On a quasilinear Schrödinger problem at resonance *

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Abstract

In this paper we establish the existence of standing wave solutions for quasilinear Schrödinger equations involving subcritical growth at resonance. By using a change of variables, the quasilinear equation is reduced to semilinear one, which associated functional is well defined in the usual Sobolev space. The "first" eigenvalue type of a nonnhomogeneous operator was studied. Using this fact and a variant of the monotone operator theorem, we show that the problem at resonance has at least one nontrivial solution.

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1 Introduction

This paper is concerned with the quasilinear elliptic equation

$$-\Delta u + V(x)u - k\Delta(u^2)u = p(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

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with $N \geq 1$, $V : \mathbb{R}^N \to \mathbb{R}$ a function called potential and $p : \mathbb{R} \to \mathbb{R}$ a continuous function. Such class of problems arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. Part of the interest is due to the fact that the solutions of (1) are related to the existence of standing waves of the following quasilinear Schrödinger equation of the form

$$i\partial_t z = -\Delta z + W(x)z - p(|z^2|)z - k\Delta \left[g(|z^2|)\right] g'(|z^2|)z, \tag{2}$$

where $W = W(x)$, $x \in \mathbb{R}^N$, is a given potential, k is a real constant and p, g are real functions.

Quasilinear equations of the form (2) have been established in several areas of physics corresponding to various types of g. For instance, the case $g(s) = s$ was used for the superfluid film equation in plasma physics in [14]. In the case $g(s) = (1+s)^{1/2}$, the equation (2) models the self-channeling of a high-power ultrashort laser in matter, see [3] and [4].

Here we consider the case where $g(s) = s$ and $k = 1$ and our special interest is in the existence of standing wave solutions, that is, solutions of type $z \equiv \psi(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that ψ satisfies (2) if and only if the function $u(x)$ solves the equation of elliptic type (1), where $V(x) := W(x) - E$ is the new potential.

In order to seek solutions to the equation (1) two variational methods have been widely used, mainly in the subcritical and critical situation. That is, for the case $p(s) = |s|^{r-1} s$ with $N \geq 3$, $\quad r + 1 \leq 22^* = 4N/(N-2)$, where 22^* behaves like a critical exponent for the equation (1), see details in [16, Remark 3.13]. For the subcritical case $r + 1 < 22^*$, in the first, which was started in [20] and extended in [15], direct variational methods, using constrained minimization arguments, were used to provide existence of positive solutions results with an unknown Lagrange multiplier λ in front of the nonlinear term. The second and more general method, which was started in [16], uses an innovative change of variables which allows to rewrite the functional in semilinear form. With this tool, they were able to overcome the problem that the functional is not well defined. Thus, critical points can be found in an associated Orlicz space and existence results are given in the case of the bounded, coercive or radial potentials. We recall that, in this new framework, the new problem become a nonhomogeneous problem. Following the strategy developed in [5] on a related problem, the authors of [6] also used a change of variables and they defined an associated equation that they called “dual”. In the recent article [17], the authors solved the equation (1) using subcritical approximations. This approach requires certain conditions on monotonicity of the structure equations.

The critical exponent case $r + 1 = 22^*$ was also considered recently, among others, by [9, 17, 19, 21, 22, 23], and in references therein. Recently, in [18] was considered this class of quasilinear Schrödinger equations and with a new perturbation approach they treated the critical exponent case giving new existence results.

We are concerned in the existence of solutions for problem (1) at resonance, and for that, the study of the first eigenvalue for the nonnhomogeneous operator $Lu = -\Delta u - \Delta(u^2)u$ was the initial difficulty and a challenge that we had to overcome. To the best of the authors knowledge there exists a few works studying of the spectrum of this operator.

To be precise, one of our objectives in this work is to study the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ 0 \not\equiv u \geq 0 & \text{on } \Omega, \end{cases} \tag{3}$$

where λ is positive constant and $\Omega \subset \mathbb{R}^N$ a bounded smooth domain.

Our main result is the following.

Theorem 1.1. *There is a number Λ with $\lambda_1 < \Lambda \leq +\infty$ such that for all $\lambda_1 \leq \lambda < \Lambda$, problem (3) admits a nonnegative solution, where λ_1 is the first eigenvalue of the Laplacian operator.*

As a consequence of the above result we obtain.

Proposition 1.1. *There exists $0 \neq v_0 \in W := H_0^1(\Omega)$, a minimizer of the minimizing problem*

$$\overline{\lambda} \equiv \overline{\lambda}(\Omega) = \inf_{0 \neq u \in W} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |f(u)|^2 \, dx},$$

where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{\sqrt{1+2f^2(t)}}, \quad \textit{in} \quad [0, +\infty), \\ f(t) &= -f(-t), \quad \textit{in} \quad (-\infty, 0]. \end{aligned} \tag{4}$$

Remark 1.1. *The number $\overline{\lambda} \equiv \overline{\lambda}(\Omega)$ is the "first" eigenvalue of the nonhomogeneous operator $Lu = -\Delta u - \Delta(u^2)u$.*

As an application of the above results, we will study a resonance problem, which was inspired by article [2]. Consider the following problem at resonance for nonhomogeneous operator:

$$\begin{cases} -\Delta u - \Delta(u^2)u = \overline{\lambda} u - g(x,u), & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{5}$$

where $\overline{\lambda}$ is that defined in Proposition1.1, $\Omega \subset \mathbb{R}^N$ a bounded smooth domain, and g satisfies the following assumptions:

(G_1) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $g(x,0) \neq 0$;

(G_2) $|g(x,s)| \leq \sigma(x) + \rho(x)\,|s|^r$, for a.e. $x \in \Omega$ and $s \in \mathbb{R}$, where $0 < r < 2(2^*) - 2$, $\sigma \in L^{(2^*)'}(\Omega)$ and $0 \leq \rho \in L^\infty(\Omega)$;

(G_3) $\inf_{s\in\mathbb{R}, s\neq 0} g(x,s)/s > \overline{\lambda}$,

we get the following result.

Theorem 1.2. *If assumptions* (G_1), (G_2), (G_3) *are fulfilled, then problem (5) has at least one weak nontrivial solution.*

This paper is organized as follows. In Section 2, we prove Theorems 1.1 - 1.2 and in Section 3 provides some abstract results that we use to prove our main results.

In this paper we make use of the following notation:
ϕ_1 denotes the first eigenfunction of the operator $-\Delta$ in $H_0^1(\Omega)$;
λ_1 is the first eigenvalue associated with eigenfunction ϕ_1;
$B_R(0)$ denotes the ball centered at the origin and radius R;
$\|u\|_s = \left(\int_\Omega |u|^s \right)^{1/s}$ denotes the usual norm in L^s-space;
$C, C_1, C_2, ..$ denote positive (possible different) constants;
$N(B)$ and $R(B)$ denote the null space and range of a linear mapping B;
dim and codim are dimension and codimension, respectively;
X^* designs the topological dual of the space X;
We denote the weak and strong convergence in X by " $\rightharpoonup$ " and " $\to$ ", respectively;
$\int_\Omega f$ denotes $\int_\Omega f(x)dx$;
W denotes $H_0^1(\Omega)$ and $|A|$ means the Lebesgue measure of $A \subset \mathbb{R}^N$.

2 Proof of the results

2.1 Proof of the Theorem 1.1

2.1.1 Reformulation of problem (3) and preliminaries

As observed , there are some technical difficulties in applying variational methods directly to the formal functional associated to problem (3) given by

$$J_\lambda(u) = \frac{1}{2} \int_\Omega (1 + 2u^2)\,|\nabla u|^2 - \frac{\lambda}{2} \int_\Omega |u|^2 .$$

The main difficulty is related to the fact that J is not well defined in the usual Sobolev space. To overcome this difficulty, we employ an argument developed in [16] (see also [6]). We make the change of variables $v = f^{-1}(u)$, where f is defined in (4).

Thus, we can write $J_\lambda(u)$ as

$$I_\lambda(v) = J_\lambda(f(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda}{2}\int_\Omega |f(v)|^2 \,,\ v \in W.$$

We observe that nontrivial critical points for I_λ are weak solutions for the problem

$$\begin{cases} -\Delta v = \lambda f'(v)f(v) & \text{in} \quad \Omega, \\ \quad v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

Lemma 2.1. *The function $f(t)$ enjoys the following properties:*

1. *f is uniquely defined C^∞ function and invertible;*
2. *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
3. *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
4. *$f(t)/2 \le tf'(t) \le f(t)$ for all $t \ge 0$;*
5. *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
6. *$|f(t)f'(t)| \le 1/\sqrt{2}$, for all $t \in \mathbb{R}$;*
7. *There exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & \text{if} \quad |t| \le 1 \\ C|t|^{1/2}, & \text{if} \quad |t| \ge 1. \end{cases}$$

Proof: The proofs of (1)-(7) can be found in [10, Lemma 2.1] (see also [16, 12] and [6]). ■

We observe that.

Proposition 2.1. *If v is a critical point of I_λ then any critical point is of class $C^{2,\alpha}(\Omega)$. Moreover, if $v \in C^{2,\alpha}(\Omega)$ is a critical point of the functional I_λ , then the function $u = f(v)$ is a classical solution of (3).*

Proof: The proof is similar to [10, Propostion 2.6]. ■

In order to seek solution to problem (6) we follow the steps found in [1]. To prove the above theorem we need to state and prove some lemmas.

Setting

$$\Lambda = \sup\{\lambda > \lambda_1 : (6) \textit{ admits a } \text{nonnegative solution}\}$$

we have

Lemma 2.2. $\Lambda > \lambda_1$.

Proof: We shall use bifurcation theory to show that (6) admits positive solutions for $\lambda > \lambda_1$ near λ_1. To do this we define $\mathcal{F} : C_0^{2,\beta}(\Omega) \times \mathbb{R} \to C^{0,\beta}(\Omega)$ by $\mathcal{F}(u,\lambda) = -\Delta u - \lambda f(u) f'(u)$. We have that $\mathcal{F}(0,\lambda) = 0$ for all λ. Moreover, by (4) in Lemma 2.1 and $f'(0) = 1$, also implies that $\mathcal{F}_u(0,\lambda_1)v = -\Delta v - \lambda_1 v$. So that,

$$\begin{aligned} N(\mathcal{F}_u(0,\lambda_1)) =& \langle \phi_1 \rangle, \\ \text{codim} R(\mathcal{F}_u(0,\lambda_1)) =& 1 \end{aligned}$$

and

$$\mathcal{F}_{\lambda,u}(0,\lambda_1)\phi_1 = -\phi_1 \notin R(\mathcal{F}_u(0,\lambda_1)).$$

It follows that $(0,\lambda_1)$ is a bifurcation point for $\mathcal{F}$(see [7]).

Thus if we decompose

$$C_0^{2,\beta}(\Omega) = \langle \phi_1 \rangle \oplus V,$$

where $V = \langle \phi_1 \rangle^{\perp}$, then by Theorem (3.1) we obtain a neighborhood U of $(0,\lambda_1)$ in $C_0^{2,\beta}(\Omega) \times \mathbb{R}$, and continuous functions $\phi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to V$ with $\phi(0) = \lambda_1$, $\psi(0) = 0$ and

$$\mathcal{F}^{-1}\{0\} \cap U = \{(\alpha\phi_1 + \alpha\psi(\alpha), \phi(\alpha)) : \alpha \in (-a,a)\} \cup \{(0,\lambda) : (0,\lambda) \in U\}.$$

Set $u_\alpha = \alpha\phi_1 + \alpha\psi(\alpha)$. Note that, in particular, $\psi(\alpha) \to 0$ in $C^{1,\beta}(\overline{\Omega})$ as $\alpha \to 0$, and which $u_\alpha > 0$ em Ω for α sufficiently small.

Next we show that $\phi(\alpha) > \lambda_1$ for all sufficiently small positive α.

Suppose, by contradiction, that there is a sequence $\alpha_n \to 0^+$ with $\phi(\alpha_n) \le \lambda_1$. Let u_n the positive solution of problem (6) associated $\lambda = \phi(\alpha_n)$. Thus,

$$-\int_\Omega \Delta u_n \phi_1 dx - \int_\Omega \phi(\alpha_n) f(u_n) f'(u_n) \phi_1 dx = 0,$$

equivalently, we have

$$\lambda_1 \int_\Omega u_n \phi_1 = \int_\Omega \phi(\alpha_n) f(u_n) f'(u_n) \phi_1 dx. \tag{7}$$

But, by (2)-(3) in Lemma 2.1, we obtain

$$\begin{aligned}\lambda_1\int_\Omega u_n\phi_1 &= \int_\Omega \phi(\alpha_n)f(u_n)f'(u_n)\phi_1 dx\\ &\le \lambda_1\int_\Omega f(u_n)f'(u_n)\phi_1 dx\\ &< \lambda_1\int_\Omega u_n\phi_1 dx,\end{aligned}$$

which gives a contradiction, since $u_n > 0,\ \forall n \in \mathbb{N}$. In the last inequality we used that $f'(t) = 1 \ \Leftrightarrow\ t = 0$, then

$$f(t)f'(t) < t,\ \forall\, t > 0.$$

Therefore, $\phi(\alpha) > \lambda_1$. ■

Lemma 2.3. *Let $\lambda \in (\lambda_1,\ \Lambda)$. Then of problem* (6) *admit a supersolution.*

Proof: It follows from the definition of Λ there exists a $\lambda_0 \in (\lambda,\ \Lambda)$ such that problem (6) admits a nonnegative solution u_+. We have that u_+ is a supersolution to (6).

Indeed for any $\phi \in W$, $\phi \ge 0$ in Ω, by (4) in Lemma 2.1, we have

$$\begin{aligned}\int_\Omega \nabla u_+\nabla\phi dx - \lambda\int_\Omega f(u_+)f'(u_+)\phi dx &= \lambda_0\int_\Omega u_+\phi dx - \lambda\int_\Omega f(u_+)f'(u_+)\phi dx\\ &\ge (\lambda_0-\lambda)\int_\Omega f(u_+)f'(u_+)\phi dx\\ &\ge 0.\end{aligned}$$

■

By Lemma 2.3 there exists a nonnegative supersolution u_+ of problem (6), thus without loss of generality, there exits R_0 such that, $u_+ \ge 0$ in $B_{R_0}(0)$. We can choose $2R < R_0$ so that $u_+(x) \ge C_o > 0$, for all $B_{2R}(0)$.

Fixed a $\epsilon > 0$, define

$$v(x) = \epsilon^\alpha u\left(\epsilon x\right),\ \alpha > 0,$$

where u is a solution of problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad B_{\epsilon R}(0) \subset \Omega \subset \mathbb{R}^N,\\ \quad u = 0 & \text{on} \quad \partial B_{\epsilon R}(0).\end{cases}$$

Thus v satisfies

$$\begin{cases} -\Delta v = \epsilon^{\alpha+2} & \text{in} \quad B_R(0) \subset \Omega \subset \mathbb{R}^N,\\ \quad v = 0 & \text{on} \quad \partial B_R(0)\end{cases}$$

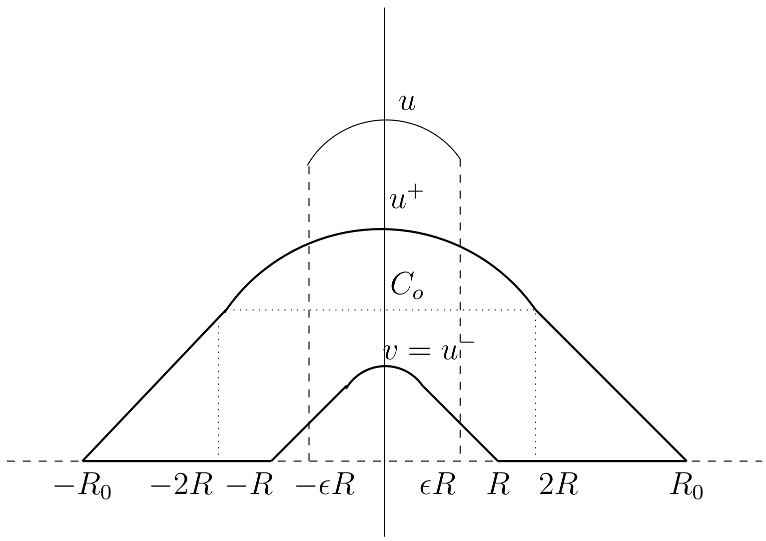
and by the maximum principle we have $v > 0$.

Lemma 2.4. *The function*

$$u_- = \begin{cases} v & in \quad B_R(0), \\ 0 & in \quad \mathbb{R}^N \setminus B_R(0), \end{cases}$$

is a subsolution for (6). And by construction, we can see that

$$u_- < u_+.$$



Proof: Assume that $\psi \in C_0^\infty(\Omega)$, with $\psi \geq 0$. We need to consider two cases:
1) supp $\psi \cap B_R(0) = \emptyset$. We have that $I'(u^-).\psi = 0$, so u^- is solution, and therefore, a subsolution of the problem.
2) supp $\psi \cap B_R(0) \neq \emptyset$. In this case, by Lemma 2.1(4)-(7),we have

$$\begin{aligned} I'(u^-)\psi = I'(v)\psi &= \int_{B_R(0)} \nabla v \nabla \psi - \lambda \int_{B_R(0)} f'(v)f(v)\psi \\ &= \epsilon^{\alpha+2} \int_{B_R(0)} \psi - \lambda \int_{B_R(0)} f'(v)f(v)\psi \\ &= \epsilon^{\alpha+2} \int_{B_R(0)} \psi - \lambda \int_{B_R(0)\cap\{v\geq 1\}} f'(v)f(v)\psi - \\ &\quad \lambda \int_{B_R(0)\cap\{v\leq 1\}} f'(v)f(v)\psi \\ &\leq \epsilon^{\alpha+2} \int_{B_R(0)} \psi - \frac{\lambda C}{2} \left(\int_{B_R(0)\cap\{v\geq 1\}} \psi + \int_{B_R(0)\cap\{v\leq 1\}} v\psi \right) \\ &\leq \epsilon^{\alpha+2} \int_{B_R(0)} \psi - \frac{\lambda C}{2} \left(\int_{B_R(0)\cap\{v\leq 1\}} v(x)\psi(x) \right), \end{aligned}$$

where C is a positive constant independent of ϵ.

Notice that for ϵ sufficiently small, we can consider

$$\int_{B_R(0)\cap\{v\leq 1\}} \psi(x) = \int_{B_R(0)} \chi_{\{v\leq 1\}}\psi(x) = \int_{B_R(0)} \psi(x).$$

In fact, we can extend u by putting zero outside of the ball $B_{\epsilon R}(0)$, which is $C^{1,\gamma}(B_{2R}(0))$, $\gamma \in (0,1)$, (see [13]). By the definition of v, for R fixed, we can choose ϵ sufficiently small, such that

$$|v(x)| = \epsilon^{\alpha}|u(\epsilon x)| \leq \epsilon^{\alpha}\max_{B_{2R}}|u(y)| = C_R\epsilon^{\alpha} \leq 1.$$

Then

$$\begin{aligned}\int_{B_R(0)} \psi &= \int_{B_R(0)} \chi_{\{v\leq 1\}}\psi(x) + \int_{B_R(0)} \chi_{\{v>1\}}\psi(x)\\ &= \int_{B_R(0)} \chi_{\{v\leq 1\}}\psi(x).\end{aligned}$$

Therefore

$$\begin{aligned}I'(u^{-})\psi = I'(v)\psi &\leq \epsilon^{\alpha+2}\int_{B_R(0)} \psi - \frac{\lambda C C_o}{2}\epsilon^{\alpha}\int_{B_R(0)\cap\{v\leq 1\}} \psi(x)\\ &= \epsilon^{\alpha+2}\int_{B_R(0)} \psi - \frac{\lambda C C_o}{2}\epsilon^{\alpha}\int_{B_R(0)} \psi(x)\\ &< 0,\end{aligned}$$

for ϵ small enough, where C_o and C are positive constants independent of ϵ.

■

Proof of Theorem 1.1: For each $\lambda \in (\lambda_1,\ \Lambda)$ we have from Lemma 2.3 that problem (6) admit a supersolution u_+. Moreover, by Lemma 2.4 there is a subsolution u_- of problem (6) satisfying $u_- < u_+$ in Ω. Therefore, there exists a nonnegative solution u of problem (6), verifying $u_- \leq u \leq u_+$ in Ω, and by Proposition 2.1 we conclude of Theorem 1.1 in this case.

Remark 2.1. *In case $\lambda = \lambda_1$ we have the first eigenfunction of the Laplacian acts as supersolution. Indeed, from (2)-(3) in Lemma 2.1, we obtain*

$$\begin{aligned}\int_{\Omega}\nabla\phi_1\nabla\phi dx - \lambda\int_{\Omega} f(\phi_1)f'(\phi_1)\phi dx &= \lambda_1\int_{\Omega}\phi_1\phi dx - \lambda_1\int_{\Omega} f(\phi_1)f'(\phi_1)\phi dx\\ &= \lambda_1\int_{\Omega}[1 - \frac{f(\phi_1)}{\phi_1}f'(\phi_1)]\phi_1\phi dx\\ &\geq 0.\end{aligned}$$

By Lemma 2.4 there is a subsolution, u_- of (6) . Since $u_- < \phi_1$ in Ω, we have, also in this case, that u is a nonnegative solution of problem (6). ■

2.2 Proof of the Proposition 1.1

We consider $\lambda_0 < \Lambda$, so that, there is u_0 a weak nonnegative solution of problem (6), that is,

$$\int_\Omega \nabla u_0 \nabla v = \lambda_0 \int_\Omega f(u_0) f'(u_0) v, \qquad \text{for all} \quad v \in W,$$

making $v = u_0$ we obtain

$$\int_\Omega |\nabla u_0|^2 = \lambda_0 \int_\Omega f(u_0) f'(u_0) u_0.$$

By (4) in Lemma 2.1 we have $\int_\Omega |\nabla u_0|^2 \leq \lambda_0 \int_\Omega f^2(u_0)$, and this implies

$$\overline{\lambda} \leq \lambda_0 < \Lambda.$$

On the other hand, by (3) in Lemma 2.1 we obtain

$$\overline{\lambda} \geq \lambda_1.$$

Considering problem (6) with $\lambda = \overline{\lambda}$, we have that

$$\int_\Omega |\nabla u|^2 \leq \overline{\lambda} \int_\Omega |f(u)|^2,$$

where u is a solution of the problem in question. Therefore we have that the infimum is attained.

2.3 The resonant problem

2.3.1 Reformulation of the problem and preliminaries

We observe that the natural functional associated to problem (5) is given by

$$J(u) = \frac{1}{2} \int_\Omega (1 + 2u^2) |\nabla u|^2 \, dx - \frac{\overline{\lambda}}{2} \int_\Omega |u|^2 \, dx - \int_\Omega G(x,u) dx.$$

Again, we use the argument developed in [16] (see also [6]), that is, we make the change of variables $v = f^{-1}(u)$, where f is defined by (4).
Thus, we can write $J(u)$ as

$$I_\lambda(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{\overline{\lambda}}{2} \int_\Omega |f(v)|^2 \, dx - \int_\Omega G(x, f(v)) dx, \;\; v \in W.$$

Moreover, nontrivial critical points of I_λ correspond precisely to the nontrivial weak solutions of the equation

$$\begin{cases} -\Delta v = \overline{\lambda} f(v)f'(v) - g(x, f(v))f'(v) & \text{in } \Omega, \\ \\ v|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

which is equivalent to find $u \in W$, with $u \neq 0$ such that

$$\int_\Omega |\nabla u| \nabla \xi dx - \overline{\lambda} \int_\Omega f(u)f'(u)\xi dx + \int_\Omega g(x, f(u))f'(u)\xi = 0, \text{ for all } \xi \in W. \tag{9}$$

We will give the proof of the existence of weak solution to problem (8) using Theorem 3.2.

Define the operators, $J, H, F : W \to W^*$, by:

$$\begin{aligned} \langle J(u), v\rangle &= \textstyle\int_\Omega |\nabla u| \nabla v dx \\ \langle H(u), v\rangle &= \textstyle\int_\Omega f(u)f'(u)v dx \\ \langle F(u), v\rangle &= \textstyle\int_\Omega g(x, f(u))f'(u)v dx \end{aligned}$$

for all $u, v \in W$. We put,

$$Tu = J - \overline{\lambda}H + F.$$

Then the operator equality $Tu = 0$ in W is equivalent to the integral identity (9).

Lemma 2.5. *Let $u_n \rightharpoonup u$ in W as $n \to \infty$. Then $H(u_n) \to H(u)$ and $F(u_n) \to F(u)$ as $n \to \infty$.*

Proof: Since $u_n \rightharpoonup u$ in W as $n \to \infty$, by Sobolev embedding, up to a subsequence, we have $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$, $2 \leq p < 2^*$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$ and there is $h \in L^p$ such that $|u_n| \leq h$ a.e. in Ω. By (1) in Lemma 2.1 we have $f(u_n(x)) \to f(u(x))$ a.e. $x \in \Omega$ and $f'(u_n(x)) \to f'(u(x))$ a.e. $x \in \Omega$.

For all $v \in W$,

$$\langle H(u_n) - H(u), v\rangle = \int_\Omega [f(u_n)f'(u_n) - f(u)f'(u)]v dx,$$

since $f(u_n)f'(u_n) \to f(u)f'(u)$ a.e. in Ω and (6) in Lemma 2.1, which together with the Lebesgue dominated convergence Theorem imply that

$$\int_\Omega f(u_n)f'(u_n)v \to \int_\Omega f(u)f'(u)v, \text{ as } n \to \infty.$$

We observe

$$\langle F(u_n) - F(u), v\rangle = \int_\Omega [g(x, f(u_n))f'(u_n) - g(x, f(u))f'(u)]v dx,$$

by (G_1) we have

$$g(x,f(u_n))f'(u_n) \to g(x,f(u))f'(u), \text{ as } n \to \infty.$$

It follows (G_2) and (5) in Lemma 2.1 that

$$|g(x,f(u_n))f'(u_n)| \leq \sigma(x) + \rho(x)\,|f(u_n)|^r \leq \sigma(x) + C\rho(x)\,|u_n|^{r/2},$$

and $g(.,f(u_n)) \to g(.,f(u))$ in $L^{(2^*)'}(\Omega)$, as $n \to \infty$, because $\rho \in L^\infty$ and so $\sigma(x)+\rho h^{r/2} \in L^{(2^*)'}(\Omega)$. ■

2.3.2 Proof of the Theorem 1.2

The proof will be done by four steps.

Step 1: We begin by proving that J, H, F are bounded operators. Let $u \in W$, such that $\|u\| \leq M$, for some $M > 0$. By the Hölder inequality we get

$$\sup_{\|v\|=1} |\langle J(u), v\rangle| \leq M.$$

It follows that J is bounded operators from W into W^*.
Using again the Hölder inequality and the Poincaré inequality we have

$$\left|\int_\Omega f(u)f'(u)vdx\right| \leq \int_\Omega |f(u)f'(u)v| \leq \frac{1}{\sqrt{2}}\int_\Omega |v|\,dx \leq C\,\|v\|\,.$$

Thus,

$$\sup_{\|v\|=1} |\langle H(u), v\rangle| \leq CM.$$

Now, by Hölder inequality, (G_2) and (5) in Lemma 2.1 we have

$$\begin{aligned}\left|\int_\Omega g(x,f(u))f'(u)vdx\right| &\leq \int_\Omega \sigma v + C\int_\Omega \rho\,|u|^{r/2}\,v\\
&\leq (\int_\Omega |\sigma|^{(2^*)'})^{\frac{1}{(2^*)'}}(\int_\Omega |v|^{(2^*)})^{\frac{1}{(2^*)}} + C(\int_\Omega |u|^{(2^*)})^{\frac{r}{(22^*)}}(\int_\Omega |\rho v|^{\frac{22^*}{22^*-r}})^{\frac{22^*-r}{(22^*)}}\\
&\leq (\int_\Omega |\sigma|^{(2^*)'})^{\frac{1}{(2^*)'}}(\int_\Omega |v|^{(2^*)})^{\frac{1}{(2^*)}} + C(\int_\Omega |u|^{(2^*)})^{\frac{r}{(22^*)}}(|\Omega|)^{\frac{1}{s}}(\int_\Omega |v|^{2^*})^{\frac{1}{2^*}},\end{aligned}$$

where $s = 2(2^*)/[2(2^*) - r - 2]$. So, by Sobolev embedding it yields,

$$\left|\int_\Omega g(x,f(u))f'(u)vdx\right| \leq C((\int_\Omega |\sigma|^{(2^*)'})^{\frac{1}{(2^*)'}} + \|u\|^r\,(|\Omega|)^{\frac{1}{s}}\,\|v\|),$$

consequently,

$$\sup_{\|v\|=1} |\langle F(u), v\rangle| \leq C((\int_\Omega |\sigma|^{(2^*)'})^{\frac{1}{(2^*)'}} + M^r(|\Omega|)^{\frac{1}{s}}) < \infty.$$

Step 2: Next, we will show that T is continuous. The continuity of the operators H and F is guaranteed by Lemma 2.5. Let $u_n, u \in W$, such that $\|u_n - u\| \to 0$, as $n \to \infty$. We have by the Hölder inequality

$$\begin{aligned} \|J(u_n) - J(u)\|_* &= \sup_{\|v\|=1} |\langle J(u_n) - J(u), v\rangle| \\ &\leq \sup_{\|v\|=1} (\int_\Omega ||\nabla u_n| - |\nabla u||^{(2^*)'} dx)^{\frac{1}{(2^*)'}} \|v\|_p . \end{aligned}$$

Therefore

$$\|J(u_n) - J(u)\|_* \to 0, \text{ as } n \to \infty.$$

Step 3: From the coercivity of T we get

$$\langle Tu, u\rangle = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \left[g(x, f(u))f'(u)u - \overline{\lambda} f(u)f'(u)u\right] dx.$$

It comes from assumption (G_3) that $g(x, f(u))f'(u)u - \overline{\lambda} f(u)f'(u)u \geq 0$ and then the coercivity of T is immediate.

Step 4: Let us define the operator $\phi : W \times W \to W^*$ by

$$\langle \phi(u, w), v\rangle = \langle J(u), v\rangle + \left\langle (F - \overline{\lambda} H)(w), v\right\rangle .$$

It is clear that $\phi(u, u) = T(u)$, for all $u \in W$. Let t_n be a real sequence such that $t_n \to 0$ and $u, v, w \in W$ then

$$\phi(u + t_n v, w) = J(u + t_n v) + (F - \overline{\lambda} H)(w).$$

Since J is a continuous operator, then $\phi(u + t_n v, w) \to \phi(u, w)$. For all $u, w \in W$ we have

$$\langle \phi(u, u) - \phi(w, u), u - w\rangle = \langle J(u) - J(w), u - w\rangle .$$

We have by the Hölder inequality

$$\begin{aligned} \langle J(u) - J(w), u - w\rangle \geq\ & \textstyle\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla w|^2 - (\int_\Omega |\nabla u|^2)^{\frac{1}{2}} (\int_\Omega |\nabla w|^2)^{\frac{1}{2}} \\ & \textstyle -(\int_\Omega |\nabla w|^2)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2)^{\frac{1}{2}} \\ =\ & (\|u\| - \|w\|)^2 \geq 0. \end{aligned}$$

Hence, $\langle \phi(u,u) - \phi(w,u), u - w \rangle \geq 0$. Let now $u_n \rightharpoonup u$ in W, as $n \to \infty$, and

$$\lim_{n\to\infty} \langle \phi(u_n,u_n) - \phi(u,u_n), u_n - u \rangle = 0.$$

It follows that $\|u_n\| \to \|u\|$, as $n \to \infty$. Thus, it follows from the continuity of the operator $(F - \overline{\lambda}H)$ that $\phi(w,u_n) \to \phi(w,u)$, as $n \to \infty$, for arbitrary $w \in W$. Let now $w \in W$, $u_n \rightharpoonup u$ in W and $\phi(w,u_n) \rightharpoonup z$, as $n \to \infty$. We have, $\langle \phi(w,u_n), u_n \rangle = \langle \phi(w,u_n), u \rangle + \langle \phi(w,u_n), u_n - u \rangle$. By assumption $\langle \phi(w,u_n), u \rangle = \langle z, u \rangle$ then we must show that $\langle \phi(w,u_n), u_n - u \rangle \to 0$ as $n \to \infty$. We have by definition

$$\begin{aligned}
\langle \phi(w,u_n), u_n - u \rangle = & \ \langle J(w), u_n - u \rangle + \left\langle (F - \overline{\lambda}H)u_n, u_n - u \right\rangle \\
= & \ \left\langle J(w) + (F - \overline{\lambda}H)u, u_n - u \right\rangle \\
& + \left\langle (F - \overline{\lambda}H)u_n - (F - \overline{\lambda}H)u, u_n - u \right\rangle.
\end{aligned}$$

We observe that $\left\langle J(w) + (F - \overline{\lambda}H)u, u_n - u \right\rangle \to 0$, as $n \to \infty$, since $u_n \rightharpoonup u$ as $n \to \infty$ and

$$\begin{aligned}
\left| \left\langle (F - \overline{\lambda}H)u_n - (F - \overline{\lambda}H)u, u_n - u \right\rangle \right| \leq & \ \left\| (F - \overline{\lambda}H)u_n - (F - \overline{\lambda}H)u \right\|_* \|u_n - u\| \\
\leq & \ C \left\| (F - \overline{\lambda}H)u_n - (F - \overline{\lambda}H)u \right\|_*,
\end{aligned}$$

since u_n is weakly convergent, so that, bounded in W. By Lemma 2.5 we have

$$\left\| (F - \overline{\lambda}H)u_n - F - \overline{\lambda}H)u, u_n - u \right\|_* \to 0 \text{ as } n \to \infty.$$

Therefore,

$$\langle \phi(w,u_n), u_n - u \rangle \to 0, \text{ as } n \to \infty$$

and then

$$\langle \phi(w,u_n), u_n \rangle \to \langle z, u \rangle.$$

It follows of Theorem 3.2 that the equation $Tu = 0$ has at least one solution in W. This solution is a weak nontrivial solution of Problem (8) because $T(0) \neq 0$.

Completing the proof of the Theorem 1.2 we observe that

Proposition 2.2. *If $v \in W$ is a critical point of I_λ then $u = f(v) \in W$ is a weak solution of (5).*

Proof: The arguments used in the proof of the Proposition 2.4 in [12] can be repeated to prove our result. ∎

3 Abstract results

We will state two abstract results, first is due to Crandall and Rabinowitz and second one is due to Leray and Lions.

Theorem 3.1. *(Lemma 1.1 [8]) Let X, Y be Banach spaces, V is an open neighborhood 0, $I = (a, b) \subset \mathbb{R}$ be an open interval and $F : I \times V \to Y$ is a twice continuously Fréchet differentiable mapping.*

Supose that $\lambda_0 \in I$ and also that

(i) $F(\lambda,\ 0) = 0$ *for* $\lambda \in I$,

(ii) $\dim N(F_x(\lambda_0,\ 0)) = \operatorname{codim} R(F_x(\lambda_0,\ 0)) = 1$,

(iii) $F_{\lambda x}(\lambda_0,\ 0)x_0 \notin R(F_x(\lambda_0,\ 0))$ *where* $x_0 \in X$ *spans* $N(F_x(\lambda_0,\ 0))$.

Let Z is any complement of span $\{x_0\}$ in X. Then there is a open interval $\widetilde{I}$ containing 0 and continuously differentiable functions $\lambda : \widetilde{I} \to \mathbb{R}$ and $\psi : \widetilde{I} \to Z$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and, if $x(s) = s\ x_0 + s\ \psi(s)$, then $F(\lambda(s), x(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near $(\lambda_0, 0)$ consists precisely of the curves $x = 0$ and $(\lambda(s), x(s))$, $s \in \widetilde{I}$.

Theorem 3.2. *(Leray- Lions [11]) Let X be a reflexive real Banach space. Let $T : X \to X^*$ be an operator satisfying the conditions:*

i.) T *is bounded;*

ii.) T *is demicontinuous;*

iii.) T *is coercive.*

Moreover, let there exist a bounded mapping $\phi : X \times X \to X^*$ such that

iv.) $\phi(u, u) = T(u)$ *for every* $u \in X$;

v.) *for all $u, w, h \in X$ and any sequence $\{t_n\}$ of real numbers such that $t_n \to 0$, we have*

$$\phi(u + t_n h, w) \rightharpoonup \phi(u, w);$$

vi.) *for all $u, w \in X$ we have*

$$\langle \phi(u, u) - \phi(w, u), u - w \rangle \geq 0;$$

(the so-called condition of monotonicity in the principal part);

vii.) *if* $u_n \rightharpoonup u$ *and*

$$\lim_{n\to\infty} \langle \phi(u_n, u_n) - \phi(u, u_n), u_n - u \rangle = 0$$

then we have

$$\phi(w, u_n) \rightharpoonup \phi(w, u) \quad \text{for arbitrary} \quad w \in X;$$

viii.) *if* $w \in X$*,* $u_n \rightharpoonup u$*,* $\phi(w, u_n) \rightharpoonup z$*, then*

$$\lim_{n\to\infty} \langle \phi(w, u_n), u_n \rangle = \langle z, u \rangle .$$

Then the equation

$$T(u) = f^*$$

has at least one solution $u \in X$ *for every* $f^* \in X^*$*.*

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