THE GENERALIZED BBM-BURGERS EQUATIONS: CONVERGENCE RESULTS FOR CONSERVATION LAW WITH DISCONTINUOUS FLUX FUNCTION

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ABSTRACT. We consider conservation laws with discontinuous flux, which are regularized with generalized BBM-Burgers equations. We study the convergence of one sequence of solutions of these equations for one solution of the associated conservation law. In the multidimensional case we prove one convergence result using the H-measures and in the one-dimensional case we use the method of compensated compactness.

1. Introduction

We consider the partial differential equations of the form

(1)
$$\partial_t u(t,x)+div_x f_\rho(t,x,u(t,x)) = \delta\sum_{j=1}^{d}\partial^3_{x_j x_j t}u(t,x)+\sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n\partial^{2n}_{x_j}u(t,x),$$

$(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, with initial data

(2)
$$u(0,x)=u_0(x), \qquad x\in\mathbb{R}^d,$$

and f_ρ satisfying certain assumptions to be listed below. We study the convergence of smooth solutions $u=u_\gamma(t,x)$ with $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ of (1) and (2), when $\gamma_1\to 0$, $\gamma_n=\gamma_n(\gamma_1)$ and $\delta=\delta(\gamma_1)$ for the associated conservation law

(3)
$$u_t+div_x f(t,x,u(t,x))=0, \quad t\geq 0, \quad x\in\mathbb{R}^d.$$

Here f is a Caratheodory function (i.e, it is continuous with respect to u and measurable with respect to t and x). Moreover,

$$\sup_{u\in\mathbb{R}}||f_\rho(.,.,u)-f(.,.,u)||_{L^p_{loc}(\mathbb{R}_+\times\mathbb{R}^d)}\to 0$$

when $\rho\to 0$, $p\geq 2$ and the flux $f\in C(\mathbb{R};BV(\mathbb{R}_+\times\mathbb{R}^d))$. Following the works by Holden, Karlsen, and Mitrovic [9], Panov [8], and ours works [13] and [14], we prove convergence results of these solutions to one solution of the problem (3), with initial data

(4)
$$u(0,x)=u_0(x), \quad x\in\mathbb{R}^d.$$

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Holden, Karlsen, and Mitrovic (see [9]) studied the convergence of solutions of the problem

$$\partial_t u + div_x f_\varrho(t,x,u(t,x)) = \epsilon\ div_x\ b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u \tag{5}$$

as $\epsilon \to 0$ and $\delta = \delta(\epsilon)$, $\rho = \rho(\epsilon) \to 0$.

In this paper, we use techniques similar to some techniques used in [9] to study the convergence of approximate solutions of generalized BBM-Burgers equations. We have studied this type of problem before, where we consider the flow being a smooth function. This paper generalizes our previous study, in our previous works (see [12]-[14]) we consider the flux $f = f(u(t,x))$ sufficiently smooth and the flux are independent of the spatial and temporal positions. Now, the flux is not smooth. Here, $f(t,x,u(t,x))$ is the Caratheodory flux vector. To perform this study we rely on the theory of approximations by H-measures (see [8] and [9]), which helps to weaken the hypotheses on the flux function f if we compare this study with what we did earlier (see [12]-[14],[15], [18]), where $f = f(u(t,x))$ was smooth and satisfying certain growth conditions at infinity. We remark that demand on controlling the flux at infinity is rather usual in the case of conservation laws with vanishing diffusion, dispersion, viscosity, etc. (see [11], [7], [15], [17]).

We wish to note that the main difference between the equations (5) and (1) is that the first have the term dispersive $u_{x_j x_j x_j}$ and have not the term $u_{x_j x_j t}$. The model under study is motived by physical considerations from fluid dynamics. The equation of type (1) is related to the well-known BBM equations which were advocated by Benjamin-Bona-Mahony [3] as a refinement of the KdV equation (see [3],[10], and [1]). The KdV equation was originaly derived for water waves and it is similary justifiable as a model for long waves in many other physical systems. It has been used to account adequately for obsevable phenomena such as the interaction of solitary waves and dissipationless, ondular shocks. The BBM equation is useful in that it describes approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems. Since the viscous term $\gamma_1 u_{xx}$ and the dissipative term $\gamma_2 u_{xxxx}$ (case $N = 2$) are of physical backgrounds [3] and [4] and, as pointed out in [4] and [15], the convergences of the solution sequences $\{u(x,t;\delta,\gamma_1,\gamma_2)\}$ as $\delta \to 0$, $\gamma_1 \to 0$, and $\gamma_2 \to 0$ correspond to some physical processes, such as vanishing viscosity, etc.

In [5] Constantin and Lannes made comparisons between the equations of BBM and KdV equations to indicate their physical significance and they study equations containing nonlinear effects (uu_x term) and dispersive effects (u_{xxx} and $uxxt$ terms) in the Camassa-Holm and Degasperis-Procesi equations. These equations also generalize BBM and KdV equations, and are known by their relevance to the modelling of wave breaking.

The remainder of this paper is divided into three sections. After this Introduction, which constitutes Section 1, we consider in Section 2 the convergence results from multidimensional case, and in Section 3 we consider the one-dimensional case.

2. The multidimensional case

We consider the problem (3)-(4), for $u = u(x,t)$, where $u_0 \in L^2(\mathbb{R}^d)$. For the flux function $f = (f_1, f_2, ..., f_d)$ we need the following assumption:
Ha) For the flux $f = f(t,x,u)$, $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ we assume that $f \in C(\mathbb{R}; BV(\mathbb{R}_+ \times \mathbb{R}^d))$ and for all $l \in \mathbb{R}_+$ we have $\max_{u\in[-l,l]} f(t,x,u) \in L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$, $p > 2$.
Hb) There exists a sequence $f_\rho = (f_{1\rho}, f_{2\rho}, ..., f_{d\rho})$, $\rho \in (0,1)$, such that $f_\rho = f_\rho(t,x,u) \in C^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R})$, satisfying for some $p \geq 2$ and every $l \in \mathbb{R}_+$:

$$\lim_{\rho\to 0} \max_{z\in[-l,l]} ||f_\rho(.,.,z) - f(.,.,z)||_{L^p(\mathbb{R}_+ \times \mathbb{R}^d)} = 0, \tag{6}$$

$$\sum_{i=1}^{d} |\partial_{x_i} f_{i\rho}(t,x,u)| \leq \frac{\mu_1(t,x)}{1+|u|^{1+\alpha}}, \quad \alpha > 0 \tag{7}$$

$$\rho^3 \sum_{i=1}^{d} |\partial_{x_i} f_{i\rho}(t,x,u)|^2 \leq \mu_2(t,x), \tag{8}$$

$$\sum_{i=1}^{d} |\partial_u f_{i\rho}(t,x,u)| \leq \frac{C}{\beta(\rho)}, \tag{9}$$

$$\sum_{i=1}^{d} |\partial^2_{x_i u} f_{i\rho}(t,x,u)| \leq \frac{\mu_3(t,x)}{1+|u|^{1+\alpha}}, \quad \alpha > 0, \tag{10}$$

where $\mu_i \in M(\mathbb{R}_+ \times \mathbb{R}^d)$, $i = 1,2,3$, are bounded measures and β is a positive function such that $\beta(\rho) = C\rho \to 0$ when $\rho \to 0$. We assume that the flux f is genuinely nonlinear, i.e., for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d - \{0\}$, the mapping

$$\mathbb{R} \ni \lambda \mapsto \sum_{j=1}^{d} f_j(t,x,\lambda)\frac{\xi_j}{|\xi|} \tag{11}$$

is non-constant on every non-degenerate interval of the real line.

In order to prove our convergence result, we shall determine a priori inequalities for solutions of problem (1)-(2) when $\rho > 0$ is fixed and f_ρ satisfies the conditions (Hb). Following Schonbek [15], we assume enough regularity on the solutions so that all formal computations can be made rigorously. It suffices to suppose, for example, $u^\gamma \in L^\infty([0,T]; H^N(\mathbb{R}^d))$. Also assume hereafter that, for some constant $C_0 > 0$ independent from δ, γ_1,..., γ_N, that the initial function u_0 satisfies

$$\|u_{0,\gamma}\|_{H^N(\mathbb{R}^d)} \leq C_0, \qquad ||u_{0,\gamma} - u_0||_{L^2(\mathbb{R}^d)} \to 0, \tag{12}$$

and

$$\|u_{0,\gamma} - u_0\|_{H^N(\mathbb{R}^d)} \leq C_0.$$

Theorem 1. *(A priori estimates)*
Let ρ fixed, suppose that the flux $f_\rho = f_\rho(t,x,u)$ satisfies the conditions (Hb). Under these conditions, the sequence of solutions $(u_\gamma)_{\gamma>0}$ of (1)-(2), satisfies the following inequalities:

(13)
$$\int_{\mathbb{R}^d} |u(T,x)|^2 dx + \delta \sum_{j=1}^{d} \int_{\mathbb{R}^d} |\partial_{x_j} u(T,x)|^2 dx + 2\sum_{j=1}^{d}\sum_{n=1}^{N} \gamma_n \int_0^T \int_{\mathbb{R}^d} |\partial^n_{x_j} u(t,x)|^2 dxdt$$
$$\leq \int_{\mathbb{R}^d} |u_0(x)|^2 dx + \delta \sum_{j=1}^{d} \int_{\mathbb{R}^d} |\partial_{x_j} u_0(x)|^2 dx + C_1.$$

$$\frac{\gamma_i}{2}\int_{\mathbb{R}^d} |\partial^i_{x_k} u(T,x)|^2\,dx + \frac{\delta\gamma_i}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d} |\partial^i_{x_k}(\partial_{x_j} u)(T,x)|^2\,dx \tag{14}$$
$$+\frac{\gamma_i^2}{3}\int_0^T\int_{\mathbb{R}^d} |\partial^{2i}_{x_k} u(t,x)|^2 dxdt + \sum_{n^*=1}^{N}\gamma_n\gamma_i\sum_{j=1}^{d}\int_0^T\int_{\mathbb{R}^d} |\partial^i_{x_k}(\partial^n_{x_j} u)(t,x)|^2\,dxdt$$
$$\leq \frac{\gamma_i}{2}\int_{\mathbb{R}^d} |\partial^i_{x_k} u_0(x)|^2\,dx + \frac{\delta\gamma_i}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d} |\partial^i_{x_k}(\partial_{x_j} u_0)(x)|^2\,dx$$
$$+C_2\left[\frac{1}{\rho^3} + \frac{1}{\gamma_1[\beta(\rho)]^2} + \frac{\delta}{\gamma_1[\beta(\rho)]^2}\right],$$

where n^* denote that $n \neq i$ if $j = k$.

$$\frac{\gamma_1}{3}\int_0^T\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2\,dxdt + \gamma_1\delta\sum_{j=1}^{d}\int_0^T\int_{\mathbb{R}^d} |\partial^2_{x_j t} u(t,x)|^2\,dxdt \tag{15}$$
$$+\sum_{j=1}^{d}\sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}^d} |\partial^n_{x_j} u(T,x)|^2\,dx \leq \sum_{j=1}^{d}\sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}^d} |\partial^n_{x_j} u_0(x)|^2\,dx$$
$$+C_3\left[\frac{\gamma_1}{\rho^3} + \frac{1}{[\beta(\rho)]^2} + \frac{\delta}{[\beta(\rho)]^2}\right].$$

Proof. We write $f = f_\rho = f_\rho(t,x,u)$, with the purpose to simplify the reading. Remember that

$$div_x f(t,x,u) = \sum_{j=1}^{d}[\partial_{x_j} f_j(t,x,u) + \partial_u f_j(t,x,u)\partial_{x_j} u].$$

We define

$$q_j(t,x,u) = \int_{-\infty}^{u} 2v\partial_v f_j(t,x,v)dv, \qquad j = 1,2,...,d.$$

We multiply the equation (1) by $2u$ and integrate in $\mathbb{R}^d$ and in $[0,T]$:

$$\int_{\mathbb{R}^d} |u(T,x)|^2 dx + \delta \sum_{j=1}^{d} \int_{\mathbb{R}^d} |\partial_{x_j} u(T,x)|^2 dx + 2\sum_{j=1}^{d}\sum_{n=1}^{N} \gamma_n \int_0^T \int_{\mathbb{R}^d} |\partial^n_{x_j} u(t,x)|^2 dxdt$$

$$= \int_{\mathbb{R}^d} |u_0(x)|^2 dx + \delta \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial_{x_j} u_0(x)|^2 dx - 2 \int_0^T \int_{\mathbb{R}^d} u(t,x) div_x f(t,x,u(t,x)) dxdt$$

From the formula $q_j(t,x,u)$ we have

$$\partial_{x_j}(q_j(t,x,u(t,x)) = \int_{-\infty}^{u(t,x)} 2v\partial^2_{vx_j} f_j(t,x,v)\, dv + 2u(t,x)\partial_u f_j(t,x,u(t,x))\partial_{x_j} u(t,x).$$

Then

$$-2\int_0^T \int_{\mathbb{R}^d} u(t,x) div_x f(t,x,u(t,x)) dxdt = -2\sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} u(t,x)\partial_{x_j} f_j(t,x,u(t,x)) dxdt + 2\sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \int_{-\infty}^{u(t,x)} v\partial^2_{vx_j} f(t,x,v) dvdxdt$$

Therefore

$$\int_{\mathbb{R}^d} |u(T,x)|^2 dx + \delta \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial_{x_j} u(T,x)|^2 dx + 2\sum_{j=1}^d \sum_{n=1}^N \gamma_n \int_0^T \int_{\mathbb{R}^d} |\partial^n_{x_j} u(t,x)|^2 dxdt$$

$$= \int_{\mathbb{R}^d} |u_0(x)|^2 dx + \delta \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial_{x_j} u_0|^2 dx$$

$$- \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \left[2u(t,x)\partial_{x_j} f_j(t,x,u(t,x)) - \int_{-\infty}^{u(t,x)} 2v\partial^2_{x_j v} f_j(t,x,v) dv \right] dxdt$$

$$= \int_{\mathbb{R}^d} |u_0(x)|^2 dx + \delta \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial_{x_j} u_0(x)|^2 dx$$

$$-2\sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \int_{-\infty}^{u(t,x)} \partial_{x_j} f_j(t,x,v) dvdxdt$$

where the latter equality formula is justified by the following partial integration

$$\int_0^T \int_{\mathbb{R}^d} \int_{-\infty}^{u(t,x)} v\partial^2_{x_j v} f_j(t,x,v) dvdxdt = \int_0^T \int_{\mathbb{R}^d} \left[u(t,x)\partial_{x_j} f_j(t,x,u(t,x)) - \int_{-\infty}^{u(t,x)} \partial_{x_j} f_j(t,x,v) dv \right] dxdt.$$

The last integral can be estimate using (7):

$$-2\sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \int_{-\infty}^{u(t,x)} |\partial_{x_j} f_j(t,x,v)| dvdxdt \le \overline{C} \int_0^T \int_{\mathbb{R}^d} \mu_1(t,x) dxdt \le C_1$$

where $\overline{C} = \int_{\mathbb{R}} \frac{1}{1+|v|^{1+\alpha}} dv < \infty$. This give (13).

We multiply the equation (1) by $(-1)^i\gamma_i\partial^{2i}_{x_k}u$, for $k\in\{1,2,...d\}$, $i\in\{1,2,...,N-1\}$, and integrate in $\mathbb{R}^d$ and in $[0,T]$:

$$\frac{\gamma_i}{2}\int_{\mathbb{R}^d}|\partial^i_{x_k}u(T,x)|^2\,dx+\frac{\delta\gamma_i}{2}\sum_{j=1}^d\int_{\mathbb{R}^d}|\partial^i_{x_k}(\partial_{x_j})u(T,x)|^2\,dx$$

$$+\sum_{n=1}^N\gamma_n\gamma_i\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|\partial^i_{x_k}(\partial^n_{x_j}u)(t,x)|^2\,dxdt$$

$$=\frac{\gamma_i}{2}\int_{\mathbb{R}^d}|\partial^i_{x_k}u_0(x)|^2\,dx+\frac{\delta\gamma_i}{2}\sum_{j=1}^d\int_{\mathbb{R}^d}|\partial^i_{x_k}(\partial_{x_j}u_0)(x)|^2\,dx$$

$$+(-1)^{i+1}\gamma_i\int_0^T\int_{\mathbb{R}^d}div_x f(t,x,u(t,x))\partial^{2i}_{x_k}u(t,x)\,dxdt.$$

The last integral can be estimate using Young's inequality we have:

$$\gamma_i\int_0^T\int_{\mathbb{R}^d}|div_x f(t,x,u(t,x))\partial^{2i}_{x_k}u(t,x)|\,dxdt$$

$$=\gamma_i\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|[\partial_{x_j}f_j(t,x,u(t,x))+\partial_u f_j(t,x,u(t,x))\partial_{x_j}u(t,x)]\partial^{2i}_{x_k}u(t,x)|dxdt$$

$$\le\frac{2\gamma_i^2}{3}\int_0^T\int_{\mathbb{R}^d}\partial^{2i}_{x_k}u(t,x)|^2\,dxdt+\sum_{j=1}^d\frac{3d}{4}\int_0^T\int_{\mathbb{R}^d}|\partial_{x_j}f_j(t,x,u(t,x))|^2\,dxdt$$

$$+\sum_{j=1}^d\frac{3d}{4}\int_0^T\int_{\mathbb{R}^d}|\partial_u f_j(t,x,u(t,x))\partial_{x_j}u(t,x)|^2\,dxdt$$

From (8), (9), (13), and (12) give (14).

We multiply the equation (1) by $\gamma_1\partial_t u(t,x)$ and integrate in $\mathbb{R}^d$ and in $[0,T]$:

$$\gamma_1\int_0^T\int_{\mathbb{R}^d}|\partial_t u(t,x)|^2\,dxdt+\gamma_1\delta\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|\partial^2_{x_jt}u(t,x)|^2\,dxdt$$

$$+\sum_{j=1}^d\sum_{n=1}^N\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}^d}|\partial^n_{x_j}u(T,x)|^2\,dx$$

$$=\sum_{j=1}^d\sum_{n=1}^N\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}^d}|\partial^n_{x_j}u_0(x)|^2\,dx-\gamma_1\int_0^T\int_{\mathbb{R}^d}\partial_t u(t,x)div_x f(t,x,u(t,x))\,dxdt.$$

The last integral can be estimated using the Young's Inequality (8), (9), (13) for $n=1$, and (12):

$$-\gamma_1\int_0^T\int_{\mathbb{R}^d}\partial_t u(t,x)div_x f(t,x,u(t,x))\,dxdt\le\frac{2}{3}\gamma_1\int_0^T\int_{\mathbb{R}^d}|\partial_t u(t,x)|^2\,dxdt$$

$$+3d\gamma_1\sum_{i=1}^d\int_0^T\int_{\mathbb{R}^d}|\partial_{x_j}f_j(t,x,u(t,x))|^2\,dxdt$$

$$+3d\gamma_1 \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} |\partial_u f_j(t,x,u(t,x)|^2 |\partial_{x_j} u(t,x)|^2 \, dxdt$$

$$\leq \frac{2\gamma_1}{3} \int_0^T \int_{\mathbb{R}^d} |\partial_t u(t,x)|^2 \, dxdt + \frac{3d\gamma_1}{\rho^3} + \frac{3dC}{\beta(\rho)^2}[C + C_1 + C\delta]$$

This give (15). The proof of theorem is completed. □

Panov, [8], stated the next theorem. Let θ the Heaviside function.

Theorem 2. *Assume that the flux vector $f = f(t,x,u)$ is genuinely non-linear in the sense of (11). Then each sequence $(v_\gamma)_{\gamma>0} \subset L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ such that for every $c \in \mathbb{R}$, the distribution*

$$\partial_t[\theta(v_\gamma - c)](v_\gamma - c) + div_x[\theta(v_\gamma - c)(f(t,x,v_\gamma) - f(t,x,c))]$$

is precompact in H^{-1}_{loc}, contains a subsequence convergent in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$.

Using the theorem above, we prove the following theorem.

Theorem 3. *Assume that the flux vector $f = f(t,x,u)$ is genuinely non-linear in the sense of (11) and that it satisfies (H). Furthermore, assume that $\rho = C\gamma_1^{\frac{1}{3}}$, $\delta = O(\gamma_1^4)$, and for $n > 1$, $\gamma_n = O(\gamma_{n-1}\gamma_1^4)$, and that $u_{0,\gamma}$ satisfies (12). Then there exists a subsequence of solutions $(u_\gamma)_{\gamma>0}$ of (1)-(2) that converges to a weak solutions of problem (3)-(4).*

Proof. We will use the Theorem 2. Since the sequence of solutions of problem (1)-(2) is not uniformly bounded, we cannot directly apply the conditions of Theorem 2. To simplify the notation, we write u instead of u_γ, when they do not cause confusion. Let $S(u), u \in \mathbb{R}$ a arbitrary function C^2. We multiply the reguralized equation (1) by $S'(u_\gamma)$. As usual, we define

$$q(t,x,u) = \int_0^u S'(v)\partial_v f_\rho dv, \qquad q = (q_1, q_2, ..., q_d).$$

Then

$$\partial_{x_j}[q_j(t,x,u(t,x))] = \int_0^{u(t,x)} S'(v)\partial^2_{vx_j} f_{j\rho}(t,x,v) \, dv$$
$$+ S'(u(t,x))\partial_u f_{j\rho}(t,x,u(t,x))\partial_{x_j} u(t,x).$$

We obtain

$$\partial_t S(u) + \sum_{j=1}^{d} S'(u)\partial_{x_j} f_{j\rho}(t,x,u(t,x)) + \sum_{j=1}^{d} \partial_{x_j}[q_j(t,x,u(t,x))] \tag{16}$$

$$-\sum_{j=1}^{d} \int_0^{u(t,x)} S'(v)\partial^2_{vx_j} f_{j\rho}(t,x,v) \, dv$$

$$= \delta \sum_{j=1}^{d} (S'(u)u_{x_j t})_{x_j} - \delta \sum_{j=1}^{d} S''(u)\partial_{x_j} u(t,x)\partial^2_{x_j t} u(t,x)$$

$$+\sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n\left[(S'(u)\partial_{x_j}^{2n-1}u(t,x))_{x_j}-S''(u)\partial_{x_j}u(t,x)\partial_{x_j}^{2n-1}u(t,x)\right].$$

We can apply this formula to different choices of $S(u)$. In order to apply the theorem 2, we consider a truncated sequence $(T_l(u_\gamma)_{\gamma>0})$, where the function truncated T_l is defined for all $l\in\mathbb{N}$ fixed, such as:

$$T_l(u)=\begin{cases}-l, & if \quad u\le -l\\ u, & if \quad -l\le u\le l\\ l, & if \quad u\ge l.\end{cases} \tag{17}$$

Our goal is to prove that the sequence $(T_l(u_\gamma))_{\gamma>0}$ is precompact for every l fixed. We denote by u_l the limit (in L^1_{loc}) of a sequence $(T_l(u_\gamma))_{\gamma>0}$, give rise to a sequence $(u_l)_{l>1}$ and prove that this sequence converges to a weak solution of (3)-(4), where $u_0\in L^2(\mathbb{R}^d)$.

To begin, we replace T_l by a regularization C^2, $T_{l,\sigma}:\mathbb{R}\to\mathbb{R}$. We define $T_{l,\sigma}:\mathbb{R}\to\mathbb{R}$ by $T_{l,\sigma}(0)=0$ and

$$T'_{l,\sigma}(u)=\begin{cases}1, & if \quad |u|<l\\ \frac{l-|u|}{\sigma}, & if \quad l\le|u|\le l+\sigma\\ 0, & if \quad |u|>l+\sigma.\end{cases}$$

Note that when $\sigma\to 0$ we have $T_{l,\sigma}(u)\to T_l(u)$ in L^p_{loc} for all $p<\infty$, where T_l is defined by (17). We assume that

$$\sigma=[\beta(\rho)]^2 \tag{18}$$

Now, we consider $U_\rho(z)$ satisfying $U_\rho(0)=0$ and

$$U'_\rho(z)=\begin{cases}0, & if \quad z<0\\ \frac{z}{\rho}, & if \quad 0<z<\rho\\ 1, & if \quad z>\rho.\end{cases}$$

Clearly, U_ρ is convex and we have $U'_\rho(z)\to\theta(z)$ in $L^p_{loc}(\mathbb{R})$ when $\rho\to 0$, for every $p<\infty$, where θ is the Heaviside function. Now we use $S(u)=U_\rho(T_{l,\sigma}(u)-C)$ in (16) we obtain

$$\begin{aligned}
&\partial_t[U_\rho(T_{l,\sigma}(u)-C)]+div_x\int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_{l,\sigma}(v)f_\rho(t,x,v)\,dv\\
&\quad=\int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_{l,\sigma}(v)div_x(\partial_v f_\rho(t,x,v))\,dv\\
&\qquad-U'_\rho(T_{l,\sigma}(u)-C)T'_{l,\sigma}(u)\sum_{j=1}^{d}\partial_{x_j}f_{j\rho}(t,x,u)\\
&\qquad+\sum_{j=1}^{d}\delta\,[U'_\rho(T_{l,\sigma}(u)-C)T'_{l,\sigma}(u)\partial^2_{jt}u]_{x_j}\\
&\qquad-\sum_{j=1}^{d}\delta\,U''_\rho(T_{l,\sigma}(u)-C)(T'_{l,\sigma}(u))^2\partial_{x_j}u\partial_{x_{jt}}u\\
&\qquad+\sum_{j=1}^{d}U'_\rho(T_{l,\sigma}(u)-C)T''_{l,\sigma}(u)\partial_{x_j}u\partial_{x_{jt}}u
\end{aligned}$$

$$+\sum_{j=1}^{d}\sum_{n=1}^{N}\gamma_n(-1)^{n+1}[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\partial_{x_j}^{2n-1}u]_{x_j}$$

$$+\sum_{j=1}^{d}\sum_{n=1}^{N}\gamma_n(-1)^{n}U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_j}^{2n-1}u$$

$$+\sum_{j=1}^{d}\sum_{n=1}^{N}\gamma_n(-1)^{n}U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)\partial_{x_j}u\partial_{x_j}^{2n-1}u.$$

Rewrite the above expression as follows

$$\partial_t[\theta(T_l(u)-C)(T_l(u)-C)]+div_x[\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))]$$

$$=\partial_t[\theta(T_l(u)-C)(T_l(u)-C)-U_\rho(T_{l,\sigma}(u)-C)]$$

$$+\ div_x[\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))$$

$$-\int_0^u U_\rho^{'}(T_{l,\sigma}(v)-C)T_{l,\sigma}^{'}\partial_v f_\rho(t,x,v)\,dv]$$

$$+\int_0^u U_\rho^{'}(T_{l,\sigma}(v)-C)T_{l,\sigma}^{'}(v)div_x\partial_v f_\rho(t,x,v)\,dv$$

$$-U_\rho^{'}(T_{l,\sigma}(v)-C)T_{l,\sigma}^{'}(v)\sum_{j=1}^{d}\partial_j f_{j\rho}(t,x,v)$$

$$+\sum_{j=1}^{d}\delta[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)u_{x_jt}]_{x_j}$$

$$-\delta\sum_{j=1}^{d}U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_jt}u$$

$$+\delta\sum_{j=1}^{d}U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)\partial_{x_j}u\partial_{x_jt}u$$

$$+\sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\partial_{x_j}^{2n-1}u]_{x_j}$$

$$-\sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_j}^{2n-1}u$$

$$-\sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)\partial_{x_j}u\partial_{x_j}^{2n-1}u$$

or
(19)

$$\partial_t[\theta(T_l(u)-C)(T_l(u)-C)]+div_x[\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))] = \sum_{i=1}^{8}\Gamma_i$$

where

$$\Gamma_1 = \partial_t[\theta(T_l(u)-C)(T_l(u)-C) - U_\rho(T_{l,\sigma}(u)-C)]$$

$$\Gamma_2 = div_x[\theta(T_l(u)-C)(f(t,x,T_l(u)) - f(t,x,C)) - \int_0^u U_\rho^{'}(T_{l,\sigma}(v)-C)T_{l,\sigma}^{'}\partial_v f_\rho(t,x,v)dv]$$

$$\Gamma_3 = \textstyle\int_0^u U_\rho^{'}(T_{l,\sigma}(v)-C)T_{l,\sigma}^{'}(v)div_x\partial_v f_\rho(t,x,v)dv - U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\sum_{j=1}^{d} f_{j\rho}(t,x,u)$$

$$\Gamma_4 = \delta\sum_{j=1}^{d}[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\partial_{x_j t}u]_{x_j}$$

$$\Gamma_5 = -\delta\sum_{j=1}^{d}U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_j t}u - \delta\sum_{j=1}^{d}U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)\partial_{x_j}u\partial_{x_j t}u$$

$$\Gamma_6 = \sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\partial_{x_j}^{2n-1}u]_{x_j}$$

$$\Gamma_7 = \sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n}\gamma_n U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_j}^{2n-1}u$$

$$\Gamma_8 = \sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n}\gamma_n U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)\partial_{x_j}u\partial_{x_j}^{2n-1}u.$$

From now on, we must prove that the sequence $(T_l(u_\gamma))_{\gamma>0}$ satisfies the conditions of the theorem 2. Thus, according to this theorem, we show that the left side of (19) is precompact in $H_{loc}^{-1}(\mathbb{R}_+\times\mathbb{R}^d)$. To make it, we will use the Lemma of Murat. More precisely, we prove that:
i) when the left side of (19) is written as $divQ_\gamma$, we have $Q_\gamma \in L_{loc}^p(\mathbb{R}_+\times\mathbb{R}^d)$, for $p\geq 2$, and
ii) the right side of (19) is of the form $M_{loc,B} + H_{loc,c}^{-1}$ where $M_{loc,B}$ denotes

the set of families which are locally bounded in measures spaces and $H^{-1}_{loc,c}$ is a set of families which are compact in H^{-1}_{loc}.
First, as $T_l(u_\gamma)$ is uniformly bounded by l, we see that (i) is satisfied. To prove (ii), we must consider each term on the right side of (19).

Let us prove that $\Gamma_1 \in H^{-1}_{loc,c}$. We have

$$\theta(T_l(u)-C)(T_l(u)-C) - U_\rho(T_{l,\sigma}(u)-C) = \theta(T_l(u)-C)(T_l(u)-C)$$
$$-\theta(T_{l,\sigma}(u)-C)(T_{l,\sigma}(u)-C) + \theta(T_{l,\sigma}(u)-C)(T_{l,\sigma}(u)-C) - U_\rho(T_{l,\sigma}(u)-C).$$

By definition the Heaviside function, the function $\theta(z-C)(z-C)$ is Lipschitz continuous in z with Lipschitz constant 1 and we write $|U_\rho(z)-\theta(z)z| = |\int_0^z U'_\rho(w)\,dw - \theta(z)z|$ and using the definition of $U'_\rho(w)$ and the Heaviside function $\theta(z)$ we obtain $|U_\rho(z)-\theta(z)z| \le \frac{3}{2}\rho$. So

$$|\theta(T_l(u)-C)(T_l(u)-C) - U_\rho(T_{l,\sigma}(u)-C)| \le |T_{l,\sigma}(u)-T_l(u)| + \frac{3}{2}\rho.$$

Then $\rho = C\gamma_1^{\frac{1}{3}}$, $\sigma = [\beta(\rho)]^2 = [C\rho]^2$, and for $\sigma \to 0$ then $T_{l,\sigma}(u) \to T_l(u)$ in L^p_{loc} for all $p<\infty$, it follows that when $\gamma_1 \to 0$ then $\theta(T_l(u)-C)(T_l(u)-C) - U_\rho(T_{l,\sigma}(u)-C) \to 0$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ for every $p<\infty$. So, (for $p=2$) we see that $\Gamma_1 \in H^{-1}_{loc,c}$. Let us show that

$\Gamma_2 \in H^{-1}_{loc,c} + M_{loc,B}$.

$$\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C)) - \int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_{l,\sigma}(v)\partial_v f_\rho(t,x,v)\,dv$$
$$= \theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))$$
$$-\ \theta(T_{l,\sigma}(u)-C)(f(t,x,T_{l,\sigma}(u))-f(t,x,C))$$
$$+\ \theta(T_{l,\sigma}(u)-C)(f(t,x,T_{l,\sigma}(u))-f(t,x,C))$$
$$-\int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_l(v)\partial_v f_\rho(t,x,v)\,dv$$
$$-\int_0^u U'_\rho(T_{l,\sigma}(v)-C)[T'_{l,\sigma}(v)-T'_l(v)]\partial_v f_\rho(t,x,v)\,dv.$$

Since $T_l(u)=u$ if $|u|\le l$ and $T'_l(u)=0$ se $|u|\ge l$ that comes

$$\int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_l(v)\partial_v f_\rho(t,x,v)\,dv$$
$$= \int_0^u U'_\rho(T_{l,\sigma}(v)-C)T'_l(v)\partial_v f_\rho(t,x,T_l(v))\,dv$$

and so

$$\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C)) - \int^u U'_\rho(T_{l,\sigma}(v)-C)T'_{l,\sigma}(v)\partial_v f_\rho(t,x,v)\,dv$$
$$= \theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))$$

$$- \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$

$$+ \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$

$$- \int_0^u U_\rho^{'}(T_{l,\sigma}(v) - C)T_l^{'}(v)\partial_v f_\rho(t,x,T_l(v))\,dv$$

$$- \int_0^u U_\rho^{'}(T_{l,\sigma}(v) - C)[T_{l,\sigma}^{'}(v) - T_l^{'}(v)]\partial_v f_\rho(t,x,v)\,dv$$

$$= \theta(T_l(u) - C)(f(t,x,T_l(u)) - f(t,x,C))$$

$$- \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$

$$+ \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$

$$- \int_0^u \theta(T_{l,\sigma}(v) - C)\partial_v[f_\rho(t,x,T_l(v))]\,dv$$

$$- \int_0^u [U_\rho^{'}(T_{l,\sigma}(v) - C) - \theta(T_{l,\sigma}(v) - C)]T_l^{'}(v)\partial_v f_\rho(t,x,T_l(v))\,dv$$

$$- \int^u U_\rho^{'}(T_{l,\sigma}(v) - C)[T_{l,\sigma}^{'}(v) - T_l^{'}(v)]\partial_v f_\rho(t,x,v)\,dv$$

$$= \Gamma_2^1 + \Gamma_2^2 + \Gamma_2^3, \tag{20}$$

with

$$\Gamma_2^1 = \theta(T_l(u) - C)(f(t,x,T_l(u)) - f(t,x,C))$$
$$- \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$

$$\Gamma_2^2 = \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))$$
$$- \int_0^u \theta(T_{l,\sigma}(v) - C)\partial_v[f_\rho(t,x,T_l(v))]\,dv$$

$$\Gamma_2^3 = - \int_0^u [U_\rho^{'}(T_{l,\sigma}(v) - C) - \theta(T_{l,\sigma}(v) - C)]T_l^{'}(v)\partial_v f_\rho(t,x,T_l(v))\,dv$$
$$- \int^u U_\rho^{'}(T_{l,\sigma}(v) - C)[T_{l,\sigma}^{'}(v) - T_l^{'}(v)]\partial_v f_\rho(t,x,v)\,dv.$$

We estimate each term on the right side of (20). As T_l is a continuous function and $T_l(u) \in [-l,l]$, the function $f(t,x,T_l(u))$ is uniformly continuous on $u \in \mathbb{R}$. Therefore, we have pointwise in $\mathbb{R}_+ \times \mathbb{R}^d$

$$|\Gamma_2^1| = |\theta(T_l(u) - C)(f(t,x,T_l(u)) - f(t,x,C))$$
$$-\theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C))| \to 0$$

when $\sigma \to 0$. As $\max\limits_{u\in[-1,1]} f(t,x,u) \in L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$, $p \geq 2$ and $T_{l,\sigma}(u) \to T_l(u)$ in L^p_{loc} when $\sigma \to 0$ then $|\Gamma^1_2|^p \in L^1_{loc}$. By the theorem of Lebesgue dominated convergence gives

$$\Gamma^1_2 \to 0$$

in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ when $\sigma \to 0$. From this we conclude that

$$div_x \Gamma^1_2 \in H^{-1}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d). \tag{21}$$

We want to estimate

$$\Gamma^2_2 = \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C)) - \int^u \theta(T_{l,\sigma}(v) - C) D_v[f_\rho(t,x,T_l(v))]dv.$$

Let us analyse separately the cases $|C| < l$ e $|C| > l$. Consider first the case where $|C| < l$, in this case we have $T_l(C) = C$ and $T_{l,\sigma}(C) = \int_0^C T'_{l,\sigma}(w)\,dw = C$, and so for $0 \leq v \leq C$ we have $T_{l,\sigma}(v) = v$ then $\theta(T_{l,\sigma}(v) - C) = 0$ for $0 \leq v \leq C$. We obtain

$$\begin{aligned}
|\Gamma^2_2| &= |\theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C)) \\
&\qquad - \int_C^u \theta(T_{l,\sigma}(v) - C)\partial_v[f_\rho(t,x,T_l(v))]dv| \\
&= |\theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C)) \\
&\qquad - \theta(T_{l,\sigma}(u) - C)\int_C^u \partial_v[f_\rho(t,x,T_l(v))]dv| \\
&= |\theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C) \\
&\qquad - \theta(T_{l,\sigma}(u) - C)(f_\rho(t,x,T_l(u)) - f_\rho(t,x,C))| \\
&\leq |\theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f_\rho(t,x,T_{l,\sigma}(u))| \\
&\qquad + |\theta(T_{l,\sigma}(u) - C)(f_\rho(t,x,T_{l,\sigma}(u)) - f_\rho(t,x,T_l(u)))| \\
&\qquad\qquad + |\theta(T_{l,\sigma}(u) - C)(f_\rho(t,x,T_l(u)) - f(t,x,C))|
\end{aligned}$$

Then

$$\|\Gamma^2_2\|_{L^p_{loc}} = o_\sigma(1) + O(\frac{\sigma}{\beta(\rho)}) + o_\sigma(1) = O(1) + o_\sigma(1)$$

where $o_\sigma(1) \to 0$ when $\sigma \to 0$. $o_\sigma(1) + O(\frac{\sigma}{\beta(\rho)}) + o_\sigma(1)$ appears due to (6) and (9) in (Hb) and remembering that $\sigma = [\beta(\rho)]^2$.

For $C > l$ we have $C \geq l + \sigma$ for a σ small enough and then $\theta(T_{l,\sigma}(u) - C) = 0$ for $u \in \mathbb{R}$ and for $C < -l$ we have $C \leq -l - \sigma$ for a σ small enough and then $\theta(T_{l,\sigma}(u) - C) = 1$ for $u \in \mathbb{R}$, because $T_{l,\sigma}(l + \sigma) = l + \frac{\sigma}{2}$ and $T_{l,\sigma}(-l - \sigma) = -l - \frac{\sigma}{2}$. Therefore for $C > l$ then $\Gamma^2_2 = 0$. Consider the case $C < -l$. We have

$$\begin{aligned}
\Gamma^2_2 &= \theta(T_{l,\sigma}(u) - C)(f(t,x,T_{l,\sigma}(u)) - f(t,x,C)) \\
&\qquad - \int_0^u \theta(T_{l,\sigma}(v) - C)\partial_v[f_\rho(t,x,T_l(v))]dv \\
&= f(t,x,T_{l,\sigma}(u)) - f_\rho(t,x,T_l(u)) + f_\rho(t,x,0) - f(t,x,C),
\end{aligned}$$

implying

$$div_x \Gamma_2^2 \in H^{-1}_{loc,c} + M_{loc,B}. \tag{22}$$

since $f(t,x,T_{l,\sigma}(u)) - f_\rho(t,x,T_l(u)) \to 0$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ for $p \geq 2$ and $f_\rho(t,x,0) - f(t,x,C) \in BV(\mathbb{R}_+ \times \mathbb{R}^d)$. It remains to estimate Γ_2^3. Recalling that $|U'_\rho| \leq 1$ and $|T'_{l,\sigma}| \leq 1$, and using (9), we obtain

$$\begin{aligned}|\Gamma_2^3| = \Big| &-\int_0^u [U'_\rho(T_{l,\sigma}(v) - C) - \theta(T_{l,\sigma}(v) - C)]T'_l(v)\partial_v f_\rho(t,x,T_l(v))dv \\ &- \int_0^u U'_\rho(T_{l,\sigma}(v) - C)[T'_{l,\sigma}(v) - T'_l(v)]\partial_v f_\rho(t,x,v)dv\Big| \\ \leq \frac{C}{\beta(\rho)} &\left[\int_{-l}^{l} |U'_\rho(T_{l,\sigma}(v) - C) - \theta(T_{l,\sigma}(v) - C)|\, dv + \int_{\mathbb{R}} |T'_{l,\sigma}(v) - T'_l(v)|\, dv\right] \\ \leq \frac{C}{\beta(\rho)}&(\rho + \sigma) \\ \leq C(1 &+ \rho)\end{aligned}$$

provided that $\sigma = [\beta(\rho)]^2$ and $\beta(\rho) = C\rho$, from which we conclude that $\Gamma_2^3 \in H^{-1}_{loc,c} + M_{loc,B}$.

The next term to be estimated is Γ_3. According to (H), as $|U'_\rho| \leq 1$ and $|T'_{l,\sigma}| \leq 1$, also using (7) and (10), we have:

$$\begin{aligned}|\Gamma_3| = \Big| &\int_0^u U'_\rho(T_{l,\sigma}(v) - C)T'_{l,\sigma}(v) div_x \partial_v f_\rho(t,x,v)dv \\ &-U'_\rho(T_{l,\sigma}(v) - C)T'_{l,\sigma}(v)\sum_{j=1}^d f_{j\rho}(t,x,u)\Big| \leq \int_{\mathbb{R}} \frac{\mu_3(t,x)}{1+|v|^{1+\alpha}}dv + \mu_1(t,x).\end{aligned}$$

So, $\Gamma_3 \in M_{loc,B}$.

Since $|U'_\rho(T_{l,\sigma}(u) - C)| \leq 1$ and $|T'_{l,\sigma}(u)| \leq 1$, to estimate

$\Gamma_4 = \delta \sum_{j=1}^d [U'_\rho(T_{l,\sigma}(u) - C)T'_{l,\sigma}(u)u_{x_{jt}}]_{x_j}$, just make the following estimate, using (15) and (12):

$$\begin{aligned}&\delta^2 \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} |\partial_{jt} u|^2 dxdt \\ &\leq \frac{\delta}{\gamma_1} C\left(\frac{\gamma_1^2}{2} + \cdots + \frac{\gamma_1\gamma_N}{2}\right) + \frac{\delta}{\gamma_1} C_4 \left(\frac{\gamma_1}{\rho^3} + \frac{1}{[\beta(\rho)]^2} + \frac{\delta}{[\beta(\rho)]^2}\right) = o(\gamma_1),\end{aligned}$$

provided that $\rho = C\gamma_1$, $\beta(\rho) = C\rho$, and $\delta = O(\gamma_1^4)$, so $\Gamma_4 \in H^{-1}_{loc,C}$. For $\Gamma_5 = -\delta \sum_{j=1}^d [U''_\rho(T_{l,\sigma}(u) - C)(T'_{l,\sigma}(u))^2 \partial_{x_j} u \partial_{x_{jt}} u + U'_\rho(T_{l,\sigma}(u) - C)T''_{l,\sigma}(u)\partial_{x_j} u \partial_{x_{jt}} u]$, we have $|U_\rho^{''}(T_{l,\sigma}(u)-C)| \leq \frac{1}{\rho}$ and $|T_{l,\sigma}^{''}(u)| \leq \frac{1}{\sigma}$, we use Young's inequality, (13), (15), and (12):

$$\int_0^T \int_{\mathbb{R}^d} |\Gamma_5|\, dxdt$$

$$= \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} \delta|[U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2 + U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)]\partial_{x_j}u\partial_{x_{jt}}u|dxdt$$

$$\leq \sum_{j=1}^{d} 2\gamma_1 \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}u|^2 dxdt + \sum_{j=1}^{d} \left(\frac{1}{\rho}+\frac{1}{\sigma}\right)^2 \frac{1}{2\gamma_1}\delta^2 \int_0^T \int_{\mathbb{R}^d} |\partial_{x_{jt}}u|^2 dxdt$$

$$\leq \sum_{j=1}^{d} 2\gamma_1 \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}u|^2 dxdt + \sum_{j=1}^{d} \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) \frac{1}{\gamma_1}\delta^2 \int_0^T \int_{\mathbb{R}^d} |\partial_{x_{jt}}u|^2 dxdt$$

$$\leq C$$

As $\delta = O(\gamma_1^4)$ and $\gamma_n = O(\gamma_{n-1}\gamma_1^4)$ we have $\Gamma_5 \in M_{loc,B}$.

For $\Gamma_6 = \sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^{n+1}\gamma_n[U_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{'}(u)\partial_{x_j}^{2n-1}u]_{x_j}$ we use (14) with $i = n-1$ and $k = j$, and (12). As $|U_\rho^{'}(T_{l,\sigma}(u)-C)| \leq 1$ and $|T_{l,\sigma}^{'}(u)| \leq 1$, just make the following estimate:

$$\sum_{j=1}^{d} \gamma_n^2 \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}^{2n-1}u|^2\, dxdt \leq C\gamma_n^2 \left[\gamma_n^{-1}\gamma_{n-1}^{-1}\left(C + \frac{1}{\rho^3} + \frac{1}{\gamma_1[\beta(\rho)]^2}\right)\right]$$

$$\leq C\gamma_n\gamma_{n-1}^{-1}(C + \gamma_1^{-1} + \gamma_1^{-\frac{5}{3}}) = o(\gamma_1),$$

provided that $\gamma_n = O(\gamma_{n-1}\gamma_1^4)$ and $\sigma = [\beta(\rho)]^2 = C\rho^2 = C\gamma_1^{\frac{2}{3}}$ by (18), where $o(\gamma_1) \to 0$ when $\gamma_1 \to 0$. So $\Gamma_6 \in H_{loc,C}^{-1}$.

For $\Gamma_7 = \sum_{j=1}^{d}\sum_{n=1}^{N}(-1)^n\gamma_n U_\rho^{''}(T_{l,\sigma}(u)-C)(T_{l,\sigma}^{'}(u))^2\partial_{x_j}u\partial_{x_j}^{2n-1}u$, remembering that $U_\rho^{''}(T_{l,\sigma}(u)-C) \leq \frac{1}{\rho}$ and $T_{l,\sigma}^{'}(u) \leq 1$, we use (13), (14), and (12):

$$\frac{\gamma_n}{\rho}\sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}u\partial_{x_j}^{2n-1}u|dxdt$$

$$\leq \frac{\gamma_n}{\rho}\left[\sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}u|^2 dxdt\right]^{\frac{1}{2}} \left[\sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}^{2n-1}u|^2 dxdt\right]^{\frac{1}{2}}$$

$$\leq C\frac{\gamma_n^{\frac{1}{2}}\gamma_1^{-\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}}{\rho}\left(C+\frac{1}{\rho^3}+\frac{1}{\gamma_1[\beta(\rho)]^2}\right)^{\frac{1}{2}}$$

$$\leq C\gamma_n^{\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}(C+\gamma_1^{-\frac{4}{3}}+\gamma_1^{-\frac{5}{3}})=o(\gamma_1)$$

if $\gamma_n = O(\gamma_{n-1}\gamma_1^4)$. Moreover, as $U_\rho^{'}(T_{l,\sigma}(u)-C)\leq 1$ and $T_{l,\sigma}^{''}(u)\leq\frac{1}{\sigma}$, follows that for $\Gamma_8=\sum_{j=1}^{d}\gamma_n(-1)^nU_\rho^{'}(T_{l,\sigma}(u)-C)T_{l,\sigma}^{''}(u)u_{x_j}\partial_{x_j}^{2n-1}u$ we have

$$\frac{\gamma_n}{\sigma}\sum_{j=1}^{d}\int_0^T\int_{\mathbb{R}^d}|u_{x_j}\partial_{x_j}^{2n-1}u|dxdt$$

$$\leq\frac{\gamma_n}{\sigma}\left[\sum_{j=1}^{d}\int_0^T\int_{\mathbb{R}^d}|u_{x_j}|^2dxdt\right]^{\frac{1}{2}}\left[\sum_{j=1}^{d}\int_0^T\int_{\mathbb{R}^d}|\partial_{x_j}^{2n-1}u|^2dxdt\right]^{\frac{1}{2}}$$

$$\leq C\frac{\gamma_n^{\frac{1}{2}}\gamma_1^{-\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}}{\sigma}\left(C+\frac{1}{\rho^3}+\frac{1}{\gamma_1[\beta(\rho)]^2}\right)^{\frac{1}{2}}$$

$$\leq C\gamma_n^{\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}(C+\gamma_1^{-\frac{5}{3}}+\gamma_1^{-2})\leq C$$

if $\gamma_n = O(\gamma_{n-1}\gamma_1^4)$. So $\Gamma_7\in H_{loc,c}^{-1}$ and $\Gamma_8\in M_{loc,B}$.

We see that (ii) is satisfied and we can use Murat's lemma to conclude that

$$\partial_t[\theta(T_l(u)-C)(T_l(u)-C)]+div_x[\theta(T_l(u)-C)(f(t,x,T_l(u))-f(t,x,C))]\in H_{loc,c}^{-1}.$$

thus we conclude that the conditions of Theorem 2 are satisfied, and find that for every $l>0$ the sequence $(T_l(u^\gamma))_{\gamma>0}$ is precompact in $L_{loc}^1(\mathbb{R}^+\times\mathbb{R}^d)$.

Let $\Omega\subset\mathbb{R}_+\times\mathbb{R}^d$ open and bounded. As $\overline{\Omega}$ is compact, we can consider that $(T_l(u_\gamma)_{\gamma>0})$ is pre-compact in $L^1(\Omega)$. By (13), $(u_\gamma)_{\gamma>0}$ is bounded in $L^2(\Omega)$. By Lemma 7 of [6], exists u such that for a subsequence

$$u_{\gamma_k}\to u \quad in \quad measure.$$

As $u_{\gamma_k}\to u$ in measure then there is a subsequence such that $u_{\gamma_{k_j}}\to u$ *a.e.* in Ω. Then $u_{\gamma_{k_j}}$ is bounded a.e. in Ω then $u\in L^1(\Omega)$ and $u_{\gamma_{k_j}}\to u$ in $L^1(\Omega)$. So, $(u_\gamma)_{\gamma>0}$ is pre-compact in $L_{loc}^1(\mathbb{R}_+\times\mathbb{R}^d)$. (For more details see [2]). And the conclusion of the theorem follows. □

Remark 4. *The technique we use to prove Theorem 3, in the multidimensional case we can only state about the existence of a weak solution, not for entropy solution.*

3. The One-dimensional Case

We examine the convergence of a sequence $\{u_\gamma\}_{\gamma>0}$ of solutions of

(23)
$$\partial_t u+\partial_x f_\rho(t,x,u(t,x)) = \delta\partial^3_{xxt}u(t,x)+\sum_{n=1}^{N}(-1)^{n+1}\gamma_n\partial_x^{2n}u(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}_+,$$

with initial data

(24)
$$u(0,x) = u_0(x), \qquad x\in\mathbb{R}.$$

As in the multidimensional case, studied in the previous section, we assume that the flux function $f = f(t,x,u)$ is continuously differentiable in the u variable. We take the initial data $u_0 = u(x,0;\delta,\gamma_1,...,\gamma_n)$ satisfying (12) for $d = 1$. Furthermore, as in [9], we need the following additional assumptions.

Ha') We assume that $f \in C^1(\mathbb{R};BV(\mathbb{R}_+\times\mathbb{R}_x)) \cap L^\infty(\mathbb{R}_+\times\mathbb{R}_x\times\mathbb{R})$ and $\partial_u f(t,x,u) \in L^\infty(\mathbb{R}_+\times\mathbb{R}_x\times\mathbb{R})$.

Hb') There is a sequence $(f_\rho)_{\rho>0}$, such that $f_\rho = f_\rho(t,x,u)$, smooth in $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and continuously differentiable in the u variable, which satisfies for any $p > 2$:

$$\lim_{\rho\to 0}\max_{z\in\mathbb{R}}||f_\rho(.,.,z)-f(.,.,z)||_{L^p_{loc}(\mathbb{R}_+\times\mathbb{R})} = 0,$$

$$|\partial_x f_\rho(t,x,u)| \le \frac{\mu_1(t,x)}{1+|u|^{1+\alpha}}$$

$$|\partial_x f_\rho(t,x,u)|^2 \le \frac{\mu_2(t,x)}{\rho^3},$$

$$|\partial^2_{xu} f_\rho(t,x,u)| \le \frac{\mu_3(t,x)}{1+|u|^{1+\alpha}},$$

$$|\partial_u f_\rho(t,x,u)| \le C,$$

where $\mu_i \in M(\mathbb{R}_+\times\mathbb{R})$, $i = 1,2,3$, are bounded measures.

In one-dimensional case, following the same ideas of the multidimensional case, we have the following a priori estimates:

Lemma 5. *The following estimates are valid:*

(25)
$$\int_{\mathbb{R}}|u(T,x)|^2\,dx+\delta\int_{\mathbb{R}}|\partial_x u(T,x)|^2\,dx+\sum_{n=1}^{N}\gamma_n\int_0^T\int_{\mathbb{R}}|\partial_x^n u(t,x)|^2\,dxdt$$
$$\le \int_{\mathbb{R}}|u_0(x)|^2\,dx+\delta\int_{\mathbb{R}}|\partial_x u_0|^2\,dx+C_4$$

(26)
$$\frac{\gamma_i^2}{2}\int_0^T\int_{\mathbb{R}}|\partial_x^{2i}u(t,x)|^2\,dxdt+\sum_{n=1,n\neq i}^{N}\gamma_n\gamma_i\int_0^T\int_{\mathbb{R}}|\partial_x^{n+i}u(t,x)|^2\,dxdt$$

$$+\frac{\gamma_i}{2}\int_{\mathbb{R}}|\partial_x^i u(T,x)|^2\,dx+\frac{\delta\gamma_i}{2}\int_{\mathbb{R}}|\partial_x^{i+1}u(T,x)|^2\,dx$$

$$\leq\frac{\gamma_i}{2}\int_{\mathbb{R}}|\partial_x^i u_0(x)|^2\,dx+\frac{\delta\gamma_i}{2}\int_{\mathbb{R}}|\partial_x^{i+1}u_0(x)|^2\,dx+C_5(\frac{1}{\gamma_1}+\frac{1}{\rho^3});$$

$$(27)\qquad \frac{\gamma_1}{3}\int_0^T\int_{\mathbb{R}}|\partial_t u(t,x)|^2dxdt+\frac{\gamma_1\delta}{2}\int_0^T\int_{\mathbb{R}}|\partial_{xt}^2 u(t,x)|^2dxdt$$

$$+\sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}}|\partial_x^n u(T,x)|^2\,dx$$

$$\leq\sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}}|\partial_x^n u_0(x)|^2\,dx+C_6\left(1+\frac{\gamma_1}{\rho^3}\right).$$

We use the following lemma which was proved in [9]:

Lemma 6. *Assume that $\{u_\gamma\}_{\gamma>0}\in L^2(\mathbb{R}_+\times\mathbb{R})$ weakly converges in $L^2(\mathbb{R}_+\times\mathbb{R})$ to a function $u\in L^2(\mathbb{R}_+\times\mathbb{R})$. Assume that $\eta(t,x,\lambda)$, $(t,x,\lambda)\in\mathbb{R}_+\times\mathbb{R}^2$ is a function such that $\eta\in C^2(\mathbb{R}_\lambda;L^\infty\cap BV(\mathbb{R}_t^+\times\mathbb{R}_x))$. By η_m we denote the truncation of the function η:*

$$\eta_m(t,x,\lambda)=\begin{cases}\eta(t,x,\lambda), & |\lambda|<m\\ 0, & |\lambda|>2m\end{cases}$$

$(t,x)\in\mathbb{R}^+\times\mathbb{R}$, and $q_m(t,x,\lambda)$ is the corresponding entropy flux.
If for every $m\in\mathbb{N}$ we have

$$div(\eta_m(t,x,u_\gamma),q_m(t,x,u_\gamma))\in H^{-1}_{loc,c}(\mathbb{R}^+\times\mathbb{R})$$

then the limit function u is a weak solution of

$$\partial_t u(t,x)+div_x f(t,x,u(t,x))=0,\quad u=u(t,x),\quad x\in\mathbb{R},\quad t\in\mathbb{R}_+.$$

Moreover, if the flux function $f=f(t,x,\lambda)$ is twice differentiable with respect to λ, and it is genuinely nonlinear, i. e., for every $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, the mapping

$$(28)\qquad \mathbb{R}\ni\lambda\to\partial_\lambda f(t,x,\lambda)\ \ is\ \ non-constant$$

on non-degenerate intervals, then $\{u_\gamma\}_{\gamma>0}$ strongly converges to u in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R})$.

Then prove the following result:

Theorem 7. *Suppose that $\delta=O(\gamma_1^2)$, $\rho^3=\gamma_1$ and for $n>1$, $\gamma_n=O(\gamma_{n-1}\gamma_1^2)$, when $\gamma_1\to0$ and that $u_0\in H^N(\mathbb{R})$. Assume that the flux function f satisfies (H'). Then there exists a subsequence of solutions $(u_{\gamma_k})\subset(u_\gamma)$ of the problem (23)- (24) converges in the sense of distributions to a weak solution of problem (3)- (4). If the flux function $f\in C^2(\mathbb{R};BV(\mathbb{R}_+\times\mathbb{R}_x))\cap L^\infty(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_x)$, and if it is genuinely nonlinear in the sense of (28), then a subsequence of* *solutions $(u_{\gamma_k}) \subset (u_\gamma)$ to problem (23)- (24) converges strongly in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ to a weak solution of (3)- (4).*

Proof. Let $\eta(t,x,\lambda)$, $(t,x,\lambda) \in \mathbb{R}_+ \times \mathbb{R}^2$, a function such that $\eta \in C^2(\mathbb{R}_\lambda; L^\infty \cap BV(\mathbb{R}^+_t \times \mathbb{R}_x))$. Let

$$\eta_m(t,x,\lambda) = \begin{cases} \eta(t,x,\lambda), & |\lambda| < m \\ 0, & |\lambda| > 2m \end{cases}$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, and $q_m(t,x,\lambda)$ the entropy flux corresponding to η_m and f. According to Lemma 6, need to prove that for every $m \in \mathbb{N}$ we have

$$div(\eta_m(t,x,u_\gamma), q_m(t,x,u_\gamma)) \in H^{-1}_{loc,c}(\mathbb{R}_+ \times \mathbb{R}).$$

Consider now the following mollifier $\eta_{m,\gamma}(t,x,u) = \eta_m(.,.,u) * \frac{1}{\gamma_1^{\frac{1}{2}}} w(\frac{t}{\gamma_1^{\frac{1}{4}}}) w(\frac{x}{\gamma_1^{\frac{1}{4}}})$ where w is a nonnegative real function with unit mass, i. e., $\int_{\mathbb{R}} w\, dx = 1$. Denote the entropy flux corresponding to $\eta_{m,\gamma}$ and f_ρ for

$$q_{m,\gamma} = \int_0^u \partial_v \eta_{m,\gamma}(t,x,v) \partial_v f_\rho(t,x,v)\, dv.$$

Note that according to assumptions about η and the choice of mollifier $\eta_{m,\gamma}$, we have:

$$|\partial_t \eta_{m,\gamma}(t,x,u)| \le \mu(t,x),$$

$$|\partial_x \eta_{m,\gamma}(t,x,u)| \le \mu(t,x),$$

$$|\partial^2_{xu} \eta_{m,\gamma}(t,x,u)| \le \mu(t,x),$$

$$|\partial_x \eta_{m,\gamma}(t,x,u)|^2 \le \frac{\mu(t,x)}{\gamma_1},$$

$$|\partial^2_{xu} \eta_{m,\gamma}(t,x,u)|^2 \le \frac{\mu(t,x)}{\gamma_1},$$

for a locally bounded Radon measure $\mu \in M(\mathbb{R}_+ \times \mathbb{R})$.

Multiplying the equation (23) by $\partial_u \eta_{m,\gamma}(t,x,u)$, we obtain:

$$\partial_t(\eta_m(t,x,u)) + \partial_x(q_m(t,x,u)) = -\partial_u \eta_{m,\gamma}(t,x,u) \partial_x f_\rho(t,x,u) + \partial_t \eta_{m,\gamma}(t,x,u)$$

$$+ \int_0^u \partial^2_{xv} f_\rho(t,x,v) \partial_v \eta_{m,\gamma}(t,x,v) dv + \int_0^u \partial_v f_\rho(t,x,v) \partial^2_{xv} \eta_{m,\gamma}(t,x,v) dv$$

$$+\delta(\partial_u \eta_{m,\gamma}(t,x,u) u_{xt})_x - \delta \partial^2_{xu} \eta_{m,\gamma}(t,x,u) \partial^2_{xt} u - \delta \partial^2_{uu} \eta_{m,\gamma}(t,x,u) \partial_x u \partial^2_{xt} u$$

$$+ \sum_{n=1}^N (-1)^{n+1} \gamma_n (\partial_u \eta_{m,\gamma}(t,x,u) \partial_x^{2n-1} u)_x$$

$$+ \sum_{n=1}^N (-1)^n \gamma_n \partial^2_{xu} \eta_{m,\gamma}(t,x,u) \partial_x^{2n-1} u$$

$$+\sum_{n=1}^{N}(-1)^{n+1}\gamma_n\partial^2_{uu}\eta_{m,\gamma}(t,x,u)u_x\partial_x^{2n-1}u$$

$$+(-q_{m,\gamma}(t,x,u)+q_m(t,x,u))_x+(-\eta_{m,\gamma}(t,x,u)+\eta_m(t,x,u))_t.$$

Now we apply a similar procedure as in the multidimensional case. According (Hb'), we have for a constant C depends only on η_m,

$$\left|\int_0^u[\partial^2_{xv}f_\rho(t,x,v)\partial_v\eta_{m,\gamma}(t,x,v)+\partial_v f(t,x,v)\partial^2_{xv}\eta_{m,\gamma}(t,x,v)]dv\right|$$

$$\leq C(\mu_3(t,x)+\mu(t,x))$$

implying limitation in the sense of measure. Similarly, for a constant $\overline{C}$,

$$|-\partial_u\eta_{m,\gamma}(t,x,u)\partial_x f_\rho(t,x,u)-\partial_t\eta_{m,\gamma}(t,x,u)|\leq\overline{C}[\mu_1(t,x)+\mu(t,x)].$$

Now let $\phi\in C_c^\infty(\mathbb{R}\times[0,T])$. We have

$$\left|\delta\int_0^T\int_{\mathbb{R}}\partial_x[\partial_u\eta_{m,\gamma}(t,x,u)\partial^2_{xt}u]\phi\ dxdt\right|\leq C\delta||\partial^2_{xt}u||_{L^2}||\phi_x||_{L^2}$$

$$\leq C\delta\delta^{-\frac{1}{2}}\gamma_1^{-\frac{1}{2}}=o(\gamma_1)$$

by (27), if $\delta=O(\gamma_1^2)$.
Also, if $\delta=O(\gamma_1^2)$ then using (Hb') (25), (26), and (27). We obtain:

$$\delta\int_0^T\int_{\mathbb{R}}|\partial^2_{xu}\eta_{m,\gamma}(t,x,u)\partial^2_{xt}u\phi|\ dxdt$$

$$\leq\delta\left[\int_0^T\int_{\mathbb{R}}|\partial^2_{xt}u|^2\ dxdt\right]^{\frac{1}{2}}\left[\int_0^T\int_{\mathbb{R}}|\partial^2_{xu}\eta_{m,\gamma}(t,x,u)\phi|^2\ dxdt\right]^{\frac{1}{2}}$$

$$\leq C\delta\delta^{-\frac{1}{2}}\gamma_1^{-\frac{1}{2}}\gamma_1^{-\frac{1}{2}}\leq C$$

and

$$\delta\left|\int_0^T\int_{\mathbb{R}}\partial^2_{uu}\eta_{m,\gamma}(t,x,u)\partial_x u\partial^2_{xt}\phi\ dxdt\right|$$

$$\leq C\delta\left[\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\ dxdt\right]^{\frac{1}{2}}\left[\int_0^T\int_{\mathbb{R}}|\partial^2_{xt}u|^2\ dxdt\right]^{\frac{1}{2}}$$

$$\leq C(\delta\gamma_1^{-2})^{\frac{1}{2}}\leq C.$$

For $n>1$, we consider $\gamma_n=O(\gamma_{n-1}\gamma_1^2)$. Using (Hb'), (25), and (26), we obtain:

$$\gamma_n\left|\int_0^T\int_{\mathbb{R}}\partial_x[\partial_u\eta_{m,\gamma}(t,x,u)\partial_x^{2n-1}u]\phi\ dxdt\right|$$

$$\leq C\gamma_n\left[\int_0^T\int_{\mathbb{R}}|\partial_x^{2n-1}u|^2\ dxdt\right]^{\frac{1}{2}}||\phi_x||_{L^2}$$

$$\leq C[\gamma_n\gamma_{n-1}^{-1}\gamma_1^{-1}]^{\frac{1}{2}}=o(\gamma_1);$$

$$\gamma_n \left| \int_0^T \int_{\mathbb{R}} \partial^2_{xu}\eta_{m,\gamma}(t,x,u)\partial_x^{2n-1}u \; dxdt \right|$$
$$\leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}} |\partial^2_{xu}\eta_{m,\gamma}(t,x,u)|^2 \; dxdt \right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}} |\partial_x^{2n-1}u|^2 \; dxdt \right]^{\frac{1}{2}}$$
$$\leq C\gamma_n^{\frac{1}{2}}\gamma_1^{-1}\gamma_{n-1}^{-\frac{1}{2}} \leq C;$$
$$\gamma_n \left| \int_0^T \int_{\mathbb{R}} \partial^2_{uu}\eta_{m,\gamma}(t,x,u)\partial_x u \partial_x^{2n-1}u\phi \; dxdt \right|$$
$$\leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}} |\partial_x u|^2 \; dxdt \right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}} |\partial_x^{2n-1}u|^2 \; dxdt \right]^{\frac{1}{2}}$$
$$\leq C(\gamma_n\gamma_1^{-1}\gamma_{n-1}^{-1})^{\frac{1}{2}} \leq C.$$

For the case $n = 1$, we using (25) and verified that the same results are valid.

Now, due to (Hb') and the definitions of q_m and $q_{m,\gamma}$,

$$|q_{m,\gamma}(t,x,u) - q_m(t,x,u)|$$
$$= \left| \int_0^u \partial_v\eta_m(t,x,v)\partial_v f(t,x,v)dv - \int_0^u \partial_v\eta_{m,\gamma}(t,x,v)\partial_v f_\rho(t,x,v)dv \right|$$
$$\leq 4mC \max_{-2m<v<2m} |f_\rho(t,x,v) - f(t,x,v)| \to 0$$

in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$ as $\gamma_1 \to 0$, implying compactness in H^{-1}_{loc} of the sequence $\partial_x[-q_{m,\gamma}(t,x,u) + q_m(t,x,u)]$. Similarly it is shown that $\partial_t[-\eta_{m,\gamma}(t,x,u) + \eta_m(t,x,u)] \in H^{-1}_c(\mathbb{R}_+ \times \mathbb{R})$, where $H^{-1}_c(\mathbb{R}_+ \times \mathbb{R})$ is a set of families which are compact in $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$. Therefore conclude using the Murat's lemma that

$$div(\eta_m(t,x,u_\gamma), q_m(t,x,u_\gamma)) \in H^{-1}_{loc,c}(\mathbb{R}_+ \times \mathbb{R}).$$

This completes the proof.

□

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