Stable equilibria to a singularly perturbed reaction-diffusion equation in a degenerated heterogeneous environment

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Abstract

We address the problem $u_t = \epsilon^2 \Delta u + f(u,x)$ in $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ under boundary condition $\partial_\nu u = 0$ where $f(u,x) = -(u-a(x))(u-\theta(x))(u-b(x))$, $\theta(x) = [a(x)+b(x)]/2$ and $a \leq b$ in Ω. The novelty here lies in the fact that the roots of f are allowed to degenerate in the sense that $a = \theta = b$ in $\Omega \setminus D$ where $D \subset \Omega$ is such that $D = D_1 \cup D_2$, $\overline{D_1} \cap \overline{D_2} = \emptyset$, D_1 and D_2 are non-empty open connected sets with Lipschitz-continuous boundaries and $a < b$ in D. For ϵ small, we prove existence of four families of stable stationary solutions u_ϵ approaching the roots of f in the topology of L^1. Our approach is variational and based on $\Gamma-$convergence theory.

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1. Introduction and Main Result

The main concern in this paper is to prove existence of nonconstant stable stationary solutions -herein referred to as patterns, for short- to the reaction-diffusion problem

$$\left.\begin{array}{ll} u_t = \epsilon^2 \Delta u + f(u,x), & (t,x) \in \mathbb{R}^+ \times \Omega \\ \frac{\partial u}{\partial \nu} = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega \end{array}\right\} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open connected bounded set with C^2 boundary, ϵ is a small positive parameter, ν denotes the outer unit normal to $\partial\Omega$ and $f(u,x)$

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is defined by

$$f(u,x) = -(u - a(x))(u - \theta(x))(u - b(x)) \tag{2}$$

where $\theta(x) = [a(x) + b(x)]/2$ and $a \le b$ in Ω. The novelty here lies in the fact that the roots of f are allowed to degenerate in the sense described below.

Let $D \subset \Omega$ be such that $D = D_1 \cup D_2$, $\overline{D_1} \cap \overline{D_2} = \emptyset$, where D_1 and D_2 are open and connected sets with Lipschitz-continuous boundaries and set

$$D_3 \stackrel{\text{def}}{=} \Omega \setminus \overline{D}.$$

The roots of f are required to satisfy

(f_1) $a, \theta, b \in C(\Omega) \cap C^1(\cup_{i=1}^3 D_i)$, C^1-bounded in D_1, D_2 and D_3 and $a = \theta = b$ in D_3,

(f_2) $a < b$ in D.

Note that a and b may not be differentiable across $\Omega \cap \partial D_3$ and the sets D_1 and D_2 may be located in Ω in such way that one of the following situations may occur

- $\partial D_j \cap \partial\Omega$ $(j = 1, 2)$ are sets of positive measure,
- $\partial D_j \cap \partial\Omega = \emptyset$ $(j = 1, 2)$ or, e.g.,
- $\partial D_1 \cap \partial\Omega$ is a set of positive measure and $\partial D_2 \cap \partial\Omega = \emptyset$.

Given the vast bibliography concerning problem (1) in the case when a, b, θ are constant functions, in order to set our work in perspective we will only mention those works closely related to ours which consider the roots of f to be non-constant functions, i.e., spatially heterogeneous.

In [9] the author showed existence of patterns for the corresponding one-dimensional problem $u_t - \varepsilon^2 u_{xx} = u[\alpha(x) - u^2]$ in $(0\,,1)$ under the boundary conditions $u_x(0) = u_x(1) = 0$ where $\alpha \in C(0, 1)$ is a positive function assuming a minimum on an interval $I \subset (0\,,1)$ satisfying $\alpha'(x) = 0$, $x \in I$. Hence α assumes a positive minimum value all over I. The method utilized was a Brezis-Nirenberg sub-supersolution type adapted to Neumann boundary condition.

In [7] the author also studied the same unidimensional problem when α is smooth and, as opposed to [9], nondegenerate, i.e. $\alpha'' > 0$ at each local minimum of α.

In [8] the authors considered $\epsilon^2 \Delta u + (\alpha(x)^2 - u^2)u = 0$, $x \in \Omega$ with zero Neumann boundary condition in the special case $\alpha(x) = \chi_D(x)$ where $D \subset \Omega$ is a sub-domain satisfying $D = D_1 \cup D_2, \overline{D_1} \cap \overline{D_2} = \emptyset$, $\overline{\partial D \cap \Omega} \subset \Omega$ and ∂D_1, ∂D_2 are of class C^2. For sufficiently small $\epsilon > 0$, they proved existence of a local minimizer u_ϵ of the corresponding energy functional $J(u)$ on $H^1(\Omega)$ with the following asymptotic behavior: u_ϵ converges to 1 uniformly on any compact subset of D_1, converges to -1 uniformly on any compact subset of D_2 and converges to 0 uniformly on any compact subset of $\Omega \backslash (\overline{D})$.

The results we present herein generalizes [8] in many ways; in particular since we do not require $\overline{\partial D \cap \Omega} \subset \Omega$, this answer a question raised at the end of [8] as to whether this condition is necessary.

Given a function $u \in L^1(\Omega)$ we will use the following notation

$$u^{D_l} \stackrel{\text{def}}{=} u\,|_{D_l}\;, \quad l \in \{1,2,3\}\,.$$

Recalling that $a = \theta = b$ on D_3, our main result states as follows.

Theorem 1. *If f satisfies (f_1) and (f_2) then $\exists\ \epsilon_0 > 0$ and four families of classical stable stationary solutions $\left\{u_\epsilon^j\right\}_{0<\epsilon\le\epsilon_0}$ $(j = 1,\ldots,4)$ to (1) such that*

- $\|u_\epsilon^1 - u_0^1\|_{L^1(\Omega)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^1 = a\chi_{D_1} + b\chi_{D_2} + \theta\chi_{D_3}$
- $\|u_\epsilon^2 - u_0^2\|_{L^1(\Omega)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^2 = b\chi_{D_1} + a\chi_{D_2} + \theta\chi_{D_3}$
- $\|u_\epsilon^3 - u_0^3\|_{L^1(\Omega)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^3 \equiv a$
- $\|u_\epsilon^4 - u_0^4\|_{L^1(\Omega)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^4 \equiv b$

For the sake of definiteness, let us say what we mean by a stable stationary solution to (1). If $u(t,x;u_0)$ stands for the solution to (1) which satisfies $u(0,x;u_0) = u_0$ then by setting $T(t)u_0 = u(t,x;u_0)$, $T(t)$ defines a dynamical system -non-linear semigroup- on $\mathcal{H} \stackrel{\text{def}}{=} L^\infty_{a,b} \cap C(\Omega)$, where $L^\infty_{a,b} = \{\varphi \in L^\infty(\Omega) : a \le \varphi \le b, \text{ a.e. in } \Omega\}$.

A stationary solution $\overline{u}$ to (1) is said to be stable -in the sup norm- if for any $\mu > 0$, there exists a $\delta > 0$ such that $T(t)\psi$ exists for all $t > 0$ and $\|T(t)\psi - \overline{u}\|_{L^\infty(\Omega)} < \mu$, $0 < t < \infty$, for any $\psi \in \mathcal{H}$ which satisfies $\|\psi - \overline{u}\|_{L^\infty(\Omega)} < \delta$.

Theorem 1 also generalizes [2] where the degenerated set D_3 was reduced to a $(n-1)$-dimensional Lipschitz hypersurface without boundary which partitions Ω in two open disjoint connected components. Our approach is based on the theory of Γ- convergence. In our case, in order to find patterns to (1) it suffices to seek local minimisers of the corresponding family of energy functionals E_ϵ whose critical points are stationary solutions to (1) . We do that by taking the $\Gamma-$limit, as $\epsilon \to 0$, of this family of energy functionals and ending up with a more tractable geometric problem of minimising the $\Gamma-$limit functional E_0 in the space $BV(\Omega)$ of functions of bounded variations in Ω.

Although this approach has been used in some of our works it should be mentioned that in the present one the $\Gamma-$limit requires further arguments since it consists of the usual $n-$dimensional perimeter functional with a weight function which vanishes on D_3 and in this sense is degenerated.

2. Preliminaries on $BV(\Omega)$ and Γ-convergence

In the sequel some definitions, notations and results about functions of bounded variations are presented. The interested reader is referred to [11] and [4], for instance, for more on this matter.

Definition 2. The space of *functions of bounded variation in* Ω, denoted by $BV(\Omega)$, consists of all functions $v \in L^1(\Omega)$ whose distributional gradient Dv is a Radon measure with finite total variation in Ω given by

$$|Dv|(\Omega) = \sup\left\{ \int_\Omega v(x)\,\mathrm{div}\,\sigma(x)dx : \sigma \in C_0^1(\Omega, \mathrm{I\!R}^n),\quad |\sigma| \leq 1 \right\} < \infty.$$

Given a Borel set $B \subset \mathrm{I\!R}^n$, with characteristic function χ_B, it is said to have *finite perimeter* in the open set Ω if

$$\mathrm{Per}_\Omega(B) = |D\chi_B|(\Omega) < \infty.$$

If h is a continuous non-negative function and $u \in BV(\Omega)$ then the integral of h with respect to the measure $|Du|$ can be expressed as

$$\int_\Omega h(x)\,|Du| = \sup\left\{ \int_\Omega u(x)\,\mathrm{div}\,\sigma(x)dx : \sigma \in C_0^1(\Omega, \mathrm{I\!R}^n),\quad |\sigma(x)| \leq h(x) \right\}.$$

For a set $B \subset \mathrm{I\!R}^n$ with finite perimeter in Ω we will often work with the integral of h with respect to the Radon measure $|D\chi_B|$, i.e., $\int_\Omega h(x)\,|D\chi_B|$.

If ∇ stands for the gradient operator and $\mathcal{L}^n$ for the Lebesgue measure then for $v \in W^{1,1}_{loc}$ the variation measure satisfies

$$|Dv| = |\nabla v|\,d\mathcal{L}^n. \tag{3}$$

If $B(x,r)$ denotes the ball of radius r and center x then the *density* of a set A in a point x is defined as

$$D(A,x) = \lim_{r\to 0} \frac{\mathcal{L}^n(A \cap B(x,r))}{\mathcal{L}^n(B(x,r))}$$

whenever the limit exists. The *essential boundary* of a set $E \subset \mathrm{I\!R}^n$ is the set $\partial_* E$ of all points in Ω where E has neither density 1 nor density 0. When the topological boundary is Lipschitz regular it coincides with the essential boundary.

If $u \in BV(\Omega)$ and f is a continuous function in Ω then

$$\int_\Omega f\,|Du| = \int_{-\infty}^{\infty} \left[\int_{\Omega \cap \partial_* \{x \in \Omega : u(x) > \xi\}} f d\mathcal{H}^{n-1} \right] d\xi. \tag{4}$$

is called the *co-area formula* and will be of great assistance.

The following working definition of the Γ-convergence of a family of functionals, with respect to the L^1 topology, will be utilized.

Definition 3. A family $\{E_\epsilon\}_{\epsilon>0}$ of real-extended functionals defined in $L^1(\Omega)$ Γ*-converges, as* $\epsilon \to 0$*, to a functional* E_0, denoted by

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(v) = E_0(v),$$

if:

(i) $\forall\ v \in L^1(\Omega)$ and $\forall\ \{v_\epsilon\} \subset L^1(\Omega) : v_\epsilon \to v$ in $L^1(\Omega)$, as $\epsilon \to 0 \quad \Rightarrow$ $E_0(v) \le \lim_{\epsilon\to 0}\inf E_\epsilon(v_\epsilon)$.

(ii) $\forall\ v \in L^1(\Omega)$, $\exists\ \{v_\epsilon\}$ in $L^1(\Omega)\ : v_\epsilon \to v$ in $L^1(\Omega)$, as $\epsilon \to 0$, and $E_0(v) \ge \lim_{\epsilon\to 0}\sup E_\epsilon(v_\epsilon)$.

Definition 4. We shall call $v_0 \in L^1(\Omega)$ a L^1*-local minimizer of* E_0 if there is $\mu > 0$ such that

$$E_0(v_0) \le E_0(v) \quad \text{whenever} \quad 0 < \|v - v_0\|_{L^1(\Omega)} < \mu.$$

Moreover if $E_0(v_0) < E_0(v)$, for $0 < \|v - v_0\|_{L^1(\Omega)} < \mu$, then v_0 is called an *isolated* L^1*-local minimizer of* E_0.

The following theorem which can be found in [6] is essential to our analysis.

Theorem 5 ([6]). *Suppose that* $\Gamma - \lim_{\epsilon\to 0} E_\epsilon = E_0$ *and*

(i) Any sequence $\{v_\epsilon\}_{\epsilon>0}$ *such that* $E_\epsilon(v_\epsilon) \le C < \infty$, $\forall\, \epsilon > 0$ *and* $C > 0$*, is compact in* L^1*.*

(ii) There exists an isolated L^1*-local minimizer* u_0 *of* E_0*.*

Then $\exists\, \epsilon_0 > 0$ *and a family* $\{v_\epsilon\}_{0<\varepsilon<\epsilon_0}$ *such that*

- v_ϵ *is an* L^1*-local minimizer of* E_ϵ *and*
- $\|v_\epsilon - v_0\|_{L^1} \to 0$*, as* $\epsilon \to 0$*.*

In the sequel $BV(\Omega, \{a, b\})$ will denote the space of functions $u \in BV(\Omega)$ such that $u \in \{a, b\}$ a.e. in Ω. If $u \in BV(\Omega, \{a, b\})$ then $\partial_* \{u(x) = a(x)\} \cap \Omega$ is a rectifiable set.

3. Computation of the Γ-limit and its isolated minimizers

In order to prove Theorem 1 our procedure utilizes Theorem 5 and therefore our task will be accomplished once we

- find the Γ-limit of the family of functionals whose critical points are stationary solutions to (1) and
- prove existence of four isolated L^1-local minimizers of the Γ-limit.

Step 1: The family of functionals $E_\epsilon : L^1(\Omega) \to \mathrm{I\!R} \cup \{\infty\}$ whose critical points are stationary solutions to (1) is defined by

$$E_\epsilon(v) = \begin{cases} \displaystyle\int_\Omega \left[\frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon}F(v,x)\right] dx, & v \in H^1(\Omega) \\ \infty, & \text{otherwise,} \end{cases} \tag{5}$$

where

$$F(v,x) = -\int_{a(x)}^{v} f(\xi,x)d\xi \tag{6}$$

with f as in (2).

Theorem 6. *Consider $E_0 : L^1(\Omega) \to \mathrm{I\!R} \cup \{\infty\}$ defined by*

$$E_0(v) = \begin{cases} \displaystyle\int_\Omega h(x)\left|D\chi_{\{v=a\}}\right|, & v \in BV(\Omega, \{a,b\}) \\ \infty, & \textit{otherwise} \end{cases}$$

where

$$h(x) = \sqrt{2}\int_{a(x)}^{b(x)} \sqrt{F(s,x)}ds. \tag{7}$$

Then

$$\Gamma^- \lim_{\epsilon \to 0} E_\epsilon(v) = E_0(v), \quad \forall \quad v \in L^1(\Omega).$$

Note that, as a consequence of (f_1), $h = 0$ on D_3 and therefore the $\Gamma-$limit E_0 is degenerated since the regular case requires $h > 0$ in Ω.

The lemmas below will be useful in the proof of Theorem 6.

Lemma 7 ([3]). *Assume $U \subset \Omega$ is open and bounded with ∂U Lipschitz. Let $f_1 \in BV(U)$ and $f_2 \in BV(\Omega \setminus \overline{U})$. Define*

$$\overline{f}(x) = \begin{cases} f_1(x), & x \in U \\ f_2(x), & x \in \Omega \setminus \overline{U}. \end{cases}$$

Then $\overline{f} \in BV(\Omega)$.

Lemma 8. *Let $E_{\epsilon,l} : L^1(D_l) \to \mathrm{I\!R} \cup \{\infty\}$, $l \in \{1,2,3\}$, be defined by*

$$E_{\epsilon,l}(u) = \begin{cases} \displaystyle\int_{D_l} \left[\frac{\epsilon}{2}|Du|^2 + \frac{1}{\epsilon}F(u,x)\right] dx, & u \in H^1(D_l) \\ \infty, & \textit{otherwise.} \end{cases}$$

Then

- $E_\epsilon(u) \geq E_{\epsilon,1}(u^{D_1}) + E_{\epsilon,2}(u^{D_2}) + E_{\epsilon,3}(u^{D_3})$ *if* $u \in L^1(\Omega) \setminus H^1(\Omega)$*;*
- $E_\epsilon(u) = E_{\epsilon,1}(u^{D_1}) + E_{\epsilon,2}(u^{D_2}) + E_{\epsilon,3}(u^{D_3})$ *if* $u \in H^1(\Omega)$*.*

Proof. Let $u \in L^1(\Omega)$. If $u \in H^1(\Omega)$ then $u^{D_l} \in H^1(D_l)$, $l \in \{1,2,3\}$ and $E_\epsilon(u) = E_{\epsilon,1}(u^{D_1}) + E_{\epsilon,2}(u^{D_2}) + E_{\epsilon,3}(u^{D_3})$. If $u \notin H^1(\Omega)$ then $E_\epsilon(u) = \infty \geq E_{\epsilon,1}(u^{D_1}) + E_{\epsilon,2}(u^{D_2}) + E_{\epsilon,3}(u^{D_3})$.

Lemma 9. *Let* $E_{0,l} : L^1(D_l) \to \mathbb{R} \cup \{\infty\}$*,* $l \in \{1,2,3\}$ *be defined by*

$$E_{0,l}(u) = \begin{cases} \displaystyle\int_{D_l} h(x) \left|D\chi_{\{v=a\}}\right|, & u \in BV(D_l, \{a,b\}) \\ \infty, & \text{otherwise} \end{cases}$$

for $l \in \{1,2\}$ *and*

$$E_{0,3}(u) = \begin{cases} 0, & u = \theta, \text{ a.e. in } D_3 \\ \infty, & \text{otherwise.} \end{cases}$$

Then E_0*, given in Theorem 6, satisfies*

$$E_0(u) = E_{0,1}(u^{D_1}) + E_{0,2}(u^{D_2}) + E_{0,3}(u^{D_3}).$$

Proof. We first prove that $BV(\Omega, \{a,b\}) = \Xi$ where

$$\Xi \stackrel{\text{def}}{=} \left\{ v \in L^1(\Omega) : v^{D_l} \in BV(D_l, \{a,b\}), l \in \{1,2\}, \right.$$
$$\left. v^{D_3} = \theta \text{ a.e. in } D_3 \right\}.$$

- $BV(\Omega, \{a,b\}) \subset \Xi$.

If $v \in BV(\Omega, \{a,b\})$ then $v^{D_3} = a$ a.e. in D_3 (recall that $a \equiv b$ in D_3) and $v^{D_l} \in \{a,b\}$ a.e. in D_l, $l \in \{1,2\}$. If $v^{D_1} \notin BV(D_1, \{a,b\})$ then given any $M > 0$ there exists $g_M \in C_0^1(D_1, \mathbb{R}^n)$ such that $|g_M| \leq 1$ and $\int_{D_1} v \, \mathrm{div} g_M \, dx > M$. Letting

$$\overline{g}_M = \begin{cases} g_M, & \text{in } D_1 \\ 0, & \text{in } \Omega \setminus D_1 \end{cases}$$

then $\overline{g}_M \in C_0^1(\Omega, \mathbb{R}^n)$, $|\overline{g}_M| \leq 1$ and $\int_\Omega v \, \mathrm{div}\overline{g}_M \, dx > M$. This prove that

$$\int_\Omega |Dv| = \sup \left\{ \int_\Omega v \, \mathrm{div} g \, dx : g \in C_0^1(\Omega, \mathbb{R}^n), \quad |g| \leq 1 \right\} = \infty,$$

which contradicts our hypotheses.

Analogously we prove that $v^{D_2} \in BV(D_2, \{a,b\})$. It follows that $v \in \Xi$.

- $BV(\Omega, \{a,b\}) \supset \Xi$

Indeed if $v \in \Xi$ then using the fact that D and $\Omega \setminus D$ have Lipschitz boundaries we resort to Lemma 7 to conclude that $v \in BV(\Omega, \{a,b\})$.

Take now $v \in L^1(\Omega)$. If $v \notin BV(\Omega, \{a, b\})$ then either $v^{D_1} \notin BV(D_1, \{a, b\})$ or $v^{D_2} \notin BV(D_2, \{a, b\})$ or it is false that $v = a$ a.e. in D_3. It follows that $E_0(v) = \infty = E_{0,1}(v^{D_1}) + E_{0,2}(v^{D_2}) + E_{0,3}(v^{D_3})$.

If $v \in BV(\Omega, \{a, b\})$ then $E_{0,3}(v^{D_3}) = 0$ and $h = 0$ a.e. in D_3 by Theorem 6 (see (7)). Using the co-area formula -see (4)- we compute

$$\begin{aligned}
E_0(v) &= \int_\Omega h(x) \left|D\chi_{\{v=a\}}\right| = \int_{-\infty}^{\infty} \left[\int_{\Omega \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1}\right] d\xi \\
&= \sum_{j=1}^{3} \int_{-\infty}^{\infty} \left[\int_{D_j \cap \partial_* \left\{\chi_{\{v=a\}} > \xi\right\}} h(x) d\mathcal{H}^{n-1}\right] d\xi \\
&= \sum_{j=1}^{3} \int_{D_j} h(x) \left|D\chi_{\{v=a\}}\right| \\
&= \int_{D_1} h(x) \left|D\chi_{\{v=a\}}\right| + \int_{D_2} h(x) \left|D\chi_{\{v=a\}}\right| \\
&= \int_{D_1} h(x) \left|D\chi_{\left\{v^{D_1}=a\right\}}\right| + \int_{D_2} h(x) \left|D\chi_{\left\{v^{D_2}=a\right\}}\right| \\
&= E_{0,1}(v^{D_1}) + E_{0,2}(v^{D_2}).
\end{aligned}$$

Therefore the proof is complete.

Lemma 10. *It holds that*

$$\Gamma^- \lim_{\epsilon \to 0} E_{\epsilon,l}(v) = E_{0,l}(v), \quad \forall \quad v \in L^1(D_l), \quad l \in \{1, 2, 3\}.$$

For $l \in \{1, 2\}$ the proof of this lemma can be rendered in a similar fashion to that found in [10] since $a < \theta < b$ in $D_1 \cup D_2$ and, according to (f_1), are C^1-bounded in D_1 and D_2. When $l = 3$ we have a degenerate case and another argument must be used.

First we prove that if $\liminf_{\epsilon \to 0} E_{\epsilon,3}(v_\epsilon) < \infty$ and $v_\epsilon \to v$ in $L^1(D_3)$ then $v = a$ a.e.. Indeed, there is a subsequence $\left\{v_{\epsilon_j}\right\}$ such that $v_{\epsilon_j} \to v$ a.e. in D_3. It follows that $F(v_{\epsilon_j}, \cdot) \to F(u, \cdot)$ a.e. and then, by Fatou's lemma, we have

$$\begin{aligned}
0 \le \int_{D_3} F(v(x), x) dx &= \int_{D_3} \lim_{j \to \infty} F(v_{\epsilon_j(x)}, x) dx \\
&\le \liminf_{j \to \infty} \int_{D_3} F(v_{\epsilon_j(x)}, x) dx \\
&\le \liminf_{j \to \infty} \epsilon_j E_{\epsilon_j,3}(v_{\epsilon_j}) = 0.
\end{aligned}$$

The last equality holds due to the assumption $\liminf_{\epsilon \to 0} E_{\epsilon,3}(v_\epsilon) < \infty$. As F vanishes over D_3 we have that $v = a$ a.e. in D_3.

In the sequel we verify the requirements of the Definition 3 of Γ-convergence. Let $v \in L^1(D_3)$ and $v_\epsilon \to v$ in $L^1(D_3)$. If $\liminf_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon) = \infty$ then obviously $E_{0,3}(v) \le \liminf\limits_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon)$ and if $\liminf_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon) < \infty$ then $v = a$ a.e. in D_3 and $E_{0,3}(v) = 0$. This proves (i) of Definition 3. Regarding (ii), given $v \in L^1(D_3)$ we may consider $v = a$ a.e. in D_3 (otherwise $E_{0,3}(v) = \infty$) and, in this case, we can take $v_\epsilon \equiv a$ which implies that $\limsup_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon) = 0$. We have proved that $\Gamma^- \lim\limits_{\epsilon\to 0} E_{\epsilon,3}(v) = E_{0,3}(v)$, for all $v \in L^1(D_3)$.

Now we are ready to prove Theorem 6.

Proof of Theorem 6. Again we verify the requirements of the Definition 3.

(i) Let $v \in L^1(\Omega)$ and a sequence $\{v_\epsilon\} \subset L^1(\Omega)$ such that $v_\epsilon \to v$ in $L^1(\Omega)$. Then

$$\begin{aligned}
E_0(v) &\overset{(*)}{=} E_{0,1}(v^{D_1}) + E_{0,2}(v^{D_2}) + E_{0,3}(v^{D_3}) \\
&\overset{(**)}{\le} \liminf_{\epsilon\to 0} E_{\epsilon,1}(v_\epsilon^{D_1}) + \liminf_{\epsilon\to 0} E_{\epsilon,2}(v_\epsilon^{D_2}) + \liminf_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon^{D_3}) \\
&\le \liminf_{\epsilon\to 0} \big(E_{\epsilon,1}(v_\epsilon^{D_1}) + E_{\epsilon,2}(v_\epsilon^{D_2}) + E_{\epsilon,3}(v_\epsilon^{D_3})\big) \\
&\overset{(***)}{\le} \liminf_{\epsilon\to 0} E_\epsilon(v_\epsilon).
\end{aligned}$$

Here Lemmas 9, 10 and 8 were used in $(*)$,$(**)$ and $(***)$, respectively.

(ii) Given $v \in L^1(\Omega)$ we may consider $v \in BV(\Omega, \{a,b\})$ since otherwise $E_0(v) = \infty$. We have that $v^{D_3} = \theta$ a.e. in D_3, $v^{D_l} \in BV(D_l, \{a,b\})$, $l \in \{1,2\}$, and by Lemma 10 there exists $\{u_{\epsilon,l}\} \in L^1(D_l)$ such that $u_{\epsilon,l} \to v^{D_l}$ in $L^1(D_l)$ and

$$E_{0,l}(v^{D_l}) \ge \limsup_{\epsilon\to 0} E_{\epsilon,l}(u_{\epsilon,l}) \quad (l \in \{1,2,3\}). \tag{8}$$

Consider

$$v_\epsilon(x) = \begin{cases} u_{\epsilon,1}(x), & x \in D_1 \\ u_{\epsilon,2}(x), & x \in D_2 \\ \theta(x), & x \in D_3. \end{cases} \tag{9}$$

It follows that $v_\epsilon \to v$ in $L^1(\Omega)$. In order to complete the proof it remains to show that

$$E_0(v) \ge \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon).$$

Claim: The functions $u_{\epsilon,l}$, $l \in \{1,2\}$, may be constructed in such a way that

$$v_\epsilon \in H^1(\Omega). \tag{10}$$

We take it for granted for now and prove it later on. Thus it follows that

$$
\begin{aligned}
E_0(v) &\overset{(*)}{=} E_{0,1}(v^{D_1}) + E_{0,2}(v^{D_2}) + E_{0,3}(v^{D_3}) \\
&\overset{(**)}{\geq} \limsup_{\epsilon\to 0} E_{\epsilon,1}(u_{\epsilon,1}) + \limsup_{\epsilon\to 0} E_{\epsilon,2}(u_{\epsilon,2}) + \limsup_{\epsilon\to 0} E_{\epsilon,3}(a) \\
&= \limsup_{\epsilon\to 0} E_{\epsilon,1}(v_\epsilon^{D_1}) + \limsup_{\epsilon\to 0} E_{\epsilon,2}(v_\epsilon^{D_2}) + \limsup_{\epsilon\to 0} E_{\epsilon,3}(v_\epsilon^{D_3}) \\
&\geq \limsup_{\epsilon\to 0} \left(E_{\epsilon,1}(v_\epsilon^{D_1}) + E_{\epsilon,2}(v_\epsilon^{D_2}) + E_{\epsilon,3}(v_\epsilon^{D_3})\right) \\
&\overset{(***)}{=} \limsup_{\epsilon\to 0} E_\epsilon(v_\epsilon).
\end{aligned}
$$

In $(*)$ and in $(**)$, Lemma 9 and (8) were used respectively and finally in $(***)$ we resorted to (10) and Lemma 8 as well.

Proof of the Claim. Our goal is to construct families of functions $\{u_{\epsilon,1}\}$ and $\{u_{\epsilon,2}\}$ so that $\{v_\epsilon\}$ (defined by (9)) satisfies (10).

First we construct the family $\{u_{\epsilon,1}\}$ defined in D_1 following the steps of [10]. Let

$$A \overset{\text{def}}{=} \{x \in \Omega : v(x) = a(x)\} \tag{11}$$

and assume that $\partial A \cap \Omega$ is C^2; in the conclusion of the proof we will show that this implies no loss of generality. Hence the signed distance function has the same regularity.

Consider

$$
\begin{aligned}
A_1 &= \{x \in D_1 : v(x) = a(x)\} \\
A_2 &= \{x \in D_1 : v(x) = b(x)\}
\end{aligned}
$$

and the signed distance function

$$
d_1(x) = \begin{cases} dist(x, \partial A \cap D_1), & x \in A_1 \\ -dist(x, \partial A \cap D_1), & x \in A_2. \end{cases}
$$

Only the case when $\partial A \cap D_1 \neq \emptyset$ and $\partial A \cap D_2 \neq \emptyset$ will be addressed since the other ones are easier to prove. It follows from the above hypothesis that

$$\partial A \cap D_1 \tag{12}$$

is C^2. Now consider Z_1 the solution of the following initial value problem

$$
\left.
\begin{aligned}
\frac{\partial Z}{\partial s}(x,s) &= \sqrt{F\left(Z(x,s),x\right)}, \quad (x,s) \in D_1 \times \mathbb{R} \\
Z(x,0) &= \theta(x), \quad x \in D_1
\end{aligned}
\right\} \tag{13}
$$

Recall that in D_1 we have $a < \theta < b$. Thus the smooth solution Z_1 given in (13) satisfies:

$$a(x) < Z_1(x,s) < b(x), \quad \forall\ (x,s) \in D_1 \times \mathbb{R}. \tag{14}$$

As $a(x) = b(x)$ on D_3 it follows that

$$\lim_{x \to x_0} Z_1(x,s) = a(x_0), \quad \forall\ (x_0, s) \in (\partial D_1 \cap \Omega) \times \mathbb{R}. \tag{15}$$

Finally we consider $u_{\epsilon,1} : D_1 \to \mathbb{R}$ defined as follows

$$u_{\epsilon,1}(x) \stackrel{\text{def}}{=}$$

$$\begin{cases} a(x), & d_1(x) > 2\sqrt{\epsilon} \\ [a(x) - Z_1(x, 1/\sqrt{\epsilon})] \frac{\left(d_1(x) - 2\sqrt{\epsilon}\right)}{\sqrt{\epsilon}} + a(x), & \sqrt{\epsilon} \le d_1(x) \le 2\sqrt{\epsilon} \\ Z_1(x, d_1(x)/\epsilon), & |d_1(x)| < \sqrt{\epsilon} \\ [Z_1(x, -1/\sqrt{\epsilon}) - b(x)] \frac{\left(d_1(x) + 2\sqrt{\epsilon}\right)}{\sqrt{\epsilon}} + b(x), & -2\sqrt{\epsilon} \le d_1(x) \le -\sqrt{\epsilon} \\ b(x), & d_1(x) < -2\sqrt{\epsilon}. \end{cases}$$

Now, as shown in [10], $u_{\epsilon,1} \in H^1(D_1)$, $u_{\epsilon,1} \to v^{D_1}$ in $L^1(D_1)$,

$$E_{0,1}(v^{D_1}) \ge \limsup_{\epsilon \to 0} E_{\epsilon,1}(u_{\epsilon,1})$$

and by (15)

$$\lim_{x \to x_0} u_{\epsilon,1}(x) = a(x_0),\ \forall\ x_0 \in \partial D_1 \cap \Omega.$$

In a similar fashion we construct $\{u_{\epsilon,2}\}$ defined in D_2 such that $u_{\epsilon,2} \in H^1(D_2)$, $u_{\epsilon,2} \to v^{D_2}$ in $L^1(D_2)$, $E_{0,2}(v^{D_2}) \ge \limsup_{\epsilon \to 0} E_{\epsilon,2}(u_{\epsilon,2})$ and

$$\lim_{x \to x_0} u_{\epsilon,2}(x) = a(x_0),\ \forall\ x_0 \in \partial D_2 \cap \Omega.$$

This proves that v_ϵ is continuous in Ω (see (9)). Thus, as $u_{\epsilon,l} \in H^1(D_l)$, $l \in \{1,2,3\}$, is not difficult to see that $v_\epsilon \in H^1(\Omega)$. The claim is proved.

It remains to prove that there is no loss of generality to assume that $\partial A \cap \Omega$ is C^2. But this can be accomplished by following the procedure utilized in [10].

Hence Step 1 is complete and in order to prove Step 2 the following lemma, whose proof is straightforward, will be useful.

Lemma 11. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with Lipschitz - continuous boundary. Let $E \subset \Omega$ be a set of finite perimeter such that*

$$0 < \mathcal{H}^n(E) < \mathcal{H}^n(\Omega). \tag{16}$$

Then $Per_\Omega(E) > 0$.

PROOF. If
$$\mathrm{Per}_\Omega(E)=0 \tag{17}$$
then for $x_0\in\Omega$ and any $r>0$ satisfying $B(x_0,r)\subset\Omega$ we would have $\mathrm{Per}_{B(x_0,r)}(E)=0$.

This is so because if $x_0\in\Omega$ and $r>0$ are such that $B(x_0,r)\subset\Omega$ and $\mathrm{Per}_{B(x_0,r)}(E)>0$ then

$$\mathrm{Per}_\Omega(E)=\mathcal{H}^{n-1}(\partial_*E\cap\Omega)=\mathcal{H}^{n-1}(\partial_*E\cap(\Omega\backslash B(x_0,r)\cup B(x_0,r)))$$
$$\geq\mathcal{H}^{n-1}(\partial_*E\cap B(x_0,r))=\mathrm{Per}_{B(x_0,r)}E>0,$$

thus contradicting (17).

Let us recall the classical isoperimetric inequality -see [11], e.g. - . If $E\subset\mathbb{R}^n$ is a bounded set of finite perimeter then there exists a constant $C=C(n)$ such that
$$\mathcal{H}^n(E)^{(n-1)/n}\leq C\mathrm{Per}E.$$

Moreover, for each ball $B(r)\subset\mathbb{R}^n$,

$$\min\left\{\mathcal{H}^n(B(r)\cap E),\mathcal{H}^n(B(r)\setminus E)\right\}^{(n-1)/n}\leq C\mathrm{Per}_{B(r)}(E).$$

This inequality would imply that either

$$\mathcal{H}^n(B(x_0,r)\cap E)=0 \ \text{ or } \ \mathcal{H}^n(B(x_0,r)\setminus E)=0.$$

Next by considering the sets

$$O_1=\{x\in\Omega:\mathcal{H}^n(B(x,r)\cap E)=0, \text{ for some } B(x,r)\subset\Omega\}$$

and
$$O_2=\{x\in\Omega:\mathcal{H}^n(B(x,r)\setminus E)=0, \text{ for some } B(x,r)\subset\Omega\},$$

it would imply that $O_1\cap O_2=\emptyset$, $\Omega=O_1\cup O_2$ and O_1 and O_2 are open sets. Since Ω is connected we have either $O_1=\emptyset$ or $O_2=\emptyset$, thus contradicting (16). Hence $\mathrm{Per}_\Omega(E)>0$.

Theorem 12. *Each of the functions $u_0^1,\ldots,u_0^4:\Omega\to\mathbb{R}$ defined by*

- $u_0^1=a\chi_{D_1}+b\chi_{D_2}+\theta\chi_{D_3}$,
- $u_0^2=b\chi_{D_1}+a\chi_{D_2}+\theta\chi_{D_3}$,
- $u_0^3=a$ *and*
- $u_0^4=b$.

is an isolated L^1-local minimizer of E_0.

Proof. We render the proofs for u_0^1 and u_0^3 only since the other cases can be dealt with in a similar fashion. Consider $\rho > 0$ such that

$$\rho < \min \left\{ \int_{D_1} (b-a)dx, \int_{D_2} (b-a)dx \right\}. \tag{18}$$

It suffices to show that if $u \in L^1(\Omega)$ satisfies

$$0 < \|u - u_0^1\|_{L^1(\Omega)} < \rho \tag{19}$$

then $E_0(u) > E_0(u_0^1)$.

On the account that

$$\Omega \cap \partial_* \left\{ \chi_{\{u_0^1=a\}} > \xi \right\} = \begin{cases} \emptyset, & \xi \notin [0,1) \\ \partial D_2 \cap \Omega, & \xi \in [0,1) \end{cases}$$

the co-area formula yields

$$\begin{aligned} E_0(u_0^1) = \int_\Omega h(x) \left| D\chi_{\{u_0^1=a\}} \right| &= \int_{-\infty}^{\infty} \int_{\Omega \cap \partial_* \left\{ \chi_{\{u_0^1=a\}} > \xi \right\}} h(x) d\mathcal{H}^{n-1} d\xi \\ &= \int_0^1 \int_{\partial D_2 \cap \Omega} h(x) d\mathcal{H}^{n-1} d\xi = 0 \end{aligned}$$

using the fact that $h = 0$ on $\partial D_2 \cap \Omega$.

If $u \notin BV(\Omega, \{a,b\})$ then $E_0(u) = \infty$ and obviously $E_0(u) > E_0(u_0^1)$ since $E_0(u_0^1) = 0$. Therefore consider $u \in BV(\Omega, \{a,b\})$ satisfying (19) and define the set

$$U \stackrel{\text{def}}{=} \{x \in \Omega; \ \ u(x) = a(x)\}.$$

Then U is a set of finite perimeter and we have two possibilities:

(i) $\mathcal{H}^{n-1}(\partial_* U \cap D) = 0$ or

(ii) $\mathcal{H}^{n-1}(\partial_* U \cap D) > 0$.

Recall that $D = D_1 \cup D_2$. If (i) occurs then we claim that

- $u(x) = u_0^1(x)$ a.e. in Ω, or
- $u(x) = b(x)\chi_{D_1} + a(x)\chi_{D_2} + \theta(x)\chi_{D_3}$ a.e. in Ω, or
- $u(x) = a(x)$ a.e. in Ω, or
- $u(x) = b(x)$ a.e. in Ω.

Indeed suppose that none of the above four cases occurs. Then, without loss of generality, by setting $K = U \cap D_2$ we can assume that

$$0 < \mathcal{H}^n(K) < \mathcal{H}^n(D_2).$$

As D_2 is open, bounded, connected, with Lipschitz-continuous boundary and $K \subset D_2$ is a set of finite perimeter, we can apply Lemma 11 to conclude that $\text{Per}_{D_2} K > 0$. But $(\partial_* K \cap D_2) \subset (\partial_* U \cap D))$ and therefore

$$\mathcal{H}^{n-1}(\partial_* U \cap D) \geq \mathcal{H}^{n-1}(\partial_* K \cap D_2) = \text{Per}_{D_2} K > 0,$$

which is a contradiction.

Now, it is not difficult to see that if (i) occurs then u does not satisfy (19). Therefore we must have (ii). Note that

$$\Omega \cap \partial_* \left\{ \chi_{\{u=a\}} > \xi \right\} = \begin{cases} \emptyset, & \xi \notin [0,1) \\ \partial_* U \cap \Omega, & \xi \in [0,1). \end{cases}$$

Hence

$$\begin{aligned} E_0(u) = \int_\Omega h(x) \left| D\chi_{\{u=a\}} \right| &= \int_{-\infty}^{\infty} \int_{\Omega \cap \partial_* \left\{ \chi_{\{u=a\}} > \xi \right\}} h(x) d\mathcal{H}^{n-1} d\xi \\ &= \int_0^1 \int_{\partial_* U \cap \Omega} h(x) d\mathcal{H}^{n-1} d\xi \\ &\geq \int_0^1 \int_{\partial_* U \cap D} h(x) d\mathcal{H}^{n-1} d\xi > 0 \end{aligned}$$

using (ii) and the fact that $h(x) > 0$ for all $x \in D$. It follows that $E_0(u) > E_0(u_0^1)$ which implies that u_0^1 is an isolated L^1-local minimizer of E_0.

Let us now consider $u_0^3(x)$. Taking $\rho > 0$ given by (18) we use that

$$\Omega \cap \partial_* \left\{ \chi_{\{u_0^3=a\}} > \xi \right\} = \emptyset$$

to compute

$$E_0(u_0^3) = \int_\Omega h(x) \left| D\chi_{\{u_0^3=a\}} \right| = \int_{-\infty}^{\infty} \int_{\Omega \cap \partial_* \left\{ \chi_{\{u_0^3=a\}} > \xi \right\}} h(x) d\mathcal{H}^{n-1} d\xi = 0.$$

Then if u satisfies $0 < ||u - u_0^3||_{L^1(\Omega)} < \rho$ similarly to the previous case we conclude that $E_0(u) > 0$ and therefore $E_0(u) > E_0(u_0^3) = 0$.

This proves Theorem 12.

Hence Step 2 is complete.

4. Proof of the Main Result

Finally we are ready to prove Theorem 1 and it is accomplished by a direct application of Theorem 5.

Proof of Theorem 1. After calculating the Γ-limit E_0 of the family of functionals given in (5) (Step 1) and proving that $\left\{u_0^j\right\}$ $(j = 1, \ldots, 4)$ are four isolated L^1-local minimizer of E_0 (Step 2) it remains to verify hypothesis (i) of Theorem 5. This hypothesis would follow if the minimizers were uniformly bounded in L^∞ (see [10], Proposition 3 and Remark (1.35), for instance). This is actually the case as can be seen by an application of the maximum principle following the procedure used in [1], for instance.

Hence an application of Theorem 5 implies that, for some $\epsilon_0 > 0$, the family of functionals given by (5) possesses four families of local minima $\left\{u_\epsilon^j\right\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \ldots, 4)$ satisfying $\|u_\epsilon^j - u_0^j\|_{L^1(\Omega)} \xrightarrow{\epsilon\to 0} 0$.

It remains to verify the stability of the solutions $\left\{u_\epsilon^j\right\}_{0<\epsilon\leq\epsilon_0}$ $(j = 1, \ldots, 4)$. The argument is standard by now and is given for the reader's convenience.

The second variation of the energy functional E_ϵ at u_ϵ^j is nonnegative. If $\lambda_1(u_\epsilon^j)$ is the first eigenvalue of the linearized problem

$$\left.\begin{array}{ll} \epsilon^2\Delta\varphi + \partial_1 f(u_\epsilon^j, x)\varphi = -\lambda\varphi & x \in \Omega \\ \partial_n\varphi = 0 & x \in \partial\Omega \end{array}\right\} \tag{20}$$

then $\lambda_1(u_\epsilon^j) \geq 0$, due to its variational characterization. It is well-known that if $\lambda_1(u_\epsilon^j) > 0$ then u_ϵ^j is asymptotically stable.

If $\lambda_1(u_\epsilon^j) = 0$ then it is a simple eigenvalue and there is a local one-dimensional critical manifold $W(u_\epsilon^j)$, tangent to principal eigenfunction corresponding to the zero eigenvalue, such that if u_ϵ^j is stable in $W(u_\epsilon)$ then it is also stable in $H^1(\Omega)$ (see [5], Theorem 6.2.1, for instance). Now the stability of u_ϵ^j in $W(u_\epsilon^j)$ follows from the existence of a Lyapunov functional and the fact that $W(u_\epsilon^j)$ is one-dimensional.

The theorem is proved.

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