Nonlinear perturbations of a periodic Schrödinger equation with supercritical growth

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Abstract. In this paper we establish the existence of positive solutions for a class of quasilinear Schrödinger equations involving supercritical growth. By using a change of variables, the quasilinear equation is reduced to a semilinear equation.Then, variational method is used together with a truncation argument used in [10] and concentration compactness principle given in [9].

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1. Introduction

Recently many researches have been studied the quasilinear equations of the form

$$-\Delta u + V(x)u - k\Delta(u^2)u = p(u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

with $N \geq 1$,$k \in \mathbb{R}$, $V : \mathbb{R}^N \to \mathbb{R}$ a function called potential and $p : \mathbb{R} \to \mathbb{R}$ a continuous function.

The solutions of (1.1) are related to the existence of standing waves of the following quasilinear Schrödinger equation:

$$i\partial_t z = -\Delta z + W(x)z - f(|z^2|)z - k\Delta\left[g(|z^2|)\right] g'(|z^2|)z, \tag{1.2}$$

where W is a given potential, $k \in \mathbb{R}$ and f, g are real functions.

Quasilinear equations of the form (1.2) have been established in several areas of physics corresponding to various types of g. For instance, the case $g(s) = s$ was used in [17]for the superfluid film equation in plasma physics. In the case $g(s) = (1+s)^{1/2}$, the equation (1.2) models the self-channeling of a high-power ultrashort laser in matter, see [5] and [6]. The equation (1.2) also appears in condensed matter theory, see [23].

Here we consider the case where $g(s) = s$ and $k = 1$ and our special interest is in the existence of standing wave solutions, that is, solutions of type $\psi(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that ψ satisfies (1.2) if and only if the function u solves the equation (1.1) , with $V(x) := W(x) - E$ is a new potential.

In order to seek solution to the equation (1.1) two variational methods have been widely used, mainly in the subcritical and critical situation. That is, when p has a behavior at infinity like $|s|^{r-1} s$, with $r + 1 \leq 22^* = \dfrac{4N}{N-2}$, with $N \geq 3$. This number is the critical exponent for the equation (1.1) [21, Remark 3.13]. For the subcritical case $r + 1 < 22^*$, in the first, which was started in [27] and extended in [20], variational methods and constrained minimization arguments were used to provide existence of positive solutions results with an unknown Lagrange multiplier λ in front of the nonlinear term. The second and more general method, which was started in [21], uses an innovative change of

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variables which allows to rewrite the functional in semilinear form. With this tool, they were able to work with functional well defined in a usual Sobolev space. We recall that, in this new framework, the new problem become a nonhomogeneous problem bringing a new difficulty to handle this equation. See also [3, 7, 8].

The critical case $r+1=22^*$ was also considered recently, among others, in [11, 19, 25, 28, 30, 31]. Recently, in [22] was considered this class of quasilinear Schrödinger equations and with a new perturbation approach they treated the critical exponent case giving new existence results (see also [19] for the subcritical case). We recall that this approach requires certain monotonicity conditions for the structure of the equations.

Now, we turn out attention to the supercritical case of the problem (1.1), that is, the function p has supercritical growth.

When $k \neq 0$, to the best of our knowledgement, there are few works in this direction. In [26] the author has obtained the existence of positive solutions, by assuming, among other conditions, p is a nonnegative function, and the potential function V is radial and $V(x)=0$ in a subdomain of $\mathbb{R}^N$, $N \geq 2$. In [24], the authors establish the existence of solutions for quasilinear Schrödinger equations involving supercritical growth with nonlinearities indefinite in sign.

The purpose of this article is to investigate the existence of positive solutions for the quasilinear elliptic problem

$$-\Delta u - \Delta(u^2)u + V(x)u = p(u) \quad x \in \mathbb{R}^{\mathrm{N}}, \quad N \geq 3, \tag{1.3}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying:

(V_0) There exists $\beta > 0$ such that $V(x) \geq \beta > 0$, $\forall x \in \mathbb{R}^N$.

(V_1) $V(x) = V(x+y)$, $\forall x \in \mathbb{R}^N$, $y \in \mathbb{Z}^N$.

The function $p \in C(\mathbb{R}, \mathbb{R})$ is written as

$$p(s) = f_0(s) + \epsilon g(s),$$

where ϵ is a real parameter and f_0, g are locally Hölder continuous functions satisfying the following:

(F_1) $f_0(0) = g(0) = 0$ and $g(s) \geq 0$ for all $s \neq 0$.

(F_2) $\displaystyle\lim_{|s|\to 0^+} \frac{f_0(s)}{s} = 0$ and $\displaystyle\lim_{|s|\to 0^+} \frac{g(s)}{s} = 0$.

(F_3) There exists $q \in (4, 22^*)$ such that $|f_0(s)| \leq |s|^{q-1}$, $\forall s \in \mathbb{R}$.

(F_4) $\displaystyle\lim_{|s|\to\infty} \frac{F_0(s)}{s^4} = \infty$ where $F_0(s) = \int_0^s f_0(t)dt$.

(F_5) There exists a sequence of positive real numbers (M_n) diverging to $+\infty$ such that

$$\frac{g(s)}{s^{q-1}} \leq \frac{g(M_n)}{M_n^{q-1}} \text{ for all } s \in [0, M_n], \quad n \in \mathbb{N}.$$

(F_6) For $\beta > 0$ given by (V_0) there exists $l > 2$ and $\sigma \in (0, (\frac{l}{2} - 1)\beta)$ such that

$$\frac{1}{2}sf_0(s) - lF_0(s) \geq -\sigma s^2 \text{ and } \frac{1}{2}sg(s) - lG(s) \geq 0 \text{ for all } s \neq 0,$$

where $G(s) = \int_0^s g(t)dt$.

Our first result is the following.

Theorem 1.1. *Suppose that V satisfies $(V_0) - (V_1)$ and $(F_1) - (F_6)$ hold. Then there is a $\epsilon_0 > 0$ such that (1.3) possesses a positive solution for all $0 < \epsilon \leq \epsilon_0$.*

For the next result, we suppose that potential V is a small perturbation of a periodic potential, more exactly, we assume

(V_2) There exist $W_0 > 0$ and $W \in C(\mathbb{R}^N, \mathbb{R}) \cap L^{N/2}(\mathbb{R}^N)$ such that $V(x) = V_p(x) - W(x) \geq W_0$, with V_p verifying (V_1) and $W(x) \geq 0, x \in \mathbb{R}^N$, where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$.

Also we assume other conditions on p.

(F_6') For $W_0 > 0$ given by (V_2) there exists $l > 2$ and $\sigma \in (0, (\frac{l}{2} - 1)W_0)$ such that

$$\frac{1}{2}sf_0(s) - lF_0(s) \geq -\sigma s^2 \text{ and } \frac{1}{2}sg(s) - lG(s) \geq 0 \text{ for all } s \neq 0,$$

where $G(s) = \int_0^s g(t)dt$.

(F_7) The function $s \longmapsto \dfrac{p(s)}{s^3}$ is increasing on $(0, +\infty)$.

We establish the main result of the work.

Theorem 1.2. *Suppose that V satisfies (V_2), and in addition to $(F_1) - (F_5)$, p verifies (F_6') and (F_7). Then there is a $\epsilon_0 > 0$ such that (1.3) possesses a positive solution for all $0 < \epsilon \leq \epsilon_0$.*

We observe that for all $\epsilon > 0$, sufficiently small, the function $p(t) = t^{q-1} + \epsilon t^{r-1}$, for all $r > 22^* > q$, satisfies the above conditions.

The underlying idea for proving our main results is motivated by the argument used in [1, 2, 4, 18]. First, we are going to prove that the periodic problem involving the subcritical exponent possesses a positive solution. Since the associated Euler-Lagrange functional on $H^1(\mathbb{R}^N)$ is not well defined in general, we can not apply variational methods directly, as for the operator well as for the supercritical nonlinearity. To overcome these difficulties motivated by the argument used in [21], we use a change of variable to reformulate the problem obtaining a semilinear which has an associated functional well defined in the Sobolev space $H^1(\mathbb{R}^N)$. Considering the associated functional with the modified problem, we use a version of the Mountain Pass Theorem without compactness condition (see [28]) to get a Cerami sequence associated with the minimax level. Next, we use this sequence and a technical result due to Lions (see [9]) to get a nontrivial critical point for the functional associated to the periodic problem. After that, we construct a sequence of cutoff functions, by using a truncation argument made in [10], and modify the nonlinearity in (1.3), in order to satisfy subcritical growth. Then we obtain a family of functionals of class C^1. Finally, we provide an estimate involving the L^∞-norm of a solution related to a subcritical problem. Then, we shall prove that for the modified nonperiodic problem has a positive solution.

Since we intend to prove the existence of positive solutions, we are going to consider $p : \mathbb{R} \to \mathbb{R}$ satisfying $(F_1) - (F_7)$ on $[0, +\infty)$ and defined as zero on $(-\infty, 0]$.

Notation: In this paper we use the following notations:

- In all integrals we omit the symbol "dx".
- $\|u\|_s = \Big(\int_\Omega |u|^s\Big)^{1/s}$ denotes the usual norm in L^s-space.
- $C, C_1, C_2, ..$ denote positive (possible different) constants.
- $\|u\| = \Big(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Big)^{1/2}$ denotes the usual norm in $H^1(\mathbb{R}^N)$-space.
- We denote the weak and strong convergence in X, as $n \to \infty$, by "$w_n \rightharpoonup w$" and "$w_n \to w$", respectively.
- $\|u\|_\infty = sup\,|u|$ denote the usual norm in L^∞-space.
- E' denote the dual of space E.

2. The periodic Case

2.1. Auxiliary Problem

We will investigate the existence of positive solution for the quasilinear equation involving subcritical growth. This result will be useful for obtaining our main result. More precisely, we study the following equation:

$$\begin{cases} -\Delta u - \Delta(u^2)u + V(x)u = h(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is bounded locally Hölder continuous function that satisfies $(V_0) - (V_1)$ and the function $h \in C(\mathbb{R}^+, \mathbb{R})$ verifies:

(H_1) $h(0) = 0$.

(H_2) $\lim_{s \to 0^+} \frac{h(s)}{s} = 0$.

(H_3) There exists $C > 0$ and $q \in (4, 22^*)$ such that $|h(s)| \le C(|s| + |s|^{q-1})$ for all $s \in \mathbb{R}^+$.

(H_4) $\lim_{|s| \to \infty} \frac{H(s)}{s^4} = \infty$ where $H(s) = \int_0^s h(t)dt$.

(H_5) For $\beta > 0$ given by (V_0) there exist $l > 2$ and $\sigma \in (0, (\frac{l}{2} - 1)\beta)$ such that $\frac{1}{2}sh(s) - lH(s) \ge -\sigma s^2$ for all $s \neq 0$.

We observe that formally the problem (2.1) is the Euler-Lagrange equation associated to the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 + 2u^2) |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} H(u).$$

From the variational point of view, the first difficult associated to problem (2.1) is finding an appropriate function space where the functional J is well defined. In order to avoid such difficulty, we make use of the change of variable introduced by [21], that is, we consider $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{\sqrt{1+2f^2(t)}}, \quad \text{on} \quad [0, +\infty), \\ f(t) &= -f(-t), \quad \text{on} \quad (-\infty, 0] \end{aligned} \tag{2.2}$$

having the following properties, which have been proved in [12], [15].

Lemma 2.1. *The function f enjoys the following properties:*

1. *f is uniquely defined C^∞ function and invertible.*
2. *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$.*
3. *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$.*
4. *$\frac{f(t)}{t} \to 1$ as $t \to 0$.*
5. *$\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to \infty$.*
6. *$\frac{f(t)}{2} \le tf'(t) \le f(t)$ for all $t \ge 0$.*
7. *$|f(t)| \le 2^{\frac{1}{4}} |t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$.*
8. *The function $f^2(t)$ is strictly convex.*
9. *There exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & \text{if} \quad |t| \le 1 \\ C|t|^{\frac{1}{2}}, & \text{if} \quad |t| \ge 1. \end{cases}$$

10. *There exist positive constants C_1 and C_2 such that $|t| \le C_1 |f(t)| + C_2 |f(t)|^2$, for all $t \in \mathbb{R}$.*

11. $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$, *for all* $t \in \mathbb{R}$.
12. *For each* $\lambda > 1$ *we have* $f^2(\lambda t) \leq \lambda^2 f^2(t)$, *for all* $t \in \mathbb{R}$.

As consequence of Lemma 2.1, the following has been proved in [12].

Corollary 2.2. i) *The function* $\frac{f(t)f'(t)}{t}$ *is decreasing for all* $t > 0$.

ii) *The function* $\frac{f^3(t)f'(t)}{t}$ *is increasing for all* $t > 0$.

By, using the above change of variables $u = f(v)$ from J, we obtain the following functional: $I : E \to \mathbb{R}^N$ defined by

$$I(v) = J(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} H(f(v)), \tag{2.3}$$

which is well defined in E and belongs to C^1 under hypotesis (V_0), where $E = H^1(\mathbb{R}^N)$ is endowed with the equivalent norm to the usual norm in $H^1(\mathbb{R}^N)$ given by

$$\|v\|_E^2 = \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2).$$

Moreover, the critical points of I are weak solutions of the problem

$$-\Delta v + V(x)f(v)f'(v) = h(f(v))f'(v), \quad x \in \mathbb{R}^N, \tag{2.4}$$

that is,

$$I'(v)w = \int_{\mathbb{R}^N} \nabla v \nabla w + \int_{\mathbb{R}^N} V(x)f(v)f'(v)w - \int_{\mathbb{R}^N} h(f(v))f'(v)w = 0,$$

for all $v, w \in H^1(\mathbb{R}^N)$. We observe that $v \in H^1(\mathbb{R}^N)$ is the critical point of the functional I, then the function $u = f(v)$ is a weak solution of (2.1). By the elliptic regularity theory and using a standard bootstrap argument, we have that $u \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, that is, $u = f(v)$ is a classical solution of (2.1).

2.2. The mountain pass geometry

In this section, we will prove that I satisfies the geometry conditions of the variant Mountain Pass Theorem (see [28]).

Let E be a real Banach space and $I : E \to \mathbb{R}$ a functional of class C^1. We say that $(v_n) \subset E$ is a Cerami sequence at $c \in \mathbb{R}$ ($(C_e)_c$ for short) for I if (v_n) satisfies

(i) $I(v_n) \to c$.
(ii) $\|I(v_n)\|_{E'} (1 + \|v_n\|_E) \to 0$,

as $n \to \infty$. We say that I satisfies the Cerami condition at c, if any Cerami sequence at c possesses a convergent subsequence.

Lemma 2.3. *Assume that* (V_0), $(H_1) - (H_4)$ *hold and* $V \in L^\infty(\mathbb{R}^N)$. *Let* I *be the functional defined in* (2.3). *Then there is a closed subset* S *of* E *such that*

- S *disconnects (arcwise)* E *into distinct connected components* E_1 *and* E_2.
- *The functional* I *satisfies* $I(0) = 0$.
- $0 \in E_1$ *and there is* $\tau > 0$ *such that* $I|_S \geq \tau > 0$;
- *There is* $e \in E_2$ *such that* $I(e) \leq 0$.

Proof. See [29, Lemma 3.1]. □

Corollary 2.4. *Assume that* (V_0), $(H_1) - (H_5)$ *hold and* $V \in L^\infty(\mathbb{R}^N)$. *Then the functional* I *possesses a* $(Ce)_c$ *sequence, with* $c \geq \alpha > 0$ *given by*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{2.5}$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : I(0) = 0, \qquad I(\gamma(1)) < 0 \text{ and } I(\gamma(1)) < 0\}.$$

Proof. Consequence of Lemma 2.3 and Mountain Pass Theorem (see [28]). □

Lemma 2.5. *Suppose* (V_0), $(H_1)-(H_5)$ *and* $V \in L^\infty(\mathbb{R}^N)$. *Then every Cerami sequence* (v_n) *associated to the functional* I *is bounded in* E.

Proof. Let (v_n) be a $(Ce)_c$ sequence:

$$I(v_n) \to c \text{ and } (1 + \|v_n\|_E) I'(v_n) \to 0, \text{ as } n \to \infty.$$

We can assume that v_n is nonnegative. By property (6) in Lemma 2.1, (H_5) and (V_0) we have

$$\begin{aligned} lI(v_n) - I'(v_n)v_n \geq \;& (\frac{l}{2} - 1)[\textstyle\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(v_n)] \\ & -l \textstyle\int_{\mathbb{R}^N} H(f(v_n)) + \int_{\mathbb{R}^N} h(f(v_n) f'(v_n) v_n \\ \geq \;& (\frac{l}{2} - 1)[\textstyle\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(v_n)] - \sigma \int_{\mathbb{R}^N} f^2(v_n) \\ \geq \;& (\frac{l}{2} - 1) \textstyle\int_{\mathbb{R}^N} |\nabla v_n|^2 + (\frac{l}{2} - 1 - \frac{\sigma}{\beta}) \int_{\mathbb{R}^N} V(x) f^2(v_n). \end{aligned}$$

This implies that there exists $M > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(v_n) \leq M.$$

We still need to show that (v_n) is bounded in $H^1(\mathbb{R}^N)$. It suffices to prove that $\int_{\mathbb{R}^N} v_n^2$ is bounded. We write

$$\int_{\mathbb{R}^N} v_n^2 = \int_{\{|v_n(x)| \leq 1\}} v_n^2 + \int_{\{|v_n(x)| \geq 1\}} v_n^2.$$

By condition (V_0) and boundeness of V, as well as, (9) in Lemma (2.1) and the Sobolev Embedding Theorem, we find a constant $C > 0$ such that

$$\int_{\{|v_n(x)| \leq 1\}} v_n^2 \leq \frac{1}{C^2} \int_{\{|v_n| \leq 1\}} f^2(v_n(x)) \leq \frac{1}{C^2 \beta} \int_{\{|v_n| \leq 1\}} V(x) f^2(v_n(x)) \leq \frac{M}{\beta C^2}$$

and

$$\int_{\{|v_n(x)| \geq 1\}} v_n^2 \leq \int_{\{|v_n| \geq 1\}} |v_n|^{2^*} \leq C_0 \left(\int_{\{|v_n| \geq 1\}} |\nabla v_n(x)|^2 \right)^{2^8/2} \leq C_0 M^{2^*/2}.$$

Therefore,

$$\int_{\mathbb{R}^N} v_n^2 = \int_{\{|v_n(x)| \leq 1\}} v_n^2 + \int_{\{|v_n(x)| \geq 1\}} v_n^2 \leq \frac{M}{a_0 C^2} + C_0 M^{2^*/2},$$

that is, (v_n) is bounded in E. □

Lemma 2.6. *Suppose that* $(V_0), (V_1)$ *and* $(H_1) - (H_4)$ *are satisfied. Let* $(v_n) \subset H^1(\mathbb{R}^N)$ *be a* $(Ce)_c$, *with* c *given by (2.5), and* $v_n \rightharpoonup 0$ *weakly in* $H^1(\mathbb{R}^N)$. *Then there exist a sequence* $(y_n) \subset \mathbb{R}^N$ *and* $r, \eta > 0$ *such that* $|y_n| \to \infty$ *and*

$$\limsup_{n \to \infty} \int_{B_r(y_n)} |v_n|^2 \geq \eta > 0.$$

Proof. See [29, Lemma 3.4]. □

Proposition 2.7. *Suppose that V and h satisfy $(V_0)-(V_1)$ and $(H_1)-(H_5)$, respectively. Then* (2.4) *has a positive solution u such that*

$$\|v\|_E \leq C_1,$$

where C_1 depends on β, σ, l and the minimax level associated to (2.4).

Proof. By Corollary 2.4, there exists $(v_n) \subset E$ a $(Ce)_c$ sequence. By Lemma 2.5, there exists $v \in E$ such that $v_n \rightharpoonup v$ in E. From this and $(H_2)-(H_3)$, we have that v is a critical point of I, that is, $I'(v) = 0$. Effectively, because $C_0^\infty(\mathbb{R}^N)$ is dense in E, it suffices to show that, $I'(v)\psi = 0$ for every $\psi \in C_0^\infty(\mathbb{R}^N)$. Note that

$$I'(v_n)\psi - I'(v)\psi = \int_{\mathbb{R}^N} (\nabla v_n - \nabla v)\nabla\psi + \int_{\mathbb{R}^N} V(x)[f(v_n)f'(v_n) - f(v)f'(v)]\psi$$
$$+ \int_{\mathbb{R}^N} [h(f(v))f'(v) - h(f(v_n))f'(v_n)]\psi.$$

Since $v_n \rightharpoonup v$ weakly in E, we have that $v_n \to v$ in $L^r_{loc}(\mathbb{R}^N)$, as $n \to \infty$, with $r \in [2, 2^*)$. Then, up to a subsequence,

$$v_n(x) \longrightarrow v(x) \text{ a.e. in } K := supp \ \ \psi, \text{ as } n \to \infty$$
$$|v_n(x)| \leq |z_r(x)| \text{ a.e. in } K \text{ with } z_r \in L^r(K).$$

Consequently,

$$f(v_n)f'(v_n) \to f(v)f'(v) \text{ a.e. in } K, \text{ as } n \to \infty$$
$$h(f(v_n))f'(v_n) \to h(f(v))f'(v) \text{ a.e. in } K, \text{ as } n \to \infty.$$

Furthemore, from $(H_2)-(H_3)$ we have

$$|h(f(v))f'(v)| \leq \epsilon\,|f(v)|\,|f'(v)| + C\,|f(v)|^{q-1}\,|f'(v)|\,.$$

Therefore,

$$|h(f(v))f'(v)\psi| \leq \epsilon\,|z_2|\,|\psi| + C\left|z_{\frac{q}{2}-1}\right|^{\frac{q}{2}-1}|\psi|\,,$$

by the Lebesgue Dominated Convergence Theorem we have

$$\int_{\mathbb{R}^n} h(f(v_n)f'(v_n)\psi \to \int_{\mathbb{R}^n} h(f(v)f'(v)\psi, \text{ as } n \to \infty.$$

From the properties of V we have,

$$V(x)f(v_n)f'(v_n)\psi \to V(x)f(v)f'(v)\psi, \text{ a.e. in } \mathbb{R}^N$$

and

$$|V(x)f(v_n)f'(v_n)\psi| \leq \|V\|_\infty\,|v_n|\,|\psi| \leq \|V\|_\infty\,|z_2|\,|\psi|\,.$$

Again by the Lebesgue Dominated Convergence Theorem we have,

$$\int_{\mathbb{R}^n} V(x)f(v_n)f'(v_n)\psi \to \int_{\mathbb{R}^n} V(x)f(v)f'(v)\psi, \text{ as } n \to \infty.$$

So, $I'(v_n)\psi - I'(v)\psi \to 0$, since $I'(v_n) \to 0$, we conclude that $I'(v) = 0$. We need to show that $v \not\equiv 0$. Suppose by contradiction that $v = 0$. By Lemma 2.6, there exist a sequence $(y_n) \subset \mathbb{R}^N$ and $r, \eta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 > \eta > 0. \tag{2.6}$$

Without loss of generality we may assume that $y_n \in \mathbb{Z}^N$. Then, defining $\widetilde{v}_n(x) = v_n(x+y_n)$, $n \in \mathbb{N}$, we have $\|\widetilde{v}_n\|_E = \|v_n\|_E$ for all $n \in \mathbb{N}$. Thus, taking a subsequence if necessary, there exists $\widetilde{v} \in E$ such that $\widetilde{v}_n \rightharpoonup \widetilde{v}$ in E, $\widetilde{v}_n \to \widetilde{v}$ in $L^2_{Loc}(\mathbb{R}^N)$ and $\widetilde{v}_n(x) \to \widetilde{v}(x)$ almost everywhere in $\mathbb{R}^N$. From (2.6), we have $\widetilde{v} \neq 0$. By the arguments used above, we deduce $I'(\widetilde{v})\psi = 0$ for every $\psi \in E$. Therefore $\widetilde{v}$ is a solution for (2.4) .

We must verify that $\widetilde{v}$ is bounded. By (H_5) we observe that

$$lI(\widetilde{v}_n) - I'(\widetilde{v}_n)\widetilde{v}_n \geq (\frac{l}{2}-1)\int_{\mathbb{R}^N} |\nabla \widetilde{v}_n|^2 + (\frac{l}{2}-1-\frac{\sigma}{\beta})\int_{\mathbb{R}^N} V(x)f^2(\widetilde{v}_n),$$

so that, applying Fatou's Lemma we have

$$\frac{2lc\beta}{l\beta-2\beta-2\sigma} \geq \int_{\mathbb{R}^N} |\nabla \widetilde{v}|^2 + \int_{\mathbb{R}^N} V(x)f^2(\widetilde{v}).$$

Arguing as in Lemma 2.5, we have

$$\|\widetilde{v}\|_E \leq M,$$

where M depends on β, l, σ and c. □

3. Proof of Theorem 1.1

In this proof we will explore a truncation argument introduced in [10] and [21]. We define a sequence of functions g_n by letting

$$g_n(t) = \begin{cases} \dfrac{g(M_n)}{M_n^{q-1}} t^{q-1}, & \text{if } t \geq M_n, \\ g(t), & \text{if } 0 \leq t \leq M_n, \\ 0, & \text{if } t \leq 0. \end{cases} \tag{3.1}$$

By (F_5), for M_n fixed and $\epsilon > 0$ sufficiently small, we have

$$|\epsilon g_n(t)| \leq \epsilon \frac{g(M_n)}{M_n^{q-1}} |t|^{q-1} \leq |t|^{q-1}, \forall t \in \mathbb{R}. \tag{3.2}$$

Denoting $f_{\epsilon,n}(t) = f_0(t) + \epsilon g_n(t)$, by (F_3) and (5.1), we have

$$|f_{\epsilon,n}(t)| \leq 2\,|t|^{q-1}, \ \forall t \in \mathbb{R}. \tag{3.3}$$

From Proposition 2.7, there exists $v_{\epsilon,n} \in E$ a solution of problem

$$-\Delta v + V(x)f(v)f'(v) = f_{\epsilon,n}(f(v))f'(v), \quad x \in \mathbb{R}^N \tag{3.4}$$

such that

$$\|v_{\epsilon,n}\| \leq C_{\epsilon,n}, \tag{3.5}$$

where $C_{\epsilon,n}$ depends on β, σ, l and the minimax level $c_{\epsilon,n}$ associated to the functional given by

$$I_{\epsilon,n}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} F_{\epsilon,n}(f(v)),$$

where $F_{\epsilon,n}(t) = \int_0^t f_{\epsilon,n}(s)ds$.

Since $F_{\epsilon,n} = F_0 + \epsilon G_n$, where $F_0(t) = \int_0^t f_0(s)ds$ and $G_n(t) = \int_0^t g_n(s)ds$ we have

$$I_{\epsilon,n}(v) \leq I_0(v) \ \text{ for all } \ v \in E,$$

where $I_0 : E \to \mathbb{R}$ given by

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} F_0(f(v)).$$

Thanks of the last inequality, we conclude that $c_{\epsilon,n} \leq c_0$, where c_0 is the minimax level associated to the functional I_0.

Therefore, there exists $K_1 > 0$ independent of ϵ and n, such that

$$\|v_{\epsilon,n}\| \leq K_1. \tag{3.6}$$

Now using Proposintion 6.1, there exists $K = K(q, K_1) > 0$ such that

$$\|v_{\epsilon,n}\|_\infty \leq K K_1^{\frac{2^*-2}{2^*-q/2}} \leq M_n,$$

for n sufficiently large. Completing the proof of the Theorem 1.1 we observe that by using the arguments found in the proof of the [15, Propostion 2.4], we can conclude that $u = f(v) \in E \cap L^\infty(\mathbb{R}^\mathbb{N})$ is a weak solution of (1.3).

4. The nonperiodic Case

First of all, we will investigate the existence of positive solution for the equation involving subcritical growth. This result will be useful for obtaining our main result. More precisely, we study the following equation

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u), \quad x \in \mathbb{R}^N, \tag{4.1}$$

where $V : \mathbb{R}^N \to R$ is bounded locally Hölder continuous function that verifies (V_2) and the function $h \in C(\mathbb{R}^+, \mathbb{R})$ satisfies, in addition to $(H_1) - (H_4)$ of problem (2.1), the following conditions:

(H_5') For $W_0 > 0$ given by (V_2) there exist $l > 2$ and $\sigma \in (0, W_0)$ such that

$$\frac{1}{2} sh(s) - lH(s) \geq -\sigma s^2 \text{ for all } s \neq 0.$$

(H_6) The function $s \longmapsto \dfrac{h(s)}{s^3}$ is increasing on $(0, +\infty)$.

As before, making the change of variables $v = f^{-1}(u)$, where f is defined in (2.2), consider the functional $I : E \to \mathbb{R}^N$ associated to problem (4.1) defined by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v) - \int_{\mathbb{R}^N} H(f(v)),$$

where $E = H^1(\mathbb{R}^N)$ is endowed with the norm equivalent to the usual norm in $H^1(\mathbb{R}^N)$ given by

$$\|v\|_E^2 = \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2).$$

We observe that nontrivial critical points of I are precisely weak solutions of the semilinear problem

$$\begin{cases} -\Delta v + V(x) f(v) f'(v) = h(f(v)) f'(v), & x \in \mathbb{R}^N, \\ v \in E, \end{cases} \tag{4.2}$$

and by the regularity theory (see [14]), we can infer that $v \in C^2(\mathbb{R}^N)$.

Proposition 4.1. *Suppose that V and h satisfies (V_2) and $(H_1)-(H_5')$, respectively. Then (4.2) possesses a positive solution v such that*

$$\|v\|_E \leq C_1,$$

where C_1 depends on W_0, l, σ, and the minimax level associated by (4.2).

We postpone the proof for while, to give some important lemmas. Consider the following functional $I_p : E \to \mathbb{R}^N$ associated to problem (4.1), with $V = V_p$, defined by

$$I_p(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_p(x) f^2(v) - \int_{\mathbb{R}^N} H(f(v)),$$

The Nehari manifold associated to I_p is given by

$$M = \left\{ v \in H^1(\mathbb{R}^N) : I_p'(v)v = 0 \right\}.$$

Next, we consider the numbers

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_p(\gamma(t)),$$

$$\widetilde{c} = \inf_M I_p$$

and

$$\bar{c} = \inf_{v \in E-\{0\}} \max_{t \geq 0} I_p(tv),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I_p(\gamma(1)) < 0\}.$$

We have the following lemma:

Lemma 4.2. *Assume that $(H_1)-(H_6)$, and (V_2) hold. Then, for each $v \in E$, with $v_+ \neq 0$, there exists a unique $t_v > 0$ such that $t_v \in M$ and $I_p(t_v) = \max_{t\geq 0} I_p(tv)$. Moreover, $\widetilde{c} = \bar{c} = c$.*

Proof. Let $v \in E \setminus 0$ be fixed and define the function $\eta(t) = I_p(tv)$, for $t \geq 0$. Note that if $tv \in M$ then $n'(t) = 0$. By Lemma 2.3, there exists t_v such that $\eta(t_v) = \max_{t\geq 0} \eta(t) = \max_{t \geq 0} I_p(t_v v)$.

Hence,

$$\eta'(t_v) = t_v \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V_p(x) f(t_v v) f'(t_v v) t_v v - \int_{\mathbb{R}^N} h(f(t_v v)) f'(t_v v) t_v v = 0,$$

implying that $t_v v \in M$.

In the sequel, we will show that t_v is unique. To this end, we suppose that there exists $s > 0$, such that $sv \in M$, $\eta'(s) = 0$. This way,

$$0 = \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} \left[\frac{h(f(t_v v)) f'(t_v v)}{t_v v} - \frac{V_p(x) f(t_v v) f'(t_v v)}{t_v v} \right] v^2 \right\}$$

and

$$0 = \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} \left[\frac{h(f(sv)) f'(sv)}{sv} - \frac{V_p(x) f(sv) f'(sv)}{sv} \right] v^2 \right\}.$$

Hence

$$\int_{\mathbb{R}^N} \left[\frac{V_p(x) f(sv) f'(sv)}{sv} - \frac{V_p(x) f(t_v v) f'(t_v v)}{t_v v} \right] v^2 = \int_{\mathbb{R}^N} \left[\frac{h(f(sv)) f'(sv)}{sv} - \frac{h(f(t_v v)) f'(t_v v)}{t_v v} \right] v^2.$$

From Corollary 2.2 and (H_6) it follows $s = t_v$. Now, the proof follows by using similar arguments found [13, Lemma 3.8]. □

Proof of Proposition 4.1 Notice that from (V_2), it follows that I satisfies the Mountain Pass Geometry (see Lemma 2.3), so that there exists a $(Ce)_d$ sequence $(v_n) \subset H^1(\mathbb{R}^N)$ such that

$$I(v_n) \to d > 0 \text{ e } I'(v_n)(1 + \|v_n\|) \to 0 \text{ as } n \to \infty,$$

where

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \quad I(\gamma(1)) < 0\}.$$

It follows from Lemma 4.2 there exists $v \in M$ such that $I_p(v) = c$ and $I_p'(v)\phi = 0$, $\forall \phi \in E$. Then, choosing $t^* \in \mathbb{R}$ such that

$$0 < d \leq \sup_{t \geq 0} I(tv) = I(t^* v).$$

By using (V_2) and recalling that $v \in M$, we have

$$0 < d \leq I(t^* v) < I_p(t^* v) \leq \sup_{t \geq 0} I_p(tv) = c.$$

Hence,

$$d < c.$$

Proceeding as in Lemma 2.5, the sequence v_n is bounded in E. Then, up to a subsequence, we obtain $v_n \rightharpoonup v$ in E. Arguing as in the proof of Proposition 2.7, we conclude that v is a weak solution of (4.2).

We are going to prove that $v \not\equiv 0$.

Suppose on the contrary that $v \equiv 0$, that is, $v_n \rightharpoonup 0$ weakly in E, as $n \to \infty$. Since $W \in L^{N/2}(\mathbb{R}^N)$, using a result by Brezis-Lieb (see [16]) we obtain

$$\int_{\mathbb{R}^N} W(x)f^2(v_n) \to 0 \quad \text{as } n \to \infty.$$

Thus, as $n \to \infty$,

$$|I(v_n) - I_p(v_n)| = \left| \int_{\mathbb{R}^N} W(x)f^2(v_n) \right| = o_n(1),$$

therefore,

$$I_p(v_n) \to d.$$

On the other hand, noting that $W \geq 0$ and taking $\phi \in H^1(\mathbb{R}^N)$, with $\|\phi\| \leq 1$, we obtain

$$\begin{aligned} \left|(I'(v_n) - I'_p(v_n))\phi\right| = &\ \left|\int_{\mathbb{R}^N} W^{1/2}(x)f(v_n)W^{1/2}(x)\phi\right| \\ \leq &\ (\int_{\mathbb{R}^N} W(x)f(v_n)^2)^{1/2}(\int_{\mathbb{R}^N} W^{N/2}(x)\phi^{2N/N-2})^{N-2/2N} \\ \leq &\ C(\int_{\mathbb{R}^N} W(x)f(v_n)^2)^{1/2}, \end{aligned}$$

for some constant $C > 0$, and on the first inequality we used the Hölder's inequality.

It follows that

$$I'_p(v_n) = o_n(1) \quad \text{as } n \to \infty.$$

Let $t_n > 0$ such that $t_n v_n \in M$. Using the same argument found in [3, Lemma 3.8], it follows $\limsup_{n\to\infty} t_n \leq 1$.

Therefore, $c \leq I_p(t_n v_n) \leq I(t_n v_n) + o_n(1) = d + o_n(1)$.

Letting $n \to \infty$, we get

$$c \leq d,$$

obtaining a contradiction. This completes the proof.

5. Proof of Theorem 1.2

The proof is analogous to that made in Theorem 1.1, but for the sake of completeness of this work we will give a sketch of the proof. To obtain the existence of a solution to (1.3), we will explore a truncation argument made in [10] again, by using the sequence of functions g_n defined in (3.1). By (F_5), for M_n fixed and $\epsilon > 0$ sufficiently small, we have

$$|\epsilon g_n(t)| \leq \epsilon \frac{g(M_n)}{M_n^{q-1}} |t|^{q-1} \leq |t|^{q-1}, \forall t \in \mathbb{R}. \tag{5.1}$$

Denoting $f_{\epsilon,n}(t) = f_0(t) + \epsilon g_n(t)$, (F_3) and (5.1), we have

$$|f_{\epsilon,n}(t)| \leq 2|t|^{q-1}, \ \forall t \in \mathbb{R}. \tag{5.2}$$

From Proposition 4.1, there exists $v_{\epsilon,n} \in E$ a solution of problem

$$-\Delta v + V(x)f(v)f'(v) = f_{\epsilon,n}(f(v))f'(v), \quad x \in \ \mathbb{R}^N \tag{5.3}$$

such that

$$\|v_{\epsilon,n}\| \leq C_{\epsilon,n}, \tag{5.4}$$

where $C_{\epsilon,n}$ depends on W_0, σ, l and the minimax level $c_{\epsilon,n}$ associated to the functional given by $I_{\epsilon,n} : E \to \mathbb{R}$,

$$I_{\epsilon,n}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} F_{\epsilon,n}(f(v)),$$

where $F_{\epsilon,n}(t) = \int_0^t f_{\epsilon,n}(s)ds$.

Since $F_{\epsilon,n} = F_0 + \epsilon G_n$, where $F_0(t) = \int_0^t f_0(s)ds$ and $G_n(t) = \int_0^t g_n(s)ds$ we have

$$I_{\epsilon,n}(v) \leq I_0(v) \quad \text{for all} \quad v \in E,$$

where $I_0 : E \to \mathbb{R}$ given by

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} F_0(f(v)).$$

Thanks of the last inequality, we conclude that $c_{\epsilon,n} \leq c_0$, where c_0 is the minimax level associated to the functional I_0.

Therefore, there exists $K_1 > 0$ independent of ϵ and n, such that

$$\|v_{\epsilon,n}\| \leq K_1. \tag{5.5}$$

Now using Proposintion 6.1, there exists $K = K(q, K_1) > 0$ such that

$$\|v_{\epsilon,n}\|_\infty \leq K K_1^{\frac{2^*-2}{2^*-q/2}} \leq M_n,$$

for n sufficiently large. Completing the proof of the Theorem 1.2 we observe that by using the arguments found in the proof of the [15, Propostion 2.4], we can conclude that $u = f(v) \in E \cap L^\infty(\mathbb{R}^{\mathbb{N}})$ is a weak solution of (1.3).

6. Appendix

Proposition 6.1. *Let $v \in E$ be a weak nonnegative solution of the problem*

$$-\Delta v + b(x)f(v)f'(v) = h(x, f(v))f'(v), \quad x \in \mathbb{R}^N, \tag{6.1}$$

where $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous functions verifying $|h(x,s)| \leq C\,|s|^{q-1}$, for all $s > 0$ and for some $4 < q < 2.2^*$, b is a bounded nonnegative function in $\mathbb{R}^N$ and f is defined by (2.2). Then, there exists a constant $K = K(q, C) > 0$ such that $\|v\|_\infty \leq K\,\|v\|^{\frac{2^*-2}{2^*-q/2}}$.

Proof. Let v_0 a weak solution of (6.1), that is

$$\int_{\mathbb{R}^N} \nabla v_0 \nabla \phi + \int_{\mathbb{R}^N} b(x)f(v_0)f'(v_0)\phi - \int_{\mathbb{R}^N} h(f(v_0))f'(v_0)\phi = 0 \tag{6.2}$$

for all $\phi \in E$. For each $k > 0$ we define the functions

$$v_k = \begin{cases} v_0 & \text{if } v_0 \leq k, \\ k, & \text{if } v_0 \geq k, \end{cases}$$

$\widetilde{v}_k = v_k^{2(\beta-1)}v_0$ and $w_k = v_0 v_k^{\beta-1}$ with $\beta > 1$ to be determined later. Note that using Lemma 2.1, items (6) and (7) we obtain

$$|h(x, f(v_0))f'(v_0)\widetilde{v}_k| = \left|h(x, f(v_0))f'(v_0)v_0 v_k^{2(\beta-1)}\right| \leq \left|\frac{h(x, f(v_0))f(v_0)}{v_0} v_0 v_k^{2(\beta-1)}\right| \leq C|v_0|^{\frac{q}{2}-2}w_k^2. \tag{6.3}$$

Taking $\widetilde{v}_k$ as a test function in (6.2), using estimate (6.3) we obtain

$$\int_{\mathbb{R}^N} v_k^{2(\beta-1)}\,|\nabla v_0|^2 + 2(\beta-1)\int_{\mathbb{R}^N} v_0 v_k^{2(\beta-1)-1}\nabla v_0 \nabla v_k \leq \int_{\mathbb{R}^N} C\,|v_0|^{\frac{q}{2}-2}\,w_k^2. \tag{6.4}$$

Because of the second integral in the left hand side of the previous inequality is nonnegative, and using properties (6) and (7) in Lemma 2.1 we conclude that

$$\int_{\mathbb{R}^N} v_k^{2(\beta-1)}\nabla v_0 \leq \int_{\mathbb{R}^N} 2\,|v|^{\frac{q}{2}-2}\,w_k^2.$$

Let $r = q/2$. By the Gagliardo-Nirenberg inequality and (6.5) it follows that

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N} v_k^{2^*}\right)^{2/2^*} &\le C\int_{\mathbb{R}^N} |\nabla v_k|^2\\
&\le C_2\int_{\mathbb{R}^N} v_k^{2(\beta-1)}|\nabla v_0|^2 + C_3(\beta-1)^2\int_{\mathbb{R}^N} v_0^2 v_k^{2(\beta-2)}|\nabla v_k|^2\\
&\le C_4\beta^2\int_{\mathbb{R}^N} v_k^{2(\beta-1)}|\nabla v_0|^2\\
&\le C_5\beta^2\int_{\mathbb{R}^N} |v|^{r-2} v_k^2.
\end{aligned}
$$

where we have used that $1 \le \beta^2$, $v_k \le v_0$ and $(1-\beta)^2 \le \beta^2$. Using the Hölder inequality we get

$$
\left(\int_{\mathbb{R}^N} w_k^{2^*}\right)^{2/2^*} \le C\beta^2 C_5 \|v_0\|_{2^*}^{r-2}\left(\int_{\mathbb{R}^N} |w_k|^{\frac{22^*}{2^*-r+2}}\right)^{\frac{2^*-r+2}{2^*}}.
$$

Since that $|w_k| \le |v_0|^\beta$ in $\mathbb{R}^N$, by definition of w_k, by continuity of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ we obtain

$$
\left(\int_{\mathbb{R}^N} (v_0 v_k^{\beta-1})^{2^*}\right)^{2/2^*} \le C\beta^2 C_6 \|v_0\|^{r-2}\left(\int_{\mathbb{R}^N} |v_0|^{\frac{2\beta 2^*}{2^*-r+2}}\right)^{\frac{2^*-r+2}{2^*}}.
$$

Choosing $\beta = 1 + (2^* - r)/2$, we have $\frac{2\beta 2^*}{2^*-r+2} = 2^*$. Thus,

$$
\left(\int_{\mathbb{R}^N} (v_0 v_k^{\beta-1})^{2^*}\right)^{2/2^*} \le C\beta^2 C_6 \|v_0\|^{r-2}\|v_0\|_{\beta\alpha^*}^{2\beta}
$$

where $\alpha^* = \dfrac{22^*}{2^*-r+2}$. By Fatou's Lemma, we obtain

$$
\|v_0\|_{\beta 2^*} \le \beta^{1/\beta}(CC_6\|v_0\|^{r-2})^{\frac{1}{2\beta}}\|v_0\|_{\beta\alpha^*} \tag{6.5}
$$

For each $m = 0, 1, 2...$ let us define $\beta_{m+1}\alpha^* = 2^*\beta_m$ with $\beta_0 = \beta$. From (6.5), it follows

$$
\begin{aligned}
\|v_0\|_{\beta_1 2^*} &\le (\beta_1^2 CC_6\|v_0\|^{r-2})^{1/2\beta_1}\|v_0\|_{\beta_1\alpha^*}\\
&\le (\beta_1^2 CC_6\|v_0\|^{r-2})^{1/2\beta_1}(\beta^2 CC_6\|v_0\|^{r-2})^{1/2\beta}\|v_0\|_{\beta\alpha^*}\\
&\le (CC_6\|v_0\|^{r-2})^{1/2\beta_1+1/2\beta}\beta^{1/\beta}(\beta_1)^{1/\beta_1}\|v_0\|_{2^*}.
\end{aligned}
$$

Observing that $\beta_m = \chi^m\beta$, where $\chi = \frac{2^*}{\alpha^*}$, by interation we obtain

$$
\|v_0\|_{\beta_m 2^*} \le (CC_6\|v_0\|^{r-2})^{1/2\beta\sum_{i=0}^m \chi^{-i}}\beta^{1/\beta\sum_{i=0}^m \chi^{-i}}\chi^{1/\beta\sum_{i=0}^m i\chi^{-i}}\|v_0\|_{2^*}
$$

Since $\chi > 1$, $\sum_{i=0}^\infty \chi^{-i}$ and $\sum_{i=0}^\infty i\chi^{-i}$ are convergent, we can pass to the limit as $m \to \infty$ to conclude that

$$
\|v_0\|_\infty \le K\|v_0\|^{\frac{2^*-2}{2^*-r}}.
$$

□

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