Convergence of solutions to some elliptic equations in bounded Neumann thin domains

César R. de Oliveira and Alessandra A. Verri

Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil

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Abstract

Consider the (elliptic) stationary nonlinear reaction-diffusion equation in a sequence of bounded Neumann tubes in space that is squeezed to a reference curve. It is supposed that the forcing term is square integrable and the nonlinear one satisfies some growth and dissipative conditions. A norm convergence of the resolvents of the operators associated with the linear terms of such equations is proven, which is then used to provide new and simpler proofs of the asymptotic behaviour of the solutions to the full nonlinear equations (previously known in similar singular problems).

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1 Introduction

The possible effects of domain squeezing and dimensional reduction have been of great interest, particularly questions about effective operators, spectral behaviour and on the dynamics of reaction-diffusion equations. Roughly, as some directions are squeezed, the first task is to find possible remaining subspaces of the Hilbert space the operator acts; next one investigates the behaviour of the systems restricted to such subspaces and also possible contribution from the process of discarding dimensions. Usually it is not an easy task to carry out all needed estimates, since they involve extremely singular limits and effective operators and dynamics are also sensitive to the boundary conditions on the border of the regions of interest [2, 3, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17].

Here we concentrate on solutions to (elliptic linear and nonlinear) stationary reaction-diffusion equations in some bounded domains (in plane and space) with Neumann boundary conditions . Our main goals are twofold: by considering certain domains, we will rigorously reinforce the choice of the remaining subspace after dimensional reduction that is usually done in the literature, and then present simpler proofs of the limit of the sequence of solutions to such equations in the squeezing process. Our techniques are mainly based on [12, 2] and previous works by the present authors [7, 5].

Let $s \in I$ denote the arc-length parameter of the curve $r(s)$ in $\mathbb{R}^3$ of class C^3, with $I \subset \mathbb{R}$ an open and bounded interval, and let $k(s)$ and $\tau(s)$ be its curvature and torsion at the point $r(s)$, respectively. Pick $S \neq \emptyset$; an open, bounded, smooth and connected subset of $\mathbb{R}^2$. Build a tube (waveguide) in $\mathbb{R}^3$ by properly moving the region S along $r(s)$; at each point $r(s)$ the cross-section region S may present a (continuously differentiable) rotation angle $\alpha(s)$ (see details in Section 2). For each $\varepsilon > 0$ small enough, one can realize this same construction with the region εS and so obtaining a sequence of tubes Ω_ε in $\mathbb{R}^3$ that is squeezed to the reference curve r as $\varepsilon \to 0$.

Let $-\Delta$ be the (negative) Laplacian in Ω_ε with Neumann condition at the boundary $\partial\Omega_\varepsilon$. Pick $\tilde{g}_\varepsilon \in \mathrm{L}^2(\Omega_\varepsilon)$, and consider the stationary reaction-diffusion equation

$$\begin{aligned} -\Delta u + u + f(u) &= \tilde{g}_\varepsilon, \quad \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial u}{\partial \eta_\varepsilon} &= 0, \quad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{1}$$

where η_ε denotes the outward unit normal vector field to $\partial\Omega_\varepsilon$, and $f : \mathbb{R} \to \mathbb{R}$ is either the null function or a C^2-function satisfying some growth and dissipative conditions (see Section 5). We are interested on the limit behaviour of one sequence of solutions to (1) as $\varepsilon \to 0$. This subject has received the attention of several authors [1, 4, 13, 16, 18].

We note that equation (1) is elliptic and actually involves no diffusion, since there is no time dependence. Before we present the main results of this paper, we briefly recall some results that motivated our questions and studies.

Let Λ be an arbitrary smooth bounded domain in $\mathbb{R}^m \times \mathbb{R}^n$. Write (x, y) for a generic point of $\mathbb{R}^m \times \mathbb{R}^n$ and, for each $\varepsilon > 0$, put $\Lambda_\varepsilon = \{(x, \varepsilon y) : (x, y) \in \Lambda\}$. Prizzi and Rybakowski [16] have considered the following reaction-diffusion equation

$$\begin{aligned} u_t - \Delta u + f(u) &= 0, \quad \text{in } \Lambda_\varepsilon, \\ \frac{\partial u}{\partial \eta_\varepsilon} &= 0, \quad \text{on } \partial\Lambda_\varepsilon, \end{aligned} \tag{2}$$

where, as before, η_ε denotes the outward unit vector field to $\partial\Lambda_\varepsilon$ and $f:\mathbb{R}\to\mathbb{R}$ is a nonlinearity satisfying some growth and dissipative conditions assuring that (2) generates a semiflow on the Sobolev space $H^1(\Lambda_\varepsilon)$.

By considering the stationary linear equation associated with (2) (i.e., $f\equiv 0$) and performing a dilation of the domain Λ_ε by a factor ε in the y-direction, one gets

$$\begin{aligned} -\Delta_x u - \frac{1}{\varepsilon^2}\Delta_y u = 0, &\qquad \text{in } \Lambda,\\ \langle \nabla_x u, \eta_x\rangle + \frac{1}{\varepsilon^2}\langle \nabla_y u, \eta_y\rangle = 0, &\qquad \text{on } \partial\Lambda, \end{aligned} \tag{3}$$

with (η_x,η_y) denoting the components of the outward unitary normal vector field to $\partial\Lambda$; consider the associated quadratic form

$$b_\varepsilon(u) = \int_\Lambda \left(|\nabla_x u|^2 + \frac{|\nabla_y u|^2}{\varepsilon^2}\right) \mathrm{d}x\,\mathrm{d}y, \qquad u\in H^1(\Lambda),$$

and the subspace

$$\mathcal{K} := \{u\in H^1(\Lambda) : \nabla_y u = 0 \ \text{ almost everywhere in } \ \Lambda\}. \tag{4}$$

By direct inspection as $\varepsilon\to 0$, one gets

$$b_\varepsilon(u) \longrightarrow \begin{cases} \int_\Lambda |\nabla_x u|^2 \mathrm{d}x\,\mathrm{d}y & \text{if} \quad u\in\mathcal{K},\\ \infty & \text{if} \quad u\in H^1(\Lambda)\backslash\mathcal{K}. \end{cases}$$

Thus, in order to establish a limit for $b_\varepsilon(u)$ [16, 18] one just restricts the studies to the subspace $\mathcal{K}$. Note that $\mathcal{K}$ is directly related to the fact that the first eigenvalue of the Neumann Laplacian in a bounded region is zero (so no need of a renormalization; see ahead), and the constant functions are the corresponding eigenfunctions.

By studying equation (3), Prizzi and Rybakowski have proven that, in some strong sense, (2) has a limit equation

$$u_t + Au + f(u) = 0,$$

as $\varepsilon\to 0$, where A is the self-adjoint operator associated with the quadratic form

$$\int_\Lambda |\nabla_x u|^2 \mathrm{d}x, \qquad u\in\mathcal{K}.$$

By using the same notation above, given a bounded sequence $(h_\varepsilon)_\varepsilon$ in $\mathrm{L}^2(\Lambda)$, Silva [18] has performed a similar analysis of the stationary linear problem

$$\begin{aligned} -\Delta_x u - \frac{1}{\varepsilon^2}\Delta_y u = h_\varepsilon, &\qquad \text{in } \Lambda,\\ \langle \nabla_x u, \eta_x\rangle + \frac{1}{\varepsilon^2}\langle \nabla_y u, \eta_y\rangle = 0, &\qquad \text{on } \partial\Lambda, \end{aligned} \tag{5}$$

and studied the limit behaviour of its solutions as $\varepsilon \to 0$. In comparison to [16], Silva has presented a simpler and direct proof of the operator convergence of resolvents.

We call attention to the fact that this strategy to select the correct action of the limit quadratic form may not be convenient in some settings. A simple example of dimensional reduction involving the Laplacian with Dirichlet boundary condition illustrates our point. Consider the sequence of tubes Ω_ε, as defined at the beginning of this Introduction, and the particular case in which $r(s)$ is a piece of a circumference of radius $a > 0$ (and $\alpha \equiv 0$); the quadratic form associated with the problem is $\int_{\Omega_\varepsilon} |\nabla u|^2 \mathrm{d}s\mathrm{d}y$ with domain $H_0^1(\Omega_\varepsilon)$. In this case it is necessary to perform a renormalization. More exactly, since the first eigenvalue λ_0 of the Dirichlet Laplacian in $H_0^1(S)$ is greater than zero, one considers

$$\int_{\Omega_\varepsilon} \left(|\nabla u|^2 - \frac{\lambda_0}{\varepsilon^2} |u|^2 \right) \mathrm{d}s\mathrm{d}y,$$

and after a natural change of variables it is transformed into

$$\int_{I\times S} \left[\frac{1}{\gamma_\varepsilon} |u'|^2 + \frac{\gamma_\varepsilon}{\varepsilon^2} \left(|\nabla_y u|^2 - \lambda_0 |u|^2 \right) \right] \mathrm{d}s\, \mathrm{d}y, \qquad u \in H_0^1(I\times S), \tag{6}$$

where u' denotes the derivative with respect to variable s, $\gamma_\varepsilon(s, y) = 1 - \varepsilon y_1/a$ (here $(s, y_1, y_2) = (s, y)$ denotes points of $I \times S$).

If $u_0 = u_0(y)$ is the positive and normalized eigenfunction associated with λ_0, the subspace corresponding to $\mathcal{K}$ (see equation (4)) in this setting is

$$\mathcal{J} := \{wu_0 : w \in H_0^1(I)\} \subset \mathrm{L}^2(I \times S). \tag{7}$$

Restricted to $\mathcal{J}$, in the limit as $\varepsilon \to 0$, the quadratic form (6) becomes (see Appendix A for details of this convergence)

$$\int_I |w'|^2 \mathrm{d}s, \qquad w \in H_0^1(I), \tag{8}$$

and one could naively guess that (8) would be the correct limiting form. However, by some careful considerations, in [2, 5, 8] it was proven that although (7) is really the correct remaining subspace, the curvature $k = 1/a$ explicitly appears in the action of the limit quadratic form, which is actually given by

$$\int_I \left(|w'|^2 - \frac{|w|^2}{4a^2} \right) \mathrm{d}s, \qquad w \in H_0^1(I). \tag{9}$$

Further, this limit is also obtained in the sense of norm resolvent convergence of the associated self-adjoint operators [5]. Such additional (curvature) term with respect to (8) may be interpreted as a contribution from the discarded dimensions in the process of squeezing, so that special caution must be exercised in the selection of the action of the limit quadratic forms.

In view of the just mentioned example, one of the points of this work is to present a more complete justification—through an important norm resolvent convergence—that for some regions (i.e., particular cases of (2) and (5)), under dimensional reduction with Neumann boundary conditions, one may actually restrict the analysis to suitable “remaining” subspaces (e.g., (4)) and with *no* net contribution from the discarded dimensions in the process of squeezing (in contrast to the Dirichlet case (9)). This is the main content of Theorem 1 below.

Returning to equation (1) in tubes, to study the limit behaviour of a sequence of its solutions as $\varepsilon \to 0$, we perform a change of variables which takes Ω_ε onto the fixed domain $\Omega := I \times S$ (we keep the notation (s, y_1, y_2) for a point of Ω). Thus, we obtain the equivalent equations

$$\begin{aligned} A_\varepsilon u + f(u) &= g_\varepsilon, \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \langle B_\varepsilon u, \nu \rangle, \qquad \text{on } \partial\Omega, \end{aligned} \tag{10}$$

where $g_\varepsilon(s, y) = \tilde{g}_\varepsilon(F_\varepsilon(s, y))$ (for the definitions of B_ε and F_ε and details of this change of variables, see Section 2), ν it the unit outward normal to $\partial\Omega$ and A_ε is the self-adjoint operator associated with the quadratic form

$$a_\varepsilon(u) = \int_\Omega \left(|u' + \langle \nabla_y u, R\, y \rangle (\tau + \alpha')|^2 + \frac{\beta_\varepsilon}{\varepsilon^2} |\nabla_y u|^2 + |u|^2 \right) \mathrm{d}s\, \mathrm{d}y, \qquad u \in H^1(\Omega).$$

The definition of $\beta_\varepsilon(s, y)$ also appears in the Section 2. We denote by $\mathcal{E}_\varepsilon$ the set of solutions to equation (10).

If $1 = 1(y)$ denotes the constant function on the cross-section S, let

$$\mathcal{L} := \{ w1 : w \in \mathrm{L}^2(I) \},$$

with the decomposition $\mathrm{L}^2(\Omega) = \mathcal{L} \oplus \mathcal{L}^\perp$, and denote by P the orthogonal projection onto the subspace $\mathcal{L}$. In the Hilbert space $\mathrm{L}^2(I)$, let A_0 be the self-adjoint operator associated with the quadratic form

$$a_0(w) := \int_I \left(|w'|^2 + |w|^2 \right) \mathrm{d}s, \qquad w \in H^1(I). \tag{11}$$

Theorem 1. *There are two positive numbers C and ε_0, so that, for $0 < \varepsilon < \varepsilon_0$,*

$$\| A_\varepsilon^{-1} - A_0^{-1} P \|_{\mathrm{L}^2(\Omega)} \le C \varepsilon.$$

To compare this theorem with the results of [2, 5, 8] mentioned above, we note that in the Neumann case some geometric characteristics of the tube (as curvature, torsion and rotation angle) have no influence on effective operators. Further, at least for the regions we consider here, this norm resolvent convergence of Theorem 1 supports that one may just restrict the initial problem to the subspace $\mathcal{L}$ for small ε, since it is the subspace that "remains" in the process, and the quadratic form (11) is the restriction of the form a_ε to $\mathcal{L}$; no contribution from the discarded dimensions shows up.

Now we pass to convergence of solutions to the stationary problem (10). We begin with null function f. Theorem 1 allows us to weaken the condition that $(g_\varepsilon)_\varepsilon$ is a bounded sequence in $\mathrm{L}^2(\Omega)$, as done in [18], and still obtaining a similar result, that is

Theorem 2. *In the problem* (10) *with null function f, let $(g_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{L}^2(\Omega)$ such that $\|Pg_\varepsilon\|_{\mathrm{L}^2(\Omega)} < c$, for all $\varepsilon > 0$ small enough and some constant $c > 0$. Consider a sequence $(u_\varepsilon)_\varepsilon$, $u_\varepsilon \in \mathcal{E}_\varepsilon$. Then, there exist $g_0 \in \mathcal{L}$, $u_0 := A_0^{-1} g_0 \in \mathrm{dom}\, A_0$, and a subsequence of $(u_\varepsilon)_\varepsilon$, denoted by the same symbol $(u_\varepsilon)_\varepsilon$, such that*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{\mathrm{L}^2(\Omega)} = 0. \tag{12}$$

However, if $\|g_\varepsilon\|_{\mathrm{L}^2(\Omega)} \le c$, for all $\varepsilon > 0$ small enough, then more can be said, i.e.,

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{H^1(\Omega)} = 0. \tag{13}$$

In the general case, i.e., with nonzero f, we have got two possibilities, discussed in Theorems 3 and 4.

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear C^2-function satisfying*

$$f(0) = 0, \qquad -f'(x) \le K_1, \qquad \forall x \in \mathbb{R}, \tag{14}$$

$$\limsup_{|x|\to\infty} \frac{-f(x)}{x} \le 0, \tag{15}$$

$$|f'(x)| \le K_2(1 + |x|^\gamma), \qquad \forall x \in \mathbb{R}, \tag{16}$$

with some $0 \le \gamma \le 2$; assume also that it is bounded and Lipschitz. Let $(g_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{L}^2(\Omega)$ such that $\|Pg_\varepsilon\|_{\mathrm{L}^2(\Omega)} < c$, for all $\varepsilon > 0$ small enough, for some constant c. Consider a sequence $(u_\varepsilon)_\varepsilon$, $u_\varepsilon \in \mathcal{E}_\varepsilon$. Then, there exist $g_0 \in \mathrm{L}^2(I)$, $u_0 \in \mathrm{dom}\, A_0 \subset \mathrm{L}^2(I)$, so that, $u_0 = A_0^{-1} P(-f(u_0) + g_0)$, and a subsequence of $(u_\varepsilon)_\varepsilon$, denoted by the same symbol $(u_\varepsilon)_\varepsilon$, such that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{\mathrm{L}^2(\Omega)} = 0. \tag{17}$$

Note that, since f is a nonlinear function, the control on the term $Pf(u)$, $u \in H^1(\Omega)$, can be far from trivial. Thus, if f is neither bounded nor Lipschitz, we will impose the condition $\|g_\varepsilon\|_{\mathrm{L}^2(\Omega)} < c$ on the forcing terms. In this case we have

Theorem 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear, bounded and Lipschitz C^2-function satisfying conditions* (14)*,* (15) *and* (16)*, and $(g_\varepsilon)_\varepsilon$ a sequence in $\mathrm{L}^2(\Omega)$ such that $\|g_\varepsilon\|_{\mathrm{L}^2(\Omega)} < c$, for all $\varepsilon > 0$ small enough. Consider a sequence $(u_\varepsilon)_\varepsilon$, $u_\varepsilon \in \mathcal{E}_\varepsilon$. Then, there exist $g_0 \in \mathrm{L}^2(\Omega)$, $u_0 \in \operatorname{dom} A_0 \subset \mathrm{L}^2(I)$, so that, $u_0 = A_0^{-1}P(-f(u_0) + g_0)$, and a subsequence of $(u_\varepsilon)_\varepsilon$, denoted by the same symbol $(u_\varepsilon)_\varepsilon$, such that*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{H^1(\Omega)} = 0. \tag{18}$$

Although the settings in Theorems 3 and 4 have been previously considered, here, due to Theorem 1, we have got simpler proofs on top of additional justification of the choices of remaining subspaces and action of limit forms. Further, in case of Theorem 3, we were able to weaken the condition on g_ε with respect to [16, 18], by only requiring that the projection $\|Pg_\varepsilon\|$ is uniformly bounded.

The above results have similar versions to strips in the plane, again built along differentiable curves r but with cross sections given by bounded intervals. The proofs are even simpler than the case of tubes, and we do not explicitly discuss them here.

In addition, we are able to say something about the asymptotic behaviour of some semilinear equations in thin planar domains as particular cases studied by Hale and Raugel [13]. Let $g : [a,b] \to \mathbb{R}$, $-\infty < a < b < +\infty$, be a C^3-function delimiting a planar region, that is, for each $\varepsilon > 0$, let

$$Q_\varepsilon := \{(s,y) \in J \times \mathbb{R} : 0 < y < \varepsilon g(s)\}, \qquad J := (a,b).$$

Consider the problem

$$\begin{aligned} &u_t - \Delta u + f(u) = 0, \qquad \text{in } Q_\varepsilon,\\ &\frac{\partial u}{\partial \nu_\varepsilon} = 0, \qquad \text{on } \partial Q_\varepsilon, \end{aligned} \tag{19}$$

where ν_ε is the exterior unit normal vector field to ∂Q_ε, and f is a nonlinearity such that this equation generates a semiflow on $H^1(Q_\varepsilon)$. In [13] it was proven that the limit semiflow, as $\varepsilon \to 0$, is the one generated by the one-dimensional boundary value problem

$$\begin{aligned} &u_t - (1/g)(gu')' - f(u) = 0, \qquad \text{in } J,\\ &u'(a) = u'(b) = 0. \end{aligned}$$

In what follows, we shall present a simpler proof, with respect to [13], of the convergence of the solutions to the stationary equation associated with (19), namely,

$$\begin{aligned} -\Delta u + f(u) = 0, \quad &\text{in } Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0, \quad &\text{on } \partial Q_\varepsilon. \end{aligned} \tag{20}$$

The first step to study equation (20) is an analysis of its linear term, which is reduced, after an appropriate change of variables (see Section 6 for details), to

$$m_\varepsilon(u) = \int_Q \left[\left(u' - \frac{g'}{2g} u \right)^2 + \frac{1}{\varepsilon^2 g^2} u_y^2 \right] \mathrm{d}s \, \mathrm{d}y,$$

where $\operatorname{dom} m_\varepsilon = H^1(Q)$ and $Q := J \times (0,1)$. We denote by M_ε the self-adjoint operator associated with $m_\varepsilon(u)$. Thus, we pass to the equation

$$\begin{aligned} M_\varepsilon u + f(u) = 0, \quad &\text{in } Q, \\ \frac{\partial u}{\partial \nu} = 0, \quad &\text{on } \partial Q. \end{aligned} \tag{21}$$

Denote by $\mathcal{E}_\varepsilon^g$ the set of solutions to (21). As before, we consider the subspace $\tilde{\mathcal{J}} = \{w1 : w \in \mathrm{L}^2(J)\}$ and the decomposition $\mathrm{L}^2(Q) = \tilde{\mathcal{J}} \oplus \tilde{\mathcal{J}}^\perp$. Denote by $\tilde{P}$ the orthogonal projection onto the subspace $\tilde{\mathcal{J}}$. Define the quadratic form

$$m(w) := \int_J \left(w' - \frac{g'}{2g} w \right)^2 \mathrm{d}s, \quad \operatorname{dom} m = H^1(J), \tag{22}$$

and denote by M the associated self-adjoint operator. In this case we have a similar result to Theorem 1 (see Theorem 7) and, as a consequence, we shall prove

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ as in Theorem 4 and take a sequence $(u_\varepsilon)_\varepsilon$, $u_\varepsilon \in \mathcal{E}_\varepsilon^g$, $\varepsilon \to 0$. Then, there exist $u_0 \in \operatorname{dom} M$, so that, $u_0 = M^{-1}\tilde{P}(-f(u_0))$, and a subsequence of $(u_\varepsilon)_\varepsilon$, denoted by the same symbol $(u_\varepsilon)_\varepsilon$, such that*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{H^1(Q)} = 0. \tag{23}$$

The rest of this paper is organized as follows. In Section 2 we present details of the construction of the tubular region Ω_ε and we study the quadratic form associated with the Neumann Laplacian in this tube. Section 3 is dedicated to the proof of Theorem 1. In Section 4 we prove Theorem 2, whereas the proofs of Theorems 3 and 4 appear in Section 5, and finally Section 6 is reserved for the proof of Theorem 5.

2 Quadratic forms

In this section we are going to construct the region where the Neumann Laplacian is considered and its associated quadratic form.

Let I be an open and bounded interval. We suppose that $r : I \to \mathbb{R}^3$ is a simple C^3 curve in $\mathbb{R}^3$ parametrized by its arc-length parameter s. The curvature of r at the position s is $k(s) := \|r''(s)\|$. We choose the usual orthonormal triad of vector fields $\{T(s), N(s), B(s)\}$, the so-called Frenet frame, given the tangent, normal and binormal vectors, respectively, moving along the curve and defined by

$$T = r'; \quad N = k^{-1}T'; \quad B = T \times N. \tag{24}$$

To justify the construction (24), it is assumed that $k > 0$, but if r has a piece of a straight line (i.e., $k = 0$ identically in this piece), usually one can choose a constant Frenet frame instead. It is possible to combine constant Frenet frames with the Frenet frame (24) to include other types of curves, for instance, curves with $k(s) > 0$ only on a compact interval of values of s (and so obtaining a global C^2 Frenet frame; see [14], Theorem 1.3.6). In each situation we assume that a global Frenet frame exists and that the Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{25}$$

where $\tau(s)$ is the torsion of $r(s)$, actually defined by (25). Let $\alpha : I \to \mathbb{R}$ be a bounded C^1 function so that $\alpha(s_0) = 0$ (s_0 is a fixed point of I), and S an open, bounded, connected and smooth (nonempty) subset of $\mathbb{R}^2$. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$\vec{x}(s, y) = r(s) + \varepsilon y_1 N(s) + \varepsilon y_2 B(s)$$

and consider the domain

$$\Omega_\varepsilon = \{\vec{x}(s, y) \in \mathbb{R}^3 : s \in I, y = (y_1, y_2) \in S\},$$

where

$$\begin{aligned} N_\alpha(s) &:= \cos\alpha(s)N(s) + \sin\alpha(s)B(s), \\ B_\alpha(s) &:= -\sin\alpha(s)N(s) + \cos\alpha(s)B(s). \end{aligned}$$

Hence, this tube Ω_ε is obtained by putting the region εS along the curve $r(s)$, which is simultaneously rotated by an angle $\alpha(s)$ with respect to the cross section at the position $s = s_0$. We suppose that $k, \tau + \alpha' \in L^\infty(I)$.

In order to study the Neumann Laplacian $-\Delta$ in Ω_ε we initially consider the corresponding family of quadratic forms

$$\hat{a}_\varepsilon(\psi) := \int_{\Omega_\varepsilon} \left(|\nabla\psi|^2 + |\psi|^2\right) \mathrm{d}\vec{x}, \qquad \psi \in \mathrm{dom}\ \hat{a}_\varepsilon = H^1(\Omega_\varepsilon). \tag{26}$$

Consider the mapping

$$\begin{array}{rccl} F_\varepsilon: & I \times S & \rightarrow & \Omega_\varepsilon \\ & (s, y) & \mapsto & r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s), \end{array}$$

that will be used to perform important change of variables ahead. The condition $k \in \mathrm{L}^\infty(I)$, is to guarantee that F_ε will be a (global) diffeomorphism for small ε. With this change of variables we work with a fixed region $I \times S$ for all $\varepsilon > 0$; more precisely, in the new variables the domain of the quadratic form (26) turns out to be $H^1(I \times S)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon$ which is induced by F_ε, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j \rangle = G_{ji}, \qquad 1 \le i, j \le 3, \tag{27}$$

where

$$e_1 = \frac{\partial F_\varepsilon}{\partial s}, \quad e_2 = \frac{\partial F_\varepsilon}{\partial y_1}, \quad e_3 = \frac{\partial F_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$W = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \beta_\varepsilon & -\varepsilon(\tau + \alpha')\langle z_\alpha^\perp, y\rangle & \varepsilon(\tau + \alpha')\langle z_\alpha, y\rangle \\ 0 & \varepsilon \cos\alpha & \varepsilon \sin\alpha \\ 0 & -\varepsilon \sin\alpha & \varepsilon\cos\alpha \end{pmatrix},$$

where

$$\beta_\varepsilon(s, y) = 1 - \varepsilon k(s)\langle z_\alpha, y\rangle, \quad z_\alpha = (\cos\alpha, -\sin\alpha), \quad \text{and} \quad z_\alpha^\perp = (\sin\alpha, \cos\alpha).$$

The inverse matrix of W is given by

$$W^{-1} = \begin{pmatrix} 1/\beta_\varepsilon & (\tau + \alpha')y_2/\beta_\varepsilon & -(\tau + \alpha')y_1/\beta_\varepsilon \\ 0 & (1/\varepsilon)\cos\alpha & -(1/\varepsilon)\sin\alpha \\ 0 & (1/\varepsilon)\sin\alpha & (1/\varepsilon)\cos\alpha \end{pmatrix}.$$

Note that $WW^t = G$ and $\det W = |\det G|^{1/2} = \varepsilon^2\beta_\varepsilon$. Since k is a bounded function, for ε small enough F_ε does not vanish in $I \times S$. Thus, $\beta_\varepsilon > 0$ and F_ε is a local diffeomorphism. In case F_ε is injective (again by requiring that $\varepsilon > 0$ is small), a global diffeomorphism is obtained.

Introducing the notation

$$\|u\|_G^2 = \int_{I\times S} |u|^2 \beta_\varepsilon \, \mathrm{d}s \, \mathrm{d}y,$$

and the unitary transformation

$$\begin{array}{rccc} U_\varepsilon : & \mathrm{L}^2(\Omega_\varepsilon) & \rightarrow & \mathrm{L}^2(I \times S, \beta_\varepsilon) \\ & u & \mapsto & \varepsilon\, u \circ F_\varepsilon \end{array},$$

we obtain, from (26), the sequence of quadratic forms

$$\tilde{a}_\varepsilon(U_\varepsilon u) := \|W^{-1}\nabla(U_\varepsilon u)\|_G^2 + \|U_\varepsilon u\|_G^2.$$

Again, since $k \in L^\infty(I)$, $\beta_\varepsilon \to 1$ uniformly in $I \times S$, as $\varepsilon \to 0$. Therefore, the spaces $\mathrm{L}^2(I\times S, \beta_\varepsilon)$ and $\mathrm{L}^2(I\times S)$ are topologically equivalent and the strong convergence in $\mathrm{L}^2(I\times S, \beta_\varepsilon)$ is equivalent to the convergence in the fixed space $\mathrm{L}^2(I\times S)$. Thus, we pass to work in $\mathrm{L}^2(I\times S)$.

After the norms are written out, and using the same notation u for $U_\varepsilon u$, we obtain

$$\tilde{a}_\varepsilon(u) = \int_{I\times S} \left(\frac{1}{\beta_\varepsilon} |u' + \langle \nabla_y u, R\, y\rangle(\tau + \alpha')|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_y u|^2 + \beta_\varepsilon |u|^2 \right) \mathrm{d}s \, \mathrm{d}y,$$

dom $\tilde{a}_\varepsilon = H^1(I \times S)$.

Now, for technical reasons, we define the quadratic form

$$a_\varepsilon(u) := \int_{I\times S} \left(|u' + \langle \nabla_y u, R\, y\rangle(\tau + \alpha')|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_y u|^2 + |u|^2 \right) \mathrm{d}s \, \mathrm{d}y,$$

dom $a_\varepsilon = H^1(I \times S)$, which is very similar to the quadratic form $\tilde{a}_\varepsilon(u)$. We denote by $\tilde{A}_\varepsilon$ and A_ε the self-adjoint operators associated with $\tilde{a}_\varepsilon$ and a_ε, respectively. Since $\beta_\varepsilon \to 1$ uniformly in $I \times S$, as $\varepsilon \to 0$, we have the following theorem, whose proof is very similar to the proof of Theorem 3.1 in [7] and for this reason will be omitted here.

Theorem 6. *There exist numbers $D > 0$ and $\varepsilon_1 > 0$ so that*

$$\|\tilde{A}_\varepsilon^{-1} - A_\varepsilon^{-1}\|_{\mathrm{L}^2(\Omega)} \leq D\varepsilon, \qquad 0 < \varepsilon < \varepsilon_1.$$

Recall that $\Omega = I \times S$. Due to the changes of variables presented above and Theorem 6, instead of studying equation (5) we pass to work with

$$\begin{aligned} A_\varepsilon u_\varepsilon + f(u_\varepsilon) &= g_\varepsilon, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \langle B_\varepsilon u, \nu\rangle, \quad \text{on } \partial\Omega, \end{aligned} \tag{28}$$

where ν is the unit outward normal to $\partial\Omega$ and $B_\varepsilon\, u = \beta_\varepsilon G^{-1}\nabla u$.

Again, for technical reasons, we define the space $H^1_\varepsilon(\Omega) := \{u \in \mathrm{L}^2(\Omega) : \|u\|_{H^1_\varepsilon(\Omega)} < \infty\}$, where

$$\|u\|_{H^1_\varepsilon(\Omega)} := \left(\int_\Omega \left(|u' + \langle \nabla_y u, R\,y\rangle(\tau+\alpha')|^2 + \beta_\varepsilon |\nabla_y u|^2 + |u|^2\right) \mathrm{d}s\,\mathrm{d}y\right)^{1/2}. \tag{29}$$

Since $k, \tau + \alpha' \in L^\infty(I)$, we can observe that $\|u\|_{H^1(\Omega)}$ and $\|u\|_{H^1_\varepsilon(\Omega)}$ are equivalent norms in $H^1(\Omega)$.

Another property that we are going to use along this text is that, since $k \in \mathrm{L}^\infty(I)$, there exist $E_1, E_2 > 0$ and $\varepsilon_2 > 0$, so that

$$E_1 < \beta_\varepsilon(s,y) < E_2, \quad \forall (s,y) \in I \times S, \quad 0 < \varepsilon < \varepsilon_2. \tag{30}$$

3 Reduction of dimension

As already mentioned in the Introduction, we consider the subspace $\mathcal{L} = \{w1 : w \in \mathrm{L}^2(I)\}$ and the orthogonal decomposition

$$\mathrm{L}^2(\Omega) = \mathcal{L} \oplus \mathcal{L}^\perp.$$

For $u \in \mathrm{L}^2(\Omega)$, we can write

$$u(s,y) = w(s) + v(s,y),$$

with $w \in \mathrm{L}^2(I)$ and $v \in \mathcal{L}^\perp$. Observe that $v \in \mathcal{L}^\perp$ implies

$$\int_S v(s,y)\,\mathrm{d}y = 0 \quad \text{a.e.[s]}. \tag{31}$$

Note that $w \in H^1(\Omega)$ if $w \in H^1(I)$. For $u \in H^1(\Omega)$, write $u = w + v$ with $w \in H^1(I)$ and $v \in H^1(\Omega) \cap \mathcal{L}^\perp$.

For $w \in H^1(I)$, we may identify $a_\varepsilon(w)$ with the "one-dimensional" quadratic form (11), that is,

$$a_0(w) = \int_I \left(|w'|^2 + |w|^2\right) \mathrm{d}s,$$

and recall that A_0 is the associated self-adjoint operator.

The method of the proof of Theorem 1 is as follows. First one identifies suitable subspaces of the Hilbert space on which the operator acts, and second, one applies the powerful functional analytic technique of Friedlander and Solomyak [12]. The essence of the proof is an application of such technique; however, showing that it can be applied, by demonstrating the required estimates, is in itself not trivial.

Proof of Theorem 1: For $u \in H^1(\Omega)$ write, as above,

$$u(s,y) = w(s) + v(s,y),$$

with $w \in H^1(I)$ and $v \in H^1(\Omega) \cap \mathcal{L}^{\perp}$. Thus, the quadratic form $a_\varepsilon(u)$ can be rewritten as

$$a_\varepsilon(w+v) = a_0(w) + 2\, a_\varepsilon(w,v) + a_\varepsilon(v).$$

Here, $a_\varepsilon(u_1,u_2)$ denotes the bilinear form associated with the quadratic form $a_\varepsilon(u)$.

We are going to show that $a_0(w), a_\varepsilon(v)$ and $a_\varepsilon(w,v)$ satisfy conditions (3.2), (3.3), (3.4) and (3.5) in Section 3 of [12], and so the theorem will follow.

Firstly, we observe that

$$a_0(w) \geq \|w\|^2_{\mathrm{L}^2(I)}, \qquad \forall w \in H^1(I),$$

and then condition (3.2) holds true.

Now, let $\lambda_1 > 0$ be the second eigenvalue of the Neumann Laplacian in $H^1(I)$. For $v \in H^1(\Omega) \cap \mathcal{L}^{\perp}$,

$$a_\varepsilon(v) \geq \frac{\lambda_1}{\varepsilon^2} \|v\|^2_{\mathrm{L}^2(\Omega)},$$

and (3.3) and (3.4) are satisfied.

Finally, we assert that there exist $C_1 > 0$ and $\varepsilon_0 > 0$, so that, for $0 < \varepsilon < \varepsilon_0$,

$$a_\varepsilon(w,v) \leq C_1\, \varepsilon\, (a_0(w))^{1/2}\, (a_\varepsilon(v))^{1/2}.$$

In fact, due to property (31),

$$a_\varepsilon(w,v) = \int_\Omega w' \,\langle \nabla_y v, R\, y\rangle (\tau + \alpha')\, \mathrm{d}s\, \mathrm{d}y.$$

By (30), we also note that

$$\frac{E_1}{\varepsilon^2} \int_\Omega |\nabla_y v|^2 \mathrm{d}s\, \mathrm{d}y \leq a_\varepsilon(v).$$

Let $C_2 := \sup_{(s,y)\in\Omega} \{\|R\, y\|(\tau + \alpha')(s)\}$. Thus,

$$\begin{aligned} \left| \int_\Omega w' \langle \nabla_y v, R\, y\rangle (\tau + \alpha')\, \mathrm{d}s\, \mathrm{d}y \right| &\leq C_2 \left(\int_I |w'|^2 \mathrm{d}s \right)^{1/2} \left(\int_\Omega |\nabla_y v|^2 \mathrm{d}s\, \mathrm{d}y \right)^{1/2} \\ &\leq \varepsilon\, (C_2/\sqrt{E_1})\, (a_\varepsilon(w))^{1/2}\, (a_\varepsilon(v))^{1/2}, \end{aligned}$$

and the condition (3.5) in [12] is also satisfied.

4 The linear problem

In this section we study the problem (10) and prove Theorem 2 presented in the Introduction. Lemma 2 in the next section will be used; note that it also holds in case of the (nonlinear) null function f.

Proof of Theorem 2: At first we prove (12). Let $\varepsilon_n \to 0$, $n \in \mathbb{N}$ $(n \to \infty)$, $A_n := A_{\varepsilon_n}$, $g_n := g_{\varepsilon_n}$ and $u_n := u_{\varepsilon_n}$ a sequence so that $u_n \in \mathcal{E}_{\varepsilon_n}$, i.e., $A_n u_n = g_n$. Theorem 1 ensures that

$$\|A_n^{-1} g_n - A_0^{-1} P g_n\|_{\mathrm{L}^2(\Omega)} \to 0, \quad \text{as} \quad n \to \infty.$$

Since $(Pg_n)_n$ is a bounded sequence and A_0^{-1} is a compact operator, the sequence $(A_0^{-1}Pg_n)_n$ has a convergent subsequence. Thus, there exist $u_0 \in \mathrm{L}^2(I)$ and an infinite subset $\mathbb{N}_1$ of $\mathbb{N}$ so that

$$A_0^{-1} P g_n \to u_0, \quad \text{in } \mathrm{L}^2(I), \quad n \in \mathbb{N}_1.$$

On the other hand, there exist $g_0 \in \mathrm{L}^2(I)$ and an infinite subset $\mathbb{N}_2$ of $\mathbb{N}_1$, so that, $Pg_n \rightharpoonup g_0$ in $\mathrm{L}^2(I)$, $n \in \mathbb{N}_2$. Thus, $A_0^{-1}Pg_n \rightharpoonup A_0^{-1}g_0$, $n \in \mathbb{N}_2$, and we conclude that $A_0^{-1}Pg_0 = u_0$, and $u_n \to u_0$ in $\mathrm{L}^2(I \times S)$, $n \in \mathbb{N}_2$.

To prove (13), we are going to show that, up to a subsequence,

$$\|u_n - u_0\|_{H^1_{\varepsilon_n}(\Omega)} \to 0. \tag{32}$$

Thus, since the norms $\|u\|_{H^1(\Omega)}$ and $\|u\|_{H^1_{\varepsilon_n}(\Omega)}$ are equivalent, this implies Theorem 2.

Suppose that $\|g_\varepsilon\| \le c$, for all $\varepsilon > 0$ small enough. There exist $\tilde{g}_0 \in \mathrm{L}^2(\Omega)$ and an infinite subset $\mathbb{N}_3$ of $\mathbb{N}_2$ so that $g_{\varepsilon_n} \rightharpoonup \tilde{g}_0$, $n \in \mathbb{N}_3$. Write $\tilde{g}_0 = P\tilde{g}_0 + P^\perp \tilde{g}_0$. In this case $P\tilde{g}_0 = g_0$. Now, by Lemma 2 and since the immersion $H^1(\Omega) \to \mathrm{L}^2(\Omega)$ is compact, we have $u_n \rightharpoonup u_0$ in $H^1(\Omega)$, $n \in \mathbb{N}_3$ (this convergence also hold in $H^1_{\varepsilon_n}(\Omega)$). To conclude (32), it is enough to show that $\|u_\varepsilon\|_{H^1_\varepsilon(\Omega)} \to \|u_0\|_{H^1(\Omega)}$; for this, just observe that

$$\begin{aligned}
\|u_0\|^2_{H^1(\Omega)} &\leq \liminf_{n\in\mathbb{N}_3} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \beta_{\varepsilon_n}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&\leq \limsup_{n\in\mathbb{N}_3} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \beta_{\varepsilon_n}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&\leq \limsup_{n\in\mathbb{N}_3} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \frac{\beta_{\varepsilon_n}}{\varepsilon_n^2}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&= \limsup_{n\in\mathbb{N}_3} \int_\Omega (A_n u_n)\, u_n \,\mathrm{d}s\,\mathrm{d}y = \limsup_{n\in\mathbb{N}_3} \int_\Omega g_n\, u_n \,\mathrm{d}s\,\mathrm{d}y \\
&= \int_\Omega (P\tilde{g}_0 + P^\perp \tilde{g}_0)\, u_0 \,\mathrm{d}s\,\mathrm{d}y = \int_\Omega g_0\, u_0 \,\mathrm{d}s\,\mathrm{d}y \\
&= \int_\Omega (A_0 u_0)\, u_0 \,\mathrm{d}s\,\mathrm{d}y = \|u_0\|^2_{H^1(\Omega)}.
\end{aligned}$$

5 The nonlinear problem

In this section we present the required conditions for the function f in equation (10). We assume that $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear C^2-function satisfying conditions conditions (14), (15) and (16) as in the statement of Theorem 3. The following related result is well known and its proof can be found in [15].

Lemma 1. *Under conditions* (14)*,* (15) *and* (16)*, the assignment $u \mapsto f \circ u$ defines a map $f : H^1(\Omega) \to \mathrm{L}^2(\Omega)$, which is Lipschitz continuous on every bounded set in $H^1(\Omega)$. Moreover, wherever $u, u_1, u_2 \in H^1(\Omega)$ and $\|u_1\|_{H^1(\Omega)}, \|u_2\|_{H^1(\Omega)} \leq L_1$ the following estimates hold:*

$$\|f(u)\|_{\mathrm{L}^2(\Omega)} \leq L_2\,(\|u\|_{\mathrm{L}^2(\Omega)} + \tilde{L}_2\|u\|^{\gamma+1}_{H^1(\Omega)}),$$

$$\|f(u_1) - f(u_2)\|_{\mathrm{L}^2(\Omega)} \leq \tilde{L}_1\,(1 + 2\,L_1^{2\gamma})\|u_1 - u_2\|_{H^1(\Omega)}.$$

Here, L_2, $\tilde{L}_2$ and $\tilde{L_1}$ are positive constants.

Note that due to condition (15), for any $\eta > 0$ there exists a positive constant C_η such that,

$$-f(x)\,x \leq \eta\,x^2 + C_\eta, \quad \forall x \in \mathbb{R}. \tag{33}$$

This inequality will be useful later on.

Proof of Theorem 3: Let $\varepsilon_n \to 0$, $n \in \mathbb{N}$ $(n \to \infty)$, $A_n := A_{\varepsilon_n}$, $g_n := g_{\varepsilon_n}$, and $u_n := u_{\varepsilon_n}$ a sequence so that $u_n \in \mathcal{E}_{\varepsilon_n}$. Since f is a bounded function we have that $(f(u_n))_n$ is a bounded sequence in $\mathrm{L}^2(\Omega)$. Thus, there exist $f_0 \in \mathrm{L}^2(\Omega)$ and an infinite subset $\mathbb{N}_4 \subset \mathbb{N}$ so that

$$f(u_n) \rightharpoonup f_0, \quad \text{ in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_4.$$

On the other hand, since $(Pg_n)_n$ is a bounded sequence, there exist $g_0 \in \mathrm{L}^2(\Omega)$ and an infinite subset $\mathbb{N}_5 \subset \mathbb{N}_4$ so that

$$Pg_n \rightharpoonup g_0, \quad \text{ in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_5.$$

By Theorem 1, we have

$$\left\| A_n^{-1}\left(-f(u_n)+g_n\right) - A_0^{-1}P\left(-f(u_n)+g_n\right)\right\|_{\mathrm{L}^2(\Omega)} \to 0, \qquad n \in \mathbb{N}_5.$$

Also note that

$$A_0^{-1}P\left(-f(u_n)+g_n\right) \rightharpoonup A_0^{-1}P\left(-f_0+g_0\right), \quad \text{ in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_5.$$

Now, since A_0^{-1} is a compact operator, $u_n^s := A_0^{-1}P(-f(u_n)+g_n)$ has a convergent subsequence. Thus, there exists $u_0 \in \mathrm{L}^2(I)$, so that,

$$u_n^s \to u_0, \quad \text{ in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_6,$$

where $\mathbb{N}_6$ is an infinite subset of $\mathbb{N}_5$. Consequently,

$$u_n \to u_0, \quad \text{ in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_6,$$

and $u_0 = A_0^{-1}P(-f_0+g_0)$. To finish the proof, since f is a Lipschitz function, we have $f(u_n) \to f(u_0)$, in $\mathrm{L}^2(\Omega)$, $n \in \mathbb{N}_6$, and so we conclude that $f_0 = f(u_0)$.

Observe that Lemma 1 was not necessary to prove Theorem 3. To prove Theorem 4, besides Lemma 1, we will need of the following result.

Lemma 2. *Suppose that there exists $c > 0$ so that $\|g_\varepsilon\| < c$, for all $\varepsilon > 0$ small enough. Then, there are two positive constants ε_3 and M, such that,*

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq M, \quad \forall u_\varepsilon \in \mathcal{E}_\varepsilon, \quad 0 < \varepsilon < \varepsilon_3.$$

Proof. Due to (33), given $\eta > 0$, there exists $C_\eta > 0$ so that

$$\begin{aligned} & \int_\Omega \left(|u_\varepsilon' + \langle \nabla_y u_\varepsilon, R\,y\rangle(\tau+\alpha')|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_y u_\varepsilon|^2 + |u_\varepsilon|^2 \right) \mathrm{d}s\,\mathrm{d}y \\ = & \int_\Omega (A_\varepsilon u_\varepsilon)\, u_\varepsilon \,\mathrm{d}s\,\mathrm{d}y = \int_\Omega (-f(u_\varepsilon) + g_\varepsilon)\, u_\varepsilon \,\mathrm{d}s\,\mathrm{d}y \\ = & \int_\Omega -f(u_\varepsilon)\, u_\varepsilon \,\mathrm{d}s\,\mathrm{d}y + \int_\Omega g_\varepsilon\, u_\varepsilon \,\mathrm{d}s\,\mathrm{d}y \\ \leq & \int_\Omega \left(\eta\, u_\varepsilon^2 + C_\eta\right) \mathrm{d}s\,\mathrm{d}y + \frac{1}{2\eta}\int_\Omega g_\varepsilon^2 \,\mathrm{d}s\,\mathrm{d}y + \frac{\eta}{2}\int_\Omega u_\varepsilon^2 \,\mathrm{d}s\,\mathrm{d}y \\ \leq & \frac{3}{2}\eta\, \|u_\varepsilon\|^2_{H^1(\Omega)} + \tilde{S}, \end{aligned}$$

where $\tilde{S} := C_\eta \,\mathrm{mes}\,(\Omega) + c^2/(2\,\eta)$. Recalling (30), we also have

$$E_1 \int_\Omega |\nabla_y u_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y \leq \int_\Omega \frac{\beta_\varepsilon}{\varepsilon^2}|\nabla_y u_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y \leq \frac{3}{2}\eta\,\|u_\varepsilon\|^2_{H^1(\Omega)} + \tilde{S}.$$

With the above inequalities, we obtain

$$\begin{aligned} E_1\|u_\varepsilon\|^2_{H^1(\Omega)} = & \; E_1 \int_\Omega \left(|u_\varepsilon'|^2 + |\nabla_y u_\varepsilon|^2 + |u_\varepsilon|^2\right) \mathrm{d}s\,\mathrm{d}y \\ \leq & \; E_1 \left[\left(\int_\Omega |u_\varepsilon' + \langle \nabla_y u_\varepsilon, R\,y\rangle(\tau+\alpha')|^2 \mathrm{d}s\,\mathrm{d}y \right)^{1/2} \right. \\ + & \left. \left(\int_\Omega |\langle \nabla_y u_\varepsilon, R\,y\rangle(\tau+\alpha')|^2 \mathrm{d}s\,\mathrm{d}y \right)^{1/2} \right]^2 + E_1 \int_\Omega \left(|\nabla_y u_\varepsilon|^2 + |u_\varepsilon|^2\right) \mathrm{d}s\,\mathrm{d}y \\ \leq & \; E_1 \left[\left(\int_\Omega |u_\varepsilon' + \langle \nabla_y u_\varepsilon, R\,y\rangle(\tau+\alpha')|^2 \mathrm{d}s\,\mathrm{d}y \right)^{1/2} + C_2 \left(\int_\Omega |\nabla_y u_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y \right)^{1/2} \right]^2 \\ + & \; E_1 \int_\Omega \left(|\nabla_y u_\varepsilon|^2 + |u_\varepsilon|^2\right) \mathrm{d}s\,\mathrm{d}y \\ \leq & \; (3E_1 + 2E_1C_2 + E_1C_2^2) \left(\frac{3}{2}\eta\,\|u_\varepsilon\|^2_{H^1(\Omega)} + \tilde{S} \right). \end{aligned}$$

Recall that $C_2 = \sup_{(s,y)\in\Omega} \{\|Ry\|(\tau+\alpha')(s)\}$. Now, it is sufficient to take $\eta > 0$ so that

$$\frac{3}{2}\eta\,(3E_1 + 2E_1C_2 + E_1C_2^2) < E_1,$$

and the proof is complete. □

Proof of Theorem 4: Let $\varepsilon_n \to 0$, $n \in \mathbb{N}$ $(n \to \infty)$, $g_n := g_{\varepsilon_n}$, $A_n := A_{\varepsilon_n}$, and $u_n := u_{\varepsilon_n}$ a sequence so that $u_n \in \mathcal{E}_{\varepsilon_n}$. Lemmas 1 and 2 ensure that $(f(u_n))_n$ is a bounded sequence in $\mathrm{L}^2(\Omega)$. Thus, there exist $f_0 \in \mathrm{L}^2(\Omega)$ and an infinite subset $\mathbb{N}_7 \subset \mathbb{N}$ with

$$f(u_n) \rightharpoonup f_0, \quad \text{in } \mathrm{L}^2(\Omega) \quad n \in \mathbb{N}_7.$$

Further, since $(g_n)_n$ is a bounded sequence, there exist $g_0 \in \mathrm{L}^2(\Omega)$ and an infinite subset $\mathbb{N}_8 \subset \mathbb{N}_7$ so that

$$g_n \rightharpoonup g_0, \quad \text{in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_8.$$

By Theorem 1, we have

$$\left\| A_n^{-1}\left(-f(u_n)+g_n\right) - A_0^{-1}P\left(-f(u_n)+g_n\right)\right\|_{\mathrm{L}^2(\Omega)} \to 0, \qquad n \in \mathbb{N}_8.$$

Also note that

$$A_0^{-1}\left(-f(u_n)+g_n\right) \rightharpoonup A_0^{-1}P\left(-f_0+g_0\right), \quad \text{in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_8.$$

Since A_0^{-1} is a compact operator, $u_n^s := A_0^{-1}P(-f(u_n)+g_n)$ has a convergent subsequence; thus, there exists $u_0 \in \mathrm{L}^2(I)$ so that

$$u_n^s \to u_0, \quad \text{in } \mathrm{L}^2(\Omega) \quad n \in \mathbb{N}_9,$$

where $\mathbb{N}_9$ is an infinite subset of $\mathbb{N}_8$. Consequently,

$$u_n \to u_0, \quad \text{in } \mathrm{L}^2(\Omega), \quad n \in \mathbb{N}_9,$$

and $u_0 = A_0^{-1}P(-f_0+g_0)$. Lemma 2 also ensures that

$$u_n' \rightharpoonup u_0' \quad \text{and} \quad \nabla_y u_n \rightharpoonup 0, \quad \text{in } \mathrm{L}^2(\Omega), \qquad n \in \mathbb{N}_9.$$

Now, since the norms $\|u\|_{H^1(\Omega)}$ and $\|u\|_{H^1_\varepsilon(\Omega)}$ are equivalent, to prove the convergence in (18) it is enough to check that $\|u_\varepsilon\|_{H^1_\varepsilon(\Omega)} \to \|u_0\|_{H^1(\Omega)}$, which follows from the following estimates

$$\begin{aligned}
\|u_0\|^2_{H^1(\Omega)} &\leq \liminf_{n\in\mathbb{N}_9} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \beta_{\varepsilon_n}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&\leq \limsup_{n\in\mathbb{N}_9} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \beta_{\varepsilon_n}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&\leq \limsup_{n\in\mathbb{N}_9} \int_\Omega \left(|u_n' + \langle \nabla_y u_n, R\,y\rangle(\tau+\alpha')|^2 + \frac{\beta_{\varepsilon_n}}{\varepsilon_n^2}|\nabla_y u_n|^2 + |u_n|^2\right) \mathrm{d}s\,\mathrm{d}y \\
&= \limsup_{n\in\mathbb{N}_9} \int_\Omega (A_n u_n)\, u_n \,\mathrm{d}s\,\mathrm{d}y = \limsup_{n\in\mathbb{N}_9} \int_\Omega \left(-f(u_n)+g_n\right)\, u_n \,\mathrm{d}s\,\mathrm{d}y \\
&= \int_\Omega \left(-f_0+g_0\right)\, u_0 \,\mathrm{d}s\,\mathrm{d}y = \int_\Omega (A_0 u_0)\, u_0 \,\mathrm{d}s = \|u_0\|^2_{H^1(\Omega)}.
\end{aligned}$$

Thus, $\|u_n - u_0\|_{H^1(\Omega)} \to 0$. This limit, combined with Lemma 1, imply that $f(u_n) \to f(u_0)$, $n \in \mathbb{N}_9$. Therefore, $u_0 = A_0^{-1}P(-f(u_0)+g_0)$.

6 Regions bounded by a smooth function

Let $g : [a,b] \to \mathbb{R}$, $-\infty < a < b < +\infty$, be a function of class C^3 satisfying,

$$\left|\frac{g'(s)}{g(s)}\right| \le X_1, \quad X_2 \le g(s) \le X_1, \quad \frac{1}{g^2(s)} - \left(\frac{g'(s)}{g(s)}\right)^2 \ge \frac{1}{2X_1^2}, \quad \forall s \in [a,b],$$

where X_1, X_2 are positive numbers.

Recall we use the notation $J = (a,b)$. As mentioned in the Introduction, we consider the domain

$$Q_\varepsilon = \{(s,y) \in J \times \mathbb{R} : 0 < y < \varepsilon g(s)\}$$

and the equation (20), that is,

$$\begin{aligned} -\Delta u + f(u) = 0, &\quad \text{in } Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0, &\quad \text{on } \partial Q_\varepsilon, \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear C^2-function satisfying the conditions (14), (15) and (16).

The linear term in equation (20) is described by the quadratic form

$$\hat{m}_\varepsilon(u) = \int_{Q_\varepsilon} |\nabla u|^2 dx, \qquad u \in H^1(Q_\varepsilon). \tag{34}$$

By performing the change of variables

$$\begin{aligned} \tilde{F}_\varepsilon : \ J \times (0,1) &\to Q_\varepsilon \\ (s,y) &\mapsto (s, \varepsilon y g(s)), \end{aligned}$$

(34) becomes

$$\tilde{m}_\varepsilon(u) = \int_Q \left[\frac{1}{g}(gu' - yg'u_y)^2 + \frac{1}{\varepsilon^2 g}u_y^2\right] \mathrm{d}s\,\mathrm{d}y, \tag{35}$$

where dom $\tilde{m}_\varepsilon = H^1(Q)$ and $Q = J \times (0,1)$. Further, dom $\tilde{m}_\varepsilon$ is a subspace of the Hilbert space $\mathrm{L}^2(Q,g)$. The details of this change of variables can be found in [13].

Now, we propose an additional change of variables in order to work in the Hilbert space $\mathrm{L}^2(Q)$, that is, consider the unitary operator

$$\begin{aligned} V_\varepsilon : \ \mathrm{L}^2(Q,g) &\to \mathrm{L}^2(Q) \\ v &\mapsto g^{1/2}\, v \end{aligned},$$

so that the quadratic form (35) becomes

$$m_\varepsilon(u) = \int_Q \left[\left(u' - \frac{g'}{2g}u\right)^2 + \frac{1}{\varepsilon^2 g^2}u_y^2\right] \mathrm{d}s\,\mathrm{d}y, \tag{36}$$

and now dom $m_\varepsilon = H^1(Q)$ as a subspace of $\mathrm{L}^2(Q)$.

Recall that M_ε is the self-adjoint operator associated with the quadratic form (36) and consider equation (21), whose set of solutions is $\mathcal{E}_\varepsilon^g$. Recall that $\tilde{\mathcal{J}} = \{w1 : w \in \mathrm{L}^2(J)\}$ and that M is the self-adjoint operator associated with the quadratic form m given by (22).

The proof of the next theorem follows the same strategy employed in the proof of Theorem 1, that is, one must identify the correct subspaces and demonstrate the required estimates for the application of the technique of [12].

Theorem 7. *There are two positive numbers N and ε_4, so that, for $0 < \varepsilon < \varepsilon_4$,*

$$\|M_\varepsilon^{-1} - M^{-1}\tilde{P}\|_{\mathrm{L}^2(Q)} \leq N\,\varepsilon,$$

where $\tilde{P}$ denotes the orthogonal projection onto the subspace $\tilde{\mathcal{J}}$.

Proof. If $u \in$ dom m_ε, write $u(s,y) = w(s) + v(s,y)$, where $w \in H^1(J)$ and $v \in \tilde{\mathcal{J}}^\perp \cap H^1(Q)$. Thus,

$$m_\varepsilon(u) = m_\varepsilon(w) + 2\,m_\varepsilon(w,v) + m_\varepsilon(v),$$

where $m_\varepsilon(u_1,u_2)$ is the bilinear form associated with quadratic form $m_\varepsilon(u)$.

Again, we are going to show that the conditions (3.2), (3.3), (3.4) and (3.5) of Section 3 of [12] are satisfied, and so the theorem follows. For this, just note that

$$m_\varepsilon(w) = m(w), \qquad \forall w \in H^1(J),$$

$$m_\varepsilon(w,v) = 0, \qquad \forall w \in H^1(J) \quad \text{and} \quad v \in \tilde{\mathcal{J}}^\perp \cap H^1(Q),$$
$$m_\varepsilon(v) \geq \frac{\pi^2}{\varepsilon^2\,X_1^2}\int_Q |v|^2 \mathrm{d}s\,dy, \qquad \forall v \in \tilde{\mathcal{J}}^\perp \cap H^1(Q),$$

and the proof is complete. □

The following lemma follows from Theorem 2.4 in [13].

Lemma 3. *There are two positive numbers T and $\varepsilon_5 > 0$, so that, for $0 < \varepsilon < \varepsilon_5$,*

$$\|u_\varepsilon\|_{H^2(Q)} \leq T, \quad \textit{for all} \quad u_\varepsilon \in \mathcal{E}_\varepsilon^g.$$

Proof of Theorem 5. Let $\varepsilon_n \to 0$, $n \in \mathbb{N}$ $(n \to \infty)$, $M_n := M_{\varepsilon_n}$, and $u_n := u_{\varepsilon_n}$ a sequence so that $u_n \in \mathcal{E}^g_{\varepsilon_n}$. Lemmas 3 and 1 (see Section 5 for Lemma 1) ensure that $(f(u_n))_n$ is a bounded sequence in $\mathrm{L}^2(Q)$. Thus, there exist $f_0 \in \mathrm{L}^2(Q)$ and an infinite subset $\mathbb{N}_{10} \subset \mathbb{N}$ so that

$$f(u_n) \rightharpoonup f_0, \quad \text{in } \mathrm{L}^2(Q), \quad n \in \mathbb{N}_{10}.$$

By Theorem 7, we have

$$\left\| M_n^{-1}\left(-f(u_n)\right) - M^{-1}\tilde{P}\left(-f(u_n)\right)\right\|_{\mathrm{L}^2(Q)} \to 0, \qquad n \in \mathbb{N}_{10},$$

and also note that

$$M_0^{-1}\left(-f(u_n)\right) \rightharpoonup M^{-1}\tilde{P}\left(-f_0\right), \quad \text{in } \mathrm{L}^2(Q), \quad n \in \mathbb{N}_{10}.$$

Since M^{-1} is a compact operator (in fact, M^{-1} is a norm limit of compact operators), $u_n^s := M^{-1}\tilde{P}(-f(u_n))$ has a convergent subsequence. Thus, there exists $u_0 \in \mathrm{L}^2(J)$, so that,

$$\|u_n^s - u_0\|_{\mathrm{L}^2(Q)} \to 0, \quad n \in \mathbb{N}_{11},$$

where $\mathbb{N}_{11}$ is an infinite subset of $\mathbb{N}_{10}$. Consequently, $u_0 = M^{-1}\tilde{P}(-f_0)$. Since the immersion $H^2(Q) \to \mathrm{L}^2(Q)$ is compact, Lemma 3 ensures that

$$\|u_n - u_0\|_{H^1(Q)} \to 0, \quad n \in \mathbb{N}_{12},$$

where $\mathbb{N}_{12}$ is an infinite subset of $\mathbb{N}_{11}$. This limit, combined with Lemma 1, imply that $f(u_n) \to f(u_0)$, $n \in \mathbb{N}_{12}$. Thus, $u_0 = M^{-1}\tilde{P}(-f(u_0))$.

A Appendix

In this appendix we present some details of the limit behaviour of the quadratic form (6), as $\varepsilon \to 0$. Recall that it is given by

$$\int_{I\times S}\left(\frac{1}{\gamma_\varepsilon}|u'|^2 + \frac{\gamma_\varepsilon}{\varepsilon^2}\left(|\nabla_y u|^2 - \lambda_0|u|^2\right)\right) \mathrm{d}s\, \mathrm{d}y,$$

where $\gamma_\varepsilon(s,y) = 1 - (\varepsilon y_1)/a$, λ_0 is the first eigenvalue of the Dirichlet Laplacian in $H_0^1(S)$ and recall that $u_0 = u_0(y)$ is the corresponding (positive) normalized eigenfunction.

Consider the subspace $\mathcal{J} = \{wu_0 : w \in H_0^1(I)\}$ and the quadratic form

$$\int_S \left(\frac{\gamma_\varepsilon}{\varepsilon^2}\left(|\nabla_y u|^2 - \lambda_0|u|^2\right)\right) \mathrm{d}s\, dy$$

restrict to $\mathcal{J}$. For this, an integration by parts shows that

$$\begin{aligned}
&\int_S \frac{(1-(\varepsilon y_1)/a)}{\varepsilon^2}\left(|\nabla_y u_0|^2 - \lambda_0 u_0^2\right) \mathrm{d}y \\
= &\int_S \frac{-y_1/a}{\varepsilon}\left(|\nabla_y u_0|^2 - \lambda_0 u_0^2\right) \mathrm{d}y \\
= &\int_S \left[\frac{\partial}{\partial y_1}\left(\frac{y_1}{a\varepsilon}\frac{\partial u_0}{\partial y_1}\right) u_0 + \frac{\partial}{\partial y_2}\left(\frac{y_1}{a\varepsilon}\frac{\partial u_0}{\partial y_2}\right) u_0 + \frac{y_1\lambda_0}{a\varepsilon} u_0^2\right] \mathrm{d}y \\
= &\int_S \left(\frac{1}{a\varepsilon}\frac{\partial u_0}{\partial y_1} u_0 + \frac{y_1}{a\varepsilon}\Delta_y u_0 + \frac{y_1}{a\varepsilon}\lambda_0 u_0^2\right) \mathrm{d}y \\
= &\int_S \left(\frac{1}{a\varepsilon}\frac{\partial u_0}{\partial y_1} u_0\right) \mathrm{d}y = 0.
\end{aligned}$$

Thus, since $\gamma_\varepsilon(s,y) \to 1$ uniformly, as $\varepsilon \to 0$, we have, for the full quadratic form,

$$\begin{aligned}
&\int_{I\times S} \left(\frac{1}{\gamma_\varepsilon}|w' u_0|^2 + |w|^2 \frac{\gamma_\varepsilon}{\varepsilon^2}|\nabla_y u_0|^2 - \lambda_0 |w|^2 |u_0|^2\right) \mathrm{d}s\, \mathrm{d}y \\
= &\int_{I\times S} \frac{1}{\gamma_\varepsilon}|w' u_0|^2 \mathrm{d}s\, \mathrm{d}y \longrightarrow \int_I |w'|^2 \mathrm{d}s.
\end{aligned}$$

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