Stable Transition Layers to Singularly Perturbed Spatially Inhomogeneous Allen-Cahn Equation

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Abstract

We use the variational concept of Γ-convergence to prove existence, stability and exhibit the geometric structure of four families of stationary solutions to the singularly perturbed parabolic equation $u_t = \epsilon^2 \triangle u + u[a^2(x) - u^2]$, for $(t, x) \in \mathbb{R}^+ \times \Omega$, where $\Omega \subset \mathbb{R}^2$ and a is positive, supplied with no-flux boundary condition. Let $\gamma \subset \Omega$ be a smooth simple closed curve and $\mathcal{N}(\gamma)$ a narrow tubular neighborhood of γ. Roughly speaking, the sufficient condition found for existence of such solutions relates the geometric profile of the function a in $\mathcal{N}(\gamma)$ to the signed curvature of γ.

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1 Introduction and statement of the main results

This work concerns existence of stable stationary solutions which develop transition layers as $\varepsilon \to 0$ -sometimes referred to as stable transition layers, for short- to the following scalar diffusion equation

$$\left.\begin{array}{ll} u_t = \epsilon^2 \triangle u + f(u,x) & (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_n u(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \partial\Omega \end{array}\right\} \tag{1.1}$$

where

$$f(u,x) = u\,[a^2(x) - u^2]$$

and $\Omega \subset \mathbb{R}^2$ is a C^2 bounded domain; ϵ is a small positive parameter; n denotes the outer unit normal to $\partial\Omega$ and $a \in C^1(\overline{\Omega})$ is a suitable positive function to be chosen.

There are some works regarding existence of stable transition layers for (1.1) when the nonlinear term is given by the prototype spatially inhomogeneous bistable functions $f(u,x) = u(u-1)(a(x)-u)$ with $0 < a(x) < 1$ (see e.g., [18], [1], [2], [9]) or $f(u,x) = a(x)u(1-u^2)$ with a a smooth positive function (see e.g., [8], [14], [3], [11]). However regarding just transition layers or, more generally, stable stationary solutions in the absence of the small parameter ε then the literature is vast.

As opposed to the above cases, problem (1.1), seems to have been addressed in fewer works. For instance, in [12] for $\Omega = (0\,,1)$ and

$$f(u,x) = -(u - a(x))(u - b(x))(u - c(x))$$

where $b = (a+c)/2$ and $a(x) < b(x) < c(x)$, $\forall\, x \in [0,\,1]$, the author used the method of upper and lower solutions to prove existence of stable transitions layers near points where $c(x) - a(x)$ has its non-degenerate strict local minima. Herein this result is going to be generalized to two-dimensional domains using the variational concept of $\Gamma-$convergence. This approach provides convergence, as $\varepsilon \to 0$, of the stable transition layers in $L^1(\Omega)$ rather than uniform convergence in compact sets outside the limiting interface.

Still for one-dimensional domains and $f(u,x) = u\,[a^2(x) - u^2]$ this very same problem was addressed in [10] for the case in which a' vanishes on an interval contained in (0, 1).

Before stating our main result we need some notation. As usual χ_A denotes the characteristic function of a set A. Let

- $\gamma(s)$, $0 \le s \le L$, be an arc-length parametrized C^2 simple closed curve in Ω,
- $\kappa(s)$, $0 \le s \le L$ denote its signed curvature,
- $\Sigma : Q_\delta \longrightarrow N_\delta$ be the change of coordinates given by $(x_1, x_2) = \Sigma(s,t) = \gamma(s) + t\nu(s)$ where N_δ denotes a tubular neighborhood around γ, $Q_\delta = \{(s,t) \in \mathbb{R}^2 : 0 < s < L,\ -\delta < t < \delta\}$ and $\nu(s)$ is the inward-pointing vector normal to γ (we fix δ small enough so that Σ is a diffeomorphism),
- $\Omega = \Omega^- \cup \gamma \cup \Omega^+$ where Ω^- denotes the open region enclosed by γ and $\Omega^+ = \Omega \setminus [\Omega^- \cup \gamma]$ the outer open region.

Our main result is described in the following theorem.

Theorem 1.1 *Suppose $a > 0$ on $\overline{\Omega}$. In Ω define the functions*

- $u_0^1 = -a\chi_{\Omega^-} + a\chi_{\Omega^+}$
- $u_0^2 = a\chi_{\Omega^-} - a\chi_{\Omega^+}$

- $u_0^3 = -a$
- $u_0^4 = a$

and set $\tilde{a}(s,t) = a(\Sigma(s,t))$ *in* Q_δ. *If*

$$\left.\begin{array}{lr} \dfrac{\partial \tilde{a}(s,0)}{\partial t} = \dfrac{1}{3}\kappa(s)\,\tilde{a}(s,0) & 0 \le s \le L \\ \dfrac{\partial^2 \tilde{a}(s,0)}{\partial t^2} > \dfrac{4}{9}\kappa^2(s)\,\tilde{a}(s,0) & 0 \le s \le L \end{array}\right\} \tag{1.2}$$

then $\exists\ \epsilon_0 > 0$ *and four families of stable stationary solutions* $\left\{u_\epsilon^j\right\}_{0<\epsilon\le\epsilon_0}$ $(j = 1,\ldots,4)$ *to (1.1) such that*

- $\|u_\epsilon^j - u_0^j\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ $(j = 1,\ldots,4)$ *and*
- $u_\epsilon^j \in C^{2,\alpha}(\overline{\Omega}),\ 0 < \alpha < 1.$

The precise geometric meaning of (1.2) is the following. For each fixed s, the function $\tilde{a}(s,\cdot)$, seen as a function of t alone, has only one minimum in Q_δ. The point where this minimum is achieved keeps switching from one side of γ to the other, as s varies from 0 to L, according to the changing of sign of the curvature of γ. In particular, wherever $\kappa(s) = 0$ the function $\tilde{a}(s,\cdot)$ will assume its minimum value on γ. It is worthwhile to note that γ_m may not be a level curve of $\tilde{a}$ in Q_δ.

Physically, conditions (1.2) mean that the limiting layer-transition curve γ acts as a barrier. It which prevents, say, a diffusing substance whose initial concentration evolves in time according to (1.1), to spread homogeneously in space and eventually settling down in an uniform concentration in Ω.

It should be pointed out that the geometric conditions (1.2) have appeared in the subject of stable transition layers in [3], [4], [15] and [8], for instance, even though the underlying analysis techniques used in these works are quite different in essence. By the way, the third and fourth solutions, $u_0^3 = -a$ and $u_0^4 = a$, are more easily constructed using sub and super-solutions method.

2 Preliminaries on $BV(\Omega)$ and Γ-convergence

In the sequel we compile some definitions and results which are going to be used herein; the interested reader is referred to [19], [13] and [7], for instance, for more on this matter.

Definition 2.1 *Let* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^n, n \ge 1$. *The space* $BV(\Omega)$ *of functions of bounded variation in* Ω *is defined as the set of all functions* $v \in L^1(\Omega)$ *whose distributional gradient* Dv *is a Radon measure with finite total variation in* Ω, *i.e.,*

$$|Dv|(\Omega) = \sup\left\{\int_\Omega v(x)\,div\,\sigma(x)dx : \sigma \in C_0^1(\Omega,\mathbb{R}^n),\ \ |\sigma| \le 1\right\} < \infty.$$

The total variation $|Dv|$ is a measure itself. A Borel set $B \subset \mathbb{R}^n$ has finite perimeter in the open set Ω if

$$\mathrm{Per}_\Omega(B) = |D\chi_B|(\Omega) < \infty,$$

where χ_B is the characteristic function of B.

If h is a continuous positive function and $u \in BV(\Omega)$ then the integral of h with respect to the measure $|Du|$ can be expressed as

$$\int_\Omega h(x)\,|Du| =$$

$$\sup\left\{\int_\Omega u(x)\,\mathrm{div}\,\sigma(x)dx : \sigma \in C_0^1(\Omega, \mathbb{R}^n),\ \ |\sigma(x)| \le h(x)\right\}.$$

Given a set $B \subset \mathbb{R}^n$ with finite perimeter in Ω we will often work with the integral of a positive function h with respect to the Radon measure $|D\chi_B|$, i.e., $\int_\Omega h(x)\,|D\chi_B|$.

We will often use that if $v \in W^{1,1}_{loc}$ and $\mathcal{L}^n$ denotes the Lebesgue measure then the variation measure satisfies $|Dv| = \mathcal{L}^n \llcorner |\nabla v|$, i.e.,

$$|Dv| = |\nabla v|\, d\mathcal{L}^n, \tag{2.1}$$

where ∇ denotes the usual gradient.

The essential boundary of a set $E \subset \mathbb{R}^n$ is the set $\partial_* E$ of all points in Ω where E has neither density 1 nor density 0. If a set $E \subset \Omega$ has finite perimeter in Ω then $\partial_* E$ is rectifiable, and we may endow it with a measure theoretic normal ν_E so that the measure derivative $D\chi_E$ is represented as

$$D\chi_E(B) = \int_{B\cap\partial_* E} \nu_E d\mathcal{H}^{n-1}$$

for every Borel set $B \subset \Omega$ where $\mathcal{H}^n$ stands for the N-dimensional Hausdorff measure.

The following version of the co-area formula will be often used: if $u \in BV(\Omega)$ and f is a continuous function in Ω then

$$\int_\Omega f\,|Du| = \int_{-\infty}^{\infty}\left[\int_{\Omega\cap\partial_*\{x\in\Omega: u(x)>\xi\}} f d\mathcal{H}^{n-1}\right]d\xi \tag{2.2}$$

In the sequel $BV(\Omega, \{\alpha, \beta\}$ will denote the space of functions of bounded variation in Ω which takes values α and β only. If $u \in BV(\Omega, \{\alpha, \beta\}$ then

$$\partial_*\{x \in \Omega : u(x) = \alpha\} \cap \Omega$$

is a rectifiable set.

Next the working definition of the $\Gamma-$ convergence of a family of functionals with respect to the L^1 topology is given.

Definition 2.2 *A family $\{E_\epsilon\}_{\epsilon>0}$ of real-extended functionals defined in $L^1(\Omega)$ is said to Γ-converge, as $\epsilon \to 0$, to a functional E_0 and we write*

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(v) = E_0(v)$$

if:

(i) $\forall\ v \in L^1(\Omega)$ *and* $\forall\ \{v_\epsilon\} \subset L^1(\Omega)$: $v_\epsilon \to v$ *in* $L^1(\Omega)$, *as* $\epsilon \to 0 \quad \Rightarrow E_0(v) \leq \liminf_{\epsilon\to 0} E_\epsilon(v_\epsilon)$.

(ii) $\forall\ v \in L^1(\Omega)$, $\exists\ \{v_\epsilon\}$ *in* $L^1(\Omega)$: $v_\epsilon \to v$ *in* $L^1(\Omega)$, *as* $\epsilon \to 0$, *and* $E_0(v) \geq \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon)$.

Definition 2.3 *We shall call $v_0 \in L^1(\Omega)$ a L^1-local minimizer of E_0 if there is $\mu > 0$ such that*

$$E_0(v_0) \leq E_0(v) \quad \textit{whenever} \quad 0 < \|v - v_0\|_{L^1(\Omega)} < \mu.$$

Moreover if $E_0(v_0) < E_0(v)$ for $0 < \|v - v_0\|_{L^1(\Omega)} < \mu$, then v_0 is called an isolated L^1-local minimizer of E_0.

The following theorem can be found in [17] and is crucial to our analysis.

Theorem 2.1 ([17]) *Let $\{E_\epsilon\}_{\epsilon>0}$ be a family of real-extended functionals defined in $L^1(\Omega)$. Suppose that $\Gamma - \lim_{\epsilon\to 0} E_\epsilon = E_0$ and the following hypotheses are satisfied*

(i) Any sequence $\{v_\epsilon\}_{\epsilon>0}$ such that $E_\epsilon(v_\epsilon) \leq C < \infty$, for all $\epsilon > 0$ and a positive constant C, is compact in L^1.

(ii) There exists an isolated L^1-local minimizer u_0 of E_0.

Then $\exists\, \epsilon_0 > 0$ and a family $\{v_\epsilon\}_{0<\varepsilon<\epsilon_0}$ such that

- *v_ϵ is an $L^1(\Omega)$-local minimizer of E_ϵ and*
- *$\|v_\epsilon - v_0\|_{L^1(\Omega)} \to 0$, as $\epsilon \to 0$.*

3 Existence and stability of stationary solutions.

The idea of the proof is to seek local minimizers of the energy functionals which will turn out to be stable stationary solution to (1.1), and to that purpose we will resort to Theorem 2.1 whose hypotheses will be verified in the sequel.

Instead of the prototype function $f(u, x) = u[a^2(x) - u^2]$ we could equally well have considered any bistable function whose three roots, say $a(x) < \theta(x) < b(x)$, satisfy $\partial_1 f(a(x), x) < 0$, $\partial_1 f(b(x), x) < 0$, $x \in \Omega$ and the equal area-condition $\int_{a(x)}^{b(x)} f(\xi, x)d\xi = 0$. The cost for this generalization would be a cumbersome notation.

Remark 3.1 For one-dimensional domains the proof is easier and thus omitted; the sufficient condition found, which would replaces (1.2), is that the root a has a isolated local minimum at the point of transition layer, as in [12]. For three-dimensional domains the proof to find L^1-local isolated minimizers of the $\Gamma-$limit problem can be rather complicated. This may be illustrated by a simpler spatial homogeneous variational problem with constraint addressed in [5].

3.1 Existence of four isolated minimizers of the Γ-limit

We start by remarking that stationary solutions to (1.1) are the critical points of the energy functional defined by penalization in $L^1(\Omega)$ by

$$E_\epsilon(u) = \begin{cases} \int_\Omega \left[\epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} F(u,x) \right] dx, & u \in H^1(\Omega) \\ \infty, & \text{otherwise,} \end{cases} \tag{3.1}$$

where

$$F(u,x) = -\int_{-a(x)}^{u} f(\xi, x)\, d\xi = \frac{1}{4}[u^2 - a^2(x)]^2.$$

According to Theorem 2.1 in order to find local minimizers to E_ϵ it suffices, in addition to the compactness condition, to find isolated local minimizers to its $\Gamma-$limit.

The next result concerns the computation of the $\Gamma-$limit of the family of functionals E_ϵ defined by (3.1); its proof can be found in [16].

Theorem 3.1 ([16]) *Consider $E_0 : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$E_0(v) = \begin{cases} \int_\Omega h(x) \left| D\chi_{\{v=a\}} \right|, & v \in BV(\Omega\,; \{-a, a\}) \\ \infty, & \textit{otherwise} \end{cases}$$

where

$$h(x) \stackrel{\text{def}}{=} \int_{-a(x)}^{a(x)} \sqrt{F(s,x)} ds = \frac{1}{\sqrt{2}} \int_{-a(x)}^{a(x)} [a^2(x) - s^2] ds = \frac{2^{3/2}}{3} a^3(x). \tag{3.2}$$

Then

$$\Gamma^- \lim_{\epsilon \to 0} E_\epsilon(v) = E_0(v), \;\; \forall \;\; v \in L^1(\Omega).$$

The functional E_0 can be thought of as a perimeter functional type with weight h. The following elementary remarks regarding the principal coordinates around γ given in the Introduction will be useful. Throughout this section given any function u defined in N_δ, the tubular neighborhood around γ, we will denote

$$\tilde{u}(s,t) = u(\Sigma(s,t)), \;\; (s,t) \in Q_\delta$$

and likewise for any other function. Moreover let

$$J_\Sigma(s,t) \stackrel{\text{def}}{=} (1 - t\kappa(s)) > 0$$

be the Jacobian of the diffeomorphism Σ given in the Introduction.

Lemma 3.1 *Suppose that a satisfies (1.2) on the curve γ and set*

$$\lambda(s,t) \stackrel{\text{def}}{=} \tilde{h}(s,t) J_\Sigma(s,t), \;\; (s,t) \in Q_\delta.$$

Then, for each $s \in [\,0,\, L)$ fixed, it holds that

$$\lambda(s,t) > \lambda(s,0), \;\; \forall\, t \in (-\delta, \delta), \;\; t \neq 0.$$

Proof. From (3.2) we have

$$\lambda(s,t) = \frac{2^{3/2}}{3}\,\tilde{a}^3(s,t)[1 - t\kappa(s)],\ \ (s,t) \in Q_\delta.$$

Hypothesis (1.2) implies that $\frac{\partial \lambda}{\partial t}(s,0) = 0$ and $\frac{\partial^2 \lambda}{\partial t^2}(s,0) > 0$, respectively, and the claim follows by taken δ small enough.

Lemma 3.2 *Let $M_\Sigma(s,t)$ be the Jacobian matrix of Σ. If $u \in C^1(N_\delta)$ then*

$$\left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\widetilde{u}\right| = \left|\left(\frac{\partial \widetilde{u}}{\partial s}\right)^2 (1 - t\kappa(s))^{-2} + \left(\frac{\partial \widetilde{u}}{\partial t}\right)^2\right|^{1/2}$$

These lemmas will be used in the next result which by its turn is the core of our procedure.

If the proof of the next theorem seems longer than expected - at least from the geometric point of view - it is because, due to the constraints imposed by the $\Gamma-$convergence approach, we have to work in L^1 which renders a rather fine topology. As a consequence while locally minimizing the arc-length functional with weight $h = \frac{2^{3/2}}{3}\,a^3$ one has to deal with competing curves which are just rectifiable and those with too small arc-length must be ruled out.

The idea of splitting the admissible functions in four classes was inspired in [17], where a simpler case is dealt with.

Recall that $\Omega = \Omega^- \cup \gamma \cup \Omega^+$ where Ω^- denotes the open region enclosed by γ and $\Omega^+ = \Omega \setminus [\Omega^- \cup \gamma]$ the outer region.

Theorem 3.2 *Under the above hypotheses the functions*

1. $u_0^1(x) = -a(x)\chi_{\Omega^-}(x) + a(x)\chi_{\Omega^+}(x),\ \ x \in \Omega,$
2. $u_0^2(x) = a(x)\chi_{\Omega^-}(x) - a(x)\chi_{\Omega^+}(x),\ \ x \in \Omega,$
3. $u_0^3(x) = -a(x),\ \ x \in \Omega,$
4. $u_0^4(x) = a(x),\ \ x \in \Omega.$

are $L^1(\Omega)$-local isolated minimizers of the $\Gamma-$limit functional E_0 defined in Theorem 3.1 acting on $BV(\Omega, \{-a, a\})$.

Proof. It suffices to work in the tubular neighborhood $N_\delta(\gamma)$; this claim will be justified at the end of the proof.

Therefore we need further notation: with Ω^- and Ω^+ as above we set

- $N_\delta^- = \Omega^- \cap N_\delta,$
- $N_\delta^+ = \Omega^+ \cap N_\delta,$
- $Q_\delta^- = \Omega^- \cap Q_\delta,$
- $Q_\delta^+ = \Omega^+ \cap Q_\delta.$

The proof is rendered for the functions u_0^1 and u_0^4 only; the other cases are similar.

- First case: $u_0^1 = -a\chi_{N_\delta^-} + a\chi_{N_\delta^+}$

It suffices to prove that for a suitable μ and $u \in BV(Q_\delta, \{-a, a\})$ it holds that

$$0 < \| u - u_0^1 \|_{L^1(N_\delta)} < \mu \Rightarrow E_0(u_0^1) < E_0(u),$$

that is

$$\int_{N_\delta} h(x)|D\chi_{\{u_0^1=a\}}| < \int_{N_\delta} h(x)|D\chi_{\{u=a\}}|$$

The co-area formula and the fact that $N_\delta \cap \partial^*\{\chi_{\{u_0^1=a\}}\} = \gamma$ yield

$$E_0(u_0^1) = \int_{N_\delta} h(x)|D\chi_{\{u_0^1=a\}}| = \int_{-\infty}^{\infty}\left[\int_{N_\delta\cap\partial_*\left\{\chi_{\{u_0^1=a\}}>\xi\right\}} h(x)d\mathcal{H}^1\right]d\xi =$$

$$\int_0^1\{\int_\gamma h(x)d\mathcal{H}^1\}d\xi = \int_0^L \tilde{h}(s,0)\,ds \tag{3.3}$$

In the sequel, for a fixed $t \in (-\delta, \delta)$, we set

$$\ell_t \stackrel{\text{def}}{=} \{(s,t) \in Q_\delta :\ 0 < s < L\}$$

Since $\chi_{\{u=a\}} \in BV(N_\delta)$ it follows that $\chi_{\{\tilde{u}=\tilde{a}\}} \in BV(Q_\delta)$ and the trace of $\tilde{u}(\cdot,t)$ is well defined on ℓ_t, for a.e. $t \in (-\delta, \delta)$. Equality of two functions along each ℓ_t means equality of its traces.

We first suppose that

(i) $\tilde{u} = \tilde{u}_0^1$ along $\ell_{\bar{t}} \cup \ell_{-\bar{t}}$, for some $\bar{t} \in (\delta/2, \delta)$.

Given that $u \in BV(N_\delta)$ there exists a sequence $\{u_j\}_{j=1}^\infty \in C^\infty(N_\delta) \cap BV(N_\delta)$ such that $u_j \longrightarrow \chi_{\{u=a\}}$ in $L^1(N_\delta)$ and

$$\int_{N_\delta} h(x)\,|D\chi_{\{u=a\}}| = \lim_{j\to\infty}\int_{N_\delta} h(x)|\nabla u_j|\,dx$$

See [7] for a proof of this property when $h \equiv 1$. Next we take a subsequence of $\{u_j\}_{j=1}^\infty$, still denoted by $\{u_j\}_{j=1}^\infty$, such that $u_j \to \chi_{\{u=a\}}$ a.e. in N_δ.

Lemmas 3.1 and 3.2, Fatou's lemma, (3.3) and (2.1) imply:

$$\begin{aligned} E_0(u) &= \int_{N_\delta} h(x)\,|D\chi_{\{u=a\}}| = \\ &= \lim_{j\to\infty}\int_{N_\delta} h(x)|\nabla u_j|\,dx \\ &= \lim_{j\to\infty}\int_{Q_\delta} (\tilde h\,J_\Sigma)(s,t)\left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\tilde{u_j}\right|\,dt\,ds \\ &\geq \liminf_{j\to\infty}\int_0^L\int_{-\delta/2}^{\delta/2}\lambda(s,t)\left|\frac{\partial\tilde{u_j}}{\partial t}\right|\,dt\,ds \\ &\geq \liminf_{j\to\infty}\int_0^L\int_{-\delta/2}^{\delta/2}\lambda(s,0)\left|\frac{\partial\tilde{u_j}}{\partial t}\right|\,dt\,ds \\ &\geq \liminf_{j\to\infty}\int_0^L\lambda(s,0)\,\mathrm{ess}\,V_{-\delta/2}^{\delta/2}[\tilde u_j(s,\cdot)]\,ds \\ &\geq \int_0^L\lambda(s,0)\,\mathrm{ess}\,V_{-\delta/2}^{\delta/2}[\chi_{\{\tilde u=\tilde a\}}(s,\cdot)]\,ds \\ &= \int_0^L\tilde h(s,0)\,ds = E_0(u_0^1), \end{aligned}$$

where we used (i) to assure that the total essential variation

$$\mathrm{ess}\,V_{-\delta/2}^{\delta/2}[\chi_{\{\tilde u=\tilde a\}}(s,\cdot)] = 1$$

for a.e. $s\in[0,L]$. Also from the theory of BV functions it holds that ess $V_{-\delta/2}^{\delta/2}[\chi_{\{\tilde u=\tilde a\}}(s,\cdot)]$ is Lebesgue integrable on $[0,L]$ for a.e. $t\in(\delta/2,\delta)$ and

$$\liminf_{j\to\infty}\ \mathrm{ess}\,V_{-\delta/2}^{\delta/2}[\tilde{u_j}(s,\cdot)] \geq \mathrm{ess}\,V_{-\delta/2}^{\delta/2}[\chi_{\{\tilde u=\tilde a\}}(s,\cdot)]$$

for a.e. $s\in[0,L]$ (see [19]).

Actually $E_0(u_0^1) < E_0(u)$ for if it were not the case then from the last chain of inequalities we would obtain

$$E_0(u) = \int_{Q_\delta\cap\partial_*\{\tilde u=\tilde a\}}\lambda(s,t)d\mathcal{H}^1 = \int_0^L\lambda(s,0)\,ds = E_0(u_0^1).$$

But by virtue of Lemma 3.1 this would hold if and only if

$$Q_\delta\cap\partial_*\{\tilde u=\tilde a\} = \gamma,$$

i.e., $u\equiv u_0^1$ which is impossible since $\|\,u-u_0^1\,\|_{L^1(N_\delta)}>0$. Hence the claim is proved.

Now if (i) does not hold then one of the following cases must occur:

(ii) $\tilde u$ is not constant a.e. along $\ell_{\bar t}$ for a.e. $\bar t\in(\delta/2,\delta)$,

(iii) $\tilde u$ is not constant a.e. along $\ell_{-\bar t}$ for a.e. $-\bar t\in(\delta/2,\delta)$,

(iv) $\tilde u(\cdot,t)=\tilde a(\cdot,t)$ a.e along $\ell_{\bar t}$ and $\tilde u(\cdot,t)=\tilde{-a}(\cdot,t)$ a.e along $\ell_{-\bar t}$, for a.e. $\bar t\in(\delta/2,\delta)$

In order to save space throughout the proof we sometimes denote

$$f\Big|_a^b \stackrel{\text{def}}{=} f(b) + f(a).$$

Next we define a set $\Delta \subset (0,\delta)$ by

$$\Delta \stackrel{\text{def}}{=} \left\{ \bar{t} \in (0,\delta) : \int_0^L |\chi_{\{\tilde{u}=\tilde{a}\}} - \chi_{\{\tilde{u}_0^1=\tilde{a}\}}| \Big|_{(s,t)}^{(s,-t)} \, ds > \frac{4\mu}{\delta} \right\}$$

where μ and δ are the radius of neighborhoods taken above. Hence $\mathcal{L}^1(\Delta) < \delta/4$. Now if (iv) holds then we can assure that

$$\int_0^L |\chi_{\{\tilde{u}=\tilde{a}\}} - \chi_{\{\tilde{u}_0^1=\tilde{a}\}}| \Big|_{(s,t)}^{(s,-t)} \, ds = 2L > \frac{4\mu}{\delta}$$

as long as

$$\mu < \frac{\delta L}{2}. \tag{3.4}$$

Hence, by the definition of Δ, we conclude that $\bar{t} \in \Delta$. Thus for a.e. $\bar{t} \in (\delta/2,\delta) \setminus \Delta$ either ii-) or iii-) holds and therefore

$$\lim_{j\to\infty} \text{ess } V_0^L[\tilde{u}_j(\cdot,\bar{t})] \geq \text{ess } V_0^L[\chi_{\{\tilde{u}=\tilde{a}\}}(\cdot,\bar{t})] = 1.$$

For the approximation functions above it holds

$$\int_0^L \left\{ \left| \frac{\partial \tilde{u}_j(s,\bar{t})}{\partial s} \right| + \left| \frac{\partial \tilde{u}_j(s,-\bar{t})}{\partial s} \right| \right\} ds \geq$$

$$\left\{ \text{ess } V_0^L[\tilde{u}_j(\cdot,\bar{t})] + \text{ess } V_0^L[\tilde{u}_j(\cdot,-\bar{t})] \right\}$$

Having in mind an estimate in $N_\delta \setminus N_{\delta/2}$, we integrate over $(\delta/2,\delta)\setminus\Delta$ and take the limit as $j \to 0$ to obtain

$$\lim_{j\to\infty} \left\{ \int_{(\delta/2,\delta)\setminus\Delta} \int_0^L \left\{ \left| \frac{\partial \tilde{u}_j(s,\bar{t})}{\partial s} \right| + \left| \frac{\partial \tilde{u}_j(s,-\bar{t})}{\partial s} \right| \right\} ds \, d\bar{t} \right\} \geq \delta/4 \tag{3.5}$$

The definition of u_0^1 provides, for $\bar{t} \in (\delta/2,\delta) \setminus \Delta$, the following inequality

$$\begin{aligned} |\chi_{\{\tilde{u}=\tilde{a}\}}(s,\bar{t}) - \chi_{\{\tilde{u}=\tilde{a}\}}(s,-\bar{t})| &\geq |\chi_{\{\tilde{u}_0^1=\tilde{a}\}}(s,\bar{t}) - \chi_{\{\tilde{u}_0^1=\tilde{a}\}}(s,-\bar{t})| - \\ \{|\chi_{\{\tilde{u}_0^1=\tilde{a}\}} &- \chi_{\{\tilde{u}=\tilde{a}\}}|(s,-\bar{t}) + |\chi_{\{\tilde{u}_0^1=\tilde{a}\}} - \chi_{\{\tilde{u}=\tilde{a}\}}|(s,\bar{t})\} \geq \\ 1 - \{|\chi_{\{\tilde{u}_0^1=\tilde{a}\}} &- \chi_{\{\tilde{u}=\tilde{a}\}}|(s,-\bar{t}) + |\chi_{\{\tilde{u}_0^1=\tilde{a}\}} - \chi_{\{\tilde{u}=\tilde{a}\}}|(s,\bar{t})\} \end{aligned} \tag{3.6}$$

It follows from the facts that $\tilde{u} \in BV(Q_\delta)$ and $\mathcal{L}^1((0,\delta/2)\setminus\Delta) \geq \delta/4$ that there exists $\bar{t} \in (0,\delta/2) \setminus \Delta$ such that $(s,\bar{t})$ and $(s,-\bar{t})$ are points of approximate continuity of $\tilde{u}$, a.e. in $(0,L)$.

This fact along with (3.4), (3.6), Lemmas 3.1, 3.2 and Fatou's Lemma yield the following estimate on $N_{\delta/2}$:

$$\begin{aligned}
\int_{N_{\delta/2}} h(x)\,|D\chi_{\{u=a\}}| &= \lim_{j\to\infty}\int_{N_{\delta/2}} h(x)|\nabla u_j|\,dx\\
&= \lim_{j\to\infty}\int_{Q_\delta}\tilde h\,J_\Sigma\left|M_\Sigma^{-1}\nabla_{s,t}\tilde u_j\right|\,dt\,ds\\
&\geq \liminf_{j\to\infty}\int_0^L\int_{(-\delta/2,\delta/2)\setminus\Delta}\lambda(s,t)\left|\frac{\partial\tilde u_j}{\partial t}\right|\,dt\,ds\\
&\geq \liminf_{j\to\infty}\int_0^L\int_{(-\delta/2,\delta/2)\setminus\Delta}\lambda(s,0)\left|\frac{\partial\tilde u_j}{\partial t}\right|\,dt\,ds\\
&\geq \liminf_{j\to\infty}\int_0^L\lambda(s,0)\,\mathrm{ess}\,V_{(-\delta/2,\delta/2)\setminus\Delta}[\tilde u_j(s,\cdot)]\,ds\\
&\geq \int_0^L\lambda(s,0)\,\mathrm{ess}\,V_{(-\delta/2,\delta/2)\setminus\Delta}[\chi_{\{\tilde u=\tilde a\}}(s,\cdot)]\,ds\\
&\geq \int_0^L\lambda(s,0)\,|\chi_{\{\tilde u=\tilde a\}}(s,\bar t)-\chi_{\{\tilde u=\tilde a\}}(s,-\bar t)|\,ds\\
&\geq \int_0^L\lambda(s,0)\,ds - h_M\int_0^L|\chi_{\{\tilde u_0^1=\tilde a\}}-\chi_{\{\tilde u=\tilde a\}}|\Big|_{(s,-\bar t)}^{(s,\bar t)}\,ds\\
&\geq E_0(u_0^1)-\frac{4\mu h_M}{\delta}
\end{aligned}$$

where

$$h_M \overset{\text{def}}{=} \max_{x\in\overline{N}_\delta} h(x)$$

Inequality (3.5) and the one above imply

$$\begin{aligned}
E_0(u) = \int_{N_\delta} h(x)\,|D\chi_{\{u=a\}}| = \lim_{j\to\infty}\int_{N_\delta} h(x)|\nabla u_j|\,dx =\\
\liminf_{j\to\infty}\int_{Q_\delta}\tilde h(s,t)J_\Sigma(s,t)\left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\tilde u_j\right|\,dt\,ds \geq\\
\liminf_{j\to\infty}\int_{(\delta/2,\delta)\setminus\Delta}\int_0^L\left\{\tilde h\left|\frac{\partial\tilde u_j}{\partial s}\right|\right\}\Big|_{(s,-t)}^{(s,t)}\,ds\,dt+\\
\liminf_{j\to\infty}\int_0^L\int_{(-\delta/2,\delta/2)\setminus\Delta}\lambda(s,t)\left|\frac{\partial\tilde u_j}{\partial t}\right|\,dt\,ds \geq\\
\frac{h_m\delta}{4}+E_0(u_0^1)-\frac{4\mu h_M}{\delta} > E_0(u_0^1)
\end{aligned}$$

as long as

$$\mu < \min\left\{\frac{\delta h_m}{2h_M},\ \frac{\delta L}{2}\right\}$$

where $h_m \overset{\text{def}}{=} \min_{x\in\overline{N}_\delta} h(x)$.

The claim is proved.

- Second case: $u_0^4(x) = a(x),\ x \in N_\delta$.

This case is trivial for since $N_\delta \cap \partial_*\{\chi_{\{a=a\}}\} = N_\delta \cap \partial N_\delta = \emptyset$ it suffices to realize that the co-area formula yields

$$E_0(u_0^4) = \int_{N_\delta} h(x)|D\chi_{\{u_0^4=a\}}| = \int_{-\infty}^{\infty}\left[\int_{N_\delta\cap\partial_*\{\chi_{\{u_0^4=a\}}>\xi\}} h(x)d\mathcal{H}^1\right]d\xi = 0$$

whereas for any admissible function $u \in BV(N_\delta, \{-a(x), a(x)\})$ it holds $E_0(u) = \int_{N_\delta} h(x)|D\chi_{\{u=a\}}| > 0 = E_0(u_0^4)$.

Let us now clarify the claim that the argument was local in the spatial variable which was necessary in order to work with principal coordinates in $N_\delta(\gamma)$.

It suffices to realize that if $u \in BV(\Omega, \{-a, a\})$ and

$$u_0^1(x) = -a(x)\chi_{\Omega^-}(x) + a(x)\chi_{\Omega^+}(x),\ x \in \Omega,$$

then

$$0 <\| u - u_0^1 \|_{L^1(\Omega)} < \mu \Rightarrow 0 <\| u - u_0^1 \|_{L^1(N_\delta)} < \mu.$$

Hence using the co-area formula and the above proven result for N_δ ,

$$E_0(u) = \int_\Omega h(x)|D\chi_{\{u=a\}}| \geq \int_{N_\delta} h(x)|D\chi_{\{u=a\}}| > \int_{N_\delta} h(x)|D\chi_{\{u_0^1=a\}}| =$$

$$\int_{-\infty}^{\infty}\left[\int_{N_\delta\cap\partial_*\left\{\chi_{\{u_0^1=a\}}>\xi\right\}} h(x)d\mathcal{H}^1\right]d\xi = \int_\gamma h(x)d\mathcal{H}^1 =$$

$$\int_\Omega h(x)|D\chi_{\{u_0^1=a\}}| = E_0(u_0^1)$$

This concludes the proof.

3.2 Proof of Theorem 1.1

The proof is accomplished by a direct application of Theorem 2.1. After calculating the Γ-limite E_0 of the family of functionals given in (3.1) and proving that each function $u_0^1, \ldots, u_0^4$ is a local minimum isolated of E_0, as shown in Theorem 3.2, it remains to verify hypothesis (i) of Theorem 2.1. This hypothesis would follow if the minimizers were uniformly bounded in L^1 (see [16], Proposition 3 and Remark (1.35), for instance). But this is really the case and it can be accomplished by an application of the maximum principle.

Hence an application of Theorem 2.1 implies that, for some $\epsilon_0 > 0$, the family of functionals given by (3.1) possesses four families of local minima $\left\{u_\epsilon^1\right\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \ldots, 4)$ satisfying $\|u_\epsilon^j - u_0^j\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ $(j = 1, \ldots, 4)$.

Therefore for $0 < \epsilon < \epsilon_0$ each H^1-minimizer u_ϵ^j, $j \in \{1, \ldots, 4\}$, is a weak stationary solution of (1.1) and regularity theory implies that $u_\epsilon^j \in C^{2,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$. It remains to verify its stability.

The second variation of the energy functional E_ϵ at u_ϵ^j is nonnegative. If $\lambda_1(u_\epsilon^j)$ is the first eigenvalue of problem (1.1) linearized around u_ϵ^j then $\lambda_1(u_\epsilon^j) \geq 0$, due to its variational characterization. It is well-known that if $\lambda_1(u_\epsilon^j) > 0$ then u_ϵ^j is asymptotically stable.

If $\lambda_1(u_\epsilon^j) = 0$ then, since it is a simple eigenvalue, there is a local one-dimensional critical manifold $W(u_\epsilon^j)$, tangent to principal eigenfunction corresponding to the zero eigenvalue, such that if u_ϵ^j is stable in $W(u_\epsilon^j)$ then it is also stable in $H^1(\Omega)$ (see [6], Theorem 6.2.1, for instance). The rest of the argument is standard by now; the stability of u_ϵ^j in $W(u_\epsilon^j)$ follows from the facts that the energy is a Liapounov functional and that $W(u_\epsilon^j)$ is one-dimensional.

The theorem is proved.

References

[1] A. S. do Nascimento, *Stable transition layers in a semilinear diffusion equation with spatial inhomogeneities in N-dimensional domains*, J. Diff. Eqns. **190** (2003), 16-38.

[2] A. S. do Nascimento, *Stable multiple-layer stationary solutions of a semilinear parabolic equation in two-dimensional domains*, Eletr. J. Diff. Eqns. **21** (1997), 1-17.

[3] A. S. do Nascimento, *Stable stationary solutions induced by spatial inhomogeneity via* Γ*-convergence*, Bulletin of the Brazilian Math. Society **29**, No.1 (1998), 75-97.

[4] A. S. do Nascimento, *Local minimizers induced by spatial inhomogeity with inner transition layer*, J. Diff. Eqns. **133** (1997), 203-223.

[5] A. S. do Nascimento, J. Biesdorf, J. Crema, *Count and symmetry of global and local minimizers of the Cahn-Hilliard energy over some cylindrical domains*, Advanced Nonlinear Studies **12** (2012), 123-148.

[6] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer Lecture Notes in Mathematics, v. 840, 1981.

[7] E. Giusti, Minimal Surfaces and Funtions of Bounded Variation, Birkhauser-Australia, 1984.

[8] F. Li and K. Nakashima, *Transition Layers for a spatially inhomogeneous Allen-Cahn equation in multi-dimensional domains*, Discrete Contin. Dyn. Syst. **32**, No. 4 (2012), 1391-1420.

[9] F. Mahmoudi, A. Malchiodi and J. Wei, *Transition layer for the heterogeneous Allen-Cahn equation*, Ann. I. H. Poincare . AN **25** (2008), 609-631.

[10] H. Matsuzawa, *Stable transition layers in a balanced bistable equation with degeneracy* , Nonlinear Analysis **58** (2004), 45-67.

[11] K. Kurata and H. Matsuzawa, *Multiple stable patterns in a balanced bistable equation with heterogeneous environments*, Applicable Analysis, **89**, No. 7 (2010), 1023-1035.

[12] K. Nakashima, *Stable transition layers in a balanced bistable equation, Differential Integral Equations* **13**, No. 7-9 (2000), 1025-1038.

[13] L. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, 1992.

[14] N. Dancer and S. Yan, *Multi-layer solutions for an elliptic problem, J. of Diff. Eqns.* **194** (2003), 382-405.

[15] N. N. Nefedov and K. Sakamoto, *Multi-dimensional stationary internal layers for spatially inhomogeneous reaction-diffusion equations with balanced nonlinearity*, Hiroshima Math. J. **33** (2003), 391-432.

[16] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rat. Mech. Anal. **101** (1988), 209-260.

[17] R. V. Kohn, P. Sternberg, *Local minimizers and singular perturbations*, Proceedings of the R. Soc. of Edinburg **111** (A) (1989), 69-84.

[18] S. B. Angenent, J. Mallet-Paret, L. A. Peletier, *Stable transition layers in a semilinear boundary value problem*, J. Diff. Eqns. **67** (1987), 212-242.

[19] W. P. Ziemer, Weakly Differentiable Functions, Springer-Verlag, (1989).