CAUCHY PROBLEM FOR A CLASS OF QUASILINEAR HYPERBOLIC SYSTEMS IS WELL POSED ON A FAMILY OF BESOV SPACES

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Abstract. In this paper we consider the two dimensional Cauchy problem for the quasi-linear systems

$$\begin{cases} \partial_t u + a(u)\partial_x u &= 0 \\ u(0,x) &= \boldsymbol{u}_0(x), \end{cases}$$

with $u = (u_1, \ldots, u_N)$ and $a(u) = \big(a_{jk}(u)\big)_{j,k=1}^N$ real matrix $N \times N$, with entries C^∞, such that the eigenvalues of $a(0)$ are real and distinct, that is, the system is hyperbolic at $u = 0$. We show that a family of Besov spaces, containing the Hölder spaces, near $u = 0$ is continuously preserved by the flow of the above Cauchy Problem.

1. **Introduction.**

In this paper we consider the following quasilinear Cauchy Problem on $\mathbb{R}^2$ given by

$$\begin{cases} \partial_t u + a(u)\partial_x u &= 0 \\ u(0,x) &= \boldsymbol{u}_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^N$, for $N \geq 1$, under the hypothesis that the system is hyperbolic at $u = 0$. Therefore, if $u = (u_1, u_2, \cdots, u_N)$ and $a(u) = (a_{jk}(u))_{j,k=1}^N$ has entries C^∞ we have that $a(u)$ has real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_N(u)$, we denote by $r_1(u), r_2(u) \ldots, r_N(u)$ the corresponding eigenvectors, which depend smoothly on u, in a neighborhood of the origin.

For the linear case the well known result, called Lax-Mizohata's theorem, shows that hyperbolicity is necessary for the well posedness of the Cauchy problem. For the case when $\mathbb{R}^N$ is replaced by $\mathbb{C}$, J. Hounie and J. R. dos Santos Filho, see [3], under a more restricted notion of well posedeness, it was showed that the equation can be reduced to a semilinear's one. Finally, we mention that G. Métivier, in [5], proved a quasilinear version of Lax-Mizohata's theorem. Those results are for analytic, C^∞ and Sobolev spaces.

In [2], Lars Hörmander showed that:

Theorem 1. *For any $\boldsymbol{u}_0 \in C^\rho$, $1 \leq \rho \in \mathbb{Z}$, with bounded derivatives of order $\leq \rho$ and those order ≤ 1 sufficiently small, the Cauchy problem (1.1) has a unique solution $u \in C^\rho$ defined in $[0,T] \times \mathbb{R}$ proved that*

$$T\|\boldsymbol{u}_0'\|_\infty \leq c$$

where c is a constant depending only on a. Moreover, for all multi-index α, $0 \leq |\alpha| \leq \rho$, there are constants $C_{|\alpha|}$, $\tilde{C}_{|\alpha|}$ such that

$$\|\partial^\alpha u(t,\cdot)\|_\infty \leq C_{|\alpha|}\|(\boldsymbol{u}_0)^{(|\alpha|)}\|_\infty \exp(\tilde{C}_{|\alpha|}t).$$

Based in the PhD's thesis of the second author, see [6], we prove extensions of the above theorem. Firstly we extend the previous result for the Hölder spaces (which, of course, can be viewed as Besov spaces), in order to make more accessible we prove it with the usual description of the norm of C^ρ instead of considering as the Besov space $B^\rho_{\infty,\infty}$, with the norm described in terms of Littlewood-Paley's decomposition. We have:

Theorem 2. *Let $\mathbf{u}_0 \in C^\rho$, where $1 < \rho$. If $\|\mathbf{u}_0\|_1$ is sufficiently small, then the Cauchy problem (1.1) has a unique solution $u \in C^\rho([0,T]\times\mathbb{R})$, whenever*

$$T\|\mathbf{u}_0'\|_\infty \leq c,$$

where c is a constant depending only on a. Moreover, there is a constant C such that

$$\|u\|_\rho \leq C\|\mathbf{u}_0\|_\rho. \tag{1.2}$$

Secondly, using paradifferential calculus, see [1] and [4], we extend Theorem 1 for other family of Besov spaces, more precisely we prove:

Theorem 3. *Let $\mathbf{u}_0 \in B^\rho_{\infty,r}$, with $2 < \rho \notin \mathbb{Z}$ and $1 \leq r \leq \infty$. If $\|\mathbf{u}_0\|_1$ is sufficiently small, then the Cauchy problem (1.1) has a unique solution $u \in C^{[\rho]}([0,T]\times\mathbb{R})$, whenever $T|\mathbf{u}_0'\|_\infty \leq c$ where c is a constant depending only on a. Moreover, $u(t,\cdot) \in B^\rho_{\infty,r}$ and there is a constant C such that*

$$\|u(t,\cdot)\|_{B^\rho_{\infty,r}} \leq C\|\mathbf{u}_0\|_{B^\rho_{\infty,r}}, \text{ for all } 0 \leq t \leq T.$$

In Section 2 we establish notation and recall results that will be useful in the proof of our results, in special results of Paradifferential's calculus. In Section 3, we prove the Theorem 2 for $\rho \in (1,2)$. In the next section, we generalize the result of Section 3 for arbitrary ρ, concluding the proof of Theorem 2. Finally, in Section 5, using Paradifferential's calculus, we prove Theorem 3.

2. **Preliminary.**

In this paper we denote α a multi-index of non-negatives integers, for $\rho \in \mathbb{R}_+$ we take $[\rho] = \max\{k \in \mathbb{Z} : k \leq \rho\}$ and $C^\rho(\mathbb{R}^n;\mathbb{R}^N)$ is the Hölder space, $0 < \rho \notin \mathbb{Z}$, with the usual norm (denoted by $\| \ \|_\rho$), which coincides with C^k (with the norm given by the sum of the sup-norms of the derivatives), if $\rho = k \in \mathbb{Z}_+$. For an detailed treatment of the results below see [1].

For the Littlewood-Paley decomposition, we consider $\mathcal{C} = \mathcal{C}(0,\frac{3}{4},\frac{8}{3})$ the ring with center at 0, of small radius 3/4 and large radius 8/3 and $\mathcal{B} = \mathcal{B}(0,\frac{4}{3})$ the ball of center 0 and radius 4/3. Let $\chi \in C_c^\infty(\mathcal{B},[0,1])$ and $\varphi \in C_c^\infty(\mathcal{C},[0,1])$ two radial functions satisfying:

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \ \ \forall \ \ \xi \in \mathbb{R}^n \setminus \{0\} \ \text{ and } \chi(\xi) + \sum_{0\leq j} \varphi(2^{-j}\xi) = 1 \ \ \forall \ \ \xi \in \mathbb{R}^n.$$

We consider the family of operators $\{\Delta_j, \ j \in \mathbb{Z}\}$ where

$$\Delta_j u = 0, \ \forall \ \ j < -1, \ \Delta_{-1}u = (\chi\hat{u})\check{} \text{ and } \Delta_j u = (\varphi(2^{-j}\cdot)\hat{u})\check{}, \forall\ 0 \leq j.$$

here $\hat{v}$ and $\check{v}$ are the Fourier transform and its inverse, respectively, for a temperate distribution v.

If $S_j u \doteq \sum_{j'\leq j-1} \Delta_{j'}u$, then $u = \lim_{j\to\infty} S_j u$ for all $u \in \mathcal{S}'$. We write

$$u = \sum_{j\in\mathbb{Z}} \Delta_j u.$$

The above decomposition is called Littlewood-Paley's decomposition of u. For u and v two temperate distributions, we have the following formal decomposition:

$$uv = \sum_{p,q\in\mathbb{Z}} \Delta_q u \Delta_{q'} v.$$

Let

$$T_u v \doteq \sum_{q\in\mathbb{Z}} S_{q-1}u\Delta_q v,\quad T_v u \doteq \sum_{q\in\mathbb{Z}} S_{q-1}v\Delta_q u \quad \text{and} \quad R(u,v) \doteq \sum_{\substack{|q-q'|\le 1\\ q,q'\in\mathbb{Z}}} \Delta_q u \Delta_{q'} v.$$

At least informally, we have the decomposition

$$uv = T_u v + T_v u + R(u,v),$$

this decomposition is called the Bony's Decomposition, the part $T_u v$ is called paraproduct of v by u, analogously $T_v u$ is the paraproduct of u by v and $R(u,v)$ is called rest of u and v. Let $(p,r)\in[1,\infty]^2$ and $s\in\mathbb{R}$. The Besov space $B^s_{p,r}$ is the space of all tempered distributions u such that

$$\|u\|_{B^s_{p,r}} \doteq \|(2^{sj}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r(\mathbb{Z})} < \infty.$$

Now we give a list of results that will be useful for the proof of our theorems. In next lemma $S(\hat{u})$ denotes the support of the distribution $\hat{u}$.

Lemma 4 (Bernstein's inequalities)**.** *Let $k\in\mathbb{Z}_+$ and $(r_1,r_2)\in\mathbb{R}^2$ satisfying $0<r_1<r_2$. There exists a constant C depending only on r_1, r_2 and N, such that*

i) $S(\hat{u})\subset\mathcal{B}(0,\lambda r_1) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q(\mathbb{R}^N)} \le C^{k+1}\lambda^{k+N(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\mathbb{R}^N)};$

ii) $S(\hat{u})\subset\mathcal{C}(0,\lambda r_1,\lambda r_2) \Rightarrow C^{-(k+1)}\lambda^k\|u\|_{L^p(\mathbb{R}^N)} \le \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p(\mathbb{R}^N)} \le C^{k+1}\lambda^k\|u\|_{L^p(\mathbb{R}^N)},$

for all $1\le p\le q\le\infty$ and $u\in L^p$.

Lemma 5 (Fatou's property)**.** *If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to u in $\mathcal{S}'$, then $u\in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}} \le \liminf \|u_n\|_{B^s_{p,r}}.$$

Theorem 6. *For all $s\in\mathbb{R}$ exists a constant C such that*

$$\|T_u v\|_{B^s_{p,r}} \le C\|u\|_{L^\infty}\|v\|_{B^s_{p,r}},\ \forall (p,r)\in[1,+\infty]^2 \text{ and } (u,v)\in L^\infty\times B^s_{p,r}.$$

Theorem 7. *If $(s_1,s_2)\in\mathbb{R}^2$ satisfies $0<s_1+s_2$, then exists a constant C for which is true*

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \le C\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}},\ \forall\ (u,v)\in B^{s_1}_{p_1,r_1}\times B^{s_2}_{p_2,r_2},$$

whenever that $(p_1,p_2,r_1,r_2)\in[1,\infty]^4$ satisfies

$$\frac{1}{p}\doteq\frac{1}{p_1}+\frac{1}{p_2}\le 1 \quad \text{and} \quad \frac{1}{r}\doteq\frac{1}{r_1}+\frac{1}{r_2}\le 1.$$

Lemma 8. *There exists a constant C such that for all f in L^p, $p\in[1,\infty]$, and g lipschitz function we have*

$$\|\,[\Delta_q, g]\,f\|_{L^p} \le C2^{-q}\|\nabla g\|_\infty\|f\|_{L^p},$$

where $[\Delta_q,g]\,f \doteq \Delta_q(gf) - g\Delta_q f$.

Lemma 9. *If a sequence of smooth applications $\psi_n:\mathbb{R}^N\to\mathbb{R}^N$ is such that $0\le\psi_n$, $\|\psi_n\|_{L^1}=1$ and $S(\psi_n)\subset\mathcal{B}\left(0,\dfrac{1}{n}\right)$ $\forall n$, then for all $u\in B^\rho_{p,r}$ the sequence $(\psi_n * u)_n$ is contained in $B^\rho_{p,r}$ and satisfies*

$$\|\psi_n * u\|_{B^\rho_{p,r}} \le \|u\|_{B^\rho_{p,r}}.$$

Finally we recall that $B^s_{p,r}\hookrightarrow B^{\tilde{s}}_{p,\tilde{r}}$ (embbeding) whenever $\tilde{s}<s$ or $\tilde{s}=s$ and $r\le\tilde{r}$. Also recall that the spaces $B^\rho_{\infty,\infty}$ and C^ρ are equal and their norms are equivalent.

3. Proof the Theorem 2 in the Case $\rho \in (1,2)$.

From Theorem 1 we need to show that the unique solution $u \in C^{[\rho]}([0,T];\mathbb{R})$ in fact belongs to $C^{\rho}([0,T];\mathbb{R})$ and satisfies (1.2). This follow from six steps below.

Step 1. Let $(\psi_n)_n$ be a smoothing compact support sequence ($S(\psi_n) \subset \mathcal{B}(0,1/n)$) approximating the delta distribution supported at the origin and φ a function C^∞ such that, $\varphi \equiv 1$ on $\mathcal{B}(0,1)$ and $S(\varphi) \subset \mathcal{B}(0,2)$. For all $\lambda \in \mathbb{N}$ we consider $\varphi_\lambda(x) \doteq \varphi(\frac{x}{\lambda})$, $\boldsymbol{u}_{0,\lambda} \doteq \boldsymbol{u}_0\varphi_\lambda$ and $\boldsymbol{u}_{0,\lambda,n} \doteq \psi_n * \boldsymbol{u}_{0,\lambda} = \psi_n * \boldsymbol{u}_0\varphi_\lambda$. We have $\boldsymbol{u}_{0,\lambda,n} \in C^\infty$ and $S(\boldsymbol{u}_{0,\lambda,n}) \subset \mathcal{B}(0,1/n) + \mathcal{B}(0,2\lambda)$. Since the derivatives of $\boldsymbol{u}_0$ up the order 1 are small, for λ sufficiently large, we have that the derivatives of $\boldsymbol{u}_{0,\lambda,n}$ of order ≤ 1 are also sufficiently small, from the Young's inequality. From the Theorem 1 we can choose a constant $0 < T$ for which the solution $u_{\lambda,n}$ of the Cauchy problem (1.1) with initial data $\boldsymbol{u}_{0,\lambda,n}$ satisfy

$$\|\partial^\alpha u_{\lambda,n}(t,\cdot)\|_\infty \leq C_{|\alpha|}\|(\boldsymbol{u}_{0,\lambda,n})^{(|\alpha|)}\|_\infty \exp(\tilde{C}_{|\alpha|}t) \tag{3.3}$$

for all $0 \leq |\alpha| \leq [\rho]$, $t \in [0,T]$, with constants $C_{|\alpha|}$ and $\tilde{C}_{|\alpha|}$, independent of $\lambda_0 \leq \lambda$ and n. Let $v_{\lambda,n,l} \doteq u_{\lambda,n+l} - u_{\lambda,n}$, therefore

$$\partial_t v_{\lambda,n,l} + a(u_{\lambda,n+l})\partial_x v_{\lambda,n,l} = -\left(a(u_{\lambda,n+l}) - a(u_{\lambda,n})\right)\partial_x u_{\lambda,n},$$

writing $v_{\lambda,n,l} \doteq \sum_{i=1}^N v_{n,l,i} r_i(u_{\lambda,n+l})$ and $L_j^{n+l} \doteq \partial_t + \lambda_j(u_{\lambda,n+l})\partial_x$ we obtain

$$\sum_{i=1}^N \left(L_i^{n+l} v_{n,l,i}\right) r_i(u_{\lambda,n+l}) = \sum_{i,j=1}^N v_{n,l,i}\phi_{ij}(u_{\lambda,n+l}, u_{\lambda,n}) r_j(u_{\lambda,n+l}) \tag{3.4}$$

where, for each i, $\phi_{ij}(u_{\lambda,n+l}, u_{\lambda,n})$ are the coordinates of

$$-\left[\left(\left(\int_0^1 a'\left(u_{\lambda,n} + \tau v_{\lambda,n,l}\right)d\tau\right) r_i(u_{\lambda,n+l})\right)\partial_x u_{\lambda,n} + \partial_t\left(r_i(u_{\lambda,n+l})\right) + a(u_{\lambda,n+l})\partial_x\left(r_i(u_{\lambda,n+l})\right)\right]$$

written in the base $\{r_1(u_{\lambda,n+l}),\ldots,r_N(u_{\lambda,n+l})\}$. From (3.4) follows that

$$L_j^{n+l} v_{n,l,j} = \sum_{i=1}^N v_{n,l,i}\phi_{ij}(u_{\lambda,n+l}, u_{\lambda,n}). \tag{3.5}$$

Let $\gamma(t) = (t, \gamma_2(t))$ a integral curve of field L_j^{n+l} and

$$M_{n+l}(t) \doteq \sup_{i,x} |v_{n,l,i}(t,x)|.$$

By integration of (3.5) over γ, we obtain

$$|v_{n,l,j}(\gamma(t))| \leq M_{n+l}(0) + C\int_0^t M_{n+l}(s)ds, \quad \forall\ j\ \forall\ \gamma\ \text{ curve}.$$

Taking the supremum over all integral curve γ and using the Gronwall's inequality, it follows that

$$M_{n+l}(t) \leq C\|\boldsymbol{u}_{0,\lambda,n+l} - \boldsymbol{u}_{0,\lambda,n}\|_\infty,$$

that is,

$$\|u_{\lambda,n+l} - u_{\lambda,n}\|_\infty \leq C\|\boldsymbol{u}_{0,\lambda,n+l} - \boldsymbol{u}_{0,\lambda,n}\|_\infty.$$

Since $\|\boldsymbol{u}_{0,\lambda,n} - \boldsymbol{u}_{0,\lambda}\| \overset{n\to\infty}{\longrightarrow} 0$, then $(u_{\lambda,n})$ is a Cauchy sequence in $C([0,T]\times\mathbb{R})$, hence there is u_λ for which holds true $u_{\lambda,n} \overset{n\to\infty}{\longrightarrow} u_\lambda$ in $C([0,T]\times\mathbb{R})$.

Step 2. From $\partial_t u_{\lambda,n} + a(u_{\lambda,n})\partial_x u_{\lambda,n} = 0$, we obtain

$$\partial_t\partial_x u_{\lambda,n} + a(u_{\lambda,n})\partial_x\partial_x u_{\lambda,n} = -\left(a'(u_{\lambda,n})\partial_x u_{\lambda,n}\right)\partial_x u_{\lambda,n}.$$

Writing $\partial_x u_{\lambda,n} = \sum_{i=1}^{N} w_{\lambda,n,i} r_i(u_{\lambda,n})$ it follows that

$$\begin{aligned}\sum_{i=1}^{N} \left(\partial_t w_{\lambda,n,i} + \lambda_i(u_{\lambda,n})\partial_x w_{\lambda,n,i}\right) r_i(u_{\lambda,n}) &= \sum_{i,l=1}^{N} w_{\lambda,n,i} w_{\lambda,n,l} \Big\{ - \left[\left(a'(u_{\lambda,n}) r_l(u_{\lambda,n})\right) r_i(u_{\lambda,n}) \right. \\ &\qquad \left. - r_i'(u_{\lambda,n}) \left(a(u_{\lambda,n}) r_l(u_{\lambda,n})\right) + a(u_{\lambda,n}) \left(r_i'(u_{\lambda,n}) r_l(u_{\lambda,n})\right)\right] \Big\} \\ &= \sum_{i,l,j=1}^{N} w_{\lambda,n,i} w_{\lambda,n,l} \phi_{ilj}(u_{\lambda,n}) r_j(u_{\lambda,n}),\end{aligned}$$

where, $\phi_{ilj}(u_{\lambda,n})$ are the coordinates of the vector between braces in the base $\{r_1(u_{\lambda,n}), \ldots, r_N(u_{\lambda,n})\}$, for each (i, l, j). Thus we have

$$L_{\lambda,n,j} w_{\lambda,n,j} = \sum_{i,l=1}^{N} w_{\lambda,n,i} w_{\lambda,n,l} \phi_{ilj}(u_{\lambda,n}), \tag{3.6}$$

with $L_{\lambda,n,j} v \doteq \partial_t v + \lambda_j(u_{\lambda,n}) \partial_x v$.

If $\Phi_{\lambda,n,j}$ is the flow of the vector field $L_{\lambda,n,j}$, by integration of (3.6) over the curve $\Phi_j(\cdot - s, (s, x))$, we have

$$\begin{aligned} w_{\lambda,n,j}(s, x) &= w_{\lambda,n,j}\Big(\Phi_{\lambda,n,j}\big(-s, (s, x)\big)\Big) \\ &+ \int_0^s \sum_{i,l=1}^{N} w_{\lambda,n,i} w_{\lambda,n,l}\Big(\Phi_{\lambda,n,j}\big(s'-s,(s,x)\big)\Big) \phi_{ilj}(u_{\lambda,n})\Big(\Phi_{\lambda,n,j}\big(s'-s,(s,x)\big)\Big) ds', \end{aligned} \tag{3.7}$$

also we have

$$|\Phi_{\lambda,n,j}(t, (s, y)) - \Phi_{\lambda,n,j}(t, (s, x))| \le \exp(At)|(s, x) - (s, y)| \tag{3.8}$$

and

$$|(s, y) - \Phi_{\lambda,n,j}(s - \tau, (\tau, x))| \le A_1|(\tau, x) - (s, y)|, \tag{3.9}$$

with A and A_1 not depending of λ, n and j.

Step 3. To simplify the notation we will remove the indices λ and n. Let (t, x), $(\tau, y) \in [0, T] \times \mathbb{R}$ two points, without loss of generality we can assume that $\tau \le t$. For estimate $\max_{j=1,\ldots,N}\{|w_j(\tau, y) - w_j(t, x)|\}$ in first we write

$$|w_j(\tau, y) - w_j(t, x)| \le |w_j(\tau, y) - w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)| + |w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big) - w_j(t, x)|. \tag{3.10}$$

Integrating from τ to t both sides of (3.6) over the curve $\Phi_j(\cdot - t, (t, x))$ we obtain

$$w_j(t, x) - w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big) = \sum_{i,l=1}^{N} \int_\tau^t w_i w_l\Big(\Phi_j\big(s-t,(t,x)\big)\Big) \phi_{ilj}(u) \Big(\Phi_j\big(s-t,(t,x)\big)\Big) ds,$$

so of (3.3) it follows that

$$|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big) - w_j(t, x)| \le C \max_{i=1,\ldots,N}\{\|w_i\|_\infty\}|(t, x) - (\tau, y)|,$$

taking the maximum in j we have

$$\max_{j=1,\ldots,N}\{|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big) - w_j(t, x)|\} \le C \max_{j=1,\ldots,N}\{\|w_j\|_\infty\}|(t, x) - (\tau, y)|. \tag{3.11}$$

The estimate for $|w_j(\tau,y) - w_j\Big(\Phi_j\big(\tau{-}t,(t,x)\big)\Big)|$ is a little more delicate. If $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the second variable, then $\pi_2\Big(\Phi_N\big(\tau{-}t,(t,x)\big)\Big) < \ldots < \pi_2\Big(\Phi_1\big(\tau{-}t,(t,x)\big)\Big)$ and one of the following situations happen:

I. $y \le \pi_2(\Phi_N(\tau - t,(t,x)))$;
II. $\pi_2(\Phi_1(\tau - t,(t,x))) \le y$;
III. $\pi_2(\Phi_N(\tau - t,(t,x))) \le y \le \pi_2(\Phi_1(\tau - t,(t,x)))$.

For the case **I** we consider for all $0 \le s \le \tau$,

$$\Lambda_{\mathbf{I}s} \doteq \Big\{ I \subset [\Phi_N(s-\tau,(\tau,y)), \Phi_1\big(s-\tau, \Phi_1\big(\tau{-}t,(t,x)\big)\big)] : I \text{ is connected and } |I| \le \exp(A(\tau - s))|(\tau,y) - \Phi_1(\tau - t,(t,x))| \Big\}$$

and

$$M_{\mathbf{I}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{I}s}} \Big\{ \max_{j=1,\ldots,N} \Big\{ \sup_{(s,a),(s,b)\in I} \{|w_j(s,a) - w_j(s,b)|\} \Big\} \Big\}.$$

Let $I \in \Lambda_{\mathbf{I}s}$ and (s,a), (s,b) in I. Applying (3.7) in the points (s,a) and (s,b) and taking the difference it follows that

$$\begin{aligned}|w_j(s,a) - w_j(s,b)| &\le |w_j\Big(\Phi_j\big(-s,(s,a)\big)\Big) - w_j\Big(\Phi_j\big(-s,(s,b)\big)\Big)| \\ &+ \int_0^s C \max_{i=1,\ldots,N} \Big\{ |w_i\Big(\Phi_j\big(s'{-}s,(s,a)\big)\Big) - w_i\Big(\Phi_j\big(s'{-}s,(s,b)\big)\Big)| \Big\} ds' \\ &+ \int_0^s C \max_{i=1,\ldots,N} \Big\{ \|w_i\|_\infty \Big\} |\Phi_j\big(s'{-}s,(s,a)\big) - \Phi_j\big(s'{-}s,(s,b)\big)| ds'.\end{aligned}$$

Since, see (3.8) and (3.9), $[\Phi_j\big(s'{-}s,(s,a)\big), \Phi_j\big(s'{-}s,(s,b)\big)] \in \Lambda_{\mathbf{I}s'}$ and $|\Phi_j\big(s'{-}s,(s,a)\big) - \Phi_j\big(s'{-}s,(s,b)\big)| \le C|(\tau,y) - (t,x)|$ from the above inequality we obtain

$$|w_j(s,a) - w_j(s,b)| \le M_{\mathbf{I}}(0) + C \max_{i=1,\ldots,N} \|w_i\|_\infty |(t,x) - (\tau,y)| + \int_0^s C M_{\mathbf{I}}(s') ds',$$

for all index j, interval I and points (s,a) and $(s,b) \in I$, so for any $s \in [0,\tau]$ we have

$$M_{\mathbf{I}}(s) \le M_{\mathbf{I}}(0) + C \max_{i=1,\ldots,N} \|w_i\|_\infty |(t,x) - (\tau,y)| + \int_0^s C M_{\mathbf{I}}(s') ds'.$$

From Gronwall's lemma it follows now that

$$M_{\mathbf{I}}(s) \le \Big(M_{\mathbf{I}}(0) + C \max_{i=1,\ldots,N} \|w_i\|_\infty |(t,x) - (\tau,y)| \Big) \exp(sC).$$

For the case **II** we define

$$\Lambda_{\mathbf{II}s} \doteq \Big\{ I \subset [\Phi_1(s{-}\tau,(\tau,y)), \Phi_N\big(s{-}\tau, \Phi_N\big(\tau{-}t,(t,x)\big)\big)] : I \text{ is connected and } |I| \le \exp(A(\tau - s))|(\tau,y) - \Phi_N(\tau - t,(t,x))| \Big\}$$

and

$$M_{\mathbf{II}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{II}s}} \Big\{ \max_{j=1,\ldots,N} \Big\{ \sup_{(s,a),(s,b)\in I} \{|w_j(s,a) - w_j(s,b)|\} \Big\} \Big\}.$$

Similarly to the case **I**, we obtain that

$$M_{\mathbf{II}}(s) \le \Big(M_{\mathbf{II}}(0) + C \max_{i=1,\ldots,N} \|w_i\|_\infty |(t,x) - (\tau,y)| \Big) \exp(sC).$$

In the case **III** defining

$$\Lambda_{\mathbf{III}s} \doteq \Big\{ I \subset [\Phi_N\big(s-\tau, \Phi_N(\tau-t,(t,x))\big), \Phi_1\big(s-\tau, \Phi_1(\tau-t,(t,x))\big)] : I \text{ is connected and } |I| \leq \exp(A(\tau-s))|\Phi_1(\tau-t,(t,x)) - \Phi_N(\tau-t,(t,x))| \Big\}$$

and

$$M_{\mathbf{III}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{III}s}} \Big\{ \max_{j=1,\dots,N} \Big\{ \sup_{(s,a),(s,b)\in I} \{|w_j(s,a) - w_j(s,b)|\} \Big\} \Big\},$$

we get

$$M_{\mathbf{III}}(s) \leq \Big(M_{\mathbf{III}}(0) + C \max_{i=1,\dots,N} \|w_i\|_\infty |(t,x) - (\tau,y)| \Big) \exp(sC).$$

Since $[(\tau,y), \Phi_j(\tau-t,(t,x))] \in \Lambda_{\mathbf{I}\tau} \cup \Lambda_{\mathbf{II}\tau} \cup \Lambda_{\mathbf{III}\tau}$ we have

$$\max_{j=1,\dots,N} \{|w_j(\tau,y) - w_j\big(\Phi_j(\tau-t,(t,x))\big)|\} \leq \max\{M_{\mathbf{I}}(\tau), M_{\mathbf{II}}(\tau), M_{\mathbf{III}}(\tau)\}.$$

From this inequality, (3.10), (3.11) and the estimates obtained for $M_{\mathbf{I}}(s)$, $M_{\mathbf{II}}(s)$ and $M_{\mathbf{III}}(s)$, it follows

$$\max_{j=1,\dots,N} \{|w_j(\tau,y) - w_j(t,x)|\} \leq C \left(\max\{M_{\mathbf{I}}(0), M_{\mathbf{II}}(0), M_{\mathbf{III}}(0)\} + \max_{j=1,\dots,N} \{\|w_j\|_\infty\} |(t,x) - (\tau,y)| \right).$$

Since for all $\mathbf{I} \in \Lambda_{\mathbf{I}0} \cup \Lambda_{\mathbf{II}0} \cup \Lambda_{\mathbf{III}0}$ we have $|I| \leq 2A_1 \exp(AT)|(t,x) - (\tau,y)|$, it follows

$$\max_{j=1,\dots,N} \{|w_j(\tau,y) - w_j(t,x)|\} \leq C \Bigg(\sup_{I\in\Lambda} \Big\{ \max_{j=1,\dots,N} \Big\{ \sup_{(0,a),(0,b)\in I} \{|w_j(0,a) - w_j(0,b)|\} \Big\} \Big\} + \max_{j=1,\dots,N} \{\|w_j\|_\infty\} |(t,x) - (\tau,y)| \Bigg),$$

where Λ is the set of $I \subset \{0\} \times \mathbb{R}$ such that: $|I| \leq 2A_1 \exp(AT)|(t,x) - (\tau,y)|$, where I are intervals. The calculations made above are independent of (t,x) and (τ,y), going back to the dependence on λ and n, defining

$$\Lambda_{(t,x),(\tau,y)} \doteq \Big\{ I \subset \mathbb{R} : I \text{ is an interval } |I| \leq 2A_1 \exp(AT)|(t,x) - (\tau,y)| \Big\}$$

and

$$M_{\lambda,n}\big((t,x),(\tau,y)\big) \doteq \sup_{I\in\Lambda_{(t,x),(\tau,y)}} \Big\{ \max_{j=1,\dots,N} \Big\{ \sup_{a,b\in I} \{|w_{\lambda,n,j}(0,a) - w_{\lambda,n,j}(0,b)|\} \Big\} \Big\}$$

we have

$$\max_{j=1,\dots,N} \{|w_{\lambda,n,j}(\tau,y) - w_{\lambda,n,j}(t,x)|\} \leq B\Big(M_{\lambda,n}\big((t,x),(\tau,y)\big) + \max_{j=1,\dots,N} \{\|w_{\lambda,n,j}\|_\infty\} |(t,x) - (\tau,y)| \Big), \tag{3.12}$$

for some constant B and all (t,x), (τ,y) in $[0,T] \times \mathbb{R}$.

Step 4. Let $0 < \epsilon$, since $\mathcal{F}_0 \doteq \{\partial_x u_{\lambda,n}(0,\cdot) : n \in \mathbb{N}\}$ is equicontinuous, $\exists\ 0 < \delta_0$ such that if $|a-b| < \delta_0$ then we have $\max_{j=1,\dots,N}\{|w_{\lambda,n,j}(0,a) - w_{\lambda,n,j}(0,b)|\} < \dfrac{\epsilon}{2B}$. Let $0 < \delta$ such that $2A_1 \exp(AT)|(t,x) - (\tau,y)| < \delta_0$ and $\max_{j=1,\dots,N} \{\|w_{\lambda,n,j}\|_\infty\} |(t,x) - (\tau,y)| < \dfrac{\epsilon}{2B}$, if $|(t,x) - (\tau,y)| < \delta$. It follows from (3.12) that

$$\max_{j=1,\dots,N} \{|w_{\lambda,n,j}(\tau,y) - w_{\lambda,n,j}(t,x)|\} < \epsilon$$

therefore, $\mathcal{F} \doteq \{\partial_x u_{\lambda,n} : n \in \mathbb{N}\}$ is equicontinuous. Since $S(\boldsymbol{u}_{0,\lambda,n}) \subset \mathcal{B}(0,1) + \mathcal{B}(0,2\lambda)$ there exists K_λ compact for which $S(u_{\lambda,n}) \subset K_\lambda$ for all n, by finite speed of propagation's argument. Hence restricting the domain of $\partial_x u_{\lambda,n}$ to K_λ we obtain that exist a subsequence $(\partial_x u_{\lambda,n_l})_{n_l}$ converging uniformly, by Arzel-Ascoli's theorem. Moreover, since u_{λ,n_l} satisfies (1.1) it follows that $(\partial_t u_{\lambda,n_l})$ also converges uniformly, thus, $\big((u_{\lambda,n_l})'\big)_{n_l}$ converge uniformly. Therefore $(u_{\lambda,n})' \overset{n\to\infty}{\longrightarrow} (u_\lambda)'$ uniformly in K_λ, by the mean value's theorem and the calculus fundamental's theorem, hence u_λ is a solution of (1.1) with initial data $\boldsymbol{u}_{0,\lambda}$.

Step 5 Given a compact $K \subset [0,T] \times \mathbb{R}$ there is λ_0 such that

$$u_{\lambda,n}(t,x) = u_{\lambda_0,n}(t,x) \quad \forall\,(t,x) \in K \quad \text{e} \quad \forall\,\lambda_0 \leq \lambda,$$

by the domain of dependence's result for hyperbolic systems. Thus, since $u_{\lambda,n} \to u_\lambda$ uniformly on compacts, we can write

$$u_\lambda(t,x) = u_{\lambda_0}(t,x) \quad \forall\,(t,x) \in K \quad \text{e} \quad \forall\,\lambda_0 \leq \lambda,$$

therefore we get the solution of the problem (1.1), with initial data $\boldsymbol{u}_0$.

Step 6. Since $C^\rho = B^\rho_{\infty,\infty}$ for to show that $u \in C^\rho([0,T]\times\mathbb{R})$ and satisfies (1.2) it is enough to prove that

$$\|u_{\lambda,n}\|_\rho \leq C\|\boldsymbol{u}_0\|_\rho \tag{3.13}$$

for some constant C, by Lemma 5. From (3.3) we have that $\{u_{\lambda,n} : \lambda, n \in \mathbb{N}\}$ is bounded therefore to check (3.13) it is sufficient to prove

$$\sup_{0<|x-y|<1} \left\{ \frac{|\partial_x u_{\lambda,n}(t,x) - \partial_x u_{\lambda,n}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \leq C\,\|\boldsymbol{u}_0\|_\rho\,,$$

or equivalently

$$\sup_{0<|x-y|<1}\ \max_{j=1,\ldots,N} \left\{ \frac{|w_{\lambda,n,j}(t,x) - w_{\lambda,n,j}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \leq C\,\|\boldsymbol{u}_0\|_\rho\,. \tag{3.14}$$

But from (3.12)

$$\begin{aligned}\max_{j=1,\ldots,N} \left\{ \frac{|w_{\lambda,n,j}(t,x) - w_{\lambda,n,j}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} &\leq B\frac{1}{|(t,x)-(\tau,y)|^{\rho-1}} M_{\lambda,n}\big((t,x),(\tau,y)\big) \\ &\quad + B\max_{j=1,\ldots,N}\{\|w_{\lambda,n,j}\|_\infty\} \frac{|(t,x)-(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-1}}.\end{aligned} \tag{3.15}$$

The second term of the right hand side can be easily estimate using (3.3)

$$\sup_{0<|x-y|<1} \left\{ B\max_{j=1,\ldots,N}\{\|w_{\lambda,n,j}\|_\infty\} \frac{|(t,x)-(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \leq C\,\|\boldsymbol{u}_{0,\lambda,n}\|_\rho\,. \tag{3.16}$$

For the first term of the right hand side, since $0 < |x-y| < 1$, by the definition of $M_{\lambda,n}$ (see previous page), for all $I \in \Lambda_{(t,x),(\tau,y)}$ we have

$$\begin{aligned}&\sup_{a,b\in I} \left\{ \frac{|w_{\lambda,n,j}(0,a) - w_{\lambda,n,j}(0,b)|}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \\ &\quad\leq \sup_{a,b\in I} \left\{ \frac{|w_{\lambda,n,j}(0,a) - w_{\lambda,n,j}(0,b)|}{|a-b|^{\rho-1}} \frac{|a-b|^{\rho-1}}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \\ &\quad\leq \sup_{a,b\in I} \left\{ \frac{|w_{\lambda,n,j}(0,a) - w_{\lambda,n,j}(0,b)|}{|a-b|^{\rho-1}} \big(2A_1\exp(AT)\big)^{\rho-1} \frac{|(t,x)-(\tau,y)|^{\rho-1}}{|(t,x)-(\tau,y)|^{\rho-1}} \right\} \\ &\quad\leq C\big(2A_1\exp(AT)\big)^{\rho-1}\,\|\boldsymbol{u}_{0,\lambda,n}\|_\rho\end{aligned}$$

therefore

$$\sup_{0<|x-y|<1}\left\{\frac{1}{|(t,x)-(\tau,y)|^{\rho-1}}M_{\lambda,n}\big((t,x),(\tau,y)\big)\right\}\leq C\big(2A_1\exp(AT)\big)^{\rho-1}\|\boldsymbol{u}_{0,\lambda,n}\|_{\rho}. \tag{3.17}$$

From (3.15), (3.16) and (3.17) follows (3.14), for $\rho \in (1,2)$, concluding the theorem in this case.

4. **Proof of Theorem 2, Case** $2<\rho\notin\mathbb{Z}$.

In this section we generalize the steps 2 to 6 of the previous section and finish the proof of the Theorem 2.

From $\partial_t u_{\lambda,n}+a(u_{\lambda,n})\partial_x u_{\lambda,n}=0$ we obtain

$$\begin{aligned}\partial_t\partial_x^{[\rho]}u_{\lambda,n}+a(u_{\lambda,n})\partial_x\partial_x^{[\rho]}u_{\lambda,n}&=-[\rho]\big(a'(u_{\lambda,n})\partial_x u_{\lambda,n}\big)\partial_x^{[\rho]}u_{\lambda,n}-\big(a'(u_{\lambda,n})\partial_x^{[\rho]}u_{\lambda,n}\big)\partial_x u_{\lambda,n}\\&\quad-\Big[\sum_{l=2}^{[\rho]}n(i_1,\dots,i_l)a^{(l)}(u_{\lambda,n})\Big(\partial_x^{i_1}u_{\lambda,n},\dots,\partial_x^{i_l}u_{\lambda,n}\Big)\Big]\partial_x u_{\lambda,n}\\&\quad-\sum_{m=2}^{[\rho]-1}\binom{[\rho]}{m}\partial_x^m(a(u_{\lambda,n}))\partial_x^{[\rho]+1-m}u_{\lambda,n}.\end{aligned}$$

Writing

$$\partial_x^{[\rho]}u_{\lambda,n}=\sum_{i=1}^N w_{\lambda,n,[\rho],i}r_i(u_{\lambda,n}),$$

we have

$$\begin{aligned}&\sum_{i=1}^N\big(\partial_t w_{\lambda,n,[\rho],i}+\lambda_i(u_{\lambda,n})\partial_x w_{\lambda,n,[\rho],i}\big)r_i(u_{\lambda,n})\\&\qquad=-\sum_{i=1}^N w_{\lambda,n,[\rho],i}\Big(\partial_t\big(r_i(u_{\lambda,n})\big)+a(u_{\lambda,n})\partial_x\big(r_i(u_{\lambda,n})\big)\Big)\\&\qquad\quad-\sum_{i=1}^N w_{\lambda,n,[\rho],i}\Big([\rho]\big(a'(u_{\lambda,n})\partial_x u_{\lambda,n}\big)r_i(u_{\lambda,n})+\big(a'(u_{\lambda,n})r_i(u_{\lambda,n})\big)\partial_x u_{\lambda,n}\Big)\\&\qquad\quad-\Big[\sum_{l=2}^{[\rho]}n(i_1,\dots,i_l)a^{(l)}(u_{\lambda,n})\Big(\partial_x^{i_1}u_{\lambda,n},\dots,\partial_x^{i_l}u_{\lambda,n}\Big)\Big]\partial_x u_{\lambda,n}\\&\qquad\quad-\sum_{m=2}^{[\rho]-1}\binom{[\rho]}{m}\partial_x^m(a(u_{\lambda,n}))\partial_x^{[\rho]+1-m}u_{\lambda,n}.\end{aligned}\tag{4.18}$$

Writing

$$\begin{aligned}-\Big(\partial_t\big(r_i(u_{\lambda,n})\big)+a(u_{\lambda,n})\partial_x\big(r_i(u_{\lambda,n})\big)+[\rho]\big(a'(u_{\lambda,n})\partial_x u_{\lambda,n}\big)r_i(u_{\lambda,n})+\big(a'(u_{\lambda,n})r_i(u_{\lambda,n})\big)\partial_x u_{\lambda,n}\Big)\\=\sum_{j=1}^N\phi_{ij}(u_{\lambda,n})r_j(u_{\lambda,n})\end{aligned}$$

and

$$-\left\{\Big[\sum_{l=2}^{[\rho]} n(i_1,\ldots,i_l)a^{(l)}(u_{\lambda,n})\Big(\partial_x^{i_1}u_{\lambda,n},\ldots,\partial_x^{i_l}u_{\lambda,n}\Big)\Big]\partial_x u_{\lambda,n}\right.$$
$$\left.+\sum_{m=2}^{[\rho]-1}\binom{[\rho]}{m}\partial_x^m(a(u_{\lambda,n}))\partial_x^{[\rho]+1-m}u_{\lambda,n}\right\}=\sum_{j=1}^{N}R_j(u_{\lambda,n})r_j(u_{\lambda,n}),$$

from (4.18) we have

$$L_{\lambda,n,j}w_{\lambda,n,[\rho],j}=\left(\sum_{i=1}^{N}w_{\lambda,n,[\rho],i}\phi_{ij}(u_{\lambda,n})\right)+R_j(u_{\lambda,n}), \tag{4.19}$$

where $L_{\lambda,n,j}$ is the same operator used in the case $1<\rho<2$. To simplify the expressions below we will remove the indices λ, $[\rho]$ and n. Integrating (4.19) from 0 to s along the curve $\Phi_j(\cdot-s,(s,x))$ we obtain

$$w_j(s,x)=w_j\big(\Phi_j(-s,(s,x))\big)+\sum_{i=1}^{N}\int_0^s w_i\big(\Phi_j(s'-s,(s,x))\big)\phi_{ij}(u)\big(\Phi_j(s'-s,(s,x))\big)ds'$$
$$+\int_0^s R_j(u)\big(\Phi_j(s'-s,(s,x))\big)ds'. \tag{4.20}$$

Consider two points $(t,x),(\tau,y)\in[0,T]\times\mathbb{R}$ and without loss of generality suppose $\tau\le t$. To get an estimate for $\max_{j=1,\ldots,N}\{|w_j(\tau,y)-w_j(t,x)|\}$, first we write

$$|w_j(\tau,y)-w_j(t,x)|\le|w_j(\tau,y)-w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)|+|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)-w_j(t,x)|. \tag{4.21}$$

It follows from (4.19), by integration along of the curve $\Phi_j(\cdot-t,(t,x))$, that

$$|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)-w_j(t,x)|\le\sum_{i=1}^{N}\int_\tau^t\Big|w_i\Big(\Phi_j\big(s-t,(t,x)\big)\Big)\phi_{ij}(u)\Big(\Phi_j\big(s-t,(t,x)\big)\Big)\Big|\,ds$$
$$+\int_\tau^t\Big|R_j(u)\Big(\Phi_j\big(s-t,(t,x)\big)\Big)\Big|\,ds,$$

using (3.3) we have

$$|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)-w_j(t,x)|\le C\max_{i=1,\ldots,N}\{\|w_i\|_\infty\}|t-\tau|+\max_{i=1,\ldots,N}\{\|R_i(u)\|_\infty\}|t-\tau|$$
$$\le C\max_{i=1,\ldots,N}\{\|w_i\|_\infty\}|(t,x)-(\tau,y)|+\max_{i=1,\ldots,N}\{\|R_i(u)\|_\infty\}|(t,x)-(\tau,y)|,$$

taking the maximum we obtain

$$\max_{j=1,\ldots,N}\Big\{|w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)-w_j(t,x)|\Big\}\le C\max_{i=1,\ldots,N}\{\|w_i\|_\infty\}|(t,x)-(\tau,y)|$$
$$+\max_{i=1,\ldots,N}\{\|R_i(u)\|_\infty\}|(t,x)-(\tau,y)|. \tag{4.22}$$

To estimate $\max_{j=1,\ldots,N}\Big\{|w_j(\tau,y)-w_j\Big(\Phi_j\big(\tau-t,(t,x)\big)\Big)|\Big\}$, we consider the cases:

- **I.** $y\le\pi_2(\phi_N(\tau-t,(t,x)))$;
- **II.** $\pi_2(\phi_1(\tau-t,(t,x)))\le y$;
- **III.** $\pi_2(\phi_N(\tau-t,(t,x)))\le y\le\pi_2(\phi_1(\tau-t,(t,x)))$.

For **I** we consider

$$\tilde{M}_{\mathbf{I}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{I}s}} \left\{ \max_{j=1,\dots,N} \left\{ \sup_{(s,a),(s,b)\in I} \{|w_j(s,a) - w_j(s,b)|\} \right\} \right\}.$$

Let $I \in \Lambda_{\mathbf{I}s}$ (see previous section) and $(s,a),(s,b)$ points in I. Applying (4.20) at the points (s,a) and (s,b) and taking the difference we have

$$\begin{aligned} |w_j(s,a) - w_j(s,b)| &\le |w_j\big(\Phi_j(-s,(s,a))\big) - w_j\big(\Phi_j(-s,(s,b))\big)| \\ &+ C\int_0^s \max_{i=1,\dots,N} \left\{ |w_i\big(\Phi_j(s'-s,(s,a))\big) - w_i\big(\Phi_j(s'-s,(s,b))\big)| \right\} ds' \\ &+ C\int_0^s \max_{i=1,\dots,N} \{\|w_i\|_\infty\} \, |\Phi_j(s'-s,(s,a)) - \Phi_j(s'-s,(s,b))| ds' \\ &+ \int_0^s \max_{i=1,\dots,N} \left\{ \|\big(R_i(u)\big)'\|_\infty \right\} |\Phi_j(s'-s,(s,a)) - \Phi_j(s'-s,(s,b))| ds'. \end{aligned} \tag{4.23}$$

Since $[\Phi_j(s'-s,(s,a)), \Phi_j(s'-s,(s,b))] \in \Lambda_{\mathbf{I}s'}$, we have

$$\max_{i=1,\dots,N} \left\{ |w_i\big(\Phi_j(s'-s,(s,a))\big) - w_i\big(\Phi_j(s'-s,(s,b))\big)| \right\} \le \tilde{M}_{\mathbf{I}}(s'),$$

also we have $|\Phi_j(s'-s,(s,a)) - \Phi_j(s'-s,(s,b))| \le C|(\tau,y)-(t,x)|$, therefore from (4.23) we get

$$\begin{aligned} \tilde{M}_{\mathbf{I}}(s) &\le \tilde{M}_{\mathbf{I}}(0)\exp(sC) \\ &+ C\left(\max_{i=1,\dots,N}\{\|w_i\|_\infty\} + \max_{i=1,\dots,N}\left\{ \|\big(R_i(u)\big)'\|_\infty \right\} \right) |(t,x)-(\tau,y)| \exp(sC). \end{aligned}$$

For the case **II** we consider

$$\tilde{M}_{\mathbf{II}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{II}s}} \left\{ \max_{j=1,\dots,N} \left\{ \sup_{(s,a),(s,b)\in I} \{|w_j(s,a) - w_j(s,b)|\} \right\} \right\}.$$

Similarly to the case **I** we obtain

$$\begin{aligned} \tilde{M}_{\mathbf{II}}(s) &\le \tilde{M}_{\mathbf{II}}(0)\exp(sC) \\ &+ C\left(\max_{i=1,\dots,N}\{\|w_i\|_\infty\} + \max_{i=1,\dots,N}\left\{ \|\big(R_i(u)\big)'\|_\infty \right\} \right) |(t,x)-(\tau,y)| \exp(sC). \end{aligned}$$

In the case **III** we consider

$$\tilde{M}_{\mathbf{III}}(s) \doteq \sup_{I\in\Lambda_{\mathbf{III}s}} \left\{ \max_{j=1,\dots,N} \left\{ \sup_{a,b\in I} \{|w_j(s,a) - w_j(s,b)|\} \right\} \right\}$$

therefore we get

$$\begin{aligned} \tilde{M}_{\mathbf{III}}(s) &\le \tilde{M}_{\mathbf{III}}(0)\exp(sC) \\ &+ \left(\max_{i=1,\dots,N}\{\|w_i\|_\infty\} + \max_{i=1,\dots,N}\left\{ \|\big(R_i(u)\big)'\|_\infty \right\} \right) |(t,x)-(\tau,y)| \exp(sC). \end{aligned}$$

Using (4.21), (4.22) and the estimates for $\tilde{M}_{\mathbf{I}}(s)$, $\tilde{M}_{\mathbf{II}}(s)$ and $\tilde{M}_{\mathbf{III}}(s)$ we obtain

$$\begin{aligned} \max_{j=1,\dots,N}\{|w_{\lambda,n,[\rho],j}(\tau,y) - w_{\lambda,n,[\rho],j}(t,x)|\} &\le B\Bigg[M_{\lambda,n,[\rho]}\big((t,x),(\tau,y)\big) \\ + \left(\max_{j=1,\dots,N}\{\|w_{\lambda,n,[\rho],j}\|_\infty\} + \max_{j=1,\dots,N}\left\{ \|\big(R_j(u_{\lambda,n})\big)'\|_\infty \right\} \right) &|(t,x)-(\tau,y)| \Bigg], \end{aligned} \tag{4.24}$$

with

$$M_{\lambda,n,[\rho]}\big((t,x),(\tau,y)\big) \doteq \sup_{I\in\Lambda_{(t,x),(\tau,y)}} \left\{ \max_{j=1,\ldots,N} \left\{ \sup_{a,b\in I}\{|w_{\lambda,n,[\rho],j}(0,a) - w_{\lambda,n,[\rho],j}(0,b)|\} \right\}\right\},$$

$$\Lambda_{(t,x),(\tau,y)} \doteq \Big\{ I\subset\mathbb{R} : I \text{ is an interval and } |I| \le 2A_1\exp(AT)|(t,x)-(\tau,y)| \Big\}$$

and B is a constant.

To verify that u satisfies (1.2) it is sufficient to show that exist a constant C, not depending the λ and n, so that

$$\sup_{0<|x-y|<1} \max_{j=1,\ldots,N} \left\{ \frac{|w_{\lambda,n,[\rho],j}(t,x) - w_{\lambda,n,[\rho],j}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\} \le C\,\|\boldsymbol{u}_0\|_\rho . \tag{4.25}$$

From (4.24) follows that

$$\begin{aligned}\max_{j=1,\ldots,N} \left\{ \frac{|w_{\lambda,n,[\rho],j}(t,x) - w_{\lambda,n,[\rho],j}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\} &\le B\frac{1}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} M_{\lambda,[\rho],n}\big((t,x),(\tau,y)\big)\\ &+ B\max_{j=1,\ldots,N}\{\|w_{\lambda,n,[\rho],j}\|_\infty\} \frac{|(t,x)-(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}}. \end{aligned}\tag{4.26}$$

Since for any x and y such that $0<|x-y|<1$ we have

$$\begin{aligned}&\sup_{a,b\in I}\left\{ \frac{|w_{\lambda,n,[\rho],j}(0,a) - w_{\lambda,n,[\rho],j}(0,b)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\}\\ &\qquad\le \sup_{a,b\in I}\left\{ \frac{|w_{\lambda,n,[\rho],j}(0,a) - w_{\lambda,n,[\rho],j}(0,b)|}{|a-b|^{\rho-[\rho]}} \frac{|a-b|^{\rho-[\rho]}}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\}\\ &\qquad\le \sup_{a,b\in I}\left\{ \frac{|w_{\lambda,n,[\rho],j}(0,a) - w_{\lambda,n,[\rho],j}(0,b)|}{|a-b|^{\rho-[\rho]}} \big(2A_1\exp(AT)\big)^{\rho-1} \frac{|(t,x)-(\tau,y)|^{\rho-[\rho]}}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\}\\ &\qquad\le C\big(2A_1\exp(AT)\big)^{\rho-1}\,\|\boldsymbol{u}_{0,\lambda,n}\|_\rho ,\end{aligned}$$

for all $I\in\Lambda_{(t,x),(\tau,y)}$. Therefore

$$\sup_{0<|x-y|<1}\left\{\frac{1}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} M_{\lambda,[\rho],n}\big((t,x),(\tau,y)\big)\right\} \le C\big(2A_1\exp(AT)\big)^{\rho-1}\|\boldsymbol{u}_{0,\lambda,n}\|_\rho . \tag{4.27}$$

Also we have

$$\sup_{0<|x-y|<1}\left\{ B\max_{j=1,\ldots,N}\{\|w_{\lambda,n,[\rho],j}\|_\infty\} \frac{|(t,x)-(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\} \le C\,\|\boldsymbol{u}_{0,\lambda,n}\|_\rho , \tag{4.28}$$

by (3.3). From (4.26), (4.27) and (4.28) we obtain

$$\sup_{0<|x-y|<1}\left\{ \max_{j=1,\ldots,N}\left\{ \frac{|w_{\lambda,n,[\rho],j}(t,x) - w_{\lambda,n,[\rho],j}(\tau,y)|}{|(t,x)-(\tau,y)|^{\rho-[\rho]}} \right\}\right\} \le C\,\|\boldsymbol{u}_{0,\lambda,n}\|_\rho \quad \forall t\in[0,T],$$

which implies (4.25), because $\|\boldsymbol{u}_{0,\lambda,n}\|_\rho \le C\|\boldsymbol{u}_0\|_\rho$.

■

5. **Proof of Theorem 3.**

From Lemma 9 and Lemma 5 we have that there is a $(\boldsymbol{u}_{0n})_{n\in\mathbb{N}}$ of smooth functions converging to $\boldsymbol{u}_0$ in $B^\rho_{\infty,r}$ and furthermore satisfies:

$$\|\boldsymbol{u}_{0n}\|_k \le \|\boldsymbol{u}_0\|_k \quad \forall\, k = 1,\ldots,[\rho]$$

and

$$\|\boldsymbol{u}_{0n}\|_{B^\rho_{\infty,r}} \le \|\boldsymbol{u}_0\|_{B^\rho_{\infty,r}}.$$

Applying the Theorem 1 to each term of the sequence $(\boldsymbol{u}_{0n})_{n\in\mathbb{N}}$ we obtain a new sequence, $(u_n)_{n\in\mathbb{N}}$, where each term u_n is a solution of the Cauchy problem

$$\left\{\begin{array}{rcl} \partial_t u_n + a(u_n)\partial_x u_n &=& 0 \\ u_n(0,x) &=& \boldsymbol{u}_{0n}(x), \end{array}\right. \tag{5.29}$$

be defined in $[0,T]\times\mathbb{R}$ and satisfies

$$\|u_n(t,\cdot)\|_k \le C_k\|\boldsymbol{u}_0\|_k \quad \forall\, k=1,\cdots,[\rho] \text{ and } n=1,2,\ldots,$$

for some constants C_k independents of n. For simplify the notation in the below computations we denote a term of the sequence $(u_n)_{n\in\mathbb{N}}$ by u. Taking $w=\partial_x u$ from (5.29) we get

$$\partial_t\partial_x u + a(u)\partial_x^2 u = -\left(a'(u)\partial_x u\right)\partial_x u. \tag{5.30}$$

Writing

$$w=\sum_{i=1}^N w_i r_i(u),$$

from (5.30) follows that

$$(\partial_t + \lambda_j(u)\partial_x)w_j = \sum_{i,l=1}^N \phi_{ilj}(u) w_i w_l, \tag{5.31}$$

where $\phi_{ilj}(u)$ satisfies

$$-\Big\{\left[a'(u)r_i(u)\right] r_l(u) - r_i'(u)[a(u)r_l(u)] + a(u)[r_i'(u)r_l(u)]\Big\} = \sum_{j=1}^N \phi_{ilj}(u) r_j(u).$$

The Bony's decomposition gives us

$$\lambda_j(u)\partial_x w_j = T_{\lambda_j(u)}\partial_x w_j + T_{\partial_x w_j}\lambda_j(u) + R(\lambda_j(u),\partial_x w_j), \tag{5.32}$$

applying Δ_q and arranging the terms appropriately we obtain

$$\begin{aligned}
\Delta_q(T_{\lambda_j(u)}\partial_x w_j) &= \Delta_q\left(\sum_{q'} S_{q'-1}\lambda_j(u)\Delta_{q'}\partial_x w_j\right)\\
&= \Delta_q\left(\sum_{|q-q'|\le 4} S_{q'-1}\lambda_j(u)\Delta_{q'}\partial_x w_j\right)\\
&= R_q^1(\lambda_j(u),\partial_x w_j) + \sum_{|q-q'|\le 2} S_{q'-1}\lambda_j(u)\Delta_q\Delta_{q'}\partial_x w_j\\
&= R_q^1(\lambda_j(u),\partial_x w_j) + R_q^2(\lambda_j(u),\partial_x w_j) + S_{q-1}\lambda_j(u)\Delta_q\partial_x w_j,
\end{aligned}$$

that is,

$$\Delta_q(T_{\lambda_j(u)}\partial_x w_j) = R_q^1(\lambda_j(u),\partial_x w_j) + R_q^2(\lambda_j(u),\partial_x w_j) + S_{q-1}\lambda_j(u)\Delta_q\partial_x w_j, \tag{5.33}$$

where

$$R_q^1(\lambda_j(u),\partial_x w_j) = \sum_{|q-q'|\le 4}[\Delta_q, S_{q'-1}\lambda_j(u)]\Delta_{q'}\partial_x w_j$$

and

$$R_q^2(\lambda_j(u),\partial_x w_j) = \sum_{|q-q'|\le 2}\left(S_{q'-1}\lambda_j(u) - S_{q-1}\lambda_j(u)\right)\Delta_{q'}\Delta_q\partial_x w_j.$$

From (5.31), (5.32) and (5.33) it follows

$$(\partial_t + S_{q-1}\lambda(u)\partial_x)\Delta_q w_j = -\Delta_q\left(T_{\partial_x w_j}\lambda_j(u)\right) - \Delta_q\left(R(\lambda_j(u),\partial_x w_j)\right) - R_q^1(\lambda_j(u),\partial_x w_j) - R_q^2(\lambda_j(u),\partial_x w_j) + \sum_{i,l=1}^{N}\Delta_q\left(\phi_{ilj}(u)w_i w_l\right). \tag{5.34}$$

Now, we will estimate each term of the right hand side of (5.34). From Bony's decomposition we have

$$\Delta_q\left(\phi_{ilj}(u)w_i w_l\right) = \Delta_q\left(T_{\phi_{ilj}(u)w_i}w_l\right) + \Delta_q\left(T_{w_l}\phi_{ilj}(u)w_i\right) + \Delta_q R\left(\phi_{ilj}(u)w_i, w_l\right). \tag{5.35}$$

Since $(\phi_{ilj}(u)w_i, w_l) \in L^\infty \times B^{\rho-1}_{\infty,r}$ it follows, from Theorem 6, that $T_{\phi_{ilj}(u)w_i}w_l \in B^{\rho-1}_{\infty,r}$ and

$$\|T_{\phi_{ilj}(u)w_i}w_l\|_{B^{\rho-1}_{\infty,r}} \le C\|\phi_{ilj}(u)w_i\|_\infty\|w_l\|_{B^{\rho-1}_{\infty,r}}.$$

So we have

$$\begin{aligned}\|\Delta_q\left(T_{\phi_{ilj}(u)w_i}w_l\right)\|_\infty &= 2^{-q(\rho-1)}\frac{2^{q(\rho-1)}\|\Delta_q\left(T_{\phi_{ilj}(u)w_i}w_l\right)\|_\infty}{\|T_{\phi_{ilj}(u)w_i}w_l\|_{B^{\rho-1}_{\infty,r}}}\|T_{\phi_{ilj}(u)w_i}w_l\|_{B^{\rho-1}_{\infty,r}}\\ &\le 2^{-q(\rho-1)}C\frac{2^{q(\rho-1)}\|\Delta_q\left(T_{\phi_{ilj}(u)w_i}w_l\right)\|_\infty}{\|T_{\phi_{ilj}(u)w_i}w_l\|_{B^{\rho-1}_{\infty,r}}}\|\phi_{ilj}(u)w_i\|_\infty\|w_l\|_{B^{\rho-1}_{\infty,r}}.\end{aligned}$$

Since $\|\phi_{ilj}(u)w_i\|_\infty \le C$ we have

$$\|\Delta_q\left(T_{\phi_{ilj}(u)w_i}w_l\right)\|_\infty \le 2^{-q(\rho-1)}Cc_q\sup_{i=1,\ldots,N}\{\|w_i\|_{B^{\rho-1}_{\infty,r}}\}, \tag{5.36}$$

with the sequence $(c_q(t))_q \in l^r$ and satisfying $\|c_q(t)\|_{l^r} = 1 \quad \forall\, t \in [0,T]$, by definition of $B^{\rho-1}_{\infty,r}$. For estimate the term $\Delta_q\left(T_{w_l}\phi_{ilj}(u)w_i\right)$, we begin observing that from Theorem 6 and Theorem 7 follow

$$\|\phi_{ilj}(u)w_i\|_{B^{\rho-1}_{\infty,r}} \le C\left(\|\phi_{ilj}(u)\|_\infty\|w_i\|_{B^{\rho-1}_{\infty,r}} + \|w_i\|_\infty\|\phi_{ilj}(u)\|_{B^{\rho-1}_{\infty,r}}\right).$$

Since the terms $\|\phi_{ilj}(u)\|_\infty$ and $\|\phi_{ilj}(u)\|_{B^{\rho-1}_{\infty,r}}$ are uniformly bounded and $\|w_i\|_\infty \le \|w_i\|_{\rho-1}$ we have

$$\|\phi_{ilj}(u)w_i\|_{B^{\rho-1}_{\infty,r}} \le C\sup_{i=1,\ldots,N}\{\|w_i\|_{B^{\rho-1}_{\infty,r}}\},$$

so

$$\begin{aligned}\|\Delta_q\left(T_{w_l}\phi_{ilj}(u)w_i\right)\|_\infty &= 2^{-q(\rho-1)}\frac{2^{q(\rho-1)}\|\Delta_q\left(T_{w_l}\phi_{ilj}(u)w_i\right)\|_{B^{\rho-1}_{\infty,r}}}{\|T_{w_l}\phi_{ilj}(u)w_i\|_{B^{\rho-1}_{\infty,r}}}\|\phi_{ilj}(u)w_i\|_{B^{\rho-1}_{\infty,r}}\\ &\le 2^{-q(\rho-1)}Cc_q(t)\sup_{i=1,\ldots,N}\{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\},\end{aligned} \tag{5.37}$$

with $(c_q(t))_q \in l^r$ and $\|(c_q(t))_q\|_{l^r} = 1$. Analogously we obtain also

$$\|\Delta_q R(\phi_{ilj}(u)w_i, w_l)(t,\cdot)\|_\infty \le 2^{-q(\rho-1)}Cc_q(t)\sup_{i=1,\ldots,N}\{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\}, \tag{5.38}$$

with $(c_q(t))_q \in l^r$ and satisfying $\|(c_q(t))_q\|_{l^r} = 1$. From (5.35), (5.36), (5.37) and (5.38) it follows that

$$\|\Delta_q(\phi_{ilj}(u)w_i w_l)(t,\cdot)\|_\infty \le 2^{-q(\rho-1)}Cc_q(t)\sup_{i=1,\ldots,N}\{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\},$$

with $(c_q(t))_q \in l^r$ and satisfying $\|(c_q(t))_q\|_{l^r} = 1\ \forall t \in [0,T]$. If we follow the same steps used to obtain the estimates (5.36) and (5.37) we get

$$\|\Delta_q\left(T_{\partial_x w_j}\lambda_j(u)\right)(t,\cdot)\|_\infty \le 2^{-q(\rho-1)}Cc_q(t)\sup_{i=1,\ldots,N}\{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\}$$

and

$$\|\Delta_q R(\lambda_j(u), \partial_x w_j)(t,\cdot)\|_\infty \leq 2^{-q(\rho-1)} C c_q(t) \sup_{i=1,\ldots,N} \{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\},$$

with $(c_q(t))_q \in l^r$ and satisfying $\|(c_q(t))_q\|_{l^r} = 1 \; \forall t \in [0,T]$.

To estimate $R^1_q(\lambda_j(u), \partial_x w_j)$ first we note (thanks to Lema 8) that there exists constant C such that

$$\| [\Delta_q, S_{q'-1}\lambda_j(u)] \, \partial_x \Delta_{q'} w_j\|_\infty \leq 2^{-q} C \|\nabla (S_{q'-1}\lambda_j(u))\|_\infty \|\partial_x \Delta_{q'} w_j\|_\infty.$$

Using the Lemma 4 we obtain $\|\Delta_{q'} \partial_x w_j\|_\infty \leq 2^{q'} C \|\Delta_{q'} w_j\|_\infty$ therefore we have

$$\| [\Delta_q, S_{q'-1}\lambda_j(u)] \, \partial_x \Delta_{q'} w_j\|_\infty \leq 2^{-q} 2^{q'} C \|\nabla (S_{q'-1}\lambda_j(u))\|_\infty \|\Delta_{q'} w_j\|_\infty.$$

Since $|q - q'| \leq 4$ and $\|\nabla (S_{q'-1}\lambda_j(u))\|_\infty$ is uniformly bounded it follows that

$$\begin{aligned} \| [\Delta_q, S_{q'-1}\lambda_j(u)] \, \partial_x \Delta_{q'} w_j\|_\infty &\leq C \|\Delta_{q'} w_j\|_\infty \\ &= C 2^{-q'(\rho-1)} 2^{q'(\rho-1)} \|\Delta_{q'} w_j\|_\infty \\ &= C 2^{-q'(\rho-1)} \frac{2^{q'(\rho-1)} \|\Delta_{q'} w_j\|_\infty}{\|w_j\|_{B^{\rho-1}_{\infty,r}}} \|w_j\|_{B^{\rho-1}_{\infty,r}}, \end{aligned}$$

therefore we have

$$\|R^1_q(\lambda_j(u), \partial_x w_j)(t,\cdot)\|_\infty \leq 2^{-q(\rho-1)} C c_q(t) \sup_{i=1,\ldots,N} \{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\},$$

with $(c_q(t))_q \in l^r$ satisfying $\|(c_q(t))_q\|_{l^r} = 1 \; \forall t \in [0,T]$.

To estimate $R^2_q(\lambda_j(u), \partial_x w_j)$ first we write

$$R^2_q(\lambda_j(u), \partial_x w_j) = -\Delta_{q-2}\lambda_j(u) \Delta_q \Delta_{q-1} \partial_x w_j + \Delta_{q-1}\lambda_j(u) \Delta_q \Delta_{q+1} \partial_x w_j.$$

From properties of the operators Δ_q and of the Lemma 4 we have

$$\begin{aligned} \|\Delta_{q-2}\lambda_j(u)\|_\infty &\leq 2^{-(q-2)(\rho-1)} \|\lambda_j(u)\|_{\rho-1} \\ &\leq 2^{-(q-2)(\rho-1)} \|\lambda_j(u)\|_{[\rho]} \\ &\leq 2^{-(q-2)(\rho-1)} C \end{aligned}$$

and

$$\|\Delta_q \Delta_{q+1} \partial_x w_j\|_\infty \leq 2^{(q-1)} C \|\Delta_q w_j\|_\infty,$$

so we have

$$\begin{aligned} \|\Delta_{q-2}\lambda_j(u) \Delta_q \Delta_{q-1} \partial_x w_j\|_\infty &\leq \|\Delta_{q-2}\lambda_j(u)\|_\infty \|\Delta_q \Delta_{q-1} \partial_x w_j\|_\infty \\ &\leq C \|\Delta_q w_j\|_\infty \\ &= 2^{-q(\rho-1)} C \frac{2^{q(\rho-1)} \|\Delta_q w_j\|_\infty}{\|w_j\|_{B^{\rho-1}_{\infty,r}}} \|w_j\|_{B^{\rho-1}_{\infty,r}}. \end{aligned}$$

Which gives us

$$\|R^2_q(\lambda_j(u), \partial_x w_j)(t,\cdot)\|_\infty \leq 2^{-q(\rho-1)} C c_q(t) \sup_{i=1,\ldots,N} \{\|w_i(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\},$$

with $(c_q(t))_q \in l^r$ and satisfying $\|(c_q(t))_q\|_{l^r} = 1 \; \forall t \in [0,T]$.

Now we will estimate the terms of Littlewood-Paley's decomposition of the w_j's. We consider $\gamma(t) = (t, \gamma_2(t))$ an integral curve of the field $\partial_t + S_{q-1}\lambda_j(u)\partial_x$. From the estimates obtained for the terms on the right side of (5.34) we obtain, integrating both sides of (5.34) over γ, that

$$|\Delta_q w_j(t, \gamma_2(t))| \leq |\Delta_q w_j(0, \gamma_2(0))| + 2^{-q(\rho-1)} C \int_0^t c_q(s) \sup_{i=1,\ldots,N} \{\|w_i(s,\cdot)\|_{B^{\rho-1}_{\infty,r}}\} ds \quad (5.39)$$

with $(c_q(s))_q \in l^r$, satisfying $\|(c_q(s))_q\|_{l^r} = 1 \; \forall s \in [0,T]$. Writing

$$\boldsymbol{u}_0' = \boldsymbol{w}_0 = \sum_{j=1}^{N} \boldsymbol{w}_{0j} r_j(u(0,\cdot))$$

it follows that $|\Delta_q w_j(0,\gamma_2(0))| = |\Delta_q \boldsymbol{w}_{0j}(\gamma_2(0))|$. Using the fact that the inequality (5.39) is true independent of γ we obtain

$$\|\Delta_q w_j(t,\cdot)\|_\infty \leq \|\Delta_q \boldsymbol{w}_{0j}\|_\infty + 2^{-q(\rho-1)} C \int_0^t c_q(s) \sup_{i=1,\ldots,N} \{\|w_i(s,\cdot)\|_{B^{\rho-1}_{\infty,r}}\} ds,$$

taking the supreme over all integral curve γ. Multiplying both sides of this inequality by $2^{q(\rho-1)}$ and taking the l^r-norm it follows

$$\|w_j(t,\cdot)\|_{B^{\rho-1}_{\infty,r}} \leq \|\boldsymbol{w}_{0j}\|_{B^{\rho-1}_{\infty,r}} + \left\| \left(C \int_0^t c_q(s) \sup_{i=1,\ldots,N} \{\|w_i(s,\cdot)\|_{B^{\rho-1}_{\infty,r}}\} ds \right)_q \right\|_{l^r}.$$

Since $\|(c_q(s))_q\|_{l^r} = 1$, from Minkowski's inequality for integrals we obtain

$$\sup_{i=1,\ldots,N} \{\|w_j(t,\cdot)\|_{B^{\rho-1}_{\infty,r}}\} \leq \sup_{i=1,\ldots,N} \{\|\boldsymbol{w}_{0i}\|_{B^{\rho-1}_{\infty,r}}\} + C \int_0^t \sup_{i=1,\ldots,N} \{\|w_i(s,\cdot)\|_{B^{\rho-1}_{\infty,r}}\} ds.$$

Finally using Gronwall's lemma we get

$$\|w(t,\cdot)\|_{B^{\rho-1}_{\infty,r}} \leq C \|\boldsymbol{w}_0\|_{B^{\rho-1}_{\infty,r}} \exp(tC).$$

From definition of $B^{\rho-1}_{\infty,r}$ and of the Lemma 4 we have

$$\|u_n(t,\cdot)\|_{B^{\rho}_{\infty,r}} \leq C \|\boldsymbol{u}_0\|_{B^{\rho}_{\infty,r}} \quad \forall\, t \in [0,T], n \in \mathbb{N}.$$

From Lemma 5 it follows that $(u_n)_{n\mathbb{N}}$ converge for $u \in B^{\rho}_{\infty,r}$ which is solution of the Cauchy problem, concluding the proof of the Theorem 3.

■

References

[1] H.Bahouri, J.Y.Chemin, R.Danchin, *Fourier Analysis and Nonlinear Partial Differential Equation*, Springer-Verlag, 2011.

[2] L.Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, 1997.

[3] J.Hounie, J.R.dos Santos Filho, *Well posed Cauchy problem for complex nonlinear equations must be semilinear*, Math.Ann., 294 (1992), pp 439-447.

[4] G.Métivier, *Para-Differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems*, Edizioni della Normale, 2008.

[5] G. Métivier, *Remarks on the Well-Posedness of the Nonlinear Cauchy Problem*, Geometric analysis of PDE and several complex variables, Comtemp. Math., 368, Amer. Math. Society (2005), pp. 337-356.

[6] R. da R. da Silva, *O Problema de Cauchy para sistemas quase-lineares é bem posto em espaços de Hölder*, Thesis presented at DM-UFSCar, see http://www.dm.ufscar.br/ppgm/attachments/article/179/4444.pdf (In portuguese).

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