lemma 7.2.** *Suppose **(P)**. Then $U \in H(\Omega)$ and $\beta(U) = 0$ if and only if $U \in H_0(\Omega)$.*

**Proof:** Suppose $U \in H(\Omega)$ and $\beta(U) = 0$. Consequently, $\|U\|_{2,\partial} = 0$, so that $U = 0$ in $\left[L^2(\partial\Omega)\right]^2$, that is, if $U = (u, v)$, then $\Sigma(U) = (\Gamma(u), \Gamma(v)) = (0, 0)$. But, by the Corollary 1.5.1.5 of [13], this implies that $u, v \in H_0^1(\Omega)$ and therefore $U \in H_0(\Omega)$.

Conversely, if $U \in H_0(\Omega)$, then arguing as in the proof of Lemma 7.1 there exists a sequence $(\Psi_n)$ in $\left[C_c^1(\Omega)\right]^2$, such that $\Psi_n \to U$ in $(H(\Omega), \|\cdot\|_H)$. It follows from the continuity of $\beta$, noticing that $\beta(\Psi_n) = 0$, $\forall\, n \in \mathbb{N}$, that $\beta(U) = 0$ and thus the Lemma 7.2 is proved. □

**Lemma 7.3.** *Suppose **(P)**. Then $(H(\Omega), \|\cdot\|_C)$ admits a decomposition of the form*

$$H(\Omega) = H_0(\Omega) \oplus_C H_0(\Omega)^{\perp},$$

*where $\oplus_C$ indicates a $C$-orthogonal direct sum.*

**Proof:** The proof of this Lemma is immediate, because $(H(\Omega), \|\cdot\|_C)$ is a Hilbert space and $H_0(\Omega)$ is a closed subspace of $(H(\Omega), \|\cdot\|_C)$. □

We now return to the proof of Theorem 3.3.

**Proof of (D-1) and (D-2):** If $U \in \mathcal{M}_l$ and $V \in \mathcal{M}_k$, with $l \neq k$, then, by $\mathcal{M}_k = \{V_1^k, V_2^k, \cdots, V_{m_k}^k\}$ be a $C$-orthonormal subset of Steklov eigenfunctions in $(H(\Omega), \|\cdot\|_C)$ associated to $\mu_k$ and by (S-2) (Theorem 3.2), we have that

$$\langle U, V\rangle_C = \mu_k \langle U, V\rangle_{2,\partial} = 0.$$

Consequently,

$$\mathcal{S} = \{V_1^1, \ldots, V_{m_1}^1, V_1^2, \ldots, V_{m_2}^2, \ldots, V_1^k, \ldots, V_{m_k}^k, \ldots\}$$

is a $C$-orthonormal subset of Steklov eigenfunctions in $(H(\Omega), \|\cdot\|_C)$.

**Proof of (D-3):** First of all, consider the sequence $\tilde{\mathbb{O}} = (W_k)$, defined by

$$W_k = \begin{cases} V_k^1 & \text{if } 1 \leq k \leq m_1, \\ V_{k-m_j}^{j+1} & \text{if } m_j < k \leq m_j + m_{j+1} \text{ and } j \in \mathbb{N}. \end{cases}$$

This sequence satisfies the following properties:

(A) $\tilde{\mathbb{O}}$ is a $C$-orthonormal sequence.

(B) $\tilde{\mathbb{O}} \subset H_0(\Omega)^\perp$.

(C) If $\tilde{U} \in H_0(\Omega)^\perp$ satisfies $\tilde{U} \perp \tilde{\mathbb{O}}$ in $(H(\Omega), \|\cdot\|_C)$, then $\tilde{U} = 0$.

The proof of (A) is immediate, because $\tilde{\mathbb{O}} \subset \mathcal{S}$. For the proof of (B), let $U \in \tilde{\mathbb{O}}$. We will prove that $U \in H_0(\Omega)^\perp$. In reason from the Lemma 7.1, $U \in H_0(\Omega)^\perp$ if and only if $U$ is a weak $H$-solution of (1), that is,

$$\langle U, \Theta \rangle_C = 0, \ \forall \ \Theta \in \left[C_c^1(\Omega)\right]^2. \tag{13}$$

Now, since $U$ is a Steklov eigenfunction, we will assume without loss of generality that $U$ is a eigenfunction associated to $\mu_k$. Hence,

$$\langle U, \Theta \rangle_C = \mu_k \langle U, \Theta \rangle_{2,\partial} = 0, \ \forall \ \Theta \in \left[C_c^1(\Omega)\right]^2.$$

As a consequence of this, $U$ satisfies (13) and thus $U \in H_0(\Omega)^\perp$. Therefore $\tilde{\mathbb{O}} \subset H_0(\Omega)^\perp$, because $U$ is an arbitrary element of $\tilde{\mathbb{O}}$.

For the proof (C), suppose, by contradiction, that there exists $\tilde{U} \in H_0(\Omega)^\perp \setminus \{0\}$ such that $\tilde{U} \perp \tilde{\mathbb{O}}$ in $(H(\Omega), \|\cdot\|_C)$. Thus, if $\tilde{V} = \frac{\tilde{U}}{\|\tilde{U}\|_C}$, then $\|\tilde{V}\|_C = 1$ and $\langle \tilde{V}, U \rangle_C = 0$, $\forall \ U \in \tilde{\mathbb{O}}$. Consequently, $\tilde{V} \in \mathbb{K}_j$, $\forall \ j \in \mathbb{N}$. Furthermore, $\beta(\tilde{V}) \geq 0$. If $\beta(\tilde{V}) = 0$, then, by the Lemma 7.2, $\tilde{V} \in H_0(\Omega)$, so that, by $\tilde{V} \in H_0(\Omega)^\perp$, $\tilde{V} = 0$ in $(H(\Omega), \|\cdot\|_C)$, which is an absurd, because $\|\tilde{V}\|_C = 1$. Hence, $\beta(\tilde{V}) > 0$. Finally, by (S-3) (Theorem 3.2), we have that $\mu_j \to +\infty$. It follows from $\mu_j = \alpha_j^{-1}$ that $\alpha_j \to 0$. Thus, there exists $J \in \mathbb{N}$ such that $\beta(\tilde{V}) > \alpha_{J+1}$, which is a contradiction with the definition of $\alpha_{J+1}$ (notice that $\tilde{V} \in \mathbb{K}_J$). Therefore (C) is valid.

Combining (A), (B) and (C), we conclude that $\tilde{\mathbb{O}}$ is a total $C$-orthonormal sequence in $H_0(\Omega)^\perp$. Hence $\tilde{\mathbb{O}}$ defines a Hilbert basis for the space $H_0(\Omega)^\perp$ in $(H(\Omega), \|\cdot\|_C)$. It follows that every $U \in H_0(\Omega)^\perp$ may be written uniquely as

$$U = \sum_{k=1}^{\infty} \langle U, W_k \rangle_C W_k \text{ and } \|U\|_C^2 = \sum_{k=1}^{\infty} |\langle U, W_k \rangle_C|^2. \tag{14}$$

Now, let $U \in H(\Omega)$. Then, by $H(\Omega) = H_0(\Omega) \oplus_C H_0(\Omega)^\perp$ (Lemma 7.3), there exist unique $U_0 \in H_0(\Omega)$ and unique $\overline{U} \in H_0(\Omega)^\perp$ such that $U = \overline{U} + U_0$. Furthermore, by $\overline{U} \in H_0(\Omega)^\perp$, there exists an unique sequence $(c_j)$ in $\mathbb{R}$ such that

$$\overline{U} = c_1 W_1 + \cdots + c_{\theta(j)} W_{\theta(j)} + \lim_{n \to +\infty} S_n,$$

where $\theta(j) \doteq \sum_{k=1}^{j} m_k$ and $S_n \doteq \sum_{k=\theta(j)+1}^{n} c_k W_k$. Thus, if we denote

$$V = c_1 W_1 + c_2 W_2 + \cdots + c_{\theta(j)} W_{\theta(j)}, Y = \lim_{n \to +\infty} S_n \text{ and } X = Y + U_0,$$

$U$ can be uniquely written as $U = V + X$ with $V \in \mathbb{V}_j$ and $X \in \mathbb{X}_j$ and thus the Theorem 3.3 is proved. $\square$

**Remark 7.1.** In view of (14), of the continuity of the trace $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$ and from the fact that $W_k$ is a Steklov eigenfunction associated to eigenvalue $\sigma_k$, where

$$\sigma_k = \begin{cases} \mu_1 & \text{if } 1 \leq k \leq m_1, \\ \mu_j & \text{if } m_{j-1} < k \leq m_j \text{ and } j \geq 2, \end{cases}$$

we obtain

$$\Sigma(U) = \sum_{k=1}^{\infty} \langle U, W_k\rangle_C \Sigma(W_k) \text{ and } \|\Sigma(U)\|_{2,\partial}^2 = \sum_{k=1}^{\infty} \sigma_k^{-1} |\langle U, W_k\rangle_C|^2. \tag{15}$$

It follows from definition of $\mathbb{V}_j$ that $dim\mathbb{V}_j = \theta(j) = m_1 + m_2 + \cdots + m_j$. Thus, if $U \in \mathbb{V}_j$, we have that $U = \sum_{k=1}^{\theta(j)} \langle U, W_k\rangle_C W_k$ and then, because of (14), (15) and (S-1) (Theorem 3.2), we find that

$$\|U\|_{2,\partial}^2 \geq \mu_j^{-1} \|U\|_C^2, \ \forall\, U \in \mathbb{V}_j.$$

Moreover, if $\overline{U} \in \mathbb{Y}_j \subset H_0(\Omega)^{\perp}$, then

$$\overline{U} = \lim_{n\to+\infty} \left[ \sum_{k=\theta(j)+1}^{n} c_k W_k \right], \ \|\overline{U}\|_{2,\partial}^2 = \lim_{n\to+\infty} \left[ \sum_{k=\theta(j)+1}^{n} \mu_k^{-1} c_k^2 \right]$$

and

$$\|\overline{U}\|_C^2 = \lim_{n\to+\infty} \left[ \sum_{k=\theta(j)+1}^{n} c_k^2 \right].$$

Combining these identities with (S-1), given in the Theorem 3.2, we conclude that

$$\|\overline{U}\|_{2,\partial}^2 \leq \mu_{j+1}^{-1} \lim_{n\to+\infty} \left[ \sum_{k=\theta(j)+1}^{n} c_k^2 \right] = \mu_{j+1}^{-1} \|\overline{U}\|_C^2, \ \forall\, \overline{U} \in \mathbb{Y}_j.$$

# 8 Proof of Theorem 3.4

First, we will prove (N-1). By the definition of $\lambda_1$, there exists a sequence $(U_j)$ in $\mathbb{L}$ such that

$$\Upsilon_C(U_j) \to \lambda_1 \text{ and } \Upsilon_C(U_j) \leq \lambda_1 + 1.$$

But, $\|U_j\|_C^2 = \Upsilon_C(U_j)$. Thus, the sequence $(U_j)$ is a bounded in $(H(\Omega), \|\cdot\|_C)$. It follows that $(U_j)$ also is bounded sequence in $(H(\Omega), \|\cdot\|_H)$ ($\|\cdot\|_H$ and $\|\cdot\|_C$ are equivalents norms in $H(\Omega)$). So that, there exists $\overline{U} \in H(\Omega)$ and subsequence $(U_{j_k})$ of $(U_j)$ such that $U_{j_k} \rightharpoonup \overline{U}$ weakly in $(H(\Omega), \|\cdot\|_H)$. Using the compact embedding of $H(\Omega)$ into $[L^2(\Omega)]^2$, we conclude that $U_{j_k} \to \overline{U}$ in $(\left[L^2(\Omega)\right]^2, \|\cdot\|_2)$ and therefore $\|U_{j_k}\|_2 \to \|\overline{U}\|_2$. Consequently, $\overline{U} \in \mathbb{L}$, because $U_{j_k} \in \mathbb{L}$.

We claim that $\Upsilon_C(\overline{U}) = \lambda_1$.

Indeed, by the Remark 4.1, we obtain that

$$\Upsilon_C(\overline{U})^{\frac{1}{2}} = \|\overline{U}\|_C \leq \liminf_{k\to+\infty} \|U_{j_k}\|_C = \lambda_1^{\frac{1}{2}}.$$

Therefore, $\Upsilon_C(\overline{U}) \leq \lambda_1$, so that $\Upsilon_C(\overline{U}) = \lambda_1$ (notice that $\overline{U} \in \mathbb{L}$). Let us denote $\overline{U}$ by $U_1$.

Now, if $\lambda_1 = 0$, then $U_1 = 0$ in $H(\Omega)$, so that $U_1 = 0$ in $[L^2(\Omega)]^2$, which is an absurd, since $U_1 \in \mathbb{L}$ and thus (N-1) is proved.

As consequence of (N-1), $\Upsilon_C(U_1)$ is an extremal for $\Upsilon_C$ constrained to

$$\mathbb{L} = G^{-1}\left(\{G(U_1)\}\right),$$

where $G(U) = \|U\|_2^2$. It is easy to see that $G \in C^1(H(\Omega), \mathbb{R})$ and moreover, by the Proposition 4.3, we get $\Upsilon_C \in C^1(H(\Omega), \mathbb{R})$. Therefore, from the Lagrange multipliers Theorem (see [18]) and from $G^{'}(U_1)(U_1) = 2\|U_1\|_2^2 = 2 \neq 0$, we conclude that there exists $\mu \in \mathbb{R}$ such that

$$\Upsilon_C^{'}(U_1)(\Theta) = \mu G^{'}(U_1)(\Theta), \ \forall\, \Theta \in H(\Omega). \tag{16}$$

Combining the identities $\Upsilon_C^{'}(U)(\Theta) = 2\langle U, \Theta\rangle_C$ and $G^{'}(U)(\Theta) = \langle U, \Theta\rangle_2$, for $\Theta, U, W \in H(\Omega)$ with (16), we obtain that

$$\langle U_1, \Theta\rangle_C = \mu\langle U_1, \Theta\rangle_2, \ \forall \ \Theta \in H(\Omega). \tag{17}$$

Furthermore, considering $\Theta = U_1$ in (17), we conclude that

$$\lambda_1 = \Upsilon_C(U_1) = \|U_1\|_C^2 = \mu\|U_1\|_2^2 = \mu.$$

Consequently, the pair $(\lambda_1, U_1)$ is a weak solution for eigensystem **(II)** and so $U_1$ is a Neumann eigenfunction associated to $\lambda_1$ (notice that $U_1 \in \mathbb{L}$).

Next, we will prove that $\lambda_1$ is the least positive eigenvalue of the Neumann eigensystem **(II)**. If not, there exist $\tilde{U} \in H(\Omega) \setminus \{0\}$ and $0 < \tilde{\lambda} < \lambda_1$ such that

$$\langle \tilde{U}, V\rangle_C = \tilde{\lambda}\langle \tilde{U}, V\rangle_2, \ \forall \ V \in H(\Omega).$$

Thus, if $V = \frac{\tilde{U}}{\|\tilde{U}\|_2^2}$, then $\Upsilon_C(V) = \tilde{\lambda}$ and $V \in \mathbb{L}$. Hence, from the definition of $\lambda_1$, $\lambda_1 \leq \tilde{\lambda}$, which is an absurd. Therefore (N-2) is valid. Notice that $\lambda_1 > 0$, since $U_1 \in \mathbb{L}$.

Finally, the proof of (N-3) is immediate by the characterization of $\lambda_1$, given in the proof of (N-1). □

# 9 Proof of Theorem 3.5

We will prove the Theorem 3.5 by induction in $k$. The Theorem 3.4 implies that the Theorem 3.5 is valid for $k = 1$. Suppose that the Theorem 3.5 holds for $1 \leq k \leq J$ and we will verify for $J + 1$.

**Claim 1:** There exists $\widehat{U} \in \mathbb{L}_J$ such that $\lambda_{J+1} = \Upsilon_C(\widehat{U})$.

Following the same argument as in the proof of the Theorem 3.4, we may assume that there exist a sequence $(U_{n_l})$ in $\mathbb{L}_J$ and $\widehat{U} \in H(\Omega)$ such that $U_{n_l} \to \widehat{U}$ in $(\left[L^2(\Omega)\right]^2, \|\cdot\|_2)$. Also, we know, by induction assumption, that

$$\langle U_{n_l}, U_i\rangle_2 = 0, \ \forall \ l \in \mathbb{N}.$$

Since $\langle U_{n_l}, U_i\rangle_2 \to \langle \widehat{U}, U_i\rangle_2$, we have that $\widehat{U} \in \mathbb{L}_J$. Therefore, $\Upsilon_C(\widehat{U}) \geq \lambda_{J+1}$. Furthermore, $\Upsilon_C(\widehat{U}) \leq \lambda_{J+1}$, because of the Remark 4.1,

$$\Upsilon_C(\widehat{U})^{\frac{1}{2}} = \|\widehat{U}\|_C \leq \liminf_{l \to +\infty} \|U_{n_l}\|_C = \lambda_{J+1}^{\frac{1}{2}}$$

and thus the Claim 1 is held. Let us denote $\widehat{U}$ by $U_{J+1}$. Notice that $U_{J+1}$ is an element of $H(\Omega) \setminus \{0\}$, because $U_{J+1} \in \mathbb{L}$.

**Claim 2:** If the pairs $(\lambda, U)$ and $(\mu, W)$ are weak solutions for the eigensystem **(II)**, with $\lambda \neq \mu$, then $\langle U, W\rangle_2 = 0$.

By hypothesis, we may assume that

$$\langle U, \Theta\rangle_C = \lambda\langle U, \Theta\rangle_2 \text{ and } \langle W, \Theta_1\rangle_C = \mu\langle W, \Theta_1\rangle_2, \ \forall \ \Theta, \Theta_1 \in H(\Omega).$$

Choosing $\Theta = W$ and $\Theta_1 = U$, we obtain that

$$(\lambda - \mu)\,\langle U, W\rangle_2 = \langle U, W\rangle_C - \langle W, U\rangle_C = 0.$$

But, $\lambda \neq \mu$, therefore $\langle U, W\rangle_2 = 0$ and thus the Claim 2 is proved.

**Claim 3:** The pair $(\lambda_{J+1}, U_{J+1})$ is a weak solution for the eigensystem **(II)**.

Let, for each $1 \leq i \leq J$ fixed, $G_i : H(\Omega) \to \mathbb{R}$, defined by $G_i(U) = \langle U, U_i\rangle_2$, for $U \in H(\Omega)$. By the Proposition 4.3, $G_i = G_{U_i}$ is an element of $C^1(H(\Omega), \mathbb{R})$ with Fréchet derivative in $U \in H(\Omega)$,

$$G_i^{'}(U)(V) = \langle V, U_i\rangle_2, \ \forall \ V \in H(\Omega).$$

Moreover, in reason of the Claim 1, $\Upsilon_C(U_{J+1}) = \lambda_{J+1}$ is an extremal of $\Upsilon_C$ constrained to

$$\mathbb{L}_J = G^{-1}\left(\{G(U_{J+1})\}\right) \cap \left[\bigcap_{i=1}^{J} G_i^{-1}\left(\{G_i(U_{J+1})\}\right)\right].$$

Since, by induction assumption and Claim 2, $detB(U_{J+1}, U_1, U_2, \cdots, U_J) = 2 \neq 0$, where

$$B(U_{J+1}, U_1, \cdots, U_J) = \begin{pmatrix} G^{'}(U_{J+1})(U_{J+1}) & G^{'}(U_{J+1})(U_1) & \dots & G^{'}(U_{J+1})(U_J) \\ G_1^{'}(U_{J+1})(U_{J+1}) & G_1^{'}(U_{J+1})(U_1) & \dots & G_1^{'}(U_{J+1})(U_J) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ G_J^{'}(U_{J+1})(U_{J+1}) & G_J^{'}(U_{J+1})(U_1) & \dots & G_J^{'}(U_{J+1})(U_J) \end{pmatrix},$$

it follows from the Lagrange multipliers Theorem (see [18]) that there exist constants $\lambda, \mu_i \in \mathbb{R}$, for $i = 1, 2, \dots, J$, such that

$$\Upsilon_C^{'}(U_{J+1})(V) = \lambda G^{'}(U_{J+1})(V) + \sum_{i=1}^{J} \mu_i G_i^{'}(U_{J+1})(V), \ \forall\ V \in H(\Omega).$$

In view of the Claim 2 and of the fact of $U_i \in \mathbb{L}$, for $i = 1, 2, \cdots, J$, we have that $\langle U_i, U_j\rangle_2 = \delta_{ij}$, for $i, j \in \{1, 2, \dots, J\}$, therefore, we obtain $\Upsilon_C^{'}(U_{J+1})(U_i) = \lambda G^{'}(U_{J+1})(U_i) + \mu_i$. Furthermore, for $1 \leq i \leq J$, $\Upsilon_C^{'}(U_{J+1})(U_i) = \Upsilon_C^{'}(U_i)(U_{J+1})$. It follows, by induction assumption, that $\Upsilon_C^{'}(U_i)(U_{J+1}) = \lambda_i G^{'}(U_i)(U_{J+1})$, $\forall\ 1 \leq i \leq J$. Hence,

$$\mu_i = \lambda_i G^{'}(U_i)(U_{J+1}) - \lambda G^{'}(U_{J+1})(U_i), \ \forall\ 1 \leq i \leq J.$$

But, as $U_{J+1} \in \mathbb{L}_J$, $\langle U_{J+1}, U_i\rangle_2 = 0$, $\forall\ 1 \leq i \leq J$ and therefore

$$G^{'}(U_i)(U_{J+1}) = G^{'}(U_i)(U_{J+1}) = 0, \ \forall\ 1 \leq i \leq J.$$

Consequently, $\mu_i = 0$, $\forall\ 1 \leq i \leq J$ and

$$\Upsilon_C^{'}(U_{J+1})(V) = \lambda G^{'}(U_{J+1})(V), \ \forall\ V \in H(\Omega).$$

Equivalently,

$$\langle U_{J+1}, V\rangle_C = \lambda \langle U_{J+1}, V\rangle_2, \ \forall\ V \in H(\Omega). \tag{18}$$

Considering $V = U_{J+1}$ in (18), we obtain $\lambda = \lambda_{J+1}$, that is, the pair $(\lambda_{J+1}, U_{J+1}) \in \mathbb{R} \times [H(\Omega) \setminus \{0\}]$ is a weak solution for the eigensystem **(II)**.

**Claim 4:** $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_J \leq \lambda_{J+1}$.

Indeed, we have $\lambda_{J+1} = \inf_{U \in \mathbb{L}_J} \Upsilon_C(U)$ and $\mathbb{L}_J \subset \mathbb{L}_{J-1}$. Thus,

$$\lambda_{J+1} = \inf_{U \in \mathbb{L}_J} \Upsilon_C(U) \geq \inf_{U \in \mathbb{L}_{J-1}} \Upsilon_C(U) = \lambda_J.$$

Furthermore, by induction assumption

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_J.$$

Consequently, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_J \leq \lambda_{J+1}$ and so the Claim 4 is valid.

Finally, following the steps similar to those used in the proof of Theorem 3.2, we conclude that $\lim_{j\to\infty} \mu_j = \infty$ and the dimension of the eigenspace associated to $\lambda_j$ is finite, for each $k$. Therefore, the proof of Theorem 3.5 is complete. $\square$

# 10 Proof of Theorem 3.6

In view of the condition (SN-2), given in the Theorem 3.5,

$$\mathcal{N} = \{U_1^1, \ldots, U_{\tau_1}^1, U_1^2, \ldots, U_{\tau_2}^2, \ldots, U_1^k, \ldots, U_{\tau_k}^k, \ldots\}$$

is an orthonormal subset of $\left[L^2(\Omega)\right]^2$.

As consequence of $dim\mathbb{F}_j < \infty$, $\mathbb{F}_j$ is a closed subspace of $\left[L^2(\Omega)\right]^2$ and thus $\left[L^2(\Omega)\right]^2 = \mathbb{F}_j \oplus \mathbb{F}_j^{\perp}$. Therefore, we obtain that $H(\Omega) = \mathbb{F}_j \oplus \left[\mathbb{F}_j^{\perp} \cap H(\Omega)\right]$. Hence, the proof of Theorem 3.6 is complete. □**Remark 10.1.** Consider the sequence $\mathbb{O} = (V_k)$, defined by

$$V_k = \begin{cases} U_k^{j+1} & if\ 1 \le k \le \tau_{j+1}, \\ U_{k-\tau_{j+l}}^{j+l+1} & if\ \tau_{j+l} < k \le \tau_{j+l} + \tau_{j+l+1},\ l \ge 1. \end{cases}$$

Then $\mathbb{O}$ is a total orthonormal sequence in $\mathbb{F}_j^{\perp} \cap H(\Omega)$ (the argument is the same as in Theorem 3.3) and thus $\mathbb{O}$ is a Hilbert basis of $\mathbb{F}_j^{\perp} \cap H(\Omega) \subset \left[L^2(\Omega)\right]^2$. Consequently, for $U \in \mathbb{F}_j^{\perp} \cap H(\Omega)$,

$$U = \sum_{k=1}^{\infty} \langle U, V_k \rangle_2 V_k \text{ and } \|U\|_2^2 = \sum_{k=1}^{\infty} \left|\langle U, V_k \rangle_2\right|^2. \tag{19}$$

**Remark 10.2.** We note that

$$\mathbb{F}_j^{\perp} \cap H(\Omega) = \{U \in H(\Omega) : \langle U, V \rangle_C = 0,\ \forall\ V \in \mathbb{F}_j\} \doteq (\mathbb{F}_j)_C^{\perp}$$

is an closed subspace of $(H(\Omega), \|\cdot\|_C)$. Also, for $V \in \mathcal{B}_k$ and $U \in H(\Omega)$, we have that

$$\langle U, V \rangle_C = \lambda_k \langle U, V \rangle_2, \tag{20}$$

so that $\mathbb{O}_C = (W_k)$, defined by $W_k = \frac{V_k}{\|V_k\|_C}$, is a total $C$-orthonormal sequence in $(\mathbb{F}_j)_C^{\perp}$ (the argument is the same as in the Theorem 3.3). Thus $\mathbb{O}_C$ is a Hilbert basis for $(\mathbb{F}_j)_C^{\perp}$. Consequently, we obtain that, for $U \in (\mathbb{F}_j)_C^{\perp} \cap H(\Omega)$,

$$U = \sum_{k=1}^{\infty} \langle U, W_k \rangle_C W_k \text{ and } \|U\|_C^2 = \sum_{k=1}^{\infty} \left|\langle U, W_k \rangle_C\right|^2. \tag{21}$$

**Remark 10.3.** Combining the identities given in (19) and (21) with (20), we obtain, for $U$ in $\mathbb{F}_j^{\perp} \cap H(\Omega) = (\mathbb{F}_j)_C^{\perp}$, that

$$\|U\|_2^2 = \sum_{k=1}^{\infty} \sigma_k^{-2} |\langle U, V_k \rangle_C|^2 \text{ and } \|U\|_C^2 = \sum_{k=1}^{\infty} \sigma_k^{-1} |\langle U, V_k \rangle_C|^2,$$

where

$$\sigma_k = \begin{cases} \lambda_{j+1} & \text{if } 1 \le k \le \tau_{j+1}, \\ \lambda_{j+l+1} & \text{if } \tau_{j+l} < k \le \tau_{j+l} + \tau_{j+l+1}, l \ge 1. \end{cases}$$

Now, since $0 < \lambda_{j+1} \le \lambda_k$, for every $k \ge j+1$, we have that $\lambda_k^{-1} \le \lambda_{j+1}^{-1}$, so that $\sigma_k \le \lambda_{j+1}^{-1}$, for every $k \ge j+1$. Therefore,

$$\|U\|_2^2 \le \lambda_{j+1}^{-1} \sum_{k=1}^{\infty} \sigma_k^{-1} |\langle U, V_k \rangle_C|_C^2 = \lambda_{j+1}^{-1} \|U\|_C^2.$$

**Remark 10.4.** Let $U \in \mathbb{F}_j$. We may assume that there exist unique constants $c_k \in \mathbb{R}$, $1 \le k \le \tau$, such that $U = \sum_{k=1}^{\tau} c_k Z_k$, where, for $1 \le k \le \tau$ and $1 \le l \le j-1$ (when $j \ge 2$),

$$Z_k = \begin{cases} V_k^1 & \text{if } 1 \le k \le \tau_1, \\ V_{k-\tau_l}^{l+1} & \text{if } \tau_l < k \le \tau_l + \tau_{l+1}, \end{cases}$$

because $\bigcup_{k=1}^{j} \mathcal{B}_k$ is a basis of $\mathbb{F}_j$. Also, by $\{Z_k : 1 \le k \le \tau\} \subset \mathcal{N}$, we conclude that

$$\|U\|_2^2 = \sum_{k=1}^{\tau} c_k^2.$$

It follows from (20) that $\langle Z_k, Z_l \rangle_C = 0$, for $1 \le k,\ l \le \tau,\ k \ne l$. Thus,

$$\|U\|_C^2 = \sum_{k=1}^{\tau} c_k^2 \|Z_k\|_C^2. \tag{22}$$

Finally, we claim that, for $U \in \mathbb{F}_j$,

$$\|U\|_2^2 \ge \lambda_j^{-1} \|U\|_C^2.$$

Indeed, if we denote, for $1 \le k \le \tau$ and $1 \le l \le j-1$ (when $j \ge 2$),

$$\varsigma_k = \begin{cases} \lambda_1 & \text{if } 1 \le k \le \tau_1, \\ \lambda_{l+1} & \text{if } \tau_l < k \le \tau_l + \tau_{l+1}, \end{cases}$$

then, using (20) in (22), we conclude that

$$\|U\|_C^2 = \sum_{k=1}^{\tau} c_k^2 \varsigma_k.$$

Consequently,

$$\|U\|_C^2 \le \lambda_j \sum_{k=1}^{\tau} c_k^2 = \lambda_j \|U\|_2^2,$$

because $\lambda_k \le \lambda_j$, for $1 \le k \le j$, $\varsigma_k \le \lambda_j$ and $1 \le k \le \tau$.

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