On Steklov-Neumann boundary value problems for some quasilinear elliptic equations*

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Abstract

We will study a class of Steklov-Neumann boundary value problems for some quasilinear elliptic equations. We obtain result ensuring the existence of solutions when resonance and nonresonance conditions occur. The result was obtained by using variational arguments.

Keywords Steklov-Neumann eigenvalue, variational methods, $p-$Laplacian.

1 Introduction

In this work, we will show existence results for the following class of Steklov-Neumann boundary value problems for some quasilinear elliptic equations

$$\begin{cases} -\triangle_p u + c(x)u = f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} = g(x,u), & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian operator, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^N$, for $N \geq 2$, is a bounded domain with $\partial\Omega \in C^{0,1}$, and $\dfrac{\partial}{\partial \eta} \doteq \eta \cdot \nabla$ is a normal derivative on $\partial\Omega$. Here the functions $c : \Omega \to \mathbb{R}$ and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verify the following conditions

(P1) $c \in L^\infty(\Omega)$, $c(x) \geq 0$, for almost everywhere $x \in \Omega$ and $\int_\Omega c(x)dx > 0$.

(P2) $f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

(P3) There exist constants $a_1, a_2 > 0$ such that

$$|g(x,u)| \leq a_1 + a_2|u|^s, \ \forall \ (x,u) \in \overline{\Omega} \times \mathbb{R},$$

with $0 < s < p^1_*(N) - 1$, where $p^1_*(N) = \frac{(N-1)p}{N-p}$ if $p < N$ and $p^1_*(N) = \infty$ if $p \geq N$.

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(P3') There exist constants $b_1, b_2 > 0$ such that

$$|f(x,u)| \leq b_1 + b_2|u|^t, \ \forall \ (x,u) \in \overline{\Omega} \times \mathbb{R},$$

with $0 < t < p_*(N) - 1$, where $p_*(N) = \frac{Np}{N-p}$ if $p < N$ and $p_*(N) = \infty$ if $p \geq N$.

Our result is established when the nonlinearity interacts in some sense with Steklov and Neumann eigenvalues for $p-$Laplacian operator. For that consider the following eigenvalue problems

$$\begin{cases} -\triangle_p u + c(x)|u|^{p-2}u = 0, & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} = \mu |u|^{p-2}u, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} -\triangle_p u + c(x)|u|^{p-2}u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is $C^{0,1}$ bounded domain, $N \geq 2$, $\triangle_p u$ is p-Laplacian operator, $p \in (1,\infty)$, $\dfrac{\partial}{\partial \eta} \doteq \eta \cdot \nabla$ is a normal derivative on $\partial\Omega$, and $c : \Omega \to \mathbb{R}$ satisfies the condition **(P1)**.

Since c verifies **(P1)**, the following norms given by

$$\|u\|_c = (\int_\Omega [|\nabla u|^p + c(x)|u|^p])^{\frac{1}{p}} \text{ and } \|u\|_{1,p} = (\int_\Omega [|\nabla u|^p + |u|^p]dx)^{\frac{1}{p}},$$

are equivalents in $W^{1,p}(\Omega)$. Moreover, $(W^{1,p}(\Omega), \|\cdot\|_{1,p})$ is a reflexive space for $p \in (1,\infty)$.

It is easy see that the following functionals $\tilde{\delta}, \beta : (W^{1,p}(\Omega), \|\cdot\|_c) \to \mathbb{R}$, defined by

$$\Upsilon_c(u) = \|u\|_c^p, \ \tilde{\delta}(u) = \|u\|_p^p - 1, \text{ and } \beta(u) = \|u\|_{p,\partial}^p,$$

belong to $C^1(W^{1,p}(\Omega), \mathbb{R})$, where

$$||u||_p = (\int_\Omega |u|^p dx)^{1/p} \text{ and } ||u||_{p,\partial} = (\int_{\partial\Omega} |u|^p dx)^{1/p},$$

are norms in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

Define

$$\alpha_1 = \sup\{\beta(u) : u \in \mathbb{K}\},$$

where $\mathbb{K} = \{u \in W^{1,p}(\Omega) : \Upsilon_c(u) \leq 1\}$ is a closed, bounded, and weakly compact set. By the Sobolev compact embedding, α_1 is attained and α_1 is positive. Moreover, $\mu_1 = \frac{1}{\alpha_1}$ is the first eigenvalue, called the first "Steklov"eigenvalue. By the characterization, we have the following inequality

$$\|u\|_c^p \geq \mu_1 \|u\|_{p,\partial}^p, \ \forall \ u \in W^{1,p}(\Omega). \tag{4}$$

Analogously define

$$\rho_1 = \inf_{u \in \mathbb{L}} \Upsilon_c(u),$$

where $\mathbb{L} = \{u \in W^{1,p}(\Omega) : \tilde{\delta}(u) = 0\}$. By the Sobolev compact embedding, ρ_1 is attained, ρ_1 is positive, and $\lambda_1 = \rho_1$ is the first eigenvalue, called the first "Neumann"eigenvalue. By the characterization we have the following inequality

$$\|u\|_c^p \geq \lambda_1 \|u\|_p^p, \ \forall\ u \in W^{1,p}(\Omega). \tag{5}$$

Eigenvalue problems were extensively studied by several researchers, we would like to mention [2, 3, 4, 10] and references therein. For eigenvalue problems involving $p-$Laplacian operator we mention for instance [12, 11, 14] and references therein.

Before enunciate our main theorem we recall the following definition.

Definition 1.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution for the problem (1) if

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \nabla v + c(x)|u|^{p-2}uv\right] dx = \int_\Omega f(x,u)v dx + \int_{\partial\Omega} g(x,u)v d\sigma, \ \forall v \in W^{1,p}(\Omega).$$

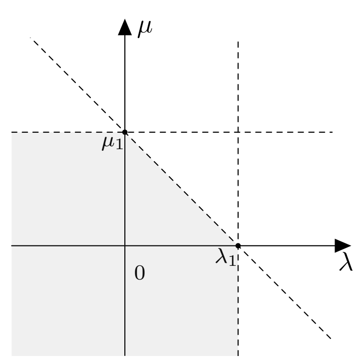
We shall establish our main result.

Theorem 1.2. *In addition to **(P1)**, **(P2)**, **(P3)**, and **(P3')**, suppose that $F, G : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, defined by $F(x,u) = \int_0^u f(x,s)ds$ and $G(x,u) = \int_0^u g(x,s)ds$, verify the following condition:*
(P4) *There exist constants $\lambda, \mu \in \mathbb{R}$ such that*

$$\limsup_{|u|\to+\infty} \frac{pG(x,u)}{|u|^p} \leq \mu < \mu_1 \ \textit{and} \ \limsup_{|u|\to+\infty} \frac{pF(x,u)}{|u|^p} \leq \lambda < \lambda_1,$$

uniformly for $x \in \overline{\Omega}$, with $\lambda_1\mu + \mu_1\lambda < \mu_1\lambda_1$. Then the problem (1) possesses at least one weak solution $u \in W^{1,p}(\Omega)$.

We have in the picture below a illustration on the cartesian plane $\lambda\mu$ of the region described by $\lambda_1\mu + \mu_1\lambda < \mu_1\lambda_1$, $\mu < \mu_1$, and $\lambda < \lambda_1$.



For $p = 2$, Auchmuty [4], proved that the Steklov eigenfunctions formed a complete orthonormal system for the space $[H_0^1(\Omega)]^\perp$ in $H^1(\Omega)$, with respect to specific inner products. In [15], for $p = 2$, the authors studied same problem with nonlinearities interacting with high order eigenvalues. Our result extends in part their work, for $p \neq 2$. In [5, 9, 13] the authors treated nonlinear Neumann problem, when $p = 2$, and the same problem involving $p-$Laplacian were studied by [6, 8, 16, 20]. There are few works treating nonlinear Steklov problem, we found only the following papers [1, 7] leading with $p = 2$ and [18] for $p > 1$.

2 Preliminaries

Since our approach is variational, we define the Euler-Lagrange functional associated to the problem (1), $I_p : (W^{1,p}(\Omega), \|\cdot\|_c) \to \mathbb{R}$ by

$$I_p(u) = \frac{1}{p}\int_\Omega [|\nabla u|^p + c(x)|u|^p]\, dx - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

where $G(x,u) = \int_0^u g(x,s)ds$ and $F(x,u) = \int_0^u f(x,s)ds$. We have that I_p belongs to $C^1(W^{1,p}(\Omega),\mathbb{R})$ and its Gâteaux derivative is given by

$$I_p'(u)v = \int_\Omega \left[|\nabla u|^{p-2}\nabla u \nabla v + c(x)|u|^{p-2}uv\right] dx - \int_\Omega f(x,u)v dx - \int_{\partial\Omega} g(x,u)v d\sigma,\ \forall u,v \in W^{1,p}(\Omega).$$

Therefore the critical points of I_p are exactly the weak solutions of problem (1).

Definition 2.1. Let $(E, \|\cdot\|)$ be a Banach space, and a functional $J \in C^1(E,\mathbb{R})$. We say that J satisfies Palais-Smale condition, (PS) in short, if every sequence (u_m) in E, such that

$$(J(u_m)) \text{ is bounded in } \mathbb{R} \text{ and } J'(u_m) \to 0 \text{ in } (E^*, \|\cdot\|^*),$$

admits a convergent subsequence in E.

The following classic abstract result can be found in [19, 17].

Proposition 2.2. *Let E a Banach space. If $J \in C^1(E,\mathbb{R})$ is bounded from below and satisfies (PS) condition, then $c \doteq \inf_E J$ is a critical value of J.*

3 Sketch of the Proof.

By continuity of F, G and **(P4)**, we have

$$G(x,u) \le \frac{1}{p}(\mu+\epsilon)|u|^p + M_\epsilon \text{ and } F(x,u) \le \frac{1}{p}(\lambda+\epsilon)|u|^p + M_\epsilon, \qquad (6)$$

for all $x \in \overline{\Omega}$ and $u \in \mathbb{R}$. Using this fact, we will prove the following claim.

Claim 1: The functional I_p is coercive on $(W^{1,p}(\Omega), \|\cdot\|_c)$, that is,

$$I_p(u) \to +\infty \text{ as } \|u\|_c \to +\infty. \quad (7)$$

Indeed, suppose $\|u\|_c \to +\infty$. Then either $\|u\|_{p,\partial} \to +\infty$ or $\|u\|_{p,\partial} \leq \tilde{K}_1$, where $\tilde{K}_1$ is a constant.
Case 1: There exists a constant $\tilde{K}_1 > 0$ such $\|u\|_{p,\partial} \leq \tilde{K}_1$.
From inequality (6), we obtain

$$I_p(u) \geq \frac{1}{p}\|u\|_c^p - \frac{1}{p}(\lambda+\epsilon)\|u\|_p^p - \frac{1}{p}(\mu+\epsilon)\|u\|_{p,\partial}^p - M_\epsilon(|\Omega| + |\partial\Omega|_\sigma). \quad (8)$$

If $\lambda < 0$, then $I_p(u) \to +\infty$, as $\|u\|_c \to +\infty$. On the other hand, if $\lambda \geq 0$, by inequalities (5) and (8), we get

$$I(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1}\right)\|u\|_c^p - \frac{1}{p}(\mu+\epsilon)\|u\|_{p,\partial}^p - M_\epsilon(|\Omega| + |\partial\Omega|_\sigma),$$

therefore, since $\lambda < \lambda_1$, choosing $\epsilon > 0$ such that $1 - \frac{\lambda}{\lambda_1} - \dfrac{\epsilon}{\lambda_1} > 0$, we conclude that $I_p(u) \to +\infty$, as $\|u\|_c \to +\infty$.
Case 2: $\|u\|_{p,\partial} \to +\infty$.
We have four situations, to consider.
• $\lambda < 0$ and $\mu < 0$.
From (8), for $\epsilon > 0$, sufficiently small, we have

$$I_p(u) \geq \frac{1}{p}\|u\|_c^p - M_\epsilon(|\Omega| + |\partial\Omega|_\sigma).$$

So, if $\|u\|_c \to +\infty$, then $I_p(u) \to +\infty$.
• $\lambda < 0$ and $\mu \geq 0$.
From (4) and (8), for $\epsilon > 0$, sufficiently small, we get

$$I_p(u) \geq \frac{1}{p}(1 - \frac{\mu}{\mu_1} - \frac{\epsilon}{\mu_1})\|u\|_c^p - M_\epsilon(|\Omega| + |\partial\Omega|_\sigma).$$

Since $\mu < \mu_1$, choosing $\epsilon > 0$ such that $\lambda + \epsilon < 0$ and $1 - \frac{\mu}{\mu_1} - \frac{\epsilon}{\mu_1} > 0$, we infer that $I_p(u) \to +\infty$, as $\|u\|_c \to +\infty$.
• $\lambda \geq 0$ and $\mu < 0$.
By hypothesis $\lambda < \lambda_1$. Choose $\epsilon > 0$ such that $\mu + \epsilon < 0$ and $1 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1} > 0$. By inequalities (5) and (8), we get

$$I_p(u) \geq \frac{1}{p}(1 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1})\|u\|_c^p - M_\epsilon(|\Omega| + |\partial\Omega|_\sigma).$$

Therefore if $\|u\|_c \to +\infty$, then $I_p(u) \to +\infty$.
• $\lambda \geq 0$ and $\mu \geq 0$.
Using inequalities (5) and (8), we have, for $C(\epsilon) \doteq M_\epsilon(|\Omega| + |\partial\Omega|_\sigma)$,

$$I_p(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1}\right)\|u\|_c^p - \frac{1}{p}(\mu+\epsilon)\|u\|_{p,\partial}^p - C(\epsilon). \quad (9)$$

By hypotheses $\lambda < \lambda_1$ and $\lambda\mu_1 + \mu\lambda_1 < \lambda_1\mu_1$. Then choosing $\epsilon > 0$ such that

$$1 - \frac{\lambda}{\lambda_1} - \frac{\epsilon}{\lambda_1} > 0 \text{ and } (1 - \frac{\lambda}{\lambda_1} - \frac{\mu}{\mu_1}) - \epsilon \left(\frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) > 0,$$

therefore, from (4) and (9), we infer

$$I_p(u) \geq \frac{\mu_1}{p}[(1 - \frac{\lambda}{\lambda_1} - \frac{\mu}{\mu_1}) - \epsilon(\frac{1}{\lambda_1} + \frac{1}{\mu_1})]\|u\|_{p,\partial}^p - C(\epsilon).$$

Then, by our assumption ($\|u\|_{p,\partial} \to +\infty$) follow that $I_p(u) \to +\infty$, as $\|u\|_c \to +\infty$.

Hence the functional I_p is coercive.

Claim 2: The functional I_p is bounded from below.

This is an immediate consequence of the Claim 1.

Claim 3: I_p verifies (PS), the Palais Smale condition.

Let (u_m) be a sequence in $(W^{1,p}(\Omega), \|\cdot\|_c)$ with $(I_p(u_m))$ bounded in $\mathbb{R}$ and $I_p'(u_m) \to 0$ in $(W^{1,p}(\Omega)^*, \|\cdot\|_c^*)$.

Since the operators $L_i : W^{1,p}(\Omega), \|\cdot\|_c) \to \mathbb{R}$ $(i = 1, 2)$, given by

$$L_1(u) = \int_\Omega F(x, u)dx \text{ and } L_2(u) = \int_{\partial\Omega} G(x, u)d\sigma,$$

are weakly continuous and their derivatives $L_i'(i = 1, 2)$ are compacts, it is enough to show that (u_m) is bounded in $(W^{1,p}(\Omega), \|\cdot\|_c)$. If not, there exists a subsequence (u_{m_k}) of (u_m) such that $\|u_{m_k}\|_c \to +\infty$, as $k \to +\infty$. Therefore, by the coercivity, I_p, $I_p(u_{m_k}) \to +\infty$, as $k \to +\infty$, which is a contradiction because $(I_p(u_m))$ bounded in $\mathbb{R}$.

Now, we can conclude the proof of the Theorem 1.2, by applying Proposition 2.2. Hence I_p has at least one critical point $u \in W^{1,p}(\Omega)$, that is, $I_p'(u) = 0$. Then u is weak solution of problem (1). The proof of Theorem 1.2 is complete.

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