AUTONOMOUS OVSYANNIKOV THEOREM AND APPLICATIONS TO NONLOCAL EVOLUTION EQUATIONS AND SYSTEMS

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Abstract. This work presents an Ovsyannikov type theorem for an autonomous abstract Cauchy problem in a scale of decreasing Banach spaces, which in addition to existence and uniqueness of solution provides an estimate about the analytic lifespan of the solution. Then, using this theorem it studies the Cauchy problem for Camassa-Holm type equations and systems with initial data in spaces of analytic functions on both the circle and the line, which is the main goal of this paper. Finally, it studies the continuity of the data-to-solution map in spaces of analytic functions.

1. Introduction and Results

We consider the following initial value problem (i.v.p.) for a nonlocal autonomous equation

$$\frac{du}{dt} = F(u), \quad u(0) = u_0, \tag{1.1}$$

and prove existence and uniqueness of solution in a space of analytic functions under appropriate conditions on $F(u)$, which is defined on a scale of Banach spaces. Furthermore, we prove an estimate for the analytic lifespan. The motivation comes from the 2003 work in [41] about the Cauchy problem of the Camassa-Holm (CH) equation with analytic initial data on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$,

$$\frac{du}{dt} = -u\partial_x u - \big(1 - \partial_x^2\big)^{-1}\partial_x\big[u^2 + \frac{1}{2}(\partial_x u)^2\big] \doteq F(u) \quad u(0) = u_0 \in C^\omega(\mathbb{T}). \tag{1.2}$$

There it was proved the following Cauchy-Kovalevski type result for CH. If $u_0(x)$ is analytic on $\mathbb{T}$, then **there exist an** $\varepsilon > 0$ and a unique solution $u(x,t)$ of the CH Cauchy problem (1.2), which is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{T}$.

While this result provides the analyticity of the solution in both the spatial and time variables (a phenomenon which does not hold for KdV, see [46] or [30]) it gives no estimate about the size of the analytic lifespan ε. Also, it provides no information about the evolution of the uniform radius of analyticity. Considering these to be important questions for CH and other nonlocal equations and systems, we shall investigate them in this paper on both the circle and the line. Furthermore, we will study the stability of their solution map.

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To do this in a unified way we shall need a refined version of the so called Ovsyannikov theorem in the autonomous case, that is the function F depends only on u. We begin by stating the following three required conditions on $F(u)$, which is defined on a scale of Banach spaces X_δ:

1. $\{X_\delta\}_{0<\delta\leq 1}$ is a scale of decreasing Banach spaces, i.e. for any $0<\delta'<\delta\leq 1$,
$$X_\delta \subset X_{\delta'},\ ||.||_{\delta'} \leq ||.||_\delta.$$
2. $F: X_\delta \to X_{\delta'}$ is a function such that for any given $u_0 \in X_1$ and $R>0$ there exist L and M positive numbers, depending on u_0 and R, such that for any $0<\delta'<\delta\leq 1$ and all $u, v \in X_\delta$ with $\|u-u_0\|_\delta < R$ and $\|v-u_0\|_\delta < R$ we have the following Lipschitz type condition
$$\|F(u)-F(v)\|_{\delta'} \leq \frac{L}{\delta-\delta'}\|u-v\|_\delta, \tag{1.3}$$
and the following bound for the X_δ norm of $F(u_0)$
$$||F(u_0)||_\delta \leq \frac{M}{1-\delta}, \quad 0<\delta<1. \tag{1.4}$$
3. For $0<\delta'<\delta<1$ and $a>0$, if the function $t \longmapsto u(t)$ is holomorphic on $\{t\in\mathbb{C} : |t|<a(1-\delta)\}$ with values in X_δ and $\sup_{|t|<a(1-\delta)}\|u(t)-u_0\|_\delta < R$, then the function $t \longmapsto F(u(t))$ is holomorphic on $\{t\in\mathbb{C} : |t|<a(1-\delta)\}$ with values in $X_{\delta'}$.

Next, we state an autonomous version of Ovsyannikov theorem, which as we mentioned earlier in addition to existence and uniqueness provides an estimate about the analytic lifespan of the solution.

Theorem 1 (Autonomous Ovsyannikov Theorem). *Assume that the scale of Banach spaces X_δ and the function $F(u)$ satisfy the above conditions (1)-(3). Then, for given $u_0 \in X_1$ and $R>0$ there exists $T>0$ such that*
$$T = \frac{R}{16LR+8M}, \tag{1.5}$$
and a unique solution $u(t)$ to the Cauchy problem (1.1)*, which for every $\delta \in (0,1)$ is a holomorphic function in the disc $D(0, T(1-\delta))$ valued in X_δ satisfying*
$$\sup_{|t|<T(1-\delta)} \|u(t)-u_0\|_\delta < R, \quad 0<\delta<1. \tag{1.6}$$

A slightly more general version of Theorem 1 (the function F depends on both u and t) but with less emphasis on the analytic lifespan T was proved by Baouendi and Goulaouic [1]. Also, in addition to the original work by Ovsyannikov [58], [59], [60], other versions of this theorem have been developed by Nirenberg [54], Nishida [55], Treves [68], [69], and Baouendi and Goulaouic [1], [2]. Here, following [1], we shall provide only an outline of this theorem's proof and use it for the continuity of the solution map of CH equations.

The autonomous Ovsyannikov theorem will help us study the Cauchy problem of Camassa-Holm (CH) type equations in a unified way. We begin by describing these equations following Vladimir Novikov's work on integrability [56]. In this paper Novikov investigated the question of integrability for CH type equations of the form
$$(1-\partial_x^2)u_t = P(u, u_x, u_{xx}, u_{xxx}, \cdots), \tag{1.7}$$

where P is a polynomial of u and its x-derivatives. Using as definition of integrability the existence of an infinite hierarchy of quasi-local higher symmetries, he produced about 20 integrable equations with quadratic nonlinearities that include the Camassa-Holm (CH) equation

$$(1-\partial_x^2)u_t = -3uu_x + 2u_xu_{xx} + uu_{xxx} \tag{1.8}$$

and the Degasperis-Procesi (DP) equation

$$(1-\partial_x^2)u_t = -4uu_x + 3u_xu_{xx} + uu_{xxx}. \tag{1.9}$$

Also, he produced about 10 integrable equations with cubic nonlinearities that include the following new one

$$(1-\partial_x^2)u_t = -4u^2u_x + 3uu_xu_{xx} + u^2u_{xxx}, \tag{1.10}$$

which is now called the Novikov equation (NE), and the Fokas-Olver-Rosenau-Qiao (FORQ) equation

$$(1-\partial_x^2)u_t = \partial_x(u^2u_{xx} - u_x^2u_{xx} + uu_x^2 - u^3), \tag{1.11}$$

that was discovered earlier independently by Fokas [28], Olver and Rosenau [57], and Qiao [61]. This equation was also derived by Fuchssteiner [29].

The CH equation arose initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [27]. However, it was written explicitly as a water wave equation by Camassa and Holm [7] in 1993, who derived it from the Euler equations of hydrodynamics using asymptotic expansions. Also, they derived its peakon solutions. DP was discovered in 1998 by Degasperis and Procesi [24]. Also, DP and CH are the only integrable members of the b-family of equations (5.1) (see Mikhailov and Novikov [52]).

Multiplying by the inverse of $(1-\partial_x^2)$, we write the Cauchy problem for CH, DP, NE and FORQ equations in the following unified way

$$u_t = (1-\partial_x^2)^{-1}P(u) \doteq F(u), \quad u(0) = u_0. \tag{1.12}$$

where, $P(u)$ is given by the right hand-sides of equations (1.8), (1.9), (1.10) and (1.11). Furthermore, for analytic initial data, to obtain precise information about the uniform radius of analyticity of the solution to the Cauchy problem (1.12) we introduce the following scale of decreasing analytic Banach spaces. For $\delta > 0$ and $s \geq 0$, in the periodic case we define

$$G^{\delta,s}(\mathbb{T}) = \{\varphi \in L^2(\mathbb{T}) : ||\varphi||^2_{G^{\delta,s}(\mathbb{T})} \doteq ||\varphi||^2_{\delta,s} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} e^{2\delta|k|}|\widehat{\varphi}(k)|^2 < \infty\}, \tag{1.13}$$

while in the nonperiodic case we define

$$G^{\delta,s}(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R}) : ||\varphi||^2_{G^{\delta,s}(\mathbb{R})} \doteq ||\varphi||^2_{\delta,s,\mathbb{R}} = \int_{\mathbb{R}} \langle \xi\rangle^{2s} e^{2\delta|\xi|}|\widehat{\varphi}(\xi)|^2 d\xi < \infty\}, \tag{1.14}$$

where $\langle k\rangle = \sqrt{1+k^2}$ and $\langle \xi\rangle = \sqrt{1+\xi^2}$.

Here, when a result holds for both the periodic and non-periodic case then we will use the notation $||\cdot||_{\delta,s}$ and $G^{\delta,s}$ for the norm and the space in both cases. Note that if $\varphi \in G^{\delta,s}(\mathbb{T})$, then φ has an analytic extension to a symmetric strip around the real axis with width δ. This δ is called the **radius of analyticity** of φ. As we shall show later, the spaces $\{G^{\delta,s}\}_{0<\delta\leq 1}$ form a scale of decreasing Banach spaces like the spaces X_δ in the autonomous Ovsyannikov theorem.

Furthermore, we shall show that the right-hand side $F(u)$ of (1.12) satisfies conditions (1)–(3) in the autonomous Ovsyannikov theorem.

The discussion above motivates our next result. For the sake of simplicity we shall assume that our initial data u_0 belong in $G^{1,s+2}$.

Theorem 2. *Let $s > \frac{1}{2}$. If $u_0 \in G^{1,s+2}$ on the circle or the line, then there exists a positive time T, which depends on the initial data u_0 and s, such that for every $\delta \in (0,1)$, the Cauchy problem* (1.12) *has a unique solution u which is a holomorphic function in the disc $D(0, T(1-\delta))$ valued in $G^{\delta,s+2}$. Furthermore, the analytic lifespan T satisfies the estimate*

$$T \approx \frac{1}{||u_0||^k_{1,s+2}}, \tag{1.15}$$

where $k = 1$ for the CH and DP equations and $k = 2$ for the NE and FORQ equations.

Remark. We would like to point out that in the case of CH and DP equations one may assume, in all results of this work, that $s > -\frac{1}{2}$.

A more precise statement of estimate (1.15) is provided in the next section, where this estimate is derived for each one of the CH equations (see (2.17), (2.27), (2.36)). For CH, DP, NE and FORQ, the proof of the analytic lifespan estimate (1.15) is based on the estimate (1.5) in the autonomous Ovsyannikov theorem and a derivation of the Lipschitz type condition (1.3) and the bound (1.4) with constants L and M expressed in terms of $||u_0||_{1,s+2}$.

Estimate (1.15) besides being interesting on its own merit, it is also the key ingredient for proving continuity for the solution map. More precisely, for the CH, DP, NE and FORQ equations we have the following important result.

Theorem 3. *If $s > \frac{1}{2}$, then the data-to-solution map $u(0) \mapsto u(t)$ of the Cauchy problem* (1.12) *for the CH equations is continuous from $G^{\delta,s+2}$ into the solutions space.*

The precise definition of the solutions space mentioned in Theorem 3 will be given later (see Theorem 5). This is an important result since it makes the CH type equations to be well-posed in the spaces $G^{\delta,s+2}$ in the sense of Hadamard. One must contrast this result with the classical Cauchy-Kovalevski theorem, where there is no continuity of the data-to-solution map. Hadamard [33] was the first to observe this instability for the solution map for the Laplace equation with analytic initial data. This led him to the definition of the so called well-posedness in the sense of Hadamard, which in addition to existence and uniqueness requires continuous dependence of the solution map on the initial data. This work demonstrates the importance of the solutions space for the stability of the solution map.

Concerning well-posedness of CH type equations in Sobolev spaces H^s, it is known that CH, DP and NE are well-posed in the sense of Hadamard for $s > 3/2$, while FORQ is well-posed for $s > 5/2$. For the well-posedness of CH for $s > 3/2$, we refer the reader to [41], [22], [51] and [63]. For the DP equaltion we refer to [71], [72], [35] and [36]. For NE we refer to [34] and [66]. For the FORQ equation we refer to [39]. While the solution map of these equations is continuous, it is not uniformly continuous (see [37] and [38] for CH, [35] for DP, [34] NE, and [39] for FORQ). For more results about well-posedness, traveling wave solutions and other properties for CH type and related nonlinear evolution equations, we refer the reader to [12], [13], [11], [15], [16], [6], [23], [49], [53], [50], [17], [45], [8], [42], [43], [44], [4], [3], [64], [18], [5], [47], [48], [10], [9], [70], and the references therein.

The rest of this paper is organized as follows. In Section 2 we summarize the basic properties of the $G^{\delta,s}$ spaces and use them together with the autonomous Ovsyannikov theorem to prove Theorem 2. In Section 3 we extent the analytic theory developed in Section 2 to Camassa-Holm systems, including the 2-component Camassa-Holm system (2CH) and the Novikov system (2NE). This is contained in Theorem 4. In Section 4 we introduce the solutions space E_a (see Definition 1) and present a sketch of the proof of the autonomous Ovsyannikov theorem (Theorem 1) following the work of Baouendi and Goulaouic [1], which help us prove the continuity of the solution map for the CH equations (Theorem 3). We provide all the details only in the case of CH, since the proof for the other equations is similar. In Section 5 we present a list of other equations for which the analytic theory described in this work is applicable. In particular, we include the Laplace equation and revisit Hadamard's example.

2. Proof of Theorem 2: Existence, Uniqueness and Lifespan

We begin with the properties of the $G^{\delta,s}$ and the estimates needed to prove the three conditions of the autonomous Ovsyannikov theorem. The next lemmas give a better understanding of the spaces $G^{\delta,s}$. One can easily prove these results.

Lemma 1. *Let $\varphi \in G^{\delta,s}$. Then, φ has an analytic extension to a symmetric strip around the real axis of width δ, for $s \geq 0$ in the periodic case and $s > \frac{1}{2}$ in the non-periodic case.*

Lemma 2. *If $0 < \delta' < \delta \leq 1$, $s \geq 0$ and $\varphi \in G^{\delta,s}$ on the circle or the line, then*

$$||\partial_x \varphi||_{\delta',s} \leq \frac{e^{-1}}{\delta - \delta'} ||\varphi||_{\delta,s} \tag{2.1}$$

$$||\partial_x \varphi||_{\delta,s} \leq ||\varphi||_{\delta,s+1} \tag{2.2}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s+2} = ||\varphi||_{\delta,s} \tag{2.3}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \leq ||\varphi||_{\delta,s} \tag{2.4}$$

$$||\partial_x(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \leq ||\varphi||_{\delta,s}. \tag{2.5}$$

Furthermore, we shall need to prove an algebra property for these spaces, which is the main result in the following lemma.

Lemma 3. *For $\varphi \in G^{\delta,s}$ on the circle or the line the following properties hold true:*

1) If $0 < \delta' < \delta$ and $s \geq 0$, then $|| \cdot ||^2_{\delta',s} \leq || \cdot ||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta',s}$.

2) If $0 < s' < s$ and $\delta > 0$, then $|| \cdot ||^2_{\delta,s'} \leq || \cdot ||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta,s'}$.

3) For $s > 1/2$ and $\varphi, \psi \in G^{\delta,s}$ we have

$$||\varphi\psi||_{\delta,s} \leq c_s ||\varphi||_{\delta,s} ||\psi||_{\delta,s}, \tag{2.6}$$

where $c_s = \sqrt{2(1+2^{2s})\sum_{k=0}^{\infty} \frac{1}{\langle k \rangle^{2s}}}$ in the periodic case and $c_s = \sqrt{2(1+2^{2s})\int_0^{\infty} \frac{1}{\langle \xi \rangle^{2s}} d\xi}$ in the non-periodic case.

Since in Theorem 2 we have assumed that the initial data u_0 is in $G^{1,s+2}$ we would like to point out that in the periodic case an analytic function belongs to a $G^{\delta_0,s}(\mathbb{T})$, for some $\delta_0 > 0$ and any $s \geq 0$. More precisely we have the following result.

Lemma 4. *If $u_0 \in C^\omega(\mathbb{T})$, there exists $\delta_0 > 0$ such that $u_0 \in G^{\delta_0,s}(\mathbb{T})$ for any $s \geq 0$.*

From now on we fix $s > 1/2$, and without loss of generality we assume that $\delta_0 = 1$. Furthermore, as we have shown above, the spaces

$$\{G^{\delta,s}\}_{0<\delta\leq 1}, \quad \text{with norm} \quad ||\cdot||_{\delta,s}$$

form a scale of decreasing Banach spaces like the spaces X_δ in condition (1) of the autonomous Ovsyannikov theorem. Also, these spaces and $F(u)$ satisfy condition (3). Therefore, assuming Theorem 1, to prove Theorem 2 it suffices to show that the right-hand side $F(u)$ of (1.12) satisfies conditions (2) of the autonomous Ovsyannikov theorem. This is contained in the following key lemma.

Lemma 5. *Let $s > 1/2$. Also, let $R > 0$ and $u_0 \in G^{1,s+2}$ be given. Then, for each one of the CH equations* (1.12) *there exist positive constants L and M, which depend on R and $||u_0||_{1,s+2}$ such that for $u, v \in G^{\delta,s+2}$, $||u - u_0||_{\delta,s+2} < R$, $||v - u_0||_{\delta,s+2} < R$ and $0 < \delta' < \delta \leq 1$ we have*

$$||F(u) - F(v)||_{\delta',s+2} \leq \frac{L}{\delta - \delta'}||u - v||_{\delta,s+2} \tag{2.7}$$

and

$$||F(u_0)||_{\delta,s+2} \leq \frac{M}{1-\delta}, \ 0 < \delta < 1. \tag{2.8}$$

Moreover, the analytic lifespan T satisfies the estimate

$$T \approx \frac{1}{||u_0||^k_{1,s+2}}, \tag{2.9}$$

where $k = 1$ for the CH and DP equations and $k = 2$ for the NE and FORQ equations.

Proof. We shall prove this lemma for each one of the CH, DP, NE and FORQ equations, beginning with CH.

The Camassa-Holm equation (CH). In this case, we shall show that for $||u - u_0||_{\delta,s+2} < R$ and $||v - u_0||_{\delta,s+2} < R$ we have

$$||F(u) - F(v)||_{\delta',s+2} \leq \frac{4e^{-1}c_s(R + ||u_0||_{1,s+2})}{\delta - \delta'}||u - v||_{\delta,s+2}, \tag{2.10}$$

which is estimate (2.7) with $L = 4e^{-1}c_s(R + ||u_0||_{1,s+2})$, where c_s is given in Lemma 3. For this we use the fact that the CH equation can be written in the following form

$$\frac{du}{dt} = F(u) \doteq -\partial_x\Big(\frac{1}{2}u^2 + \big(1 - \partial_x^2\big)^{-1}\big[u^2 + \frac{1}{2}(\partial_x u)^2\big]\Big). \tag{2.11}$$

Applying Lemma 2 and the triangle inequality we get

$$||F(u) - F(v)||_{\delta',s+2} \leq \frac{e^{-1}}{\delta - \delta'}\Big(\frac{1}{2}||u^2 - v^2||_{\delta,s+2} + ||u^2 - v^2||_{\delta,s} + \frac{1}{2}||(\partial_x u)^2 - (\partial_x v)^2||_{\delta,s}\Big).$$

Also, applying the algebra property (2.6) and inequality (2.2) we get the estimates

$$||u^2 - v^2||_{\delta,s} \leq ||u^2 - v^2||_{\delta,s+2} \leq c_s||u - v||_{\delta,s+2}||u + v||_{\delta,s+2}, \tag{2.12}$$

$$||(\partial_x u)^2 - (\partial_x v)^2||_{\delta,s} = ||\partial_x(u - v)\partial_x(u + v)||_{\delta,s} \leq c_s||u - v||_{\delta,s+2}||u + v||_{\delta,s+2}. \tag{2.13}$$

Finally, bounding $||u + v||_{\delta,s+2}$ as follows

$$||u + v||_{\delta,s+2} \leq ||u - u_0||_{\delta,s+2} + ||v - u_0||_{\delta,s+2} + 2||u_0||_{\delta,s+2} \leq 2(R + ||u_0||_{1,s+2})$$

and combining the above three inequalities gives the desired estimate (2.10).

Next we prove (2.8) for CH. Using the properties of our scale of Banach spaces $G^{\delta,s}$ stated in Lemmas 2 and 3 for $0 < \delta' < \delta \leq 1$ we have

$$||\partial_x(u_0^2)||_{\delta',s+2} \leq \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$

$$||\partial_x(1 - \partial_x^2)^{-1}(u_0^2)||_{\delta',s+2} \leq \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$

$$||\partial_x(1 - \partial_x^2)^{-1}(\partial_x u_0)^2||_{\delta',s+2} \leq \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'}.$$

Combining these we get the inequality

$$||F(u_0)||_{\delta',s+2} \leq \frac{2e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$

which, by replacing δ' by δ and δ by 1, gives the desired estimate (2.8), with

$$M = 2e^{-1}c_s||u_0||^2_{1,s+2}. \tag{2.14}$$

Now, we are ready to complete the proof of Lemma 5 for CH. For any u_0 in $G^{1,s+2}$ and $R > 0$, according to (2.10) and (2.8) estimate (2.7) is satisfied if the constant L is given by $L = C(R + ||u_0||_{1,s+2})$, where $C = 4e^{-1}c_s$. With this notation, from (2.14) we also have $M = \frac{C}{2}||u_0||^2_{1,s+2}$. Thus, thanks to Theorem 1, for

$$T = \frac{R}{16LR + 8M} = \frac{R}{16C(R + ||u_0||_{1,s+2})R + 4C||u_0||^2_{1,s+2}} \tag{2.15}$$

there exists a unique solution $u(t)$ to the Cauchy problem (1.12), which for every $\delta \in (0, 1)$ is a holomorphic function in $D(0, T(1 - \delta)) \to G^{\delta,s+2}$ and

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_{\delta,s+2} < R. \tag{2.16}$$

Thus, by letting $R = ||u_0||_{1,s+2}$ we obtain

$$T = \frac{e}{144c_s} \cdot \frac{1}{||u_0||_{1,s+2}}. \tag{2.17}$$

This completes the proof of Lemma 5 for CH.

The Degasperis-Procesi equation (DP). Writing this equation in the form

$$\frac{du}{dt} = F(u) \doteq -\partial_x\Big(\frac{1}{2}u^2 + \big(1 - \partial_x^2\big)^{-1}\big[\frac{3}{2}u^2\big]\Big) \tag{2.18}$$

we see that its two terms appear in CH and only the second has a different coefficient. Thus, estimating like in the case of CH we obtain the inequalities (2.7) and (2.8) for $||F(u) -$

$F(v)||_{\delta',s+2}$ and $||F(u_0)||_{\delta,s+2}$ respectively, with the same constants. Also, for the DP equation we obtain the same estimate for the analytic lifespan T, which is given by (2.17).

The Novikov equation (NE). It is interesting that, unlike CH and DP, we can not factor the operator ∂_x from all terms of NE. Instead, NE can be written in the following form

$$\frac{du}{dt} = F(u) \doteq -\partial_x\Big(\frac{1}{3}u^3 + (1-\partial_x^2)^{-1}\Big[u^3 + \frac{3}{2}\big(u(\partial_x u)^2\big)\Big]\Big) - (1-\partial_x^2)^{-1}\Big[\frac{1}{2}(\partial_x u)^3\Big]. \quad (2.19)$$

For $||u-u_0||_{\delta,s+2} \le R$, $||v-u_0||_{\delta,s+2} \le R$, $s > \frac{1}{2}$ and $0 < \delta' < \delta \le 1$ we shall prove that

$$||F(u) - F(v)||_{\delta',s+2} \le \frac{10c_s^2 e^{-1}(R + ||u_0||_{1,s})^2}{\delta - \delta'}||u-v||_{\delta,s+2}, \quad (2.20)$$

which is estimate (2.7) with $L = 10c_s^2 e^{-1}(R + ||u_0||_{1,s})^2$.

For this we use the properties of our scale of Banach spaces $G^{\delta,s}$ stated in Lemmas 2 and 3 to estimate $F(u) - F(v)$ as follows

$$\begin{aligned}||F(u) - F(v)||_{\delta',s+2} \le \frac{e^{-1}}{\delta - \delta'}\Big(\frac{1}{3}||u^3 - v^3||_{\delta,s+2} + ||u^3 - v^3||_{\delta,s} + \frac{3}{2}||u(\partial_x u)^2 - v(\partial_x v)^2||_{\delta,s}\Big)\\ + \frac{1}{2}||(\partial_x u)^3 - (\partial_x v)^3||_{\delta',s}. \quad (2.21)\end{aligned}$$

Also, using the algebra property we estimate the first and second term of (2.21) as follows

$$\begin{aligned}||u^3 - v^3||_{\delta,s} \le ||u^3 - v^3||_{\delta,s+2} &\le c_s^2\Big(||v||^2_{\delta,s+2} + ||v||_{\delta,s+2}||u||_{\delta,s+2} + ||u||^2_{\delta,s+2}\Big)||v-u||_{\delta,s+2}\\ &\le 3c_s^2(R + ||u_0||_{\delta,s+2})^2||u-v||_{\delta,s+2}, \quad (2.22)\end{aligned}$$

where the last inequality follows from replacing $||u||_{\delta,s+2}$ and $||v||_{\delta,s+2}$ with the bound $R + ||u_0||_{\delta,s+2}$. For the third term of (2.21), using the identity

$$u(\partial_x u)^2 - v(\partial_x v)^2 = (u-v)(\partial_x u)^2 + v\partial_x(u-v)\partial_x(u+v) \quad (2.23)$$

and the properties of the $G^{\delta,s}$-norm we have

$$\begin{aligned}||u(\partial_x u)^2 - v(\partial_x v)^2||_{\delta,s} &\le c_s^2\big(||v-u||_{\delta,s}||u||^2_{\delta,s+1} + ||u-v||_{\delta,s+1}||v||_{\delta,s}||u+v||_{\delta,s+1}\big)\\ &\le 3c_s^2(R + ||u_0||_{\delta,s+2})^2||u-v||_{\delta,s+2}. \quad (2.24)\end{aligned}$$

Finally, for the fourth term of of (2.21), using the identity

$$(\partial_x u)^3 - (\partial_x v)^3 = \Big[\partial_x u - \partial_x v\Big]\Big[(\partial_x u)^2 + (\partial_x u)(\partial_x v) + (\partial_x v)^2\Big]$$

we have

$$\begin{aligned}||(\partial_x u)^3 - (\partial_x v)^3||_{\delta',s} &\le c_s^2||\partial_x(u-v)||_{\delta',s}\Big(||\partial_x u||^2_{\delta',s} + ||\partial_x u||_{\delta',s}||\partial_x v||_{\delta',s} + ||\partial_x v||^2_{\delta',s}\Big)\\ &\le \frac{c_s^2 e^{-1}}{\delta - \delta'}||u-v||_{\delta,s}\Big(||u||^2_{\delta,s+2} + ||u||_{\delta,s+2}||v||_{\delta,s+2} + ||v||^2_{\delta,s+2}\Big)\\ &\le \frac{3c_s^2 e^{-1}(R + ||u_0||_{\delta,s+2})^2}{\delta - \delta'}||u-v||_{\delta,s+2}.\end{aligned}$$

Combining the above estimates and using the fact that $||u_0||_{\delta,s+2} \le ||u_0||_{1,s+2}$ gives the desired inequality (2.20).

To prove inequality (2.8) for (NE), using the properties of the spaces $G^{\delta,s}$ we have

$$\frac{1}{3}||\partial_x(u_0^3)||_{\delta',s+2} \leq \frac{1}{3}\frac{e^{-1}c_s^2||u_0||^3_{\delta,s+2}}{\delta-\delta'},$$

$$||\partial_x(1-\partial_x^2)^{-1}(u_0)^3||_{\delta',s+2} \leq \frac{e^{-1}c_s^2||u_0||^3_{\delta,s+2}}{\delta-\delta'},$$

$$\frac{3}{2}||\partial_x(1-\partial_x^2)^{-1}(u_0\partial_x u_0)^2||_{\delta',s+2} \leq \frac{\frac{3}{2}e^{-1}c_s^2||u_0||^3_{\delta,s+2}}{\delta-\delta'},$$

$$\frac{1}{2}||(1-\partial_x^2)^{-1}(\partial_x u_0)^3||_{\delta',s+2} \leq \frac{\frac{1}{2}e^{-1}c_s^2||u_0||^3_{\delta,s+2}}{\delta-\delta'}.$$

Combining these and replacing δ' by δ and δ by 1 we have

$$||F(u_0)||_{\delta,s+2} \leq \frac{\frac{10}{3}e^{-1}c_s^2||u_0||^3_{1\,s+2}}{1-\delta},\ 0<\delta<1, \tag{2.25}$$

which is (2.8) for (NE) with $M = \frac{10}{3}e^{-1}c_s^2||u_0||^3_{1,s+2}$.

Finally, for given u_0 in $G^{1,s+2}(\mathbb{T})$ and $R>0$, according to (2.20) and (2.25) we have $L = C(R+||u_0||_{1,s+2})^2$, and $M = \frac{C}{3}||u_0||^3_{1,s+2}$, where $C = 10e^{-1}c_s^2$. Thus, the lifespan for NE is given by

$$T = \frac{R}{16LR+8M} = \frac{R}{16C(R+||u_0||_{1,s+2})^2R + \frac{8}{3}C||u_0||^3_{1,s+2}}. \tag{2.26}$$

Choosing $R = ||u_0||_{1,s+2}$ we obtain the following estimate in terms of the initial data

$$T = \frac{3e}{2000c_s^2}\cdot\frac{1}{||u_0||^2_{1,s+2}}. \tag{2.27}$$

This completes the proof of Lemma 5 for NE.

The Fokas-Olver-Rosenau-Qiao equation (FORQ). We begin the proof of Lemma 5 for the FORQ equation by writing it in the following form

$$\frac{du}{dt} = F(u) \doteq -\partial_x(1-\partial_x^2)^{-1}\Big(u^3 - u(\partial_x u)^2 - u^2\partial_x^2 u + (\partial_x u)^2\partial_x^2 u\Big), \tag{2.28}$$

and for $||u-u_0||_{\delta,s+2} \leq R$, $||v-u_0||_{\delta,s+2} \leq R$, $s>\frac{1}{2}$ and $0<\delta'<\delta\leq 1$ we prove that

$$||F(u)-F(v)||_{\delta',s+2} \leq \frac{12e^{-1}c_s^2(R+||u_0||_{1,s+2})^2}{\delta-\delta'}||u-v||_{\delta,s+2}, \tag{2.29}$$

which is estimate (2.7) with $L = 12e^{-1}c_s^2(R+||u_0||_{1,s+2})^2$. For this, using the properties of the spaces $G^{\delta,s}$ stated in Lemmas 2 and 3 we have

$$\begin{aligned} ||F(u)-F(v)||_{\delta',s+2} \leq \frac{e^{-1}}{\delta-\delta'}\Big(&||u^3-v^3||_{\delta,s} + ||u(\partial_x u)^2 - v(\partial_x v)^2||_{\delta,s} \\ &+ ||u^2\partial_x^2 u - v^2\partial_x^2 v||_{\delta,s} + ||(\partial_x u)^2\partial_x^2 u - (\partial_x v)^2\partial_x^2 v||_{\delta,s}\Big). \end{aligned} \tag{2.30}$$

The first and second term of (2.30) are estimated like the corresponding terms for NE obtaining inequalities (2.22) and (2.24). For the third term of (2.30) using the identity

$$u^2\partial_x^2 u - v^2\partial_x^2 v = (u-v)(u+v)\partial_x^2(u) + v^2\partial_x^2(u-v) \tag{2.31}$$

and the algebra property we have

$$\begin{aligned}||u^2\partial_x^2 u - v^2\partial_x^2 v||_{\delta,s} &\le c_s^2\big(||u-v||_{\delta,s}||u+v||_{\delta,s}||u||_{\delta,s+2} + ||u-v||_{\delta,s+2}||v||_{\delta,s}^2\big)\\ &\le 3c_s^2(R+||u_0||_{\delta,s+2})^2||v-u||_{\delta,s+2}. \end{aligned} \tag{2.32}$$

Finally, for the fourth term of (2.30) using the identity

$$(\partial_x u)^2\partial_x^2 u - (\partial_x v)^2\partial_x^2 v = (\partial_x u)^2\partial_x^2(u-v) + \partial_x(u-v)\partial_x(u+v)\partial_x^2 v, \tag{2.33}$$

we have

$$\begin{aligned}||(\partial_x u)^2\partial_x^2 u - (\partial_x v)^2\partial_x^2 v||_{\delta,s} &\le c_s^2||u||_{\delta,s+1}^2||u-v||_{\delta,s+2}\\ &\quad + c_s^2||u-v||_{\delta,s+1}\Big(||u||_{\delta,s+1} + ||v||_{\delta,s+1}\Big)||v||_{\delta,s+2}\\ &\le 3c_s^2(R+||u_0||_{\delta,s+2})^2||u-v||_{\delta,s+2}. \end{aligned} \tag{2.34}$$

Now, combining the above estimates and using the fact that $||u_0||_{\delta,s+2} \le ||u_0||_{1,s+2}$ gives the desired inequality (2.29).

Next, using the properties of the $G^{\delta,s}$-norms we get the estimate

$$||F(u_0)||_{\delta',s+2} \le \frac{4e^{-1}c_s^2||u_0||_{\delta,s+2}^3}{\delta-\delta'},$$

and replacing δ' by δ and δ by 1 we have

$$||F(u_0)||_{\delta,s+2} \le \frac{4e^{-1}c_s^2||u_0||_{1,s+2}^3}{1-\delta},\ 0<\delta<1, \tag{2.35}$$

which is inequality (2.8) with $M = 4e^{-1}c_s^2||u_0||_{1,s}^3$.

Finally, using lifespan formula (1.5) with the constants present in inequalities (2.29) and (2.35), and letting $R = ||u_0||_{1,s+2}$ we obtain the following lifespan estimate

$$T = \frac{e}{800c_s^2}\frac{1}{||u_0||_{1,s+2}^2}, \tag{2.36}$$

which completes the proof of Lemma 5 for the FORQ equation. □

3. Camassa-Holm type systems

Next we shall prove analytic well-posedness for two integrable systems, one with quadratic nonlinearities and one with cubic. The first one is the 2-component Camassa-Holm (2CH) system, which can be written in the following nonlocal form

$$\partial_t u = F_1(u,v) \doteq -\partial_x\Big(\frac{1}{2}u^2 + (1-\partial_x^2)^{-1}\Big[u^2 + \frac{1}{2}(\partial_x u)^2 + \frac{\sigma}{2}v^2\Big]\Big) \tag{3.1}$$

$$\partial_t v = F_2(u,v) \doteq -\partial_x\big(uv\big) \tag{3.2}$$

where $\sigma = \pm 1$. For $v=0$ it gives the CH equation. This system is integrable (see Falqui [26] and Shabat and Alonso [65]). In the context of shallow water wave theory it was derived by Constantin and Ivanov [14] who also proved the existence of peakon traveling wave solutions. Well-posedness in Sobolev and Besov spaces was studied in [14], [25], [31] and [67].

The second system can be thought as a 2-component version of the Novikov equation (2NE)

$$(1-\partial_x^2)\partial_t u = -4uv\partial_x u + 3v\partial_x u\partial_x^2 u + uv\partial_x^3 u$$
$$(1-\partial_x^2)\partial_t v = -4uv\partial_x v + 3u\partial_x v\partial_x^2 v + vu\partial_x^3 v.$$

In fact setting $u = v$ gives the NE equation. The 2NE system was introduced recently by Geng and Xue in [32] who proved its integrability and established its Hamiltonian structure. Well-posedness in Sobolev spaces H^s for $s > 3/2$ and peakon solutions have been studied in [40].

To place the 2NE system in the framework of the autonomous Ovsyannikov theorem we write it in the following nonlocal form

$$\partial_t u = F_1(u,v) \doteq (1-\partial_x^2)^{-1}\Big(\big[-4uv\partial_x u + 2v\partial_x u\partial_x^2 u - u\partial_x v\partial_x^2 u\big] + \partial_x\big[uv\partial_x^2 u\big]\Big) \tag{3.3}$$

$$\partial_t v = F_2(u,v) \doteq (1-\partial_x^2)^{-1}\Big(\big[-4uv\partial_x v + 2u\partial_x v\partial_x^2 v - v\partial_x u\partial_x^2 v\big] + \partial_x\big[uv\partial_x^2 v\big]\Big). \tag{3.4}$$

Then, using the two-component function $F(u,v) = (F_1(u,v), F_2(u,v))$ we are able to write the Cauchy problem for both the 2CH and the 2NE system in the following unified way

$$\frac{d}{dt}(u,v) = F(u,v) = \big(F_1(u,v), F_2(u,v)\big), \qquad (u,v)(0) = (u_0,v_0), \tag{3.5}$$

where (F_1, F_2) is given by the right hand-sides of equations (3.1)-(3.2) for the 2CH system and by (3.3)-(3.4) for the 2NE system.

Next we shall study the Cauchy problem for the system (3.5) in the following scale of decreasing Banach spaces

$$\{\mathbb{G}^{\delta,s}\}_{0<\delta\le 1} = \{G^{\delta,s}\times G^{\delta,s}\}_{0<\delta\le 1}.$$

on both the circle $\mathbb{T}$ and the line $\mathbb{R}$, where for $(\varphi_1,\varphi_2)\in\mathbb{G}^{\delta,s}$ the norm is defined by

$$||(\varphi_1,\varphi_2)||_{\mathbb{G}^{\delta,s}} = ||\varphi_1||_{G^{\delta,s}} + ||\varphi_2||_{G^{\delta,s}}.$$

More precisely, we will prove the following result.

Theorem 4. *Let $s > \frac{1}{2}$. If (u_0,v_0) is in $\mathbb{G}^{1,s+2}$ on the circle or the line, then there exists a positive time T, which depends on the initial data (u_0,v_0) and s, such that for every $\delta\in(0,1)$, the Cauchy problem (3.5) has a unique solution $(u(t),v(t))$, which is a holomorphic function in $D(0,T(1-\delta))$ valued in $\mathbb{G}^{\delta,s+2}$. Furthermore, the analytic lifespan T satisfies the estimate*

$$T \approx \frac{1}{||(u_0,v_0)||^k_{\mathbb{G}^{1,s+2}}}, \tag{3.6}$$

where $k = 1$ for the 2-CH system and $k = 2$ for the 2NE system. Finally, the solution map is continuous.

Proof. Like in the proof of the corresponding part of Theorem 2, one can show that for the 2-component Camassa-Holm system (2CH) the lifespan is given by

$$T = \frac{e}{184c_s}\frac{1}{||(u_0,v_0)||_{\mathbb{G}^{1,s+2}}} \tag{3.7}$$

and for the 2-component Novikov system (2NE) the lifespan is given by

$$T = \frac{e}{2176c_s^2}\frac{1}{||(u_0,v_0)||^2_{\mathbb{G}^{1,s+2}}}. \tag{3.8}$$

The proof of the continuity of the solution map will be discussed in Section 4. □

4. Continuity of the data-to-solution map and the Ovsyannikov theorem

We start by presenting a sketch of the proof of autonomous Ovsyannikov theorem, which will be useful in the proof of the continuity of the data-to-solution map. A complete proof can be found in [1].

Ouline of proof for Theorem 1. Recall that the scale of Banach spaces X_δ and the function $F(u)$ satisfy the conditions (1)-(3) described before the statement of Theorem 1. Also, notice that for $\delta \in (0,1]$ and $v \in \mathcal{H}(|t|<b; X_\delta)$ with $b>0$, the equation

$$\frac{du}{dt} = v, \quad u(0) = u_0, \tag{4.1}$$

has a unique solution $u \in \mathcal{H}(|t|<b; X_\delta)$ given by

$$u(t) = u_0 + Kv(t) \doteq u_0 + \int_0^t v(\tau)d\tau. \tag{4.2}$$

Therefore, it follows that the existence of u in Theorem 1 is equivalent to the existence of $v \in \mathcal{H}(|t|<T(1-\delta); X_\delta)$, for every $\delta \in (0,1)$, satisfying for $|t| < T(1-\delta)$

$$\Big|\Big|\int_0^t v(\tau)d\tau\Big|\Big|_\delta < R \tag{4.3}$$

and

$$v = F(u_0 + Kv). \tag{4.4}$$

Then our initial value problem reduces to finding the fixed point of the equation (4.4). For this, we shall need a new space, which we define next.

Definition 1. *For $a>0$ we denote by $E_a = \bigcap_{0<\delta<1} \mathcal{H}(D(0,a(1-\delta)); X_\delta)$ the Banach space of all functions $t \mapsto u(t)$ which for every $0<\delta<1$ we have that*

$$u : \{t : |t| < a(1-\delta)\} \to X_\delta \quad \text{is holomorphic,} \tag{4.5}$$

and whose norm is defined by

$$|||u|||_a \doteq \sup\left\{||u(t)||_\delta (1-\delta)\sqrt{1-\frac{|t|}{a(1-\delta)}} : 0<\delta<1 \text{ and } |t|<a(1-\delta)\right\} < \infty. \tag{4.6}$$

Remark. Note that if $0<a<b$ then $E_b \hookrightarrow E_a$.

Using the spaces E_a and the norm (4.6) we have the following three lemmas.

Lemma 6. *Let $a>0$, $u \in E_a$, $0<\delta<1$ and $|t|<a(1-\delta)$. Then*

$$||Ku(t)||_\delta \le \int_0^{|t|} \Big\|u\left(\tau\frac{t}{|t|}\right)\Big\|_\delta d\tau \le 2a|||u|||_a.$$

Proof. We start by setting the path $\gamma : [0,1] \to \mathbb{C}$ given by $\gamma(\tau) = \tau t$, where t is fixed. We have

$$\begin{aligned} ||Ku(t)||_\delta &= \Big\| \int_0^t u(z)dz \Big\|_\delta = \Big\| \int_\gamma u(z)dz \Big\|_\delta = \Big\| \int_0^1 u(\gamma(\tau))\gamma'(\tau)d\tau \Big\|_\delta \\ &\le \int_0^1 ||u(\gamma(\tau))||_\delta |\gamma'(\tau)| d\tau = \int_0^1 ||u(\tau t)||_\delta |t| d\tau \\ &= \int_0^{|t|} \Big\| u\left(\tau \frac{t}{|t|}\right) \Big\|_\delta d\tau = \int_0^{|t|} \Big\| u\left(\tau \frac{t}{|t|}\right) \Big\|_\delta \frac{(1-\delta)\sqrt{1-\frac{|\tau|}{a(1-\delta)}}}{(1-\delta)\sqrt{1-\frac{|\tau|}{a(1-\delta)}}} d\tau \\ &\le |||u|||_a \int_0^{|t|} \frac{1}{(1-\delta)\sqrt{1-\frac{|\tau|}{a(1-\delta)}}} d\tau, \end{aligned}$$

since $|\tau| = \tau \le |t| < a(1-\delta)$.

Letting $\theta = \dfrac{\tau}{a(1-\delta)} = \dfrac{|\tau|}{a(1-\delta)}$, since $\tau \ge 0$ in the last integral, we have

$$||Ku(t)||_\delta \le a|||u|||_a \int_0^{\frac{|t|}{a(1-\delta)}} \frac{1}{\sqrt{1-\theta}} d\theta \le a|||u|||_a \int_0^1 \frac{1}{\sqrt{1-\theta}} d\theta = 2a|||u|||_a,$$

which completes the proof. □

Lemma 7. *For every $a > 0$, $u \in E_a$, $0 < \delta < 1$ and $|t| < a(1-\delta)$ we have*

$$\int_0^{|t|} \frac{\Big\| u\left(\tau \frac{t}{|t|}\right) \Big\|_{\delta(\tau)}}{\delta(\tau) - \delta} d\tau \le \frac{8a|||u|||_a}{1-\delta} \sqrt{\frac{a(1-\delta)}{a(1-\delta) - |t|}},$$

where $\delta(\tau) = \frac{1}{2}\left(1 + \delta - \frac{|\tau|}{a}\right)$.

Lemma 8. *Let $a > 0$, $0 < \delta < 1$, $|t| < a(1-\delta)$, $u \in E_a$, with $|||u|||_a < \frac{R}{4a}$ and $v \in E_{2a}$ with $|||v|||_{2a} < \frac{R}{8a}$. Under assumption* (1.3) *the following inequality holds:*

$$||F(u_0 + Ku(t)) - F(u_0 + Kv(t))||_\delta \le L \int_0^{|t|} \frac{\Big\| u\left(\tau \frac{t}{|t|}\right) - v\left(\tau \frac{t}{|t|}\right) \Big\|_{\delta(\tau)}}{\delta(\tau) - \delta} d\tau, \qquad (4.7)$$

where $\delta(\tau)$ is a continuous function on $[0, |t|]$ satisfying

$$\delta < \delta(\tau) \le \frac{1}{2}\Big(1 + \delta - \frac{|\tau|}{a}\Big),$$

and L is the same constant as in condition (1.3).

Let $b > 0$, $u \in E_b$ with $|||u|||_b < \dfrac{R}{4b}$ and $|t| < b(1-\delta)$, $\delta \in (0,1)$. Now, our aim is to define an appropriate Banach space such that G has a unique fixed point. We start by doing some computations. By using the fact that $K(0) = 0$ it follows from lemmas 7, 8 and our assumption

1.4 that for $\delta \in (0,1)$ we have

$$\begin{aligned} ||G(u(t))||_\delta &\leq ||F(u_0 + Ku(t)) - F(u_0 + K(0))||_\delta + ||F(u_0)||_\delta \\ &\leq L \int_0^{|t|} \frac{\left\|u\left(\tau \frac{t}{|t|}\right) - 0\right\|_{\delta(\tau)}}{\delta(\tau) - \delta} d\tau + \frac{M}{1-\delta} \\ &\leq \frac{8bL|||u|||_b}{1-\delta} \sqrt{\frac{b(1-\delta)}{b(1-\delta) - |t|}} + \frac{M}{1-\delta}. \end{aligned}$$

Since $u \in E_b$ it implies that for every $0 < \delta < 1$, $u \in \mathcal{H}(|t| < b(1-\delta); X_\delta)$ and therefore $Ku(t) = \int_0^t u(z)dz \in \mathcal{H}(|t| < b(1-\delta); X_\delta)$. The condition (3) on the function F, in our Theorem 1, imply that $G(u(t)) = F(u_0 + K(u(t))) \in \mathcal{H}(|t| < b(1-\delta); X_{\delta'})$ where $0 < \delta' < \delta$. Thus, it makes sense to compute $|||G(u)|||_b$. By using the last inequality it follows from the definition of the norm $||| \cdot |||_b$ that

$$|||G(u)|||_b \leq 8bL|||u|||_b + M. \tag{4.8}$$

Let $u \in E_a$, $v \in E_{2a}$ and $|||u|||_a < \frac{R}{4a}$, $|||v|||_{2a} < \frac{R}{8a}$; it follows from (4.8) that $G(u)$ and $G(v)$ are in E_a. Also, for $|||u|||_a < \frac{R}{4a}$, $|||v|||_{2a} < \frac{R}{8a}$ we use Lemmas 7 and 8 and we obtain

$$||G(u) - G(v)||_\delta \leq \frac{8aL|||u-v|||_a}{1-\delta} \sqrt{\frac{a(1-\delta)}{a(1-\delta) - |t|}}$$

and using this inequality in the definition of $|||G(u) - G(v)|||_a$ gives

$$|||G(u) - G(v)|||_a \leq 8aL|||u - v|||_a. \tag{4.9}$$

Assume now that

$$a < \frac{R}{16LR + 8M}. \tag{4.10}$$

Denote by E the closure in E_a of the ball $\{u \in E_{2a} : |||u|||_{2a} < \frac{R}{8a}\}$. The space E is a complete metric space and since $\{u \in E_{2a} : |||u|||_{2a} < \frac{R}{8a}\} \subset \{u \in E_a : |||u|||_a \leq \frac{R}{8a}\}$ then taking the closure in E_a we conclude that the set $E \subset \{u \in E_a : |||u|||_a \leq \frac{R}{8a}\}$. Notice that if $u \in E$ then there exists a sequence $u_n \in E_{2a}$, with $|||u_n|||_{2a} < \frac{R}{8a}$ and $u_n \to u$ in E_a. Applying (4.8) to u_n with $b = 2a$ gives

$$|||G(u_n)|||_{2a} \leq 8 \cdot 2aL|||u_n|||_{2a} + M < 8 \cdot 2aL\frac{R}{8a} + M = 2LR + M < \frac{R}{8a},$$

since $a < R/16LR + 8M$. Also, applying (4.9) with $v = u_n$ and noticing that $\frac{R}{8a} < \frac{R}{4a}$ we obtain

$$|||G(u_n) - G(u)|||_a \leq 8aL|||u_n - u|||_a \to 0, \text{ as } n \to \infty.$$

Hence, we can conclude that G maps E into E.

Finally, we will show that G is a contraction on E. Let $u, v \in E$ be given. Since there exists a sequence $v_n \in E_{2a}$, with $|||v_n|||_{2a} < \frac{R}{8a}$ and $v_n \to v$ in E_a it follows as above that $|||G(v_n) - G(v)|||_a \to 0,$ as $n \to \infty$. Also, as above, we have

$$\begin{aligned} |||G(u) - G(v)|||_a &\leq |||G(u) - G(v_n)|||_a + |||G(v_n) - G(v)|||_a \\ &\leq 8aL(|||u - v|||_a + |||v - v_n|||_a) + |||G(v_n) - G(v)|||_a \end{aligned}$$

which implies that $|||G(u)-G(v)|||_a \le 8La|||u-v|||_a$ and we have $8La < 1$ since $a < \frac{R}{16LR+8M}$. Therefore, G is a contraction on E and we can conclude that G has a fixed point $v \in E$. For $v \in E$ there exists a sequence $v_n \in E_{2a}$ with $|||v_n|||_{2a} \le \frac{R}{8a}$ and $v_n \to v$ in E_a. Therefore, since $v \in E_a$ it follows from Lemma 6 that $||\int_0^t v(z)dz||_\delta = ||Kv(t)||_\delta \le 2a|||v|||_a = 2a|||\lim_{n\to\infty} v_n|||_a = 2a \lim_{n\to\infty} |||v_n|||_a \le 2a \lim_{n\to\infty} |||v_n|||_{2a} \le 2a \lim_{n\to\infty} \dfrac{R}{8a} = \dfrac{R}{4} < R$. Therefore, v is clearly a solution of (4.3) and (4.4). Thus, $u(t) = u_0 + \int_0^t v(\tau)d\tau$ is a solution to the Cauchy problem (1.1) which, for every $\delta \in (0,1)$ is a holomorphic function in $D(0, T(1-\delta))$ valued in X_δ satisfying

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1.$$

The proof of theorem 1 is now complete.

Remark 4.1. It is clear from the proof of Theorem 1 that under the hypotheses (1), (2) and (3) that given $u_0 \in X_1$ and $R > 0$ there exists $T > 0$ and a unique solution to the Cauchy problem (1.1) in the set

$$E_{T,R} \doteq \{u(t) \in \bigcap_{0<\delta<1} \mathcal{H}(D(0,T(1-\delta)); X_\delta) \text{ and } \sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1\}.$$

Notice that if $u \in E_{T,R}$ then $u \in E_T$. Thus, from now on we endow $E_{T,R}$ with the metric $d(u,v) = |||u-v|||_T$.

Continuity of the solution map. We now prove the continuity of the data-to-solution map for initial data and solution in Theorem 3 and Theorem 4. We will do this only for the well-known Camassa-Holm equation since for the other Camassa-Holm type equations and systems the proof is similar.

We start by recalling the Camassa-Holm equation:

$$\partial_t u + u\partial_x u + \left(1 - \partial_x^2\right)^{-1} \partial_x \left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0. \tag{4.11}$$

Theorem 5. *Given $u_0 \in G^{1,s+2}$, $s > -1/2$, and $R > 0$ there exists $T = T_{u_0} > 0$, which is given by $T = C_s \frac{1}{||u_0||_{G^{1,s+2}}}$, such that the Cauchy problem for CH has a unique solution $u \in E_{T,R} \doteq \{u(t) \in \bigcap_{0<\delta<1} \mathcal{H}(D(0,T(1-\delta)); G^{\delta,s+2})$ and $\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1\}$. Moreover the data-to-solution map $G^{1,s+2} \ni u_0 \longmapsto u \in E_{T,R}$ is continuous.*

Before proving Theorem 5 we recall what means the data-to-solution map to be continuous.

Definition 2. *One says that the data-to-solution map $u(0) \longmapsto u(t)$ is continuous if for a given $u_\infty(0) \in G^{1,s+2}$ there exist $T = T(||u_\infty(0)||_{1,s+2}) > 0$ and $R > 0$ such that for any sequence of initial data $u_n(0) \in G^{1,s+2}$ converging to $u_\infty(0)$ in $G^{1,s+2}$ the corresponding solutions, $u_n(t), u_\infty(t)$ to the CH Cauchy problem for all sufficiently large n satisfy: $u_n(t), u_\infty(t) \in E_{T,R}$ and $|||u_n - u_\infty|||_T \to 0$, where*

$$|||u|||_T \doteq \sup\left\{||u(t)||_\delta (1-\delta)\sqrt{1 - \frac{|t|}{T(1-\delta)}} : 0 < \delta < 1 \text{ and } |t| < T(1-\delta)\right\} < \infty.$$

Proof of Theorem 5. Let $s > \frac{1}{2}$ and let $u_\infty(0) \in G^{1,s+2}$ be given. For $R_\infty = ||u_\infty(0)||_{1,s+2}+1$ the existence and uniqueness of the solution already has been proved. It follows from (2.15) that the lifespan of the corresponding solution to the CH Cauchy problem, $u_\infty(t) \in E_{T_\infty, R_\infty}$, is given by

$$T_{u_\infty(0)} = \frac{R_\infty}{16C(R_\infty + ||u_\infty(0)||_{1,s+2})R_\infty + 4C||u_\infty(0)||^2_{1,s+2}},$$

where $C = 4e^{-1}c_s$.

Now let $u_n(0) \in G^{1,s+2}$ be a sequence of initial data converging to $u_\infty(0)$ in $G^{1,s+2}$. By setting

$$R_n = R_\infty + ||u_n(0) - u_\infty(0)||_{1,s+2}$$

let $u_n(t) \in E_{T_n,R_n}$ be the corresponding solutions to the CH Cauchy problem, where again, according to (2.15) the lifespan of $u_n(t)$ is given by

$$T_{u_n(0)} = \frac{R_n}{16C(R_n + ||u_n(0)||_{1,s+2})R_n + 4C||u_n(0)||^2_{1,s+2}}.$$

By noticing that

$$\begin{aligned} ||u_n(0)||_{1,s+2} &\leq ||u_n(0) - u_\infty(0)||_{1,s+2} + ||u_\infty(0)||_{1,s+2} \\ &< ||u_n(0) - u_\infty(0)||_{1,s+2} + R_\infty = R_n \end{aligned}$$

we have

$$T_{u_n(0)} \geq \frac{R_n}{16C(R_n + R_n)R_n + 4CR_n^2} = \frac{1}{36CR_n} \longrightarrow \frac{1}{36CR_\infty},$$

since $||u_n(0) - u_\infty(0)||_{1,s+2} \longrightarrow 0$ as $n \to \infty$.

Thus, we have $\frac{1}{2}T_{u_n(0)} \geq \frac{1}{72CR_n} \to \frac{1}{72CR_\infty}$. Given $\epsilon = \frac{1}{144CR_\infty}$ there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$ we have

$$\frac{1}{72CR_n} > \frac{1}{144CR_\infty}.$$

By taking $\tilde{T} \doteq \min\{\frac{1}{144CR_\infty}, \frac{1}{2}T_{u_\infty(0)}\} = \frac{1}{144CR_\infty} = \frac{1}{144C(||u_\infty(0)||_{1,s+2})}$ we can conclude that $\tilde{T} \leq \frac{1}{2}\min\{T_{u_n(0)}, T_{u_\infty(0)}\}$ for $n \geq N_1$.

We now are going to determine $T > 0$ and $R > 0$ as in the Definition 2. Since $R_n \to R_\infty$ as n goes to ∞, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$ we have $R_n < 5R_\infty$. By setting $R = 10R_\infty$ and $T = \frac{R}{16C(R+||u_\infty(0)||_{1,s+2})R+4C||u_\infty(0)||^2_{1,s+2}}$ we notice that

$$\begin{aligned} T &= \frac{10R_\infty}{16C(10R_\infty + ||u_\infty(0)||_{1,s+2})10R_\infty + 4C||u_\infty(0)||^2_{1,s+2}} \\ &\leq \frac{10R_\infty}{1600CR_\infty^2} = \frac{1}{160CR_\infty} < \frac{1}{144CR_\infty} \end{aligned}$$

and therefore for $n \geq \tilde{N} \doteq \max\{N_1, N_2\}$ we have $R_n < \frac{R}{2}$ and $T < \frac{1}{2}\min\{T_{u_n(0)}, T_{u_\infty(0)}\}$.

It is easily seen that $u_\infty \in E_{T,R}$. Since $||u_n(0) - u_\infty(0)||_{1,s+2} \to 0$, as n goes to ∞ there exists $N_3 \in \mathbb{N}$ such that for $n \geq N_3$ we obtain $||u_n(0) - u_\infty(0)||_{1,s+2} < \frac{R}{2}$. Thus, for $n \geq$

$N \doteq \max\{\tilde{N}, N_3\}$ we have

$$\begin{aligned}\sup_{|t|<T(1-\delta)} \|u_n(t) - u_\infty(0)\|_{\delta,s+2} &\leq \sup_{|t|<T(1-\delta)} \|u_n(t) - u_n(0)\|_{\delta,s+2} + \sup_{|t|<T(1-\delta)} \|u_n(0) - u_\infty(0)\|_{\delta,s+2} \\ &\leq R_n + \|u_n(0) - u_\infty(0)\|_{1,s+2} < \frac{R}{2} + \frac{R}{2} = R\end{aligned}$$

and therefore we can conclude that $u_n \in E_{T,R}$ for $n \geq N$.

In order to complete the proof it suffices to prove the following

Lemma 9. *For $n \geq N$ and $s > \frac{1}{2}$ we have*

$$|||u_n - u_\infty|||_T \leq 2||u_n(0) - u_\infty(0)||_{1,s+2}. \tag{4.12}$$

Proof. Let $n \geq N$ and $s > \frac{1}{2}$ be given. We know that

$$u_\infty(t) = u_\infty(0) + K(F(u_\infty(t))), \text{ for } |t| < T_{u_\infty(0)}(1-\delta) \tag{4.13}$$

and

$$u_n(t) = u_n(0) + K(F(u_n(t))), \text{ for } |t| < T_{u_n(0)}(1-\delta). \tag{4.14}$$

Notice that for $n \in \mathbb{N}$ and $0 < \delta \leq 1$ we have

$$\{u \in G^{\delta,s+2} : ||u - u_\infty(0)||_{\delta,s+2} < R_\infty\} \subset \{v \in G^{\delta,s+2} : ||v - u_n(0)||_{\delta,s+2} < R_n\}.$$

We now set

$$T_{u_\infty(0),u_n(0)} = \frac{1}{2}\min\{T_{u_\infty(0)}, T_{u_n(0)}\}. \tag{4.15}$$

This choice of $T_{u_\infty(0),u_n(0)}$ gives the same lifespan for the solutions $u_\infty(t)$ and $u_n(t)$.

For $0 < \delta < 1$ and $|t| < T_{u_\infty(0),u_n(0)}(1-\delta)$ we have $|t| < T_{u_\infty(0)}(1-\delta)$ and $|t| < T_{u_n(0)}(1-\delta)$ and therefore, for every $\delta \in (0,1)$, $u_\infty(t)$ and $u_n(t)$ are holomorphic functions on $\{t \in \mathbb{C} : |t| < T_{u_\infty(0),u_n(0)}(1-\delta)\}$ with values in $G^{\delta,s+2}$.

Thus, for $0 < \delta < 1$ and $|t| < T_{u_\infty(0),u_n(0)}(1-\delta)$ it follows from (4.13) and (4.14) that

$$||u_\infty(t) - u_n(t)||_{\delta,s+2} - ||u_\infty(0) - u_n(0)||_{\delta,s+2} \leq ||K[F(u_\infty(t)) - F(u_n(t))]||_{\delta,s+2}.$$

Since, as in the proof of Lemma 6, we have

$$\Big\| \int_0^t [F(u_\infty(z)) - F(u_n(z))]dz \Big\|_{\delta,s+2} \leq \int_0^{|t|} \Big\| F(u_\infty(\tau\frac{t}{|t|})) - F(u_n(\tau\frac{t}{|t|})) \Big\|_{\delta,s+2} d\tau$$

we conclude that

$$||u_\infty(t)-u_n(t)||_{\delta,s+2} - ||u_\infty(0)-u_n(0)||_{\delta,s+2} \leq \int_0^{|t|} \Big\| F(u_\infty(\tau\frac{t}{|t|})) - F(u_n(\tau\frac{t}{|t|})) \Big\|_{\delta,s+2} d\tau. \tag{4.16}$$

For $0 < \delta < 1$, $|t| < T_{u_\infty(0),u_n(0)}(1-\delta)$, $0 \leq |\tau| = \tau \leq |t|$ and $\delta(\tau) = \frac{1}{2}(1 + \delta - \frac{|\tau|}{T_{u_\infty(0),u_n(0)}})$ we shall need to prove that

$$||u_\infty(\tau\frac{t}{|t|}) - u_n(0)||_{\delta(\tau),s+2} < R_n \tag{4.17}$$

and

$$||u_n(\tau\frac{t}{|t|}) - u_n(0)||_{\delta(\tau),s+2} < R_n. \tag{4.18}$$

Let us start by proving (4.17). We have

$$||u_\infty(\tau\frac{t}{|t|}) - u_n(0)||_{\delta(\tau),s+2} \leq ||u_\infty(\tau\frac{t}{|t|}) - u_\infty(0)||_{\delta(\tau),s+2} + ||u_\infty(0) - u_n(0)||_{\delta(\tau),s+2}. \quad (4.19)$$

Thus, it follows from (4.19) that (4.17) will be proved if we be able to show that

$$||u_\infty(\tau\frac{t}{|t|}) - u_\infty(0)||_{\delta(\tau),s+2} < R_\infty. \quad (4.20)$$

For this, by using (1.6) with $T, R, u_0, u(t)$ replaced by $T_{u_\infty(0)}, R_\infty, u_\infty(0), u_\infty(t)$, respectively, and noticing that $0 < \delta(\tau) < 1$ for any τ, as above, it suffices to prove that

$$\left|\frac{\tau t}{|t|}\right| < T_{u_\infty(0)}(1 - \delta(\tau)) \quad (4.21)$$

for any t, τ and $\delta(\tau)$ as above. Since $T_{u_\infty(0),u_n(0)} < T_{u_\infty(0)}$ we will prove that $\left|\frac{\tau t}{|t|}\right| < T_{u_\infty(0),u_n(0)}$ $\cdot(1 - \delta(\tau))$ for any t, τ and $\delta(\tau)$, as above. It is easily seen that $\left|\frac{\tau t}{|t|}\right| < T_{u_\infty(0),u_n(0)}(1 - \delta(\tau))$ if and only if $|\tau| < T_{u_\infty(0),u_n(0)}(1 - \delta)$ what is true since $|\tau| \leq |t| < T_{u_\infty(0),u_n(0)}(1 - \delta)$. Similarly one can prove (4.18). The proof of (4.17) and (4.18) is now complete.

Thanks to (4.16), (4.17), (4.18) and (1.3), for $0 < \delta < 1$ and $|t| < T_{u_\infty(0),u_n(0)}(1 - \delta)$ we have

$$\begin{aligned} ||u_\infty(t) - u_n(t)||_{\delta,s+2} \quad &- \quad ||u_\infty(0) - u_n(0)||_{\delta,s} \leq \int_0^{|t|} \left\| F(u_\infty(\tau\frac{t}{|t|})) - F(u_n(\tau\frac{t}{|t|}))\right\|_{\delta,s+2} d\tau \\ &\leq \quad L_n \int_0^{|t|} \frac{1}{\delta(\tau) - \delta} \left\| u_\infty(\tau\frac{t}{|t|}) - u_n(\tau\frac{t}{|t|})\right\|_{\delta(\tau),s+2} d\tau \quad (4.22) \end{aligned}$$

where $L_n = C(R_n + ||u_n(0)||_{1,s+2})$ is the constant that comes from the condition (1.3) for the ball of the center $u_n(0)$ and radius R_n and $\delta(\tau) = \frac{1}{2}(1 + \delta - \frac{|\tau|}{T_{u_\infty(0),u_n(0)}})$. Note that for $|\tau| < T_{u_\infty(0),u_n(0)}(1 - \delta)$ we have $0 < \delta < \delta(\tau) < 1$.

Since $u_\infty \in E_{T_{u_\infty(0),R_\infty}} \hookrightarrow E_{T_{u_\infty(0)}} \hookrightarrow E_{T_{u_\infty(0),u_n(0)}}$, $u_n \in E_{T_{u_n(0),R_n}} \hookrightarrow E_{T_{u_n(0)}} \hookrightarrow E_{T_{u_\infty(0),u_n(0)}}$ we conclude that $u_\infty(t) - u_n(t) \in E_{T_{u_\infty(0),u_n(0)}}$. We shall need to estimate $|||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}}$. Thanks to Lemma 7, (with $a = T_{u_\infty(0),u_n(0)}$), and (4.22), for $0 < \delta < 1$ and $|t| < T_{u_\infty(0),u_n(0)}(1 - \delta)$, we have

$$\begin{aligned} &||u_\infty(t) - u_n(t)||_{\delta,s} - ||u_\infty(0) - u_n(0)||_{\delta,s} \\ &\leq \frac{8T_{u_\infty(0),u_n(0)}L_n|||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}}}{1 - \delta} \sqrt{\frac{T_{u_\infty(0),u_n(0)}(1 - \delta)}{T_{u_\infty(0),u_n(0)}(1 - \delta) - |t|}}, \end{aligned}$$

which implies that

$$|||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}} \leq 8T_{u_\infty(0),u_n(0)}L_n|||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}} + ||u_\infty(0) - u_n(0)||_{1,s+2}$$

in turns implies that

$$(1 - 8T_{u_\infty(0),u_n(0)}L_n)|||u - v|||_{T_{u_\infty(0),u_n(0)}} \leq ||u_\infty(0) - u_n(0)||_{1,s+2}. \quad (4.23)$$

Since $T_{u_\infty(0),u_n(0)} = \frac{1}{2}\min\{T_{u_\infty(0)}, T_{u_n(0)}\} < T_{u_n(0)} < \frac{1}{16L_n}$ we have $T_{u_\infty(0),u_n(0)} < \frac{1}{16L_n}$, which implies that

$$8T_{u_\infty(0),u_n(0)}L_n < \frac{1}{2} \Rightarrow -8T_{u_\infty(0),u_n(0)}L_n > -\frac{1}{2} \Rightarrow 1 - 8T_{u_\infty(0),u_n(0)}L_n > \frac{1}{2}.$$

Hence,

$$|||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}} \leq \frac{1}{1-8T_{u_\infty(0),u_n(0)}L_n}||u_\infty(0)-u_n(0)||_{1,s+2} < 2||u_\infty(0)-u_n(0)||_{1,s+2}. \tag{4.24}$$

Since for $n \geq N$ we have $T \leq \frac{1}{2}\min\{T_{u_n(0)}, T_{u_\infty(0)}\} = T_{u_\infty(0),u_n(0)}$ the last inequality implies

$$|||u_n - u_\infty|||_T \leq |||u_\infty - u_n|||_{T_{u_\infty(0),u_n(0)}} \leq 2||u_n(0)-u_\infty(0)||_{1,s+2}.$$

The proof of Lemma 9 and therefore the proof of Theorem 5 is now complete. □

5. Further Applications and Hadamard's Example

As we have mentioned earlier, this work provides a unified approach for the study of the Cauchy problem for Camassa-Holm type equations and systems with initial data in spaces of analytic functions. For example, a result like Theorem 2 can be proved for the following more general CH type equation with quadratic nonlinearities,

$$(1-\partial_x^2)u_t = -(b+1)uu_x + bu_xu_{xx} + uu_{xxx}, \tag{5.1}$$

called the b-family equation. This equation was introduced by Degasperis, Holm and Hone [23] who pointed out that, like CH and DP, it has peakon and multipeakon traveling solutions for all b. Furthermore, it was shown by Mikhailov and Novikov [52] that it is integrable only for $b = 2$ (CH) and for $b = 3$ (DP). Also, these techniques can be applied to the Hyperelastic Rod (HR) equation [21]

$$(1-\partial_x^2)u_t = -3uu_x + \gamma\big(2u_xu_{xx} + uu_{xxx}\big), \tag{5.2}$$

which is integrable only when $\gamma = 1$ (CH). Also, it has peakon solutions only when $\gamma = 1$. It is worth noticing that the techniques presented here apply also to CH equations which may not be integrable but have appropriate nonlinearities. Equations (5.1) and (5.2) are such examples.

Furthermore, the unified method presented here can be applied to equations with mixed nonlinearities. The following

$$m_t = b_1\partial_x u + \frac{1}{2}k_1\partial_x\left[m(u^2-(\partial_x u)^2)\right] + \frac{1}{2}k_2(2m\partial_x u + (\partial_x m)u), \quad m = (1-\partial_x^2)u, \tag{5.3}$$

is an example of such an equation, where b_1, k_1 and k_2 are arbitrary constants. This equation has been considered by Qiao, Xia and Li in [62], where they studied its integrability and derived peakon and multi-peakon solutions when $b_1 = 0$ and kink and kink-peakon solutions when $b_1 \neq 0$ and $k_2 = 0$. Observe that letting $b_1 = 0$, $k_1 = 0$ and $k_2 = -2$ gives the CH equation, while, letting $b_1 = 0$, $k_1 = -2$ and $k_2 = 0$ gives the FORQ equation.

Next, we shall provide the estimates of the type (1.3) and (1.4), which are needed for the application of the autonomous Ovsyannikov theorem for equation (5.3) in the spaces $G^{\delta,s}$. For this we rewrite equation (5.3) as follows

$$\begin{aligned}\frac{du}{dt} = F(u) \doteq (1-\partial_x^2)^{-1}\partial_x\Big(b_1u - \frac{k_2}{2}u\partial_x^2u + \frac{3}{4}k_2u^2 - \frac{k_2}{4}(\partial_xu)^2 \\ + \frac{k_1}{2}u^3 - \frac{k_1}{2}u(\partial_xu)^2 - \frac{k_1}{2}u^2(\partial_x^2u) + \frac{k_1}{2}(\partial_xu)^2\partial_x^2u\Big).\end{aligned} \tag{5.4}$$

Then, assuming that $||u-u_0||_{\delta,s+2} < R$, $||v-u_0||_{\delta,s+2} < R$, $0<\delta'<\delta\le 1$ and $s>\frac{1}{2}$ and using the properties of the spaces $\{G^{\delta,s}\}_{0<\delta\le 1}$ one can show that

$$||F(u)-F(v)||_{\delta',s+2} \le \frac{L}{\delta-\delta'}||u-v||_{\delta,s+2}, \tag{5.5}$$

where $L = e^{-1}\Big(|b_1| + 3c_s|k_2|(R+||u_0||_{1,s+2}) + 6c_s^2|k_1|(R+||u_0||_{1,s+2})^2\Big)$. Furthermore, using these properties one can show that for $0<\delta<1$ we have

$$||F(u_0)||_{\delta,s+2} \le \frac{M}{1-\delta}, \tag{5.6}$$

where $M = e^{-1}||u_0||_{1,s+2}\Big(|b_1| + \frac{3}{2}c_s|k_2|\,||u_0||_{1,s+2} + 2c_s^2|k_1|\,||u_0||^2_{1,s+2}\Big)$. Finally, using the lifespan formula (1.5) of Theorem 1 gives the following estimate

$$T \approx \frac{1}{1+||u_0||_{1,s+2}+||u_0||^2_{1,s+2}},$$

for the lifespan of equation (5.4). This completes an outline for the proof of a result like Theorem 2 for equation (5.4), which has mixed degree nonlinearities.

The Laplace equation and Hadamard's example. Now, we consider the initial value problem for the Laplace equation

$$\begin{cases} u_{tt}+u_{xx}=0 \Longleftrightarrow (\partial_t - i\partial_x)(\partial_t u + i\partial_x u) = 0 \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x). \end{cases} \tag{5.7}$$

By setting $\partial_t u + i\partial_x u = v$, Cauchy problem (5.7) takes the system form

$$\begin{cases} \partial_t(u,v) = i\partial_x(-u,v)+(v,0) \doteq F(u,v) \doteq (F_1(u,v),F_2(u,v)) \\ (u,v)(x,0) = (u_0(x), u_1(x)+i\partial_x u_0(x)) \doteq (u_0(x),v_0(x)). \end{cases} \tag{5.8}$$

Like in the case of CH type systems, we use the scale of Banach spaces $\{\mathbb{G}^{\delta,s}\}_{0<\delta\le1} = \{G^{\delta,s}\times G^{\delta,s}\}_{0<\delta\le 1}$ with norm $||(\varphi_1,\varphi_2)||_{\mathbb{G}^{\delta,s}} = ||\varphi_1||_{\delta,s}+||\varphi_2||_{\delta,s}$. Then, for given $R>0$ and $(u_0,v_0)\in\mathbb{G}^{1,s}$, let $(u_1,v_1),(u_2,v_2)\in\mathbb{G}^{\delta,s}$ such that $||(u_1,v_1)-(u_0,v_0)||_{\mathbb{G}^{\delta,s}} < R$, $||(u_2,v_2)-(u_0,v_0)||_{\mathbb{G}^{\delta,s}} < R$, and $0<\delta'<\delta\le 1,\ s\ge 0$ we have

$$\begin{aligned} ||F(u_1,v_1)-F(u_2,v_2)||_{\mathbb{G}^{\delta',s}} &= ||i\partial_x(-u_1+u_2)+(v_1-v_2)||_{\delta',s} + ||i\partial_x(v_1-v_2)||_{\delta',s} \\ &\le \frac{e^{-1}+1}{\delta-\delta'}||(u_1,v_1)-(u_2,v_2)||_{\mathbb{G}^{\delta,s}}. \end{aligned} \tag{5.9}$$

That is, we have $L=e^{-1}+1$, which is independent of R. Furthermore, we have

$$||F(u_0,v_0)||_{\mathbb{G}^{\delta',s}} = ||-i\partial_x u_0+v_0||_{\delta',s}+||i\partial_x v_0)||_{\delta',s} \le \frac{e^{-1}+1}{\delta-\delta'}||(u_0,v_0)||_{\mathbb{G}^{\delta,s}},$$

which gives the estimate

$$||F(u_0,v_0)||_{\delta,s} \le \frac{M}{1-\delta}, \quad 0<\delta<1 \tag{5.10}$$

with the constant M given by $M=(e^{-1}+1)||(u_0,v_0)||_{\mathbb{G}^{1,s}}$.

Finally, substituting the constants L and M in formula (1.5) and choosing $R = ||(u_0, v_0)||_{\mathbb{G}^{1,s}}$ gives the following interesting estimate for the analytic lifespan for the Laplace equation

$$T = \frac{1}{24(1+e^{-1})}. \tag{5.11}$$

Thus, we have obtained an Ovsyannikov theorem for the Laplace equation in $\mathbb{R}^2$ with lifespan estimate (5.11), which is independent of the the initial data. We mention that doing a similar analysis of the wave equation we obtain the same analytic lifespan (5.11).

Hadamard's example. Next we revisit the question of continuity of the data-to-solution map for the Laplace equation in the context of the Ovsyannikov theorem. For this we use the following well-known example due to Hadamard [33]

$$\begin{cases} u_{tt} + u_{xx} = 0 \\ u(x,0) = 0,\ u_t(x,0) = ne^{-\sqrt{n}} sin(nx) \doteq u_1^n(x), \end{cases} \tag{5.12}$$

where $n = 1, 2, 3, \ldots$. According to (5.8) we have $u_n(0) = 0$ and $v_n(0) = u_1^n(x)$. Since

$$\widehat{u_1^n}(k) = \begin{cases} \mp \frac{i}{2} ne^{-\sqrt{n}},\ \text{if } k = \pm n \\ 0,\ \text{if } k \neq \pm n, \end{cases}$$

we have

$$||v_n(0)||^2_{\delta,s} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} e^{2\delta|k|} |\widehat{u_1^n}(k)|^2 \simeq \langle n\rangle^{2s} n^2 e^{2\delta|n|-2\sqrt{n}} < \infty,$$

for any $0 < \delta \leq 1$ and any $s \geq 0$. Therefore, $v_n(0) \in G^{\delta,s}(\mathbb{T})$ for $n = 1, 2, 3, \ldots$. Thus we can conclude that the initial data $(u_n(0), v_n(0)) \in G^{1,s}(\mathbb{T}) \times G^{1,s}(\mathbb{T})$ for any $n = 1, 2, \ldots$, **but they do not converge to zero**, since

$$\begin{aligned} ||(u_n(0), v_n(0))||_{\mathbb{G}^{1,s}(\mathbb{T})} &= ||u_n(0)||_{1,s+2} + ||v_n(0)||_{1,s} = ||v_n(0)||_{1,s} \\ &\simeq \left(\langle n\rangle^{2s} n^2 e^{2|n|-2\sqrt{n}}\right)^{1/2} \longrightarrow \infty,\ \text{as } n \longrightarrow \infty. \end{aligned}$$

This shows that the above sequence of initial data does not contradict the continuity of the solution map for the Laplace equation with data in $G^{1,s}(\mathbb{T}) \times G^{1,s}$. Note that in other spaces like the ones considered by Hadamard [33] it does. This demonstrates the importance of the solutions space for the stability of the solution map.

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