Stable equilibria of a singularly perturbed reaction-diffusion equation when the roots of the degenerate equation contact or intersect along a non-smooth hypersurface.

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Abstract. We use the variational concept of Γ-convergence to prove existence, stability and exhibit the geometric structure of four families of stationary solutions to the singularly perturbed parabolic equation $u_t = \epsilon^2 \mathrm{div}(k\nabla u) + f(u,x)$, for $(t,x) \in \mathbb{R}^+ \times \Omega$, where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, supplied with no-flux boundary condition. The novelty here lies in the fact that the roots of the bistable function f are not isolated, meaning that the graphs of its roots are allowed to have contact or intersect each other along a Lipschitz-continuous $(n-1)-$dimensional hypersurface $\gamma \subset \Omega$; across this hypersurface the stable equilibria may have corners. This case of intersecting roots stems from the phenomenon known as exchange of stability which is characterized by $f(\cdot, x)$ having only two roots.

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1. Introduction and statement of the main results

This work is a contribution to the study of the mechanisms which give rise to nontrivial stable stationary solutions -often herein referred to as patterns, for short- to bistable spatially inhomogeneous reaction-diffusion equations. It is well-known that such solutions have a leading role when determining the global dynamics determined by these evolution equations given that it defines a gradient flow. Herein the focus will be on the scalar diffusion equation

$$\left.\begin{array}{ll} u_t = \epsilon^2 \mathrm{div}[k(x)\nabla u] + f(u,x) & (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_\nu u(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \partial\Omega \end{array}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a open bounded set with C^2 boundary, ϵ is a small positive parameter, ν denotes the outer unit normal to $\partial\Omega$, $k \in C^1(\Omega)$ stands for the positive diffusivity function and $f : \mathbb{R} \times \overline{\Omega} \longmapsto \mathbb{R}$, often referred to as reaction term, is smooth in the first variable and Lipschitz-continuous in the second variable.

Regarding bistable reaction-diffusion equations some parameters, as long as acting alone, are known to provide the simplest mechanisms which are responsible for existence of patterns, namely: the geometry of $\partial\Omega$ (see e.g. [21], [17], [20], [10]), the diffusivity function k (see e.g. [13], [16], [7], [9], [8], [10]) and the spatially inhomogeneous bistable reaction term $f(x, u)$ (see e.g. [22], [11], [1], [6]). It should be emphasized that references are restricted only to the cases concerned with stable stationary solutions as the literature regarding merely stationary solutions, mainly in the context of transition layer solutions, is vast.

Our concern herein will be with patterns induced by a class of reaction terms whose prototype is the bistable function

$$f(u, x) = -(u - a(x))(u - \theta(x))(u - b(x)).$$

All existing works regarding this problem deal with a bistable term $f(u, x)$ whose roots a, θ, b are isolated, i.e., $a < \theta < b$ in $\overline{\Omega}$; the novelty here in this work lies in the fact that the graphs of these roots are allowed to have either contact or intersect over Ω along a hypersurface not necessarily smooth. One of our concern is to verify if the four families of stable layered stationary solutions, whose existence for the case when the roots of f do not intersect was proved in [11], for $n = 2$ and $f(u, x) = u[a^2(x) - u^2]$, and in [22], for $n = 1$ and $k \equiv 1$, either cease or continue to exist when the roots intersect or have contact. Remark that in case their existence is preserved the transition layers established in the case of non-intersecting roots are smoothed out and no longer exist, eventually, being replaced with patterns with corners. See comment below on the motivation for addressing such case.

Therefore throughout Section 3 we assume that

- there exists a $(n-1)$-dimensional Lipschitz hypersurface $\gamma \subset \Omega$ without boundary which partitions Ω in two open disjoint connected components Ω_a and Ω_b, i.e.
$$\Omega = \Omega_a \cup \gamma \cup \Omega_b$$
where for definiteness Ω_b denotes the open region enclosed by γ and $\Omega_a = \Omega \setminus (\Omega_b \cup \gamma)$;
- $\exists\, \theta, a, b \in C(\Omega) \cap C^1(\Omega \setminus \gamma)$, C^1-bounded in Ω_a and Ω_b such that
$$f(a(x), x) = f(b(x), x) = f(\theta(x), x) = 0,\ \forall x \in \Omega,$$
$$\partial_1 f(a(x), x) < 0, \quad \partial_1 f(b(x), x) < 0, \quad \forall x \in \Omega \setminus \gamma$$

and either

$$(f_1) \begin{cases} a > \theta > b, & \text{in} \quad \Omega \setminus \gamma \\ a = \theta = b, & \text{on} \quad \gamma \end{cases}$$

or

$$(f_2)\begin{cases} a>\theta>b, & \text{in } \Omega_a \\ a=\theta=b, & \text{on } \gamma \\ a<\theta<b, & \text{in } \Omega_b. \end{cases}$$

Under (f_1) the function $d(x)=a(x)-b(x)\geq 0$ vanishes only along γ and is continuous but may not be differentiable in the smooth parts of γ; (f_2) means that d changes sign across γ with same remark on its smoothness.

Also for the purpose of meeting a condition required by the $\Gamma-$convergence procedure we further assume the equal-area condition

$$(f_3)\left\{ \int_{\min\{a(x),b(x)\}}^{\max\{a(x),b(x)\}} f(\xi,x)d\xi=0, \quad \forall \; x\in\Omega. \right.$$

Our main result for $n-$dimensional ($n\geq 2$) domains and just one hypersurface states as follows.

Theorem 1.1. *If f satisfies (f_1) and (f_3) then $\exists\ \epsilon_0>0$ and four families of stable stationary solutions $\{u_\epsilon^j\}_{0<\epsilon\leq\epsilon_0}$ $(j=1,\ldots,4)$ to (1.1) such that*

- $\|u_\epsilon^1-u_0^1\|_{L^1(\Omega)}\overset{\epsilon\to 0}{\longrightarrow}0$ *where* $u_0^1=a\chi_{\Omega_a}+b\chi_{\Omega_b}$
- $\|u_\epsilon^2-u_0^2\|_{L^1(\Omega)}\overset{\epsilon\to 0}{\longrightarrow}0$ *where* $u_0^2=b\chi_{\Omega_a}+a\chi_{\Omega_b}$
- $\|u_\epsilon^3-u_0^3\|_{L^1(\Omega)}\overset{\epsilon\to 0}{\longrightarrow}0$ *where* $u_0^3\equiv a$
- $\|u_\epsilon^4-u_0^4\|_{L^1(\Omega)}\overset{\epsilon\to 0}{\longrightarrow}0$ *where* $u_0^4\equiv b$

Under (f_2) and (f_3) the same result holds after relabeling the stable stationary solutions $\{u_\epsilon^j\}_{0<\epsilon\leq\epsilon_0}$ $(j=1,\ldots,4)$ accordingly to limit-functions $u_0^1\equiv a$, $u_0^2\equiv b$, $u_0^3=a\chi_{\Omega_a}+b\chi_{\Omega_b}$ and $u_0^4=b\chi_{\Omega_a}+a\chi_{\Omega_b}$.

The very same result still holds with the same proof for one-dimensional domains –occasionally changing notation– and also for a finite number of Lipschitz-continuous hypersurfaces as long as they partition Ω in Lipschitz-continuous connect components (see Section 4 for this matter). A simple example in one-dimensional domain which satisfies (f_1) is given by

$$f(u,x)=u[|\cos x|^2-u^2],\ x\in(0,\,\pi),$$

where the roots $|\cos x|$ and $-|\cos x|$ have only C^0-contact at $\pi/2$; on the other hand

$$f(u,x)=u[\cos^2 x-u^2],\ x\in(0,\,\pi),$$

satisfies (f_2) with the roots $\cos x$ and $-\cos x$ intersecting transversally at $\pi/2$.

Note that under either (f_1) or (f_2) the limiting functions u_0^j $(j=1,2,3,4)$ are continuous but may not be differentiable across γ.

For $n=2$ and under (f_1) the graphs of the solutions $\xi_1=a(x)$ and $\xi_2=b(x)$ of the degenerated equation $f(\xi,x)=0$ are allowed to have C^m-contact $(m\geq 0)$ at γ –depending on the smoothness of γ, a,θ and b– whereas under (f_2) they cross each other transversally or tangentially. In other words, if γ, a and b are smooth and η denotes the normal vector field along γ then we allow $\partial_\eta(a-b)=0$ in γ; this stands in contrast to the case of exchange of stability (see definition and more comments below) addressed in [2], [19],

[18] and [3] where due to the methods utilized the non-degeneracy condition $\partial_\eta(a-b)\neq 0$ along γ is required.

Hypotheses (f_1) and (f_2) along with the fact that γ may be non-smooth are the main source of difficulty of our problem and imply that the standard theory for singularly perturbed equations cannot be applied (see [2] or [18] for example). Instead we utilize a variational procedure based on $\Gamma-$convergence that provides sufficient conditions under which the problem of finding local minimizers of the family of corresponding energy functionals is reduced to finding local minimizers of a more tractable geometric problem in the space $BV(\Omega,\{a,b\})$ of functions of bounded variation in Ω which takes values $a(x)$ and $b(x)$ only. This geometric problem, which turns out to be the $\Gamma-$limit of the family of energy functionals, is computed inspired in some results from [23] and consists of a perimeter functional with a weight that vanishes on γ and therefore, in this sense, is degenerated.

The idea of addressing the case of intersecting roots stems from the phenomenon known as exchange of stability which is characterized by $f(\cdot,x)$ having only two roots $a(x)$ and $b(x)$, the prototype being

$$f(u,x)=-[u-a(x)][u-b(x)],$$

intersecting along a simple closed smooth curve γ and satisfying $\partial_1 f(a(x),x)<0$ and $\partial_1 f(b(x),x)>0,\ x\in\Omega\setminus\gamma$; for this matter the interested reader is referred to [2], [3], [18], [19], e.g. and to [4] for a comprehensive survey.

In [18] the problem $-\varepsilon^2\triangle u+(u-a(|x|))(u-b(|x|))=0$, in the unit $n-$dimensional ball, supplied with no-flux boundary condition was addressed. Roughly speaking, under the hypothesis that the function $a(r)-b(r),\ r=|x|$, changes sign at $r_0\in(0,1)$ with $(a-b)_r(r_0)<0$, the authors prove existence of two radial solutions u_+ and u_- where the former one is an asymptotically stable stationary solution to the corresponding parabolic problem. Moreover this solution converges uniformly, as ε goes to zero, to $\max\{a,b\}$. In [19] the authors have generalized these results to any smooth domain $\Omega\subset\mathbb{R}^2$ but still considering γ a simple smooth closed curve along which $\partial_n(a-b)\neq 0$.

Still regarding the phenomenon of exchange of stability in [3] the author, for the one-dimensional domain $\Omega=(0\,,\,1)$, $k\equiv 1$ and $f(u,x)=-h(u,x)[u-a(x)][u-b(x)],\ h>0$, under the hypothesis that $d(x)=a(x)-b(x)$ changes sign at $x_0\in(0\,,\,1)$ with $d'(x_0)>0$, uses asymptotic methods, to prove existence of an asymptotically stable -Lyapunov sense- stationary solution to (1.1); this stable stationary solution converges pointwise, as $\varepsilon\to 0$, to $\max\{a,b\}$.

For a motivating example, from reaction kinetics, to study this kind of problem the interested reader is referred to [2], for instance.

2. Preliminaries

Next we give some definitions and results which are going to be used herein. The interested reader is referred to [24] and [14], for instance, for more details.

Definition 2.1. The space $BV(\Omega)$ of *functions of bounded variation in* Ω is defined as the set of all functions $v \in L^1(\Omega)$ whose distributional gradient Dv is a Radon measure with finite total variation in Ω, i.e.,

$$|Dv|(\Omega) = \sup\left\{\int_\Omega v(x)\,\mathrm{div}\,\sigma(x)dx : \sigma \in C_0^1(\Omega, \mathbb{R}^n), \quad |\sigma| \leq 1\right\} < \infty.$$

A Borel set $B \subset \mathbb{R}^n$ has finite perimeter in the open set Ω if

$$\mathrm{Per}_\Omega(B) = |D\chi_B|(\Omega) < \infty,$$

where χ_B is the characteristic function of the set B.

For a continuous non-negative function h and $u \in BV(\Omega)$ the integral of h with respect to the measure $|Du|$ is defined as

$$\int_\Omega h(x)\,|Du| = \sup\left\{\int_\Omega u(x)\,\mathrm{div}\,\sigma(x)dx : \sigma \in C_0^1(\Omega, \mathbb{R}^n), \quad |\sigma(x)| \leq h(x)\right\}.$$

For a set $B \subset \mathbb{R}^n$ with finite perimeter in Ω, the integral of h with respect to the Radon measure $|D\chi_B|$, denoted by $\int_\Omega h(x)\,|D\chi_B|$, will often appears in the context.

It turns out that if $v \in W^{1,1}_{loc}$ and $\mathcal{L}^n$ denotes the Lebesgue measure then the variation measure satisfies

$$|Dv| = |\nabla v|\, d\mathcal{L}^n, \tag{2.1}$$

where ∇ denotes the usual gradient. Let $B(x,r)$ denotes the ball of radius r and center x; then the *density* of a set A in a point x is defined as

$$D(A,x) = \lim_{r\to 0} \frac{\mathcal{L}^n(A \cap B(x,r))}{\mathcal{L}^n(B(x,r))}$$

whenever the limit exists. The essential boundary of a set $E \subset \mathbb{R}^n$ is the set $\partial_* E$ of all points in Ω where E has neither density 1 nor density 0.

For a set $E \subset \Omega$ with finite perimeter in Ω, $\partial_* E$ is a rectifiable set and it may be endowed with a measure theoretic normal ν_E so that the measure derivative $D\chi_E$ is expressed as

$$D\chi_E(B) = \int_{B\cap\partial_* E} \nu_E d\mathcal{H}^{n-1}$$

for every Borel set $B \subset \Omega$ where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure.

The following version of the co-area formula will be often used: if $u \in BV(\Omega)$ and f is a continuous function in Ω then

$$\int_\Omega f\,|Du| = \int_{-\infty}^{\infty}\left[\int_{\Omega\cap\partial_*\{x\in\Omega: u(x)>\xi\}} f d\mathcal{H}^{n-1}\right] d\xi \tag{2.2}$$

The well-known isoperimetric inequalities for sets of finite perimeter as well as an approximation theorem for sets of finite perimeter by sets with smooth boundary will be used.

Theorem 2.2 ([24]). *Let $E \subset \mathbb{R}^n$ be a bounded set of finite perimeter. Then there exists a constant $C = C(n)$ such that*

$$\mathcal{H}^n(E)^{(n-1)/n} \leq C Per E.$$

Moreover, for each ball $B(r) \subset \mathbb{R}^n$,

$$\min\{\mathcal{H}^n(B(r) \cap E), \mathcal{H}^n(B(r) \setminus E)\}^{(n-1)/n} \leq C Per_{B(r)}(E).$$

Lemma 2.3 ([23]). *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset of $\mathbb{R}^n$ with Lipschitz-continuous boundary and $A \subset \Omega$ be a set of finite perimeter in Ω with $0 < \mathcal{L}^n(A) < \mathcal{L}^n(\Omega)$. Then there exists a sequence of open sets A_j such that*

(i) $\partial A_j \in C^2$,
(ii) $\mathcal{L}^n((A_j \cap \Omega)\delta A) \to 0$ *as* $j \to \infty$,
(iii) $Per_\Omega A_j \to Per_\Omega A$ *as* $j \to \infty$,
(iv) $\mathcal{H}^{n-1}(\partial A_j \cap \partial\Omega) = 0$,
(v) $\mathcal{L}^n(A_j \cap \Omega) = \mathcal{L}^n(A)$, *for all* j *sufficiently large.*

Next the working definition of the $\Gamma-$ convergence of a family of functionals with respect to the L^1 topology is given.

Definition 2.4. A family $\{E_\epsilon\}_{\epsilon>0}$ of real-extended functionals defined in $L^1(\Omega)$ is said to Γ-*converge, as* $\epsilon \to 0$, *to a functional* E_0 and we write

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(v) = E_0(v)$$

if:

(i) $\forall\ v \in L^1(\Omega)$ and $\forall\ \{v_\epsilon\} \subset L^1(\Omega) : v_\epsilon \to v$ in $L^1(\Omega)$, as $\epsilon \to 0 \quad \Rightarrow$ $E_0(v) \leq \liminf_{\epsilon\to 0} E_\epsilon(v_\epsilon)$.
(ii) $\forall\ v \in L^1(\Omega)$, $\exists\ \{v_\epsilon\}$ in $L^1(\Omega)\ : v_\epsilon \to v$ in $L^1(\Omega)$, as $\epsilon \to 0$, and $E_0(v) \geq \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon)$.

Definition 2.5. We shall call $v_0 \in L^1(\Omega)$ a L^1-*local minimizer of* E_0 if there is $\mu > 0$ such that

$$E_0(v_0) \leq E_0(v) \quad \text{whenever} \quad 0 < \|v - v_0\|_{L^1(\Omega)} < \mu.$$

Moreover if $E_0(v_0) < E_0(v)$, for $0 < \|v - v_0\|_{L^1(\Omega)} < \mu$, then v_0 is called an *isolated* L^1*-local minimizer of* E_0.

The following theorem which can be found in [20] is essential to our analysis.

Theorem 2.6 ([20]). *Suppose that* $\Gamma - \lim_{\epsilon\to 0} E_\epsilon = E_0$ *and the following hypotheses are satisfied*

(i) Any sequence $\{v_\epsilon\}_{\epsilon>0}$ *such that* $E_\epsilon(v_\epsilon) \leq C < \infty$, $\forall\,\epsilon > 0$ *and* $C > 0$, *is compact in* L^1.
(ii) There exists an isolated L^1*-local minimizer* v_0 *of* E_0.

Then $\exists\,\epsilon_0 > 0$ *and a family* $\{v_\epsilon\}_{0<\varepsilon<\epsilon_0}$ *such that*

- v_ϵ *is an* L^1*-local minimizer of* E_ϵ *and*

- $\|v_\epsilon - v_0\|_{L^1} \to 0,\ as\ \epsilon \to 0.$

In the sequel $BV(\Omega, \{\alpha, \beta\})$ will denote the space of functions of bounded variation in Ω which takes values α and β only. If $u \in BV(\Omega, \{\alpha, \beta\})$ then $\partial_* \{u = \alpha\} \cap \Omega$ is a rectifiable set.

3. Intersection or contact along a single Lipschitz hypersurface

This section aims at proving Theorem 1.1. Roughly speaking, the first step is to figure out what the $\Gamma-$limit of the family of energy functionals is and then, after finding isolated local minimizers of the $\Gamma-$limit, we resort to Theorem 2.6. The proof to find local minimizers of the $\Gamma-$limit is completely different from the case when the roots do not intersect and has been addressed for two-dimensional domains in [11].

We first prove Theorem 1.1 under (f_1) and then as a corollary prove it under (f_2).

3.1. Computation of the Γ-limit under (f_1)

The first step in our procedure is to find the Γ-limit of the family of functionals whose critical points are stationary solutions to (1.1). To that end we define $E_\epsilon : L^1(\Omega) \to \mathrm{I\!R} \cup \{\infty\}$ by

$$E_\epsilon(v) = \begin{cases} \displaystyle\int_\Omega \left[\frac{\epsilon k(x)}{2} |\nabla v|^2 + \frac{1}{\epsilon} F(v,x)\right] dx, & v \in H^1(\Omega) \\ \infty, & \text{otherwise,} \end{cases} \tag{3.1}$$

where

$$F(v,x) = -\int_{b(x)}^{v} f(\xi, x) d\xi \tag{3.2}$$

and f as in (1.1). At this point, in order to have E_ϵ a C^2 functional we require either a growth condition on f, as in [9], or use a truncation argument as in [8]. We chose not to describe this requirement in the Introduction in order not to divert attention from the distinguished hypothesis needed for our purposes.

It follows that $F(\cdot, x)$ is C^2 and from (f_3) satisfies: $F(v,x) > 0$ for $v \in (a(x), b(x))$, $F(a(x), x) = F(b(x), x) = \partial_1 F(a(x), x) = \partial_1 F(b(x), x) = 0$ and $\partial_{11} F(a(x), x) > 0$ and $\partial_{11} F(b(x), x) > 0$. This profile of the potential F is necessary for the computation of the $\Gamma-$limit of the family of functionals given by (3.1).

Theorem 3.1. *Consider $E_0 : L^1(\Omega) \to \mathrm{I\!R} \cup \{\infty\}$ defined by*

$$E_0(v) = \begin{cases} \displaystyle\int_\Omega h(x) \left|D\chi_{\{v=a\}}\right|, & v \in BV(\Omega, \{a, b\}) \\ \infty, & \textit{otherwise} \end{cases}$$

where

$$h(x) = \sqrt{2} \int_{b(x)}^{a(x)} \sqrt{k(x) F(s,x)} ds \tag{3.3}$$

and the roots a and b satisfy (f_1).

Then

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(v) = E_0(v), \quad \forall \quad v \in L^1(\Omega).$$

Remark 3.2. Note that due to the fact that the weight h is degenerated, in the sense that $h = 0$ on $\gamma \subset \Omega$, the computation of the Γ-limit provided in [23] for the case when $h > 0$ in Ω does not apply. Instead further arguments must be given.

The lemmas below will be useful in the proof of our main result.

Lemma 3.3 ([12]). *Assume $U \subset \Omega$ is open and bounded with ∂U Lipschitz. Let $f_1 \in BV(U)$ and $f_2 \in BV(\Omega \setminus \overline{U})$. Define*

$$\overline{f}(x) = \begin{cases} f_1(x), & x \in U \\ f_2(x), & x \in \Omega \setminus \overline{U}. \end{cases}$$

Then $\overline{f} \in BV(\Omega)$.

Recall that Ω_b denotes the connected region delimited by γ and $\Omega_a = \Omega \setminus (\Omega_b \cup \gamma)$; both of them, as a consequence of our hypothesis, have Lipschitz boundaries. This fact is going to be used in the proof of Lemma 3.5.

Given a function $v \in L^1(\Omega)$ we hereafter use the following notation

$$v^l \stackrel{\text{def}}{=} v\,|_{\Omega_l}\,, \quad l \in \{a, b\}\,.$$

Lemma 3.4. *Let $E^l_\epsilon : L^1(\Omega_l) \to \mathbb{R} \cup \{\infty\}$, $l \in \{a, b\}$, be defined by*

$$E^l_\epsilon(v) = \begin{cases} \displaystyle\int_{\Omega_l} \left[\frac{\epsilon k(x)}{2} |Dv|^2 + \frac{1}{\epsilon} F(v, x)\right] dx, & v \in H^1(\Omega_l) \\ \infty, & \textit{otherwise.} \end{cases}$$

Then

- $E_\epsilon(v) \geq E^a_\epsilon(v^a) + E^b_\epsilon(v^b)$ *if* $v \in L^1(\Omega) \setminus H^1(\Omega)$.
- $E_\epsilon(v) = E^a_\epsilon(v^a) + E^b_\epsilon(v^b)$ *if* $v \in H^1(\Omega)$.

Proof. Let $v \in L^1(\Omega)$.

If $v \in H^1(\Omega)$ then $v^a \in H^1(\Omega_a)$, $v^b \in H^1(\Omega_b)$ and clearly $E_\epsilon(v) = E^a_\epsilon(v^a) + E^b_\epsilon(v^b)$. If $v \notin H^1(\Omega)$ then $E_\epsilon(v) = \infty \geq E^a_\epsilon(v^a) + E^b_\epsilon(v^b)$. □

Lemma 3.5. *Let $E^l_0 : L^1(\Omega_l) \to \mathbb{R} \cup \{\infty\}$, $l \in \{a, b\}$ be defined by*

$$E^l_0(v) = \begin{cases} \displaystyle\int_{\Omega_l} h(x) \left|D\chi_{\{v=a\}}\right|, & v \in BV(\Omega_l, \{a, b\}) \\ \infty, & \textit{otherwise}\,. \end{cases}$$

Then E_0, given in Theorem 3.1, satisfies $E_0(v) = E^a_0(v^a) + E^b_0(v^b)$.

Proof. We first prove that $BV(\Omega, \{a, b\}) = \Xi$ where

$$\Xi \stackrel{\text{def}}{=} \left\{v \in L^1(\Omega) : v^a \in BV(\Omega_a, \{a, b\}) \,\text{and}\, v^b \in BV(\Omega_b, \{a, b\})\right\}.$$

- $BV(\Omega, \{a, b\}) \subset \Xi$.

If $v \in BV(\Omega,\{a,b\})$ then given $x \in \Omega_a$ either $v^a(x) = a(x)$ or $v^a(x) = b(x)$. If $v^a \notin BV(\Omega_a,\{a,b\})$ then given any $M > 0$ there exists $g_M \in C_0^1(\Omega_a, \mathbb{R}^n)$ such that $|g_M| \le 1$ and $\int_{\Omega_a} v \,\mathrm{div} g_M \, dx > M$. Letting

$$\overline{g}_M = \begin{cases} g_M, & \text{in } \Omega_a \\ 0, & \text{in } \Omega \setminus \Omega_a \end{cases}$$

then $\overline{g}_M \in C_0^1(\Omega, \mathbb{R}^n)$, $|\overline{g}_M| \le 1$ and $\int_\Omega v \,\mathrm{div}\overline{g}_M \, dx > M$. This prove that

$$\int_\Omega |Dv| = \sup\left\{ \int_\Omega v \,\mathrm{div} g \, dx : g \in C_0^1(\Omega, \mathbb{R}^n), \quad |g| \le 1 \right\} = \infty,$$

which contradicts the fact $v \in BV(\Omega,\{a,b\})$. Analogously we prove that $v^b \in BV(\Omega_b,\{a,b\})$.

- $BV(\Omega,\{a,b\}) \supset \Xi$

Indeed if $v \in \Xi$ then using the fact that Ω_a and Ω_b have Lipschitz boundaries we resort to Lemma 3.3 to conclude that $v \in BV(\Omega,\{a,b\})$.

Take now $v \in L^1(\Omega)$. If $v \notin BV(\Omega,\{a,b\})$ then either $v^a \notin BV(\Omega_a,\{a,b\})$ or $v^b \notin BV(\Omega_b,\{a,b\})$. It follows that $E_0(v) = \infty = E_0^a(v^a) + E_0^b(v^b)$.

If $v \in BV(\Omega,\{a,b\})$ then recalling that $h \equiv 0$ on γ and using the co-area formula (see (2.2)) we compute

$$\begin{aligned}
E_0(v) &= \int_\Omega h(x) \left|D\chi_{\{v=a\}}\right| = \int_{-\infty}^{\infty} \left[\int_{\Omega \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1} \right] d\xi \\
&= \int_{-\infty}^{\infty} \left[\int_{\Omega_a \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1} \right] d\xi \\
&+ \int_{-\infty}^{\infty} \left[\int_{\gamma \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1} \right] d\xi \\
&+ \int_{-\infty}^{\infty} \left[\int_{\Omega_b \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1} \right] d\xi \\
&= \int_{\Omega_a} h(x) \left|D\chi_{\{v=a\}}\right| + \int_\gamma h(x) \left|D\chi_{\{v=a\}}\right| + \int_{\Omega_b} h(x) \left|D\chi_{\{v=a\}}\right| \\
&= \int_{\Omega_a} h(x) \left|D\chi_{\{v=a\}}\right| + \int_{\Omega_b} h(x) \left|D\chi_{\{v=a\}}\right| \\
&= \int_{\Omega_a} h(x) \left|D\chi_{\{v^a=a\}}\right| + \int_{\Omega_b} h(x) \left|D\chi_{\{v^b=a\}}\right| \\
&= E_0^a(v^a) + E_0^b(v^b).
\end{aligned}$$

Therefore the proof is complete. □

Remark 3.6. The above result may not be valid for functionals of perimeter type in general; it holds in our case due only to the fact that the weight h vanish on γ.

Lemma 3.7. *It holds that*

$$\Gamma - \lim_{\epsilon\to 0} E^l_\epsilon(v) = E^l_0(v), \quad \forall \quad v \in L^1(\Omega_l), \quad l \in \{a,b\}\,.$$

The proof of this lemma can be rendered in a similar fashion to that found in [23] for the case $F(v,x) = (v-a(x))^2(v-b(x))^2$ with $a < b$ in Ω, i.e.

$$f(v,x) = 4(v-a(x))(v-b(x))\left[v - \left(\frac{a(x)+b(x)}{2}\right)\right].$$

Indeed a careful analysis of the proof of Theorem 2 in [23] shows that the function f above satisfies all sufficient conditions to obtain the Γ-convergence in the Lipschitz-continuous sets Ω_a and Ω_b, separately, since a and b are bounded in Ω_a and Ω_b, respectively, in the C^1 topology. Note that Lemma 3.7 calculates the Γ-limit in Ω_a where $a(x) > b(x)$ and, separately, in Ω_b where $a(x) < b(x)$. Therefore each limit $\Gamma - \lim_{\epsilon\to 0} E^l_\epsilon, \quad l \in \{a,b\}$, can be computed as in [23]. Now we are ready to prove the main theorem of this section.

Proof of Theorem 3.1. In the sequel we verify the requirements of the Definition 2.4 of Γ-convergence.

(i) Let $v \in L^1(\Omega)$ and a sequence $\{v_\epsilon\} \subset L^1(\Omega)$ such that $v_\epsilon \to v$ in $L^1(\Omega)$. Then

$$\begin{aligned} E_0(v) &\overset{(*)}{=} E^a_0(v^a) + E^b_0(v^b) \\ &\overset{(**)}{\leq} \liminf_{\epsilon\to 0} E^a_\epsilon(v^a_\epsilon) + \liminf_{\epsilon\to 0} E^b_\epsilon(v^b_\epsilon) \\ &\leq \liminf_{\epsilon\to 0} \left(E^a_\epsilon(v^a_\epsilon) + E^b_\epsilon(v^b_\epsilon)\right) \\ &\overset{(***)}{\leq} \liminf_{\epsilon\to 0} E_\epsilon(v_\epsilon). \end{aligned}$$

Here Lemmas 3.5, 3.7 and 3.4 were used in $(*)$,$(**)$ and $(***)$, respectively.

(ii) Given $v \in L^1(\Omega)$ we may consider $v \in BV(\Omega, \{a,b\})$ since otherwise $E_0(v) = \infty$. We have that $v^l \in BV\{\Omega_l, \{a,b\}\}$ and by Lemma 3.7 there exists $\{u_{\epsilon,l}\} \in L^1(\Omega_l)$ such that $u_{\epsilon,l} \to v^l$ in $L^1(\Omega_l)$ and

$$E^l_0(v^l) \geq \limsup_{\epsilon\to 0} E^l_\epsilon(u_{\epsilon,l}) \quad (l \in \{a,b\}). \tag{3.4}$$

Consider

$$v_\epsilon(x) = \begin{cases} u_{\epsilon,a}(x), & x \in \Omega_a \\ a(x), & x \in \gamma \\ u_{\epsilon,b}(x), & x \in \Omega_b. \end{cases} \tag{3.5}$$

It follows that $v_\epsilon \to v$ in $L^1(\Omega)$. In order to complete the proof it remains to show that

$$E_0(v) \geq \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon).$$

Claim: The functions $u_{\epsilon,l},\ l \in \{a,b\}$, may be constructed in such a way that

$$v_\epsilon \in H^1(\Omega). \tag{3.6}$$

We take it for granted for now and prove it later on. It follows that

$$\begin{aligned}
E_0(v) &\overset{(*)}{=} E_0^a(v^a) + E_0^b(v^b) \\
&\overset{(**)}{\geq} \limsup_{\epsilon\to 0} E_\epsilon^a(u_{\epsilon,a}) + \limsup_{\epsilon\to 0} E_\epsilon^b(u_{\epsilon,b}) \\
&= \limsup_{\epsilon\to 0} E_\epsilon^a(v_\epsilon^a) + \limsup_{\epsilon\to 0} E_\epsilon^b(v_\epsilon^b) \\
&\geq \limsup_{\epsilon\to 0} \left(E_\epsilon^a(v_\epsilon^a) + E_\epsilon^b(v_\epsilon^b)\right) \\
&\overset{(***)}{=} \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon).
\end{aligned}$$

In $(*)$ and in $(**)$, Lemma 3.5 and (3.4) were used respectively and finally in $(***)$ we resorted to (3.6) and Lemma 3.4 as well.

Proof of the Claim. Our goal is to build families of functions $\{u_{\epsilon,a}\}$ and $\{u_{\epsilon,b}\}$ so that $\{v_\epsilon\}$ (defined by (3.5)) satisfies (3.6).

First we construct the family $\{u_{\epsilon,a}\}$ defined in Ω_a following the steps of [23] with the diffusivity function k introduced. Let

$$D \overset{\text{def}}{=} \{x \in \Omega : v(x) = a(x)\} \tag{3.7}$$

and assume that $\partial D \cap \Omega$ is C^2; in the conclusion of the proof we will show that this represents no loss of generality.

Consider

$$A_1 = \{x \in \Omega_a : v(x) = a(x)\}$$
$$A_2 = \{x \in \Omega_a : v(x) = b(x)\}$$

and the signed distance function

$$d_a(x) = \begin{cases} dist(x, \partial D \cap \Omega_a), & x \in A_1 \\ -dist(x, \partial D \cap \Omega_a), & x \in A_2. \end{cases}$$

Only the case when $\partial D \cap \Omega_a \neq \emptyset$ and $\partial D \cap \Omega_b \neq \emptyset$ will be addressed since the other ones are easier to prove. It follows from the above hypothesis that

$$\Lambda_a \overset{\text{def}}{=} \partial D \cap \Omega_a \tag{3.8}$$

is C^2. Now consider Z_a the solution of the following initial value problem

$$\left.\begin{array}{rcl} \frac{dZ}{ds}(s,x) & = & \sqrt{k^{-1}(x)F(s,x)}, \quad (s,x)\in \mathbb{R}\times\Omega_a \\ Z(0,x) & = & \theta(x), \quad x\in\Omega_a \end{array}\right\} \tag{3.9}$$

Recall that in Ω_a we have $a > \theta > b$. Thus the smooth solution Z_a of (3.9) -recall that a and b are smooth and C^1 bounded on each connected component- satisfies:

$$a(x) > Z_a(s,x) > b(x), \quad \forall\ (s,x)\in\mathbb{R}\times\Omega_a. \tag{3.10}$$

As $a = b$ on γ, it follows that

$$\lim_{x\to x_0} Z_a(s,x) = a(x_0), \quad \forall\ (s,x_0)\in\mathbb{R}\times\gamma. \tag{3.11}$$

Finally we consider $u_{\epsilon,a} : \Omega_a \to \mathbb{R}$ defined as follows

$$u_{\epsilon,a}(x) \stackrel{\text{def}}{=}$$

$$\begin{cases} a(x), & d_a(x) > 2\sqrt{\epsilon} \\ [a(x) - Z_a(x, 1/\sqrt{\epsilon})]\,\frac{\left(d_a(x)-2\sqrt{\epsilon}\right)}{\sqrt{\epsilon}} + a(x), & \sqrt{\epsilon}\le d_a(x) \le 2\sqrt{\epsilon} \\ Z_a(x, d_a(x)/\epsilon), & |d_a(x)| < \sqrt{\epsilon} \\ [Z_a(x, -1/\sqrt{\epsilon}) - b(x)]\,\frac{\left(d_a(x)+2\sqrt{\epsilon}\right)}{\sqrt{\epsilon}} + b(x), & -2\sqrt{\epsilon}\le d_a(x)\le -\sqrt{\epsilon} \\ b(x), & d_a(x) < -2\sqrt{\epsilon}. \end{cases}$$

Now, as shown in [23], $u_{\epsilon,a}\in H^1(\Omega_a)$, $u_{\epsilon,a}\to v^a$ in $L^1(\Omega_a)$, $E_0^a(v^a) \ge \lim_{\epsilon\to 0}\sup E_\epsilon^a(u_{\epsilon,a})$ and by (3.11)

$$\lim_{x\to x_0} u_{\epsilon,a}(x) = a(x_0),\ \forall\ x_0\in\gamma.$$

In a similar fashion we construct $\{u_{\epsilon,b}\}$ defined in Ω_b such that $u_{\epsilon,b}\in H^1(\Omega_b)$, $u_{\epsilon,b}\to v^b$ in $L^1(\Omega_b)$, $E_0^b(v^b)\ge \lim_{\epsilon\to 0}\sup E_\epsilon^b(u_{\epsilon,b})$ and

$$\lim_{x\to x_0} u_{\epsilon,b}(x) = a(x_0),\ \forall\ x_0\in\gamma.$$

This proves that v_ϵ is continuous in γ (see (3.5)). It follows that v_ϵ is continuous in Ω and its partial derivatives in the sense weak is given by

$$\partial_i v_\epsilon = \partial_i u_{\epsilon,a}\chi_{\Omega_a} + \partial_i u_{\epsilon,b}\chi_{\Omega_b}\ \ (i = 1,\dots,n).$$

Indeed, first note that since the sets $\partial\Omega_a = \gamma\cup\partial\Omega$ and $\partial\Omega_b = \gamma$ are Lipschitz-continuous their outer unit-normals $\nu_a = (\nu_a^1,\dots,\nu_a^n)$ and $\nu_b = (\nu_b^1,\dots,\nu_b^n)$, respectively, exist a.e. with respect to surface-measure and the Divergence Theorem applies. Given that $\nu_a = -\nu_b$ on γ then for all test function $\phi \in C_0^\infty(\Omega)$ we compute

$$\begin{aligned}
\int_\Omega v_\epsilon \partial_i \phi dx &= \int_{\Omega_a} u_{\epsilon,a}\partial_i\phi dx + \int_{\Omega_b} u_{\epsilon,b}\partial_i\phi dx \\
&= \int_{\partial\Omega_a} u_{\epsilon,a}\phi\nu_a^i dS - \int_{\Omega_a} \partial_i u_{\epsilon,a}\phi dx \\
&+ \int_{\partial\Omega_b} u_{\epsilon,b}\phi\nu_b^i dS - \int_{\Omega_b} \partial_i u_{\epsilon,b}\phi dx \\
&= \int_\gamma u_{\epsilon,a}\phi\nu_a^i dS - \int_\Omega \left(\partial_i u_{\epsilon,a}\chi_{\Omega_a} + \partial_i u_{\epsilon,b}\chi_{\Omega_b}\right)\phi dx + \int_\gamma u_{\epsilon,b}\phi\nu_b^i dS \\
&= -\int_\gamma a\phi\nu_b^i dS + \int_\gamma a\phi\nu_b^i dS - \int_\Omega \left(\partial_i u_{\epsilon,a}\chi_{\Omega_a} + \partial_i u_{\epsilon,b}\chi_{\Omega_b}\right)\phi \\
&= -\int_\Omega \left(\partial_i u_{\epsilon,a}\chi_{\Omega_a} + \partial_i u_{\epsilon,b}\chi_{\Omega_b}\right)\phi dx.
\end{aligned}$$

Thus, as $u_{\epsilon,l} \in H^1(\Omega_l)$, $l \in \{a,b\}$, is not difficult to see that $v_\epsilon \in H^1(\Omega)$. The claim is proved.

It remains to prove that there is no loss of generality to assume that $\partial D \cap \Omega$ is C^2. We have that $D \subset \Omega$ (see (3.10)) is a set of finite perimeter in Ω and satisfies the conditions of Lemma 2.3. Thus there exists a sequence of open sets $\{D_j\}$ satisfying conditions $(i)-(v)$ of Lemma 2.3.

Consider a sequence $\{v_j\}$ defined by

$$v_j(x) = \begin{cases} a(x) & x \in D_j \cap \Omega \\ b(x) & x \in \Omega \setminus D_j. \end{cases}$$

From property (ii), given $m \in \mathbb{N}$, there exists $j_1(m)$ such that

$$\|v_j - v\|_{L^1(\Omega)} < \frac{1}{2m}$$

for all $j \geq j_1(m)$. By (iii) and as $\|h\|_{L^\infty} < \infty$, we have that there exists $j_2(m)$ such that $|E_0(v_j) - E_0(v)| < \frac{1}{2m}$, for all $j \geq j_2(m)$. Let $j(m) := \max\{j_1(m), j_2(m)\}$.

Since ∂D_j is C^2 then, as established above, there exists a sequence $\{v_{\epsilon,j}\}$ in $H^1(\Omega)$ and $\epsilon_1(m)$ such that $\|v_{\epsilon,j} - v_j\|_{L^1(\Omega)} < \frac{1}{2m}$, for all $\epsilon \leq \epsilon_1(m)$ and there exists $\epsilon_2(m)$ such that $|E_\epsilon(v_{\epsilon,j}) - E_0(v_j)| < \frac{1}{2m}$, for all $\epsilon \leq \epsilon_2(m)$. Let $\epsilon(m) := \min\{\epsilon_1(m), \epsilon_2(m)\}$.

Take the sequence $\{v_m\}_{m\in\mathbb{N}}$ such that $v_m = v_{\epsilon(m),j(m)}$; then

$$\|v_m - v\|_{L^1(\Omega)} =$$

$$\|v_{\epsilon(m),j(m)} - v\|_{L^1(\Omega)} \leq \|v_{\epsilon(m),j(m)} - v_{j(m)}\|_{L^1(\Omega)} + \|v_{j(m)} - v\|_{L^1(\Omega)} < \frac{1}{m} \text{ and}$$

$$\left|E_{\epsilon(m)}(v_m) - E_0(v)\right| = \left|E_{\epsilon(m)}(v_{\epsilon(m),j(m)}) - E_0(v)\right| \leq$$

$$\left|E_{\epsilon(m)}(v_{\epsilon(m),j(m)}) - E_0(v_{j(m)})\right| + \left|E_0(v_{j(m)}) - E_0(v)\right| < \frac{1}{m}.$$

The two inequalities above assure us that there is no loss of generality in assuming that $\partial D \cap \Omega \in C^2$. □

3.2. Existence of four isolated minimizers for the Γ-limit

Recall that existence of isolated minimizers of the Γ-limit is required in order to use Theorem 2.6. The elementary result below will be helpful in the proof of the main result of this section. We could not find any reference of it and therefore a proof is provided for the sake of completeness.

Lemma 3.8. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with Lipschitz-continuous boundary. Let $E \subset \Omega$ be a set of finite perimeter such that*

$$0 < \mathcal{H}^n(E) < \mathcal{H}^n(\Omega). \tag{3.12}$$

Then $Per_\Omega(E) > 0$.

Proof. Suppose by contradiction that

$$\mathrm{Per}_\Omega(E) = 0. \tag{3.13}$$

If $x_0 \in \Omega$ then for any $r > 0$ such that $B(x_0, r) \subset \Omega$ it holds that $\mathrm{Per}_{B(x_0,r)}(E) = 0$.

Indeed if there existed $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$ and $\mathrm{Per}_{B(x_0,r)}(E) > 0$ then we would have

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{n-1}(\partial_* E \cap \Omega) = \mathcal{H}^{n-1}(\partial_* E \cap (\Omega \backslash B(x_0, r) \cup B(x_0, r)))$$

$$\geq \mathcal{H}^{n-1}(\partial_* E \cap B(x_0, r)) = \mathrm{Per}_{B(x_0,r)} E > 0,$$

which contradicts (3.13).

Thus by Theorem 2.2 we have that either $\mathcal{H}^n(B(x_0, r) \cap E) = 0$ or $\mathcal{H}^n(B(x_0, r) \setminus E) = 0$. Consider the sets

$$A_1 = \{x \in \Omega : \mathcal{H}^n(B(x, r) \cap E) = 0, \text{ for some } B(x, r) \subset \Omega\}$$

and

$$A_2 = \{x \in \Omega : \mathcal{H}^n(B(x, r) \setminus E) = 0, \text{ for some } B(x, r) \subset \Omega\}.$$

We conclude that $A_1 \cap A_2 = \emptyset$, $\Omega = A_1 \cup A_2$ and A_1 and A_2 are open sets. Since Ω is connected we have either $A_1 = \emptyset$ or $A_2 = \emptyset$. In each case (3.12) is contradicted. Therefore $\mathrm{Per}_\Omega(E) > 0$. □

Now we state and prove the main result of this section which is crucial to our approach.

Theorem 3.9. *Each of the functions $u_0^1, \ldots, u_0^4 : \Omega \to \mathbb{R}$ defined by*

- $u_0^1 = a\chi_{\Omega_a} + b\chi_{\Omega_b}$
- $u_0^2 = b\chi_{\Omega_a} + a\chi_{\Omega_b}$
- $u_0^3 = a$
- $u_0^4 = b$

is an isolated L^1-local minimizer of E_0.

Proof. We render the proofs for u_0^1 and u_0^3 only since the other cases can be dealt with in a similar fashion. Consider $\rho > 0$ such that

$$\rho < \min\left\{\int_{\Omega_a}(a-b)dx, \int_{\Omega_b}(b-a)dx\right\}. \tag{3.14}$$

It suffices to show that if $u \in L^1(\Omega)$ satisfies

$$0 < \|u - u_0^1\|_{L^1(\Omega)} < \rho \tag{3.15}$$

then $E_0(u) > E_0(u_0^1)$.

On the account that

$$\Omega \cap \partial_*\left\{\chi_{\{u_0^1=a\}} > \xi\right\} = \begin{cases} \emptyset, & \xi \notin [0,1) \\ \gamma, & \xi \in [0,1) \end{cases}$$

the co-area formula yields

$$\begin{aligned} E_0(u_0^1) = \int_\Omega h(x)\left|D\chi_{\{u_0^1=a\}}\right| &= \int_{-\infty}^{\infty}\int_{\Omega\cap\partial_*\left\{\chi_{\{u_0^1=a\}}>\xi\right\}} h(x)d\mathcal{H}^{n-1}d\xi \\ &= \int_0^1\int_\gamma h(x)d\mathcal{H}^{n-1}d\xi = 0 \end{aligned}$$

using once more the fact that $h = 0$ on γ.

If $u \notin BV(\Omega,\{a,b\})$ then $E_0(u) = \infty$ and obviously $E_0(u) > E_0(u_0^1)$ since $E_0(u_0^1) = 0$. Therefore consider $u \in BV(\Omega,\{a,b\})$ satisfying (3.15) and define the set

$$A \stackrel{\text{def}}{=} \{u = a \text{ a.e. in } \Omega\}.$$

Then A is a set of finite perimeter and we have two possibilities:

(i) $\mathcal{H}^{n-1}((\partial_* A \cap \Omega) \setminus \gamma) = 0$ or
(ii) $\mathcal{H}^{n-1}((\partial_* A \cap \Omega) \setminus \gamma) > 0$

Claim: If (i) occurs then $u = u_0^j$ a.e. in Ω, for some $j \in \{1,2,3,4\}$.

Indeed suppose that none of the above four cases occurs and define the set

$$B \stackrel{\text{def}}{=} \{u = b \text{ a.e. in } \Omega\}.$$

If $u \neq u_0^3(\equiv a)$ a.e. in Ω then $A \neq \Omega_a$ for $A = \Omega_a$ would imply, given that u takes on only two values a or b, that $B = \Omega_b$, i.e., $u \equiv u_0^1$, as we wish.

Using the same kind of argument the other cases can be dealt with thus yielding either $A \neq \Omega_a$ or $B \neq \Omega_b$.

Suppose first that

$$K \stackrel{\text{def}}{=} A \cap \Omega_a \neq \emptyset.$$

As Ω_a is open, bounded, connected, with Lipschitz-continuous boundary and $K \subset \Omega_a$ is a set of finite perimeter satisfying $0 < \mathcal{H}^n(K) < \mathcal{H}^n(\Omega_a)$, we can apply Lemma 3.8 to conclude that $\text{Per}_{\Omega_a} K > 0$. But $(\partial_* K \cap \Omega_a) \subset [(\partial_* A \cap \Omega) \setminus \gamma]$ and therefore

$$\mathcal{H}^{n-1}((\partial_* A \cap \Omega) \setminus \gamma) \geq \mathcal{H}^{n-1}((\partial_* K \cap \Omega_a)) = \text{Per}_{\Omega_a} K > 0,$$

which is a contradiction. The case $B \cap \Omega_b \neq \emptyset$ can be treated in a similar fashion. The claim is proved.

Next we consider all the four possibilities, implied by (i), and its implications:

- $u = u_0^1$ a.e. in $\Omega \Rightarrow \|u - u_0^1\|_{L^1(\Omega)} = 0$
- $u = u_0^2$ a.e. in $\Omega \Rightarrow \|u - u_0^1\|_{L^1(\Omega)} = \int_\Omega |b-a| > \rho$
- $u = a$ a.e. in $\Omega \Rightarrow \|u - u_0^1\|_{L^1(\Omega)} = \int_{\Omega_b} |b-a| > \rho$
- $u = b$ a.e. in $\Omega \Rightarrow \|u - u_0^1\|_{L^1(\Omega)} = \int_{\Omega_a} |b-a| > \rho$

In all four cases we reach a contradiction given that u satisfies (3.15).

Therefore we must have (ii). Now note that

$$\Omega \cap \partial_* \{\chi_{\{u=a\}} > \xi\} = \begin{cases} \emptyset, & \xi \notin [0,1) \\ \Omega \cap \partial_* A, & \xi \in [0,1) \end{cases}$$

Hence

$$\begin{aligned} E_0(u) = \int_\Omega h(x) \left|D\chi_{\{u=a\}}\right| &= \int_{-\infty}^{\infty} \int_{\Omega \cap \partial_* \{\chi_{\{u=a\}} > \xi\}} h(x) d\mathcal{H}^{n-1} d\xi \\ &= \int_0^1 \int_{\Omega \cap \partial_* A} h(x) d\mathcal{H}^{n-1} d\xi \\ &\geq \int_0^1 \int_{(\Omega \cap \partial_* A) \setminus \gamma} h(x) d\mathcal{H}^{n-1} d\xi > 0 \end{aligned}$$

using (ii) and the fact that $h(x) > 0$ for all $x \in \Omega \setminus \gamma$. It follows that $E_0(u) > E_0(u_0^1)$ thus implying that u_0^1 is an isolated local minimum of E_0.

Let us now consider u_0^3. Taking $\rho > 0$ given by (3.14) we use that

$$\Omega \cap \partial_* \left\{\chi_{\{u_0^3 = a\}} > \xi\right\} = \emptyset$$

to compute

$$E_0(u_0^3) = \int_\Omega h(x) \left|D\chi_{\{u_0^3=a\}}\right| = \int_{-\infty}^{\infty} \int_{\Omega \cap \partial_* \{\chi_{\{u_0^3=a\}} > \xi\}} h(x) d\mathcal{H}^{n-1} d\xi = 0$$

Then if u satisfies $0 < \left|u - u_0^3\right|_{L^1(\Omega)} < \rho$ similarly to the previous case we conclude that $E_0(u) > 0$ and therefore $E_0(u) > E_0(u_0^3)$.

This proves Theorem 3.9.

□

3.3. Proof of Theorem 1.1 under (f_1)

The proof is now accomplished by a direct application of Theorem 2.6. After calculating the Γ-limit E_0 of the family of functionals given in (3.1) and proving that each function $u_0^1, \ldots, u_0^4$ is a local minimum isolated of E_0, as shown in Theorem 3.9, it remains to verify hypothesis (i) of Theorem 2.6. This hypothesis would follow if the minimizers were uniformly bounded in L^1 (see [23], Proposition 3 and Remark (1.35), for instance). But this is really the case and it can be accomplished by an application of the maximum principle following the procedure used in [8], for instance.

Hence an application of Theorem 2.6 implies that, for some $\epsilon_0 > 0$, the family of functionals given by (3.1) possesses four families of local minima $\{u_\epsilon^1\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \dots, 4)$ satisfying $\|u_\epsilon^j - u_0^j\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ $(j = 1, \dots, 4)$.

Therefore for $0 < \epsilon < \epsilon_0$ each function u_ϵ^j, $j \in \{1, 2, 3, 4\}$, is a stationary solution of (1.1) in $H^1(\Omega)$ and it remains to verify its stability.

The second variation of the energy functional E_ϵ at u_ϵ^j is nonnegative. If $\lambda_1(u_\epsilon^j)$ is the first eigenvalue of the problem

$$\left.\begin{array}{ll} \epsilon^2 \mathrm{div}[k(x)\nabla\varphi] + \partial_1 f(u_\epsilon^j, x)\varphi = -\lambda\varphi & x \in \Omega \\ \partial_n \varphi = 0 & x \in \partial\Omega \end{array}\right\} \tag{3.16}$$

then $\lambda_1(u_\epsilon^j) \geq 0$, due to its variational characterization. It is well-known that if $\lambda_1(u_\epsilon^j) > 0$ then u_ϵ^j is asymptotically stable.

The case $\lambda_1(u_\epsilon^j) = 0$ is standard by now (see [5], for instance). Actually since it is a simple eigenvalue, there is a local one-dimensional critical manifold $W(u_\epsilon^j)$, tangent to principal eigenfunction corresponding to the zero eigenvalue, such that if u_ϵ^j is stable in $W(u_\epsilon^j)$ then it is also stable in $H^1(\Omega)$ (see [15], Theorem 6.2.1, for instance). Now the stability of u_ϵ^j in $W(u_\epsilon^j)$ follows from the existence of a Lyapunov functional and the fact that $W(u_\epsilon^j)$ is one-dimensional.

The theorem is proved.

3.4. Proof of Theorem 1.1 under (f_2)

Corollary 3.10. *Theorem* 1.1 *still holds by replacing* (f_1) *with assumption* (f_2).

Proof. Indeed this is accomplished by defining new continuous functions p and n by

$$p(x) = \begin{cases} a(x), & x \in \Omega_a \\ b(x), & x \in \Omega_b \end{cases}$$

and

$$n(x) = \begin{cases} b(x), & x \in \Omega_a \\ a(x), & x \in \Omega_b \end{cases}$$

Since $a, b \in C^0(\Omega) \cap C^1(\Omega \setminus \gamma)$, the continuous functions p and n clearly satisfy

- $p, n \in C^1(\Omega \setminus \gamma)$
- $f(p(x), x) = f(n(x), x) = f(\theta(x), x) = 0,\ \forall\, x \in \Omega$
- $p > n$ in $\Omega \setminus \gamma$ and $p = \theta = n$ on γ
- $\partial_1 f(p(x), x) < 0$ and $\partial_1 f(n(x), x) < 0$, $\forall\ \ x \in \Omega \setminus \gamma$.

Therefore Theorem 1.1 under (f_1) applies for the roots n, θ, p and the proof is established after relabeling the stable stationary solutions $\{u_\epsilon^j\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \dots, 4)$ accordingly, i.e., $u_0^1 \equiv a$, $u_0^2 \equiv b$, $u_0^3 = a\chi_{\Omega_a} + b\chi_{\Omega_b}$ and $u_0^4 = b\chi_{\Omega_a} + a\chi_{\Omega_b}$. □

4. Intersection and/or contact along many hypersurfaces

Theorem 1.1 can be generalized in a natural fashion to a finite number of Lipschitz hypersurfaces in many different ways. For the sake of brevity we indicate how this can be done for just two Lipschitz hypersurfaces.

Given two Lipschitz-continuous hypersurfaces without boundaries γ_1 and γ_2 let $\mathcal{O}_j$ denote the open region enclosed by γ_j ($j = 1, 2$). With Ω as before we say that these hypersurfaces are nested in Ω if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \Omega$, $\partial\mathcal{O}_1 = \gamma_1$, $\partial\mathcal{O}_2 = \gamma_1 \cup \gamma_2$ and $\partial(\Omega \setminus \mathcal{O}_2) = \gamma_2 \cup \partial\Omega$.

Furthermore we define the sets

$$\Omega_1 = \mathcal{O}_1, \quad \Omega_2 = \mathcal{O}_2 \setminus \overline{\mathcal{O}}_1 \text{ and } \Omega_3 = \Omega \setminus \overline{\mathcal{O}}_2$$

where $\overline{\mathcal{O}}$ stands for he closure of $\mathcal{O}$.

Assume there exist two nested $(n-1)$-dimensional hypersurfaces in Ω such that Ω_j ($j = 1, 2$) are Lipschitz and three continuous functions θ, a, b satisfying $\theta, a, b \in C^1(\Omega \setminus \gamma_1 \cup \gamma_2)$ and C^1-bounded in Ω_1,Ω_2 and Ω_3. Moreover suppose that $f(a(x), x) = f(b(x), x) = f(\theta(x), x) = 0, \forall x \in \Omega$, $\partial_1 f(a(x), x) < 0$, $\partial_1 f(b(x), x) < 0, \forall x \in \Omega \setminus (\gamma_1 \cup \gamma_2)$ and one of the following three hypotheses hold

$$(f_4) \begin{cases} a > \theta > b, & \text{in } \Omega \setminus (\gamma_1 \cup \gamma_2) \\ a = \theta = b, & \text{on } \gamma_1 \cup \gamma_2 \end{cases}$$

or

$$(f_5) \begin{cases} a > \theta > b \text{ in } \Omega_1 \\ b > \theta > a \text{ in } \Omega_2 \\ a > \theta > b \text{ in } \Omega_3 \\ a = \theta = b \text{ in } \gamma_1 \cup \gamma_2 \end{cases}$$

or

$$(f_6) \begin{cases} a > \theta > b \text{ in } \Omega_1 \\ b > \theta > a \text{ in } \Omega_2 \cup \Omega_3 \\ a = \theta = b \text{ in } \gamma_1 \cup \gamma_2 \end{cases}$$

Note that under (f_4) the function $d(x) = a(x) - b(x)$ is non-negative in Ω and vanishes only along $\gamma_1 \cup \gamma_2$; (f_5) means that d changes sign across γ_1 and γ_2 whereas (f_6) is a combination of both of them.

Theorem 4.1. *If f satisfies (f_3) and one of the hypotheses (f_4), (f_5) or (f_6) then $\exists\, \epsilon_0 > 0$ and eight families of stable stationary solutions $\{u_\epsilon^j\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \ldots, 8)$ to (1.1) such that*

$$\|u_\epsilon^j - u_0^j\|_{L^1(\Omega)} \xrightarrow{\epsilon\to 0} 0 \; (j = 1, \ldots, 8).$$

For brevity we omit the description of the limiting function u_0^j $(j = 1, \ldots, 8)$.

Remark 4.2. Although we did not care for, Theorem 1.1 and 4.1 also hold for one-dimensional domains by using the same approach. Just for the sake of illustration a simple one-dimensional example which satisfies hypothesis (f_5)

of Theorem 4.1 is given by

$$\begin{array}{ll} u_t = \epsilon^2[k(x)u_x]_x + u[\cos^2(x) - u^2] & (t,x) \in \mathbb{R}^+ \times (-\pi,\, \pi) \\ u_x(t,-\pi) = u_x(t,\pi) = 0 & t \in \mathbb{R}^+ \end{array}$$

Here the roots of $f(x,\,\cdot)$, $\cos x$ and $-\cos x$, intersect transversally at the points $-\pi/2$ and $\pi/2$.

Furthermore $f(u,x) = u[|\cos x|^2 - u^2]$, $x \in (-\pi,\, \pi)$, is a simple example which satisfies (f_4) and the roots $|\cos x|$ and $-|\cos x|$ have just C^0-contact at $-\pi/2$ and $\pi/2$.

One can easily check that if the two roots of $f(x,\,\cdot)$ intersect along m nested hypersurfaces then (1.1) possesses $2^{(m+1)}$ stable stationary solutions converging in the L^1-norm, as $\varepsilon \to 0$, to the corresponding local isolated minimizers of the $\Gamma-$limit problem.

Remark 4.3. In Theorem 4.1 instead of having γ_1 and γ_2 two nested Lipschitz hypersurfaces we could have allowed $\gamma_1 \cap \gamma_2 \neq \emptyset$ as long as all connected components of $\Omega \setminus \gamma_1 \cup \gamma_2$ are Lipschitz sets. Also one of the hypersurfaces could have boundary as long as it intersect $\partial\Omega$ transversally in such way that the connected components of $\Omega \setminus \gamma_1 \cup \gamma_2$ are also Lipschitz sets. Of course the number of patterns and its limiting functions will change accordingly.

5. Conclusions

Note that all the patterns $\{u^j_\epsilon\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \ldots, 4)$ obtained in Theorem 1.1 do not exhibit transition layers due to the fact that the limit functions $u^j_0\,(j = 1, \ldots, 4)$, as $\epsilon \to 0$, are continuous across γ. This is true for the patterns obtained in Theorem 4.1 as well.

In the sequel we make some considerations about the regularity of the patterns obtained in Theorem 1.1. As opposed to the existing results on the phenomenon of exchange of stability (see e.g. [18], [19], [2]) which treat only the cases of one and two-dimensional domains and γ a simple smooth closed curve our present result, due to the approach utilized, holds for $n-$dimensional domains and allows γ to be non-smooth. The price paid for these gains is the lack of information on the regularity of the patterns $u^j_\epsilon\,(j = 1, \ldots, 4)$ and convergence of these patterns to the limiting functions only in $L^1(\Omega)$. Based on results for the case of exchange of stability we expected, under (f_1) for the case when $a, b\, \in C^1(\Omega \setminus \gamma)$ have only C^0-contact along γ, that all the patterns converge uniformly in $\overline{\Omega}$, as $\epsilon \to 0$, with u^1_ϵ and u^2_ϵ being C^2 outside a narrow tubular neighborhood around γ with a corner layer along γ; the regularity of u^3_ϵ and u^4_ϵ will depend on the regularity of γ and of the functions a, b in Ω so that if $a, b\, \notin C^1(\Omega)$ they would have a corner layer as well. Now still under (f_1) if a, b are smooth across a smooth γ then in addition to the uniform convergence to the limiting functions we expect all patterns to be in $C^2(\Omega)$.

In [11] the case in which $\Omega \subset \mathbb{R}^2$ and the zeros of $f(u,x) = u\,[a^2(x) - u^2]$ do not intersect, i.e. $a > 0$ in $\overline{\Omega}$, was addressed. More precisely let

- $\gamma(s)$, $0 \le s \le L$, be an arc-length parametrized C^2 simple closed curve in Ω,
- $\kappa(s)$, $0 \le s \le L$ denote its signed curvature

and consider the principal coordinate system (s,t) in a narrow tubular neighborhood around γ, i.e., t is the signed distance from a point (x_1, x_2) to γ. Set $\tilde{a}(s,t) = a(x_1(s,t), x_2(s,t))$ and suppose that

$$\left.\begin{array}{ll} \dfrac{\partial \tilde{a}(s,0)}{\partial t} = \dfrac{1}{3}\kappa(s)\,\tilde{a}(s,0) & 0 \le s \le L \\ \dfrac{\partial^2 \tilde{a}(s,0)}{\partial t^2} > \dfrac{4}{9}\kappa^2(s)\,\tilde{a}(s,0) & 0 \le s \le L \end{array}\right\} \tag{5.1}$$

Then in [11], under (5.1), we proved existence of four stable stationary solutions to (1.1) satisfying the conclusions of Theorem 1.1 where in the definition of the limit functions u_0^j $(j = 1, \dots, 4)$ the function b is replaced with $-a$. In that case the solutions u_ε^1 and u_ε^2 exhibited transition layer at the interface γ as they jump from $-a(x)$ to $a(x)$ across γ and vice-versa. What Theorem 1.1 states is that by moving the graphs of $-a(x)$ and $a(x)$ towards each other so that they touch along γ -thus satisfying (f_1)- the number of such solutions remains the same but the transition layers cease to exist since now both roots coincide along γ. Therefore in this sense this is a continuous process.

Although in [11] we utilized the same Γ-convergence procedure the fact that in the present work the weight h of the Γ-limit E_0 vanishes on γ allows us not only to get rid of condition (5.1) but also to have a proof for $n-$dimensional $(n \ge 1)$ domains.

An important geometric feature required in the case of non-intersecting roots (see [11] when $n = 2$ and [22] when $n = 1$ and $k \equiv 1$) which seems to be hidden in the present work is the following. Equations (5.1) mean that the function $d(s,\cdot) = \tilde{a}(s,\cdot) - [-\tilde{a}(s,\cdot)] = 2\tilde{a}(s,\cdot)$, seen as a function of t alone, reaches its non-degenerated strict local minima along a curve $\overline{\gamma}$ which lies in a tubular neighborhood of γ and, in particular, keeps switching from one side of γ to the other, as s varies from 0 to L, according to the changing of sign of the curvature of γ; wherever $\kappa(s) = 0$ the function $\tilde{a}(s,\cdot)$ assumes its minimum value on γ. In [22] for one-dimensional domains and $f(u,x) = -(u - a(x))(u - \theta(x))(u - b(x))$ it was explicitly required that $d(x) = a(x) - b(x)$ assumes its non-degenerated strict local minima at a finite number of point in (0, 1). Although this condition seems to be disguised in the present case note that hypothesis (f_1) implies that $d(x) = a(x) - b(x)$ assumes its strict minima – either degenerated or not – along γ as well.

Theorems 1.1 and 4.1 still hold for the slightly more general equation which appears as a selection-migration model in population genetics (see [15], e.g.)

$$\left.\begin{array}{ll} u_t = \epsilon^2 \mathrm{div}[k(x)\nabla u] + s(x) f(u,x) & (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_\nu u(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \partial\Omega \end{array}\right\} \tag{5.2}$$

where the continuous function $s > 0$ in Ω stands for the local relative selective advantage of the gene position. In this case the Γ-limit E_0 is the same as in Theorem 3.1 except that the weight is given by

$$h(x) = \sqrt{2} \int_{b(x)}^{a(x)} \sqrt{s(x)k(x)F(s,x)}ds.$$

In order to see that all the proofs still carry over to this case it suffices to realize that the selective advantage s has the same role as the diffusivity function k in the weight h as they appear multiplying each other.

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