

THE ANALYTIC TORSION OF THE FINITE METRIC CONE OVER A COMPACT MANIFOLD

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ABSTRACT. We give explicit formulas for the L^2 analytic torsion of the finite metric cone over an oriented compact connected Riemannian manifold. We provide an interpretation of the different factors appearing in these formulas. We prove that the analytic torsion of the cone is the finite part of the limit obtained collapsing one of the boundaries, of the ratio of the analytic torsion of the frustum to a regularising factor. We show that the regularising factor comes from the set of the non square integrable eigenfunctions of the Laplace Beltrami operator on the cone.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let (M, g) be a compact connected oriented Riemannian manifold without boundary of dimension n with metric g . Let Δ denotes the Hodge-Laplace operator on M relative to the metric g . Then, Δ has a non negative discrete spectrum $\text{Sp}\Delta$ and the *zeta function* of Δ is well defined by

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in \text{Sp}_+ \Delta^{(q)}} \lambda^{-s},$$

for $\text{Re}(s) > \frac{n}{2}$, and by analytic continuation elsewhere, and is regular at $s = 0$. The *analytic torsion* $T(M, g)$ of the pair (M, g) is defined by

$$\log T(M, g) = \frac{1}{2} \sum_{q=1}^n (-1)^q q \zeta'(0, \Delta^{(q)}).$$

If the manifold M has a boundary ∂M , the Laplace operator is assumed to be defined by suitable boundary conditions BC [18, Section 3]. In such a case, it is convenient to split the logarithm of the analytic torsion into two parts, the first being a global term and the second a local one, defined on a neighborhood of the boundary [4, Section 3]. The first term coincides with the Reidemeister torsion [14] of either M or the pair $(M, \partial M)$, with the Ray and Singer homology basis [18], by the Cheeger Müller theorem [4] [15], so

$$(1.1) \quad \begin{aligned} \log T_{\text{abs}}(M, g) &= \log \tau(M, g) + \log T_{\text{bound,abs}}(\partial M), \\ \log T_{\text{rel}}(M, g) &= \log \tau((M, \partial M), g) + \log T_{\text{bound,rel}}(\partial M), \end{aligned}$$

the second term splits as

$$\log T_{\text{bound,BC}}(\partial M) = \frac{1}{4} \chi(\partial M) \log 2 + A_{\text{BM,BC}}(\partial M),$$

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where χ is the Euler characteristic, and the last term is called the anomaly boundary term, and was described in the more general case in [1] and [2].

It is clear that what is necessary in order to define the analytic torsion $T(M, g)$ is that the spectrum of the Hodge-Laplace operator satisfies some assumptions that guarantee the possibility of defining the zeta function and of proving its regularity at $s = 0$. It is also clear that this follows by some spectral properties of the Hodge-Laplace operator. There are several approaches to describe these properties. We will follow the one of Spreafico, introduced in [21] and [22]: so we require that $\text{Sp}_+(\Delta)$ is a graded regular sequence of spectral type (of non positive order), as defined in [22, Definitions 2.1, 2.6]. The fact that this is true for the Hodge-Laplace operator on a compact manifold is well known.

The given definition of the analytic torsion extends easily considering forms with values in some vector bundle V_ρ associated to some orthogonal representation ρ of the fundamental group of W [18, Section 1]. Under our approach what is necessary is that the spectrum of the resulting operator is a regular sequence of spectral type of non positive order, and again this is well known. Since the results of this paper are independent of these extensions, we will consider the simpler case of the Hodge-Laplace operator itself.

An other possible generalization of the given definition of analytic torsion, and this is the case that we will consider here, is when the underlying space is no longer a compact manifold, but some type of open manifold, or manifold with singularities. In this paper we consider the case of the cone over a manifold $C(W)$, as defined below.

Definition 1.1. *Let (W, g) be an oriented compact connected Riemannian manifold of dimension m without boundary with metric g . Let $0 \leq l_1 \leq l_2$ be real numbers. Consider the space $C_{[l_1, l_2]}(W) = [l_1, l_2] \times W$ with the metric (defined for $x > 0$ when $l_1 = 0$)*

$$dx \otimes dx + x^2 g.$$

We call $C_{[l_1 > 0, l_2]}(W)$ the finite metric frustum over W ; we call $C_{(0, l]}(W)$ the finite metric cone over W , and we denote it by $C_l(W)$.

It is clear that in order to obtain a suitable extension of the definition of the analytic torsion to the cone, we need first a suitable definition of the Hodge-Laplace operator. Spectral analysis on cones was developed by Cheeger [5] [6], that in particular showed that the formal Hodge-Laplace operator Δ has a self adjoint extension on the space of square integrable forms. A complete set of solutions of the eigenvalues equation for Δ can be described in terms of a complete discrete resolution of the Hodge-Laplace operator on the section of the cone, see Lemma 3.1 [5, Section 3]. Considering square integrable forms and applying the boundary conditions, we obtain an explicit description of the spectrum of Δ in terms of the spectrum of the Hodge-Laplace operator on the section, see Lemmas 3.2.

Remark 1.1. *It is important to observe here that beside the usual (either absolute or relative) boundary conditions, due to the presence of a non empty boundary, when the section W has even dimension $m = 2p$, further boundary conditions, called ideal boundary conditions, were introduced by Cheeger [5], as we recall here. Assume that there exists a decomposition $\mathcal{H}^p(W) = V_\alpha \oplus V_\tau$, where V_α and V_τ are maximal self annihilating subspaces for the cup product pairing. Then, a p dimensional form belongs to the domain of the Laplace operator if its components in V_α and V_τ satisfy Neumann and Dirichlet conditions respectively at $x = 0$ (see [5] pg. 580 for details). Observe that our determination of the spectrum of the cone is obtained assuming this decomposition and these conditions. Moreover, where ever not explicitly stated, the above decomposition and the*

ideal conditions will be assumed. We conclude this remark recalling that ideal boundary conditions are necessary to guarantee Poincaré duality on the cone.

We can then prove that $\mathrm{Sp}_+(\Delta)$ is a regular sequence of spectral type and therefore the analytic torsion of the cone is well defined. We are in the position of extending the above definition of the analytic torsion to this setting, and we call the resulting object the L^2 analytic torsion of the cone, $T_{\mathrm{abs,ideal}}(C_l(W))$ (we use the same notation and we restrict to absolute BC, since the relative torsion follows by Poincaré duality [10, Section 4]). Of course it is not obvious at all if the invariant obtained in this way as some geometric or topological meaning. Since the spaces of the L^2 harmonic forms on W as proved to be isomorphic to the intersection cohomology of W , the natural candidate for the L^2 analytic torsion is the intersection torsion. It was proved in [11] that indeed L^2 analytic torsion and intersection torsion of the cone over an odd dimensional manifold coincide. The case of an even dimensional section is not clear yet, due to some difficulty in producing a natural definition for the intersection torsion in this case. There is work in progress in this direction. Here we tackle the analytic side of the problem.

The main purpose of this work is to compute $T_{\mathrm{abs,ideal}}(C_l(W))$, or more precisely to give formulas for it in terms of other either geometric or spectral invariants, and in particular invariants of the section (W, g) . A key point here is the following. The description of the spectrum of the Hodge-Laplace operator on the cone in terms of the spectrum of the Hodge-Laplace operator on the section makes possible to express (or decompose) all the spectral functions (in particular the zeta function and the logarithmic Gamma function, see Subsection 5) on the cone in terms of the spectral functions on the sections, and to apply the technique introduced in [22] in order to tackle the derivative at zero of a class of double series. This is the main technical point, and in fact this is the reason that permits to obtain the final results. We did follow the same approach first in [9], where we gave the formula for the torsion of the cone over a sphere, and then in [10], where we considered as section any compact connected manifold of odd dimension. These results are superseded by the formulas obtained in the present work, where the section can have any dimension. We stress the fact that, as observed in [10], the calculation are exactly the same and there is no more difficulty to deal with the even dimensional section case that with the odd dimensional one. Moreover, while the formulas in the odd dimensional section case have a clear geometric interpretation in terms of intersection torsion, due to [12], the even dimensional section case is still quite obscure. However, due to the recent interest in the subject (see [16]), we decided to present the formulas for the general case and the details of the calculation. Indeed, the presentation followed in this paper adds some insights in the possible interpretation of the result, as we explain at the end of this section, but first we present our first main result in the following theorem.

Theorem 1.1. *Let $(W^{(m)}, g)$ be an oriented compact connected Riemannian manifold of dimension m with metric g . The L^2 analytic torsion $T_{\mathrm{abs,ideal}}(C_l(W))$ of the cone over W with absolute boundary and ideal conditions is as follows, where $r_q = \mathrm{rk}H_q(W)$, and $m_{\mathrm{cex},q,n}$, α_q and $\mu_{q,n}$ are defined in Lemma 3.1, $p \leq 1$:*

$$\begin{aligned} \log T_{\mathrm{abs}}(C_l(W^{(2p-1)})) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q (2p-2q)r_q \log l + \frac{1}{2} \log T(W, g) - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2(p-q)) \\ &\quad + A_{\mathrm{BM,abs}}(\partial C_l(W)), \end{aligned}$$

$$\begin{aligned}
\log T_{\text{abs,ideal}}(C_l(W^{(2p)})) &= \left(\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q (2p - 2q + 1) r_q + (-1)^p \frac{1}{4} r_p \right) \log l \\
&\quad - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p - 2q + 1) ((2p - 2q - 1)!)^2 \\
&\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \frac{1 + \frac{\alpha_q}{\mu_{q,n}}}{1 - \frac{\alpha_q}{\mu_{q,n}}} \\
&\quad + \frac{1}{2} \chi(W) \log 2 + A_{\text{BM,abs}}(\partial C_l(W)).
\end{aligned}$$

Next, we give an interpretation of the formulas given in Theorem 1.1. This is done in two steps. In the first, described in whole details in Section 2, we show that all the sums appearing in the formula of the torsion in the odd case, and some of these sums in the even case, coincide with the determinant of the change of basis between the basis of the harmonic forms on the cone and the basis of the harmonic forms on the sections, see Theorem 2.1 of Section 2. This is a classical geometric contribution in the Reidemeister torsion of the cone (see [12]), and therefore this interpretation clarifies completely the appearance of these sums. In particular, this describes completely the L^2 -analytic torsion of a cone over an odd dimensional manifold, and also suggests that the Euler part of the boundary contribution is anomalous in the analytic torsion (see Remark 2.1 at the end of Section 2). In the other case, i.e. for a cone over an even dimensional manifold, other sums appear in the formula for the analytic torsion. An interpretation of these further sums, collected into the two terms B_1 and B_2 in Theorem 2.1, is the second step of our interpretation of the formulas for the analytic torsion, and is obtained in the final section of the paper. The interpretation is suggested after proving that the formulas for the torsion can be obtained as some limit case of the formula for the torsion of the frustum. We postpone the discussion of these results to the last section.

We conclude this introductory section with a few remarks on the anomaly term appearing in the even case. We observe that the anomaly term can be also written in terms of residues of some spectral functions, see Subsection 8.1.2, or Theorem 1.1 of [9]. Formulas for particular sections, as spheres of discs, are given in [9] and [8] [7].

Remark 1.2. *By duality:*

$$\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \frac{1 + \frac{\alpha_q}{\mu_{q,n}}}{1 - \frac{\alpha_q}{\mu_{q,n}}} = \frac{1}{2} \sum_{q=0}^{2p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 + \frac{\alpha_q}{\mu_{q,n}} \right).$$

Remark 1.3. *Using Proposition 2.9 of [21] the first term in the last line of the second formula in the theorem reads*

$$\begin{aligned}
\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \frac{1 + \frac{\alpha_q}{\mu_{q,n}}}{1 - \frac{\alpha_q}{\mu_{q,n}}} &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q (\zeta'_q(0, \alpha_q) - \zeta'_q(0, -\alpha_q)), \\
&= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \log \frac{\det_{\zeta} \left((\Delta_{(W,g)}^{(q)} + \alpha_q^2)^{\frac{1}{2}} - \alpha_q \right)}{\det_{\zeta} \left((\Delta_{(W,g)}^{(q)} + \alpha_q^2)^{\frac{1}{2}} + \alpha_q \right)},
\end{aligned}$$

where $\Delta_{(W,g)}$ is the Hodge-Laplace operator on the section of the cone (W, g) , and

$$\zeta_q(s, x) = \sum_{n=1}^{\infty} m_{\text{cex},q,n} (\mu_{q,n} + x)^{-s}.$$

2. A GEOMETRIC INTERPRETATION OF THE ANALYTIC TORSION

The spaces of harmonic forms on the frustum (with absolute and mixed BC) were computed in [11]. The space of harmonic forms on the cone (with absolute and relative BC) were computed [12]. In particular, in the even case $m = 2p$ the result changes if we assume the Cheeger ideal BC, as described in the introduction. A simple calculation, proceeding as in the proof of Lemma 4.1 of [12], gives the following result.

Lemma 2.1. *We have the following isomorphisms of vector spaces induced by the extension of the inclusion of the forms:*

$$\begin{aligned} \mathcal{H}_{\text{abs}}^q(C_l(W)) &= \begin{cases} \mathcal{H}^q(W), & 0 \leq q \leq p-1, \\ \{0\}, & p \leq q \leq 2p-1, \end{cases} & \dim W = 2p-1, \\ \mathcal{H}_{\text{abs}}^q(C_l(W)) &= \begin{cases} \mathcal{H}^q(W), & 0 \leq q \leq p, \\ \{0\}, & p+1 \leq q \leq 2p+1, \end{cases} & \dim W = 2p, \\ \mathcal{H}_{\text{abs,ideal}}^q(C_l W) &= \begin{cases} \mathcal{H}^q(W) & \text{if } 0 \leq q \leq p-1, \\ V_a & \text{if } q = p, \\ 0 & \text{if } p+1 \leq q \leq 2p+1, \end{cases} & \dim W = 2p, \\ \mathcal{H}_{\text{abs}}^q(C_{[l_1, l_2]}(W)) &= \mathcal{H}^q(W). \end{aligned}$$

Proceeding as in Section 3.5 of [11], we have the following commutative diagram of isomorphisms of vector spaces (we give here the diagram for the cone, the one for the frustum is in [11]), for $0 \leq q \leq [\frac{m}{2}]$,

$$\begin{array}{ccccccc} \mathcal{H}_{\text{abs}}^q(C_l(W)) & \xrightarrow{\star_{C_l(W)}} & \mathcal{H}_{\text{rel}}^{m-q+1}(C_l(W)) & \xrightarrow{\mathcal{A}_{C_l(W), \text{rel}}^{m-q+1}} & H^{m-q+1}(\widehat{C_l(W)}, \partial \widehat{C_l(W)}) & \xleftarrow{\mathcal{P}} & H_q(C_l(W)) \\ \uparrow \frac{\ddot{\omega}}{\sqrt{\gamma_q}} & & \uparrow \frac{(-1)^q x^{m-2q} dx \wedge \cdot}{\sqrt{\gamma_q}} & & \uparrow & & \uparrow (-1)^q \sqrt{\gamma_q} \\ \mathcal{H}^q(W) & \xrightarrow{\star_W} & \mathcal{H}^{m-q}(W) & \xrightarrow{\mathcal{A}_W^{m-q}} & H^{m-q}(\widehat{W}) & \xleftarrow{\mathcal{P}} & H_q(W) \end{array}$$

where \mathcal{A} is the de Rham map on the cone, \mathcal{P} Poincaré duality, \star the Hodge isomorphism, and the factor γ_q is the ratio between the L^2 norm of the constant extension of a form ω on the cone and the L^2 norm of ω (the double dot indicates the constant extension):

$$\gamma_q = \frac{\|\ddot{\omega}_{C_l}\|_{C_l(W)}^2}{\|\omega\|_W^2} = \int_0^l x^{m-2q} dx = \frac{l^{m-2q+1}}{m-2q+1}.$$

The same analysis on the frustum gives (see [11, Section 3.5] for all definitions and details)

$$\Gamma_q = \frac{\|\ddot{\omega}_F\|_{C_{[l_1, l_2]}(W)}^2}{\|\omega\|_W^2} = \int_{l_1}^{l_2} x^{m-2q} dx = \begin{cases} \frac{1}{m+1-2q} (l_2^{m+1-2q} - l_1^{m+1-2q}) & \text{if } m+1-2q \neq 0, \\ \ln \frac{l_2}{l_1}, & \text{if } m+1-2q = 0. \end{cases}$$

Note that for all $q < [\frac{m}{2}]$,

$$\lim_{l_1 \rightarrow 0^+, l_2 = l} \Gamma_q = \gamma_q.$$

Recall that for a real vector space V we denote $\Lambda^{\dim V} V$ by $\det V$, and we call this line the determinant line of V ($\det 0 := \mathbb{R}$). If $\mathbf{v} = \{v_1, \dots, v_{\dim V}\}$ is a basis for V , we use the notation $\det \mathbf{v}$ for $v_1 \wedge \dots \wedge v_{\dim V}$. For a finite dimensional graded vector space $V_\bullet = (V_q)_{q \in \mathbb{Z}} = \bigoplus_{q \in \mathbb{Z}} V_q$, set

$$\text{Det}_n^m V_\bullet := \bigotimes_{q=n}^m (\det V_q)^{\dagger^q},$$

where $V^{\dagger^{2p}} = V$ and $V^{\dagger^{2p+1}} = V^\dagger$. This one dimensional vector space is called the determinant line of V_\bullet .

With this notation, if α is a (graded) basis for the harmonic forms on W , we have that the ratio between the determinant of the homology graded basis induced by an harmonic basis of the frustum and of its section, and of the cone and of this section are, respectively:

$$\begin{aligned} \frac{\text{Det} \ddot{\alpha}_F}{\text{Det} \alpha} &= \prod_{q=0}^m \Gamma_q^{\frac{(-1)^q r_q}{2}}, & \text{absolute BC} \\ \frac{\text{Det}_0^{\lfloor \frac{m-1}{2} \rfloor} \ddot{\alpha}_{C_l}}{\text{Det}_0^{\lfloor \frac{m-1}{2} \rfloor} \alpha} &= \prod_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} \gamma_q^{\frac{(-1)^q r_q}{2}}, & \text{absolute BC} \\ \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_l}}{\text{Det}_0^{p-1} \alpha} &= \gamma_p^{\frac{(-1)^p r_p}{4}} \prod_{q=0}^{p-1} \gamma_q^{\frac{(-1)^q r_q}{2}}, & m = 2p, \text{ absolute, ideal BC.} \end{aligned}$$

For completeness we gave here also the result without ideal BC here, however in the following the formulas are all given assuming ideal BC. A simple calculation completes the proof of the following corollary of Theorem 1.1 (a similar formula holds in the case $m = 0$).

Theorem 2.1. *The analytic torsion of the cone reads:*

$$\begin{aligned} \log T_{\text{abs,ideal}}(C_l(W^{(m)})) &= \frac{1}{2} \log T(W, g) + \log \frac{\text{Det}_0^{\lfloor \frac{m-1}{2} \rfloor} \ddot{\alpha}}{\text{Det}_0^{\lfloor \frac{m-1}{2} \rfloor} \alpha} + \frac{1}{4} \chi(W) \log 2 + A_{\text{BS,abs}}(\partial C_l(W)) \\ &\quad + B_1^{(m)}(C_l(W)) + B_2^{(m)}(C_l(W)), \end{aligned}$$

where $B_{1/2}^{(m)}(C_l(W))$ are the following anomaly terms (vanishing when $m = 2p - 1$ is odd):

$$\begin{aligned} B_1^{(2p)}(C_l(W)) &= - \sum_{q=0}^{p-1} (-1)^q r_q \log(2p - 2q - 1)!! + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \frac{1 + \frac{\alpha_q}{\mu_{q,n}}}{1 - \frac{\alpha_q}{\mu_{q,n}}}, \\ B_2^{(2p)}(C_l(W)) &= \frac{1}{4} \chi(W) \log 2. \end{aligned}$$

The analytic torsion of the frustum reads:

$$\log T_{\text{abs}}(C_{[l_1, l_2]}(W)) = \log T(W, g) + \log \frac{\text{Det} \ddot{\alpha}}{\text{Det} \alpha} + \frac{1}{2} \chi(W) \log 2 + A_{\text{BS,abs}}(\partial C_{[l_1, l_2]}(W)).$$

Remark 2.1. *As announced in the introduction, the formula in Theorem 2.1 completely describe the analytic torsion of a cone over an odd dimensional manifold in terms of topological and geometric quantities. In particular, the splitting between global and boundary terms is evident (compare with the formula in equation (1.1) for a regular manifold, and the discussion in Section 6). In the even case, the same analysis suggests the description of the torsion given in the formula, with two new*

anomaly terms, called B_1 and B_2 . If now we read the formula for the torsion in the even case looking for the global and the boundary contributions, we find out part of the boundary contribution can be interpret as anomalous. This is justified as follow. Recall that the boundary term splits into two parts: one is the classical part, see the work of Lück [13] and depends on the Euler characteristic of the boundary, the other is the anomaly boundary term, computed by Brüning and Ma [1, 2]. Then, we realise that (collecting together all the terms where the Euler characteristic appears), the classical boundary term in the cone coincides with the one of the frustum, even if the boundary is different. More precisely, it seems that the analytic torsion of the cone does not see that the boundary at $x = 0$ collapse to a point. Following this interpretation, we splitter the term into two parts: the first is

$$\frac{1}{4}\chi(W) \log 2,$$

and is classical contribution of the boundary, the second one is $B_2(W)$, and is understood as an anomaly boundary term.

3. SPECTRAL PROPERTIES OF THE HODGE-LAPLACE OPERATOR

Consider the metric in Definition 1.1 either on the cone $C_l(W)$ or on the frustum $C_{[l_1 > 0, l_2]}(W)$. Let $\omega \in \Omega^q(C_l(W))$ ($\omega \in \Omega^q(C_{[l_1 > 0, l_2]}(W))$), set

$$\omega(x, y) = f_1(x)\omega_1(y) + f_2(x)dx \wedge \omega_2(y),$$

with smooth functions f_1 and f_2 , and $\omega_j \in \Omega(W)$. Then (we denote operators acting on the section by a tilde),

$$\star\omega(x, y) = x^{m-2q+2}f_2(x)\tilde{\star}\omega_2(y) + (-1)^q x^{m-2q}f_1(x)dx \wedge \tilde{\star}\omega_1(y),$$

$$\begin{aligned} d\omega(x, y) &= f_1(x)\tilde{d}\omega_1(y) + \partial_x f_1(x)dx \wedge \omega_1(y) - f_2(x)dx \wedge d\omega_2(y), \\ d^\dagger\omega(x, y) &= x^{-2}f_1(x)\tilde{d}^\dagger\omega_1(y) - ((m-2q+2)x^{-1}f_2(x) + \partial_x f_2(x))\omega_2(y) \\ &\quad - x^{-2}f_2(x)dx \wedge \tilde{d}^\dagger\omega_2(y), \end{aligned}$$

$$\begin{aligned} \Delta\omega(x, y) &= (-\partial_x^2 f_1(x) - (m-2q)x^{-1}\partial_x f_1(x))\omega_1(y) + x^{-2}f_1(x)\tilde{\Delta}\omega_1(y) - 2x^{-1}f_2(x)\tilde{d}\omega_2(y) \\ &\quad + dx \wedge (x^{-2}f_2(x)\tilde{\Delta}\omega_2(y) + \omega_2(y)(-\partial_x^2 f_2(x) - (m-2q+2)x^{-1}\partial_x f_2(x) \\ &\quad + (m-2q+2)x^{-2}f_2(x)) - 2x^{-3}f_1(x)\tilde{d}^\dagger\omega_1(y)). \end{aligned}$$

Lemma 3.1 (Cheeger [5]). *Let $\{\varphi_{\text{har}}^{(q)}, \varphi_{\text{cex}, n}^{(q)}, \varphi_{\text{ex}, n}^{(q)}\}$ be an orthonormal basis of $\Omega^q(W)$ consisting of harmonic, coexact and exact eigenforms of $\tilde{\Delta}^{(q)}$. Let $\lambda_{q, n}$ denotes the eigenvalue of $\varphi_{\text{cex}, n}^{(q)}$ and $m_{\text{cex}, q, n}$ its multiplicity. Let J_ν be the Bessel function of index ν . Define*

$$\alpha_q = \frac{1}{2}(1 + 2q - m), \quad \mu_{q, n} = \sqrt{\lambda_{q, n} + \alpha_q^2}.$$

Then, assuming that $\mu_{q,n}$ is not an integer, the solutions of the equation $\Delta u = \lambda^2 u$, with $\lambda \neq 0$, are of the following six types:

$$\begin{aligned}
\psi_{\pm,1,n,\lambda}^{(q)} &= x^{\alpha_q} J_{\pm\mu_{q,n}}(\lambda x) \varphi_{\text{cex},n}^{(q)}, \\
\psi_{\pm,2,n,\lambda}^{(q)} &= x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} + \partial_x(x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x)) dx \wedge \varphi_{\text{cex},n}^{(q-1)} \\
\psi_{\pm,3,n,\lambda}^{(q)} &= x^{2\alpha_{q-1}+1} \partial_x(x^{-\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x)) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
&\quad + x^{\alpha_{q-1}-1} J_{\pm\mu_{q-1,n}}(\lambda x) dx \wedge \tilde{d}^\dagger \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
\psi_{\pm,4,n,\lambda}^{(q)} &= x^{\alpha_{q-2}+1} J_{\pm\mu_{q-2,n}}(\lambda x) dx \wedge \tilde{d}\varphi_{\text{cex},n}^{(q-2)} \\
\psi_{\pm,E,\lambda}^{(q)} &= x^{\alpha_q} J_{\pm|\alpha_q|}(\lambda x) \varphi_{\text{har}}^{(q)} \\
\psi_{\pm,O,\lambda}^{(q)} &= \partial_x(x^{\alpha_{q-1}} J_{\pm|\alpha_{q-1}|}(\lambda x)) dx \wedge \varphi_{\text{har}}^{(q-1)}.
\end{aligned}$$

When the index is an integer the – solutions must be modified including some logarithmic term (see for example [23] for a set of linear independent solutions of the Bessel equation).

Following [5] and [3], the formal Hodge-Laplace operator in equation (3.2) defines a concrete self adjoint operator with domain in the space of the square integrable forms on the cone (see [9] for details). This operator (that we denote by the same symbol Δ) has a pure point spectrum as described in the following lemmas, whose proof follows applying BC and square integrability to the solutions described in Lemma 3.1.

Lemma 3.2. *The positive part of the spectrum of the Hodge-Laplace operator on $C_1(W)$, with absolute boundary conditions on $\partial C_1(W)$ is as follows, where $0 \leq q \leq m+1$. If $m = \dim W = 2p-1$, $p \geq 1$:*

$$\begin{aligned}
\text{Sp}_+ \Delta_{\text{abs}}^{(q)} &= \left\{ m_{\text{cex},q,n} : \hat{j}_{\mu_{q,n},\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-2,n} : \hat{j}_{\mu_{q-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ m_{\text{har},q} : \hat{j}_{|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1} : \hat{j}_{|\alpha_{q-1}|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty.
\end{aligned}$$

If $m = \dim W = 2p$, $p \geq 1$:

$$\begin{aligned}
\text{Sp}_+ \Delta_{\text{abs}}^{(q \neq p, p+1)} &= \left\{ m_{\text{cex},q,n} : \hat{j}_{\mu_{q,n},\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-2,n} : \hat{j}_{\mu_{q-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ m_{\text{har},q} : \hat{j}_{|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1} : \hat{j}_{|\alpha_{q-1}|,\alpha_{q-1},k}^2 / l^2 \right\}_{k=1}^\infty,
\end{aligned}$$

$$\begin{aligned}
\text{Sp}_+ \Delta_{\text{abs}}^{(p)} &= \left\{ m_{\text{cex},p,n} : \hat{j}_{\mu_{p,n},\alpha_p,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p-1,n} : \hat{j}_{\mu_{p-1,n},\alpha_{p-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ m_{\text{cex},p-1,n} : \hat{j}_{\mu_{p-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p-2,n} : \hat{j}_{\mu_{p-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\
&\cup \left\{ \frac{1}{2} m_{\text{har},p} : \hat{j}_{\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ \frac{1}{2} m_{\text{har},p} : \hat{j}_{-\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},p-1} : \hat{j}_{|\alpha_{p-1}|,\alpha_{p-1},k}^2 / l^2 \right\}_{k=1}^\infty,
\end{aligned}$$

$$\begin{aligned} \text{Sp}_+ \Delta_{\text{abs}}^{(p+1)} &= \left\{ m_{\text{cex}, p+1, n} : \hat{j}_{\mu_{p+1}, n, \alpha_{p+1}, k}^2 / l^2 \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, p, n} : \hat{j}_{\mu_p, n, \alpha_p, k}^2 / l^2 \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{cex}, p, n} : j_{\mu_p, n, k}^2 / l^2 \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, p-1, n} : j_{\mu_{p-1}, n, k}^2 / l^2 \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{har}, p+1} : \hat{j}_{|\alpha_{p+1}|, \alpha_{p+1}, k}^2 / l^2 \right\}_{k=1}^{\infty} \cup \left\{ \frac{1}{2} m_{\text{har}, p} : j_{-\frac{1}{2}}^2 / l^2 \right\}_{k=1}^{\infty} \cup \left\{ \frac{1}{2} m_{\text{har}, p} : j_{\frac{1}{2}}^2 / l^2 \right\}_{k=1}^{\infty}, \end{aligned}$$

where the $j_{\mu, k}$ are the positive zeros of the Bessel function $J_{\mu}(x)$, the $\hat{j}_{\mu, c, k}$ are the positive zeros of the function $\hat{J}_{\mu, c}(x) = cJ_{\mu}(x) + xJ'_{\mu}(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q, n}$ are defined in Lemma 3.1.

Lemma 3.3. *The positive part of the spectrum of the Hodge-Laplace operator on $C_{[l_1, l_2]}(W)$, $l_1 > 0$, with relative boundary conditions on $\partial_1 C_{[l_1, l_2]}(W)$ and absolute boundary conditions on $\partial_2 C_{[l_1, l_2]}(W)$ is ($0 \leq q \leq m+1$):*

$$\begin{aligned} \text{Sp}_+ \Delta_{\text{rel } \partial_1, \text{abs } \partial_2}^{(q)} &= \left\{ m_{\text{cex}, q, n} : \hat{f}_{\mu_{q, n}, \alpha_q, k}^2(l_1, l_2) \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, q-1, n} : \hat{f}_{\mu_{q-1, n}, \alpha_{q-1}, k}^2(l_1, l_2) \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{cex}, q-1, n} : \hat{f}_{\mu_{q-1, n}, -\alpha_{q-1}, k}^2(l_2, l_1) \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, q-2, n} : \hat{f}_{\mu_{q-2, n}, -\alpha_{q-2}, k}^2(l_2, l_1) \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{har}, q} : \hat{f}_{|\alpha_q|, \alpha_q, k}^2(l_1, l_2) \right\}_{k=1}^{\infty} \cup \left\{ m_{\text{har}, q-1} : \hat{f}_{|\alpha_{q-1}|, \alpha_{q-1}, k}^2(l_1, l_2) \right\}_{k=1}^{\infty}, \end{aligned}$$

where the $\hat{f}_{\mu, c, k}(a, b)$ are the zeros of the function

$$\hat{F}_{\mu, c}(x; l_1, l_2) = J_{\mu}(l_1 x)(cY_{\mu}(l_2 x) + l_2 x Y'_{\mu}(l_2 x)) - Y_{\mu}(l_1 x)(cJ_{\mu}(l_2 x) + l_2 x J'_{\mu}(l_2 x)),$$

with real $c \neq 0$, and α_q and $\mu_{q, n}$ as defined in Lemma 3.1.

Remark 3.1. *Application of the BC in Lemma 3.1 would give the zeros of the function*

$$J_{\mu}(l_1 x)(cJ_{-\mu}(l_2 x) + l_2 x J'_{-\mu}(l_2 x)) - J_{-\mu}(l_1 x)(cJ_{\mu}(l_2 x) + l_2 x J'_{\mu}(l_2 x)) = -\sin(\pi\mu) \hat{F}_{\mu, c}(x; l_1, l_2),$$

however, for the following analysis, it is much more convenient to work with the function Y instead that with the function J_- .

Lemma 3.4. *The positive part of the spectrum of the Hodge-Laplace operator on $C_{[l_1, l_2]}(W)$, $l_1 > 0$, with absolute boundary conditions is ($0 \leq q \leq m+1$):*

$$\begin{aligned} \text{Sp}_+ \Delta_{\text{abs}}^{(q)} &= \left\{ m_{\text{cex}, q, n} : \hat{v}_{\mu_{q, n}, \alpha_q, k}^2 \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, q-1, n} : \hat{v}_{\mu_{q-1, n}, \alpha_{q-1}, k}^2 \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{cex}, q-1, n} : v_{\mu_{q-1, n}, k}^2 \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\text{cex}, q-2, n} : v_{\mu_{q-2, n}, k}^2 \right\}_{n, k=1}^{\infty} \\ &\cup \left\{ m_{\text{har}, q} : \hat{v}_{|\alpha_q|, \alpha_q, k}^2 \right\}_{k=1}^{\infty} \cup \left\{ m_{\text{har}, q-1} : \hat{v}_{|\alpha_{q-1}|, \alpha_{q-1}, k}^2 \right\}_{k=1}^{\infty}, \end{aligned}$$

where the $v_{\mu, k}$ are the zeros of the function

$$\Upsilon_{\mu}(x) = J_{\mu}(l_2 x)Y_{\mu}(l_1 x) - Y_{\mu}(l_1 x)J_{\mu}(l_2 x),$$

and the $\hat{v}_{\mu, c, k}$ are the zeros of the function

$$\hat{\Upsilon}_{\mu, c}(x) = (cJ_{\mu}(l_2 x) + l_2 x J'_{\mu}(l_2 x))(cY_{\mu}(l_1 x) + l_1 x Y'_{\mu}(l_1 x)) - (cY_{\mu}(l_2 x) + l_2 x Y'_{\mu}(l_2 x))(cJ_{\mu}(l_1 x) + l_1 x J'_{\mu}(l_1 x)),$$

with real $c \neq 0$, and α_q and $\mu_{q, n}$ as defined in Lemma 3.1.

4. SIMPLIFYING THE TORSION ZETA FUNCTION

We define the *torsion zeta function* by

$$t_{M^{(n)}}(s) = \frac{1}{2} \sum_{q=1}^n (-1)^q q \zeta(s, \Delta^{(q)}),$$

and the analytic torsion is:

$$\log T(M^{(n)}, g) = t'_{M^{(n)}}(0).$$

In this section, we consider the torsion zeta function for the cone with absolute BC and for the frustum with mixed and absolute BC respectively. We proceed to some simplifications of it. We present the proof in the case of the cone, the proof for the frustum is analogous.

Lemma 4.1. *The torsion zeta function of the cone is:*

$$t_{\text{Cone}}^{(m)}(s) = t_0^{(m)}(s) + t_1^{(m)}(s) + t_2^{(m)}(s) + t_3^{(m)}(s),$$

with

$$\begin{aligned} t_0^{(m)}(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((Z_q(s) - \hat{Z}_{q,+}(s)) + (-1)^{m-1} (Z_q(s) - \hat{Z}_{q,-}(s)) \right), \\ t_1^{(2p-1)}(s) &= (-1)^{p-1} \frac{l^{2s}}{2} \left(Z_{p-1}(s) - \hat{Z}_{p-1,0}(s) \right), & t_1^{(2p)}(s) &= 0 \\ t_2^{(m)}(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (z_{q-1,-}(s) + (-1)^m z_{q,-}(s)), \\ t_3^{(2p)}(s) &= (-1)^{p+1} m_{\text{har},p} \frac{l^{2s}}{4} (z_{p,+}(s) + z_{p,-}(s)), & t_3^{(2p-1)}(s) &= 0, \end{aligned}$$

where $\hat{Z}_{q,0}$ denotes $\hat{Z}_{q,\pm}$ with $\alpha_q = 0$, and

$$Z_q(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_q,n,k}^{-2s}, \quad \hat{Z}_{q,\pm}(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_q,n,\pm\alpha_q,k}^{-2s}, \quad z_{q,\pm}(s) = \sum_{k=1}^{\infty} j_{\pm\alpha_q,k}^{-2s}.$$

Proof. Rearranging the sums and isolating the case $q = p$, $\alpha_p = \frac{1}{2}$ when $m = 2p$, we obtain: if $m = 2p - 1$,

$$t_{\text{Cone}}^{(2p-1)}(s) = \frac{l^{2s}}{2} \sum_{q=0}^{2p-1} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_q,n,k}^{-2s} - \hat{j}_{\mu_q,n,\alpha_q,k}^{-2s} \right) + \frac{l^{2s}}{2} \sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s},$$

if $m = 2p$,

$$\begin{aligned} t_{\text{Cone}}^{(2p)}(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{2p-1} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_q,n,k}^{-2s} - \hat{j}_{\mu_q,n,\alpha_q,k}^{-2s} \right) + \frac{l^{2s}}{2} \sum_{q=0, q \neq p}^{2p} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} \\ &\quad + (-1)^{p+1} \frac{l^{2s}}{4} \sum_{k=1}^{\infty} m_{\text{har},p} \left(j_{\frac{1}{2},k}^{-2s} + j_{-\frac{1}{2},k}^{-2s} \right). \end{aligned}$$

Using Hodge duality on coexact q -forms on the section, we obtain the following identities:

$$\begin{aligned} m_{\text{cex},q,n} &= m_{\text{cex},m-1-q,n}, \\ \lambda_{q,n} &= \lambda_{m-1-q,n}, \\ \alpha_q &= -\alpha_{m-1-q}, \\ \mu_{q,n} &= \mu_{m-1-q,n}, \end{aligned}$$

and hence the first term in the previous equations reads:

$$\begin{aligned} & \frac{l^{2s}}{2} \sum_{q=0}^{m-1} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \right) \\ (4.1) \quad &= \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(\left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \right) + (-1)^{m-1} \left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s} \right) \right) \\ &+ \left(\text{term with } q = \left\lfloor \frac{m}{2} \right\rfloor \right), \end{aligned}$$

where the last term appears only if $\lfloor \frac{m}{2} \rfloor$ is an integer, namely if $m = 2p - 1$ is odd, and has the following form. Since when $q = p - 1$, $m = 2p - 1$, $\alpha_{p-1} = 0$, (and $j_{\nu,0,k} = \lambda_{p-1,n} = j'_{\nu,k}$), then

$$\left(\text{term with } q = \left\lfloor \frac{m}{2} \right\rfloor \right) = (-1)^{p-1} \frac{l^{2s}}{2} \sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} \left(j_{\mu_{p-1,n},k}^{-2s} - \hat{j}_{\mu_{p-1,n},0,k}^{-2s} \right).$$

The sign in front to the second term in equation (4.1) is the key difference between the even and the odd case.

Next consider the term involving the harmonics, i.e. the sums

$$k^{(2p-1)}(s) = \frac{1}{2} \sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}, \quad k^{(2p)}(s) = \frac{1}{2} \sum_{q=0, q \neq p}^{2p} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}.$$

Consider the function

$$\hat{J}_{|\alpha_q|,\alpha_q}(x) = \alpha_q J_{|\alpha_q|}(x) + lx J'_{|\alpha_q|}(x),$$

Since

$$z Z'_\mu(z) = -z Z_{\mu+1}(z) + \mu Z_\mu(z), \quad \text{and} \quad z Z'_\mu(z) = z Z_{\mu-1}(z) - \mu Z_\mu(z),$$

where Z is either J_+ or J_- , it follows that

$$\begin{aligned} \hat{Z}_{-\alpha_q,\alpha_q}(z) &:= \alpha_q Z_{-\alpha_q}(z) + z Z'_{-\alpha_q}(z) = -z Z_{-\alpha_q+1}(z) = -z Z_{-\alpha_{q-1}}(z), \\ \hat{Z}_{\alpha_q,\alpha_q}(z) &:= \alpha_q Z_{\alpha_q}(z) + z Z'_{\alpha_q}(z) = z Z_{\alpha_q-1}(z) = z Z_{\alpha_{q-1}}(z). \end{aligned}$$

This permit to simplify $\hat{J}_{|\alpha_q|,\alpha_q}$ as follows. If α_q is negative, then

$$\hat{J}_{|\alpha_q|,\alpha_q}(x) = -x J_{-\alpha_{q-1}}(x),$$

and this means that $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{-\alpha_{q-1},k}$, for these q . If is positive, then

$$\hat{J}_{|\alpha_q|,\alpha_q}(x) = x J_{\alpha_{q-1}}(x),$$

and this means that $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{\alpha_{q-1},k}$, for these q . When $\alpha_q = 0$ (and this happens only if $m = 2p - 1$, $q = p - 1$), we have $\hat{J}_{0,0}(x) = x J_{\pm 1}(x)$, and hence we can use either $j_{1,k}$ or $j_{-1,k}$. Next,

if $m = 2p - 1$, α_q is negative for $0 \leq q \leq p - 2$, $\alpha_{p-1} = 0$, and α_q is positive for $p \leq q \leq 2p - 1$, whence

$$k^{(2p-1)}(s) = \frac{1}{2} \sum_{q=0}^{p-2} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q}}{j_{-\alpha_{q-1},k}^{2s}} + \frac{1}{2} (-1)^p \sum_{k=1}^{\infty} \frac{m_{\text{har},1}}{j_{\alpha_1,k}^{-2s}} + \frac{1}{2} \sum_{q=p}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q}}{j_{\alpha_{q-1},k}^{-2s}},$$

if $m = 2p$, α_q is negative for $0 \leq q \leq p - 1$, and α_q is positive for $p \leq q \leq 2p - 1$, whence

$$k^{(2p)}(s) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q}}{j_{-\alpha_{q-1},k}^{2s}} + \frac{1}{2} \sum_{q=p}^{2p} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q}}{j_{\alpha_{q-1},k}^{-2s}},$$

and since by Poincaré duality $m_{\text{har},q} = m_{\text{har},m-q}$, and $\alpha_{m-q} = -\alpha_{q-1}$, we have the thesis (note that when $m = 2p$, $\alpha_p = \frac{1}{2}$). \square

For the frustum, we define (for $c > 0$) the function

$$F_\nu(x; l_1, l_2) = J_\nu(l_1 x) Y_{\nu-1}(l_2 x) - Y_\nu(l_1 x) J_{\nu-1}(l_2 x),$$

and let $f_{\nu,k}(l_1, l_2)$ denote its zeros. Then, we have the following result.

Lemma 4.2. *The torsion zeta function of the frustum with mixed BC is*

$$t_{\text{Frustum,mix}}^{(m)}(s) = w_0^{(m)}(s) + w_1^{(m)}(s) + w_2^{(m)}(s) + w_3^{(m)}(s),$$

with

$$\begin{aligned} w_0^{(m)}(s) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\hat{D}_{q,-}(s; l_2, l_1) - \hat{D}_{q,+}(s; l_1, l_2)) \right. \\ &\quad \left. + (-1)^{m-1} (\hat{D}_{q,+}(s; l_2, l_1) - \hat{D}_{q,-}(s; l_1, l_2)) \right), \\ w_1^{(2p-1)}(s) &= (-1)^{p-1} \frac{1}{2} \left(\hat{D}_{p-1,0}(s; l_2, l_1) - \hat{D}_{p-1,0}(s; l_1, l_2) \right), & w_1^{(2p)}(s) &= 0, \\ w_2^{(m)}(s) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} (d_q(s; l_2, l_1) + (-1)^m d_q(s; l_1, l_2)), \\ w_3^{(2p)}(s) &= (-1)^{p+1} \frac{1}{2} d_p(s; l_1, l_2), & w_3^{(2p-1)}(s) &= 0, \end{aligned}$$

where $\hat{D}_{p-1,0}$ is given either by $\hat{D}_{p-1,+}$ or $\hat{D}_{p-1,-}$, since $\alpha_{p-1} = 0$, when $m = 2p - 1$, and

$$\hat{D}_{q,\pm}(s; l_1, l_2) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{f}_{\mu_{q,n},\pm\alpha_{q,k}}^{-2s}(l_1, l_2), \quad d_q(s; l_1, l_2) = m_{\text{har},q} \sum_{k=1}^{\infty} f_{-\alpha_{q-1},k}^{-2s}(l_1, l_2).$$

Lemma 4.3. *The torsion zeta function on the frustum with absolute BC reads*

$$t_{\text{Frustum,abs}}^{(m)}(s) = y_0^{(m)}(s) + y_1^{(m)}(s) + y_2^{(m)}(s) + y_3^{(m)}(s),$$

where:

$$\begin{aligned}
y_0^{(m)}(s) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((E_q(s) - \hat{E}_{q,+}(s)) + (-1)^{m-1} (E_q(s) - \hat{E}_{q,-}(s)) \right), \\
y_1^{(2p-1)}(s) &= (-1)^{p-1} \frac{1}{2} \left(E_{p-1}(s) - \hat{E}_{p-1,0}(s) \right), & y_1^{(2p)}(s) &= 0, \\
y_2^{(m)}(s) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (e_{q-1,-}(s) + (-1)^m e_{q,-}(s)), \\
y_3^{(2p)}(s) &= (-1)^{p+1} m_{\text{har},p} \frac{1}{2} e_{p,-}(s), & y_3^{(2p-1)}(s) &= 0,
\end{aligned}$$

where $\hat{E}_{q,0}$ denotes $\hat{E}_{q,\pm}$ with $\alpha_q = 0$, and

$$E_q(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} v_{\mu_{q,n},k}^{-2s}, \quad \hat{E}_{q,\pm}(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{v}_{\mu_{q,n},\pm\alpha_q,k}^{-2s}, \quad e_{q,-}(s) = \sum_{k=1}^{\infty} v_{-\alpha_q,k}^{-2s}.$$

5. ZETA DETERMINANTS

We recall in this section the main points of the technique that we will use to compute the derivative at $s = 0$ of the zeta functions appearing in the torsion zeta functions introduced in Section 4. This section is essentially contained in Section 4 of [9], to which we refer for details, and based on [22]. Given a sequence $S = \{a_n\}_{n=1}^{\infty}$ of spectral type, we define the *zeta function* by

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\text{Re}(s) > \mathfrak{e}(S)$, and by analytic continuation otherwise, and for all $\lambda \in \rho(S) = \mathbb{C} - S$, we define the *Gamma function* by the canonical product,

$$(5.1) \quad \frac{1}{\Gamma(-\lambda, S)} = \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n} \right) e^{\sum_{j=1}^{\mathfrak{g}(S)} \frac{(-1)^j}{j} \frac{(-\lambda)^j}{a_n^j}}.$$

Given a double sequence $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ of non vanishing complex numbers with unique accumulation point at the infinity, finite exponent $s_0 = \mathfrak{e}(S)$ and genus $p = \mathfrak{g}(S)$, we use the notation $S_n (S_k)$ to denote the simple sequence with fixed $n (k)$, we call the exponents of S_n and S_k the *relative exponents* of S , and we use the notation $(s_0 = \mathfrak{e}(S), s_1 = \mathfrak{e}(S_k), s_2 = \mathfrak{e}(S_n))$; we define *relative genus* accordingly.

Definition 5.1. *Let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence with finite exponents (s_0, s_1, s_2) , genus (p_0, p_1, p_2) , and positive spectral sector Σ_{θ_0, c_0} . Let $U = \{u_n\}_{n=1}^{\infty}$ be a totally regular sequence of spectral type of infinite order with exponent r_0 , genus q , domain $D_{\phi, d}$. We say that S is spectrally decomposable over U with power κ , length ℓ and asymptotic domain $D_{\theta, c}$, with $c = \min(c_0, d, c')$, $\theta = \max(\theta_0, \phi, \theta')$, if there exist positive real numbers κ , ℓ (integer), c' , and θ' , with $0 < \theta' < \pi$, such that:*

- (1) *the sequence $u_n^{-\kappa} S_n = \left\{ \frac{\lambda_{n,k}}{u_n^{\kappa}} \right\}_{k=1}^{\infty}$ has spectral sector $\Sigma_{\theta', c'}$, and is a totally regular sequence of spectral type of infinite order for each n ;*

(2) the logarithmic Γ -function associated to S_n/u_n^κ has an asymptotic expansion for large n uniformly in λ for λ in $D_{\theta,c}$, of the following form

$$(5.2) \quad \log \Gamma(-\lambda, u_n^{-\kappa} S_n) = \sum_{h=0}^{\ell} \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} + \sum_{l=0}^L P_{\rho_l}(\lambda) u_n^{-\rho_l} \log u_n + o(u_n^{-r_0}),$$

where σ_h and ρ_l are real numbers with $\sigma_0 < \dots < \sigma_\ell$, $\rho_0 < \dots < \rho_L$, the $P_{\rho_l}(\lambda)$ are polynomials in λ satisfying the condition $P_{\rho_l}(0) = 0$, ℓ and L are the larger integers such that $\sigma_\ell \leq r_0$ and $\rho_L \leq r_0$.

Define the following functions, ($\Lambda_{\theta,c} = \{z \in \mathbb{C} \mid |\arg(z-c)| = \frac{\theta}{2}\}$, oriented counter clockwise):

$$(5.3) \quad \Phi_{\sigma_h}(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt.$$

By Lemma 3.3 of [22], for all n , we have the expansions:

$$(5.4) \quad \begin{aligned} \log \Gamma(-\lambda, S_n/u_n^\kappa) &\sim \sum_{j=0}^{\infty} a_{\alpha_j,0,n} (-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} a_{k,1,n} (-\lambda)^k \log(-\lambda), \\ \phi_{\sigma_h}(\lambda) &\sim \sum_{j=0}^{\infty} b_{\sigma_h,\alpha_j,0} (-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} b_{\sigma_h,k,1} (-\lambda)^k \log(-\lambda), \end{aligned}$$

for large λ in $D_{\theta,c}$. We set (see Lemma 3.5 of [22])

$$(5.5) \quad \begin{aligned} A_{0,0}(s) &= \sum_{n=1}^{\infty} \left(a_{0,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h,0,0} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \\ A_{j,1}(s) &= \sum_{n=1}^{\infty} \left(a_{j,1,n} - \sum_{h=0}^{\ell} b_{\sigma_h,j,1} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \quad 0 \leq j \leq p_2, \end{aligned}$$

where the notation \sum' means that only the terms such that $\zeta(s, U)$ has a pole at $s = \sigma_h$ appear in the sum.

Theorem 5.1. *Let S be spectrally decomposable over U as in Definition 5.1. Assume that the functions $\Phi_{\sigma_h}(s)$ have at most simple poles for $s = 0$. Then, $\zeta(s, S)$ is regular at $s = 0$, and*

$$\begin{aligned} \zeta(0, S) &= -A_{0,1}(0) + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U), \\ \zeta'(0, S) &= -A_{0,0}(0) - A'_{0,1}(0) + \frac{\gamma}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) \\ &\quad + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_0 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) + \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_0 \zeta(s, U), \end{aligned}$$

where the notation \sum' means that only the terms such that $\zeta(s, U)$ has a pole at $s = \sigma_h$ appear in the sum.

Remark 5.1. *We call regular part of $\zeta(0, S)$ the first term appearing in the formula given in the theorem, and regular part of $\zeta'(0, S)$ the first two terms. The other terms we call singular part.*

Corollary 5.1. *Let $S_{(j)} = \{\lambda_{(j),n,k}\}_{n,k=1}^{\infty}$, $j = 1, \dots, J$, be a finite set of double sequences that satisfy all the requirements of Definition 5.1 of spectral decomposability over a common sequence U , with the same parameters κ , ℓ , etc., except that the polynomials $P_{(j),\rho}(\lambda)$ appearing in condition (2) do not vanish for $\lambda = 0$. Assume that some linear combination $\sum_{j=1}^J c_j P_{(j),\rho}(\lambda)$, with complex coefficients, of such polynomials does satisfy this condition, namely that $\sum_{j=1}^J c_j P_{(j),\rho}(\lambda) = 0$. Then, the linear combination of the zeta function $\sum_{j=1}^J c_j \zeta(s, S_{(j)})$ is regular at $s = 0$ and satisfies the linear combination of the formulas given in Theorem 5.1.*

We conclude recalling some formulas for the zeta determinants of some simple sequences. The results are known to specialists, and can be found in different places. We will use the formulation of [19]. For positive real l and q , define the *non homogeneous quadratic Bessel zeta function* by

$$z(s, \nu, q, l) = \sum_{k=1}^{\infty} \left(\frac{j_{\nu,k}^2}{l^2} + q^2 \right)^{-s},$$

for $\operatorname{Re}(s) > \frac{1}{2}$. Then, $z(s, \nu, q, l)$ extends analytically to a meromorphic function in the complex plane with simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots$. The point $s = 0$ is a regular point and

$$(5.6) \quad z(0, \nu, q, l) = -\frac{1}{2} \left(\nu + \frac{1}{2} \right), \quad z'(0, \nu, q, l) = -\log \sqrt{2\pi l} \frac{I_{\nu}(lq)}{q^{\nu}}.$$

In particular, taking the limit for $q \rightarrow 0$,

$$(5.7) \quad z'(0, \nu, 0, l) = -\log \frac{\sqrt{\pi} l^{\nu + \frac{1}{2}}}{2^{\nu - \frac{1}{2}} \Gamma(\nu + 1)}.$$

6. DECOMPOSITION OF THE TORSION ZETA FUNCTION

Inspection of the formulas in the lemmas of Section 4 shows that the torsion zeta function is the finite sum of some simple and some double series:

$$\begin{aligned} t_{\text{Cone,abs}}(s) &= t_0(s) + t_1(s) + t_2(s) + t_3(s), \\ t_{\text{Frustum,mixed}}(s) &= w_0(s) + w_1(s) + w_2(s) + w_3(s), \\ t_{\text{Frustum,abs}}(s) &= y_0(s) + y_1(s) + y_2(s) + y_3(s), \end{aligned}$$

where t_0, t_1, w_0, w_1 , and y_0, y_1 are double series, and the others simple series. Simple series can be treated by using the formulas appearing at the end of Section 5. In order to deal with the double series, applying the spectral decomposition Theorem 5.1, the derivative at zero of t_0, t_1, w_0, w_1 , and y_0, y_1 decomposes into two parts, called regular and singular contribution (see Remark 5.1), and this gives the following decomposition of the analytic torsion (in the following we assume $l_1 > 0$),

$$\begin{aligned} \log T_{\text{abs,ideal}}(C_l(W)) &= t'_{0,\text{reg}}(0) + t'_{1,\text{reg}}(0) + t'_{0,\text{sing}}(0) + t'_{1,\text{sing}}(0) + t'_2(0) + t'_3(0), \\ \log T_{\text{mixed}}(C_{[l_1, l_2]}(W)) &= w'_{0,\text{reg}}(0) + w'_{1,\text{reg}}(0) + w'_{0,\text{sing}}(0) + w'_{1,\text{sing}}(0) + w'_2(0) + w'_3(0), \\ \log T_{\text{abs}}(C_{[l_1, l_2]}(W)) &= y_{0,\text{reg}}^{(m)}(s) + y_{1,\text{reg}}^{(m)}(s) + y_{0,\text{sing}}^{(m)}(s) + y_{1,\text{sing}}^{(m)}(s) + y_2^{(m)}(s) + y_3^{(m)}(s). \end{aligned}$$

On the other side, for the frustum, that is a manifold, we also have the decomposition

$$\begin{aligned} \log T_{\text{mixed}}(C_{[l_1, l_2]}(W)) &= \log \tau(C_{[l_1, l_2]}(W), \{l_1\} \times W) \\ &\quad + \frac{1}{4} \chi(\partial C_{[l_1, l_2]}(W)) \log 2 + A_{\text{BM, mixed}}(\partial C_{[l_1, l_2]}(W)), \\ \log T_{\text{abs}}(C_{[l_1, l_2]}(W)) &= \log \tau(C_{[l_1, l_2]}(W)) \\ &\quad + \frac{1}{4} \chi(\partial C_{[l_1, l_2]}(W)) \log 2 + A_{\text{BM, abs}}(\partial C_{[l_1, l_2]}(W)). \end{aligned}$$

We will prove in Section 9 that

$$(6.1) \quad \begin{aligned} \log \tau(C_{[l_1, l_2]}(W), \{l_1\} \times W) + \frac{1}{2} \chi(W) \log 2 &= w'_{0, \text{reg}}(0) + w'_{1, \text{reg}}(0) + w'_2(0) + w'_3(0), \\ A_{\text{BM, mixed}}(\partial C_{[l_1, l_2]}(W)) &= w'_{0, \text{sing}}(0) + w'_{1, \text{sing}}(0), \end{aligned}$$

$$(6.2) \quad \begin{aligned} \log \tau(C_{[l_1, l_2]}(W)) + \frac{1}{2} \chi(W) \log 2 &= y'_{0, \text{reg}}(0) + y'_{1, \text{reg}}(0) + y'_2(0) + y'_3(0), \\ A_{\text{BM, abs}}(\partial C_{[l_1, l_2]}(W)) &= y'_{0, \text{sing}}(0) + y'_{1, \text{sing}}(0). \end{aligned}$$

This suggests to introduce a similar decomposition for the cone

$$\log T_{\text{abs, ideal}}(C_l(W)) = \log T_{\text{global}}(C_l(W)) + \frac{1}{4} \chi(\partial C_l(W)) \log 2 + A_{\text{BM, abs}}(\partial C_l(W)),$$

with

$$(6.3) \quad \begin{aligned} \log T_{\text{global}}(C_l(W)) + \frac{1}{4} \chi(\partial C_l(W)) \log 2 &= t'_{0, \text{reg}}(0) + t'_{1, \text{reg}}(0) + t'_2(0) + t'_3(0), \\ A_{\text{BM, abs}}(\partial C_l(W)) &= t'_{0, \text{sing}}(0) + t'_{1, \text{sing}}(0). \end{aligned}$$

We will call the first term, namely global plus Euler, the *regular part* of the torsion, and the second term the *anomaly boundary term*. We will prove in Section 10 that the notation is justified by the fact that $t'_{0, \text{sing}}(0) + t'_{1, \text{sing}}(0)$ coincides with the term that we would obtain if we would apply the formula of Brüning and Ma for the cone as if it were a regular manifold.

7. CALCULATIONS I: APPLICATION OF DEFINITION 5.1

In both the cases of the cone and of the frustum, we prove that the double series are spectrally decomposable according to Definition 5.1 (see detail in [10]) on the same simple sequence, that we discuss now. The simple sequence is $U_q = \{m_{q, n} : \mu_{q, n}\}_{n=1}^{\infty}$. This is a totally regular sequence of spectral type with infinite order, $\mathbf{e}(U_q) = \mathbf{g}(U_q) = m = \dim W$, and the associated zeta functions is

$$\zeta(s, U_q) = \zeta_{\text{cex}} \left(\frac{s}{2}, \tilde{\Delta}^{(q)} + \alpha_q^2 \right).$$

The possible poles of $\zeta(s, U_q)$ are at $s = m - h$, $h = 0, 2, 4, \dots$, and the residues are completely determined by the residues of the function $\zeta_{\text{cex}}(s, \tilde{\Delta}^{(q)})$ [10, Lemma 5.2].

We give in this section complete calculations for the cone and for the frustum with mixed BC, we omit the calculations for the frustum with absolute BC that are very similar to ones for the frustum with mixed BC.

7.1. **The cone.** We consider the double series

$$Z_q(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_{q,n},k}^{-2s}, \quad \hat{Z}_{q,\pm}(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\pm\alpha_q,k}^{-2s},$$

appearing in Lemma 4.1. These are the zeta functions associated to the double sequences $S_q = \{m_{\text{cex},q,n} : j_{\mu_{q,n},k}^2\}$ and $\hat{S}_{q,\pm} = \{m_{\text{cex},q,n} : \hat{j}_{\mu_{q,n},\pm\alpha_q,k}^2\}$. It is easy to see that these sequences have power $\kappa = 2$. By Definition 5.1, the relevant spectral function associated to S_q is the function (compare with equation (5.1)):

$$\begin{aligned} \log \Gamma(-\lambda, S_q) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{j_{\mu_{q,n},k}^2}\right) \\ &= -\log I_{\mu_{q,n}}(\sqrt{-\lambda}) + \mu_{q,n} \log \sqrt{-\lambda} - \mu_{q,n} \log 2 - \log \Gamma(\mu_{q,n} + 1), \end{aligned}$$

and the relevant spectral function associated to $\hat{S}_{q,\pm}$ is the function:

$$\begin{aligned} \log \Gamma(-\lambda, \hat{S}_{q,\pm}) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{j}_{\mu_{q,n},\pm\alpha_q,k}^2}\right) \\ &= -\log \hat{I}_{\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda}) + \mu_{q,n} \log \sqrt{-\lambda} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right), \end{aligned}$$

where (for $-\pi < \arg(z) \leq \frac{\pi}{2}$)

$$\hat{I}_{\nu,c}(z) = e^{-\frac{\pi}{2}i\nu} \hat{J}_{\nu,c}(iz).$$

First, we need uniform asymptotic expansions of the function $\log \Gamma$ for large n . Such expansions can be obtained from those of the Bessel function (see [10, Section 5.1] for details). For large n , uniformly in λ , (compare with equation (5.2))

$$\begin{aligned} \log \Gamma(-\lambda, S_q/\mu_{q,n}^2) &= \mu_{q,n} \log \mu_{q,n} \sqrt{-\lambda} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n} + 1) - \mu_{q,n} \sqrt{1-\lambda} \\ (7.1) \quad &- \mu_{q,n} \log \sqrt{-\lambda} + \mu_{q,n} \log(1 + \sqrt{1-\lambda}) + \frac{1}{2} \log 2\pi\mu_{q,n} + \frac{1}{4} \log(1-\lambda) \\ &+ \sum_{j=1}^m \frac{\phi_{q,j}(\lambda)}{\mu_{q,n}^j} + O(\mu_{q,n}^{-2p}), \end{aligned}$$

where $\phi_{q,j}(\lambda)$ is the coefficient of $\mu_{q,n}^{-j}$ in the asymptotic expansion of (for the definition of the function U and W see Lemma 7.1 below)

$$\log \left(1 + \sum_{k=1}^{\infty} U_k(\lambda) \mu_{q,n}^{-k}\right),$$

for large $\mu_{q,n}$, i.e.

$$(7.2) \quad \log \left(1 + \sum_{k=1}^{\infty} U_k(\lambda) \mu_{q,n}^{-k}\right) = \sum_{j=1}^{\infty} \phi_{q,j}(\lambda) \mu_{q,n}^{-j}.$$

and

$$(7.3) \quad \begin{aligned} \log \Gamma(-\lambda, \hat{S}_{q,\pm}/\mu_{q,n}^2) &= \mu_{q,n} \log \mu_{q,n} \sqrt{-\lambda} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n} + 1) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right) \\ &\quad - \mu_{q,n} \sqrt{1-\lambda} - \mu_{q,n} \log \sqrt{-\lambda} + \mu_{q,n} \log(1 + \sqrt{1-\lambda}) \\ &\quad + \frac{1}{2} \log 2\pi \mu_{q,n} - \frac{1}{4} \log(1-\lambda) + \sum_{j=0}^m \frac{\hat{\psi}_{q,j,\pm}(\lambda)}{\mu_{q,n}^j} + O(\mu_{q,n}^{-2p}), \end{aligned}$$

where $\hat{\psi}_{q,j,\pm}(\lambda)$ is the coefficient of $\mu_{q,n}^{-j}$ in the asymptotic expansion of

$$\log \left(1 + \sum_{k=1}^{\infty} W_{\pm\alpha_q,k}(\lambda) \mu_{q,n}^{-k} \right),$$

for large $\mu_{q,n}$, i.e.

$$\log \left(1 + \sum_{k=1}^{\infty} W_{\pm\alpha_q,k}(\lambda) \mu_{q,n}^{-k} \right) = \sum_{j=1}^{\infty} \hat{\psi}_{q,j,\pm}(\lambda) \mu_{q,n}^{-j},$$

therefore the complete coefficient of $\mu_{q,n}^{-j}$ in $\log \Gamma(-\lambda, \hat{S}_{q,\pm}/\mu_{q,n}^2)$ is

$$(7.4) \quad \hat{\phi}_{q,j,\pm}(\lambda) = \hat{\psi}_{q,j,\pm}(\lambda) + \frac{(-1)^{j+1}}{j} (\pm\alpha_q)^j.$$

Second, we need asymptotic expansions of the function $\log \Gamma$ and of the functions $\phi_{q,j}$ and $\hat{\phi}_{q,j,\pm}$ for large λ . The expansion of $\log \Gamma$ can be obtained from those of the Bessel function [10, pg.s 641, 642] (compare with equation (5.4))

$$(7.5) \quad \begin{aligned} \log \Gamma(-\lambda, S_q/\mu_{q,n}^2) &= \frac{1}{2} \log 2\pi + \left(\mu_{q,n} + \frac{1}{2} \right) \log \mu_{q,n} - \mu_{q,n} \log 2 \\ &\quad - \log \Gamma(\mu_{q,n} + 1) + \frac{1}{2} \left(\mu_{q,n} + \frac{1}{2} \right) \log(-\lambda) + O(e^{-\mu_{q,n} \sqrt{-\lambda}}), \end{aligned}$$

$$(7.6) \quad \begin{aligned} \log \Gamma(-\lambda, \hat{S}_{q,\pm}/\mu_{q,n}^2) &= \mu_{q,n} \sqrt{-\lambda} + \frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) \\ &\quad + \frac{1}{2} \left(\mu_{q,n} - \frac{1}{2} \right) \log(-\lambda) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right) + O(e^{-\mu_{q,n} \sqrt{-\lambda}}). \end{aligned}$$

About the functions $\phi_{q,j}(\lambda)$ and $\hat{\phi}_{q,j,\pm}(\lambda)$ we have the following facts.

Lemma 7.1. *For $j = 0$,*

$$\begin{aligned} \phi_{q,0}(\lambda) &= \frac{1}{2} \log 2\pi + \frac{1}{4} \log(1-\lambda), \\ \hat{\phi}_{q,0,\pm}(\lambda) &= \frac{1}{2} \log 2\pi - \frac{1}{4} \log(1-\lambda). \end{aligned}$$

For $j > 0$, the functions $\phi_{q,j}(\lambda)$ and $\hat{\phi}_{q,j,\pm}(\lambda)$ are polynomial of order $3j$ in $\frac{1}{\sqrt{1-\lambda}}$, whose parity only depends on the index j , and with minimal monomial of degree j , and satisfy the following

formulas:

$$\begin{aligned} \left(2\phi_{q,2k-1}(\lambda) - \hat{\phi}_{q,2k-1,+}(\lambda) - \hat{\phi}_{q,2k-1,-}(\lambda)\right)\Big|_{\lambda=0} &= 0, \\ \left(\hat{\phi}_{q,2k,-}(\lambda) - \hat{\phi}_{q,2k,+}(\lambda)\right)\Big|_{\lambda=0} &= 0. \end{aligned}$$

Proof. The case $j = 0$ follows by inspection of the formulas in equations (7.1) and (7.3). Next, assume $j > 0$. Recall the definition of the functions $U_j(\lambda)$ and $V_j(\lambda)$ [17, (7.10) and Ex. 7.2]. Let

$$\begin{aligned} u_0(w) &= 1, & u_{j+1}(w) &= \frac{1}{2}w^2(1-w^2)u'_j(w) + \frac{1}{8}\int_0^w (1-5y^2)u_j(y)dy, \\ v_0(w) &= 1, & v_{j+1}(w) &= u_{j+1}(w) - \frac{1}{2}w(1-w^2)u_j(w) - w^2(1-w^2)u'(w), \end{aligned}$$

then

$$U_j(\lambda) = u_j\left(\frac{1}{\sqrt{1-\lambda}}\right), \quad V_j(\lambda) = v_j\left(\frac{1}{\sqrt{1-\lambda}}\right).$$

By these formulas, it is clear that the functions U_j and V_j are polynomials in $\frac{1}{\sqrt{1-\lambda}}$ with the stated properties. Since, [10, pg. 639]

$$W_{\pm\alpha_q,j}(\lambda) = V_j(\lambda) \pm \frac{\alpha_q}{\sqrt{1-\lambda}}U_{j-1}(\lambda),$$

the same is true for $W_j(\lambda)$. Whence the first part of the statement follows by the very definition of the functions $\phi_{q,j}(\lambda)$ and $\hat{\phi}_{q,j,\pm}(\lambda)$, since in the functions $\phi_{q,2k-1}(\lambda)$ and $\hat{\phi}_{q,2k-1,\pm}(\lambda)$ only U_{2k-1} and V_{2k-1} appear, and in the $\phi_{q,2k}(\lambda)$ and $\hat{\phi}_{q,2k,\pm}(\lambda)$ only U_{2k} and V_{2k} appear. Next consider the situation in more details: by definition

$$\log\left(1 + \sum_{k=1}^{\infty} U_k(\lambda)\mu_{q,n}^{-k}\right) = \sum_{j=1}^{\infty} \phi_{q,j}(\lambda)\mu_{q,n}^{-j}, \quad \hat{\phi}_{q,j,\pm}(\lambda) = \hat{\psi}_{q,j,\pm}(\lambda) + \frac{(-1)^{j+1}}{j}(\pm\alpha_q)^j,$$

where

$$\log\left(1 + \sum_{k=1}^{\infty} \left(V_k(\lambda) \pm \frac{\alpha_q}{\sqrt{1-\lambda}}U_{k-1}(\lambda)\right)\mu_{q,n}^{-k}\right) = \sum_{j=1}^{\infty} \hat{\psi}_{q,j,\pm}(\lambda)\mu_{q,n}^{-j}.$$

Since, again by the very definition, $V_j(0) = U_j(0)$ for all j , by comparing the two formulas above, we realize that, if $\lambda = 0$, in each odd term of the expansion of $2\phi_{q,2k-1}(\lambda) - \hat{\phi}_{q,2k-1,+}(\lambda) - \hat{\phi}_{q,2k-1,-}(\lambda)$ in $\mu_{q,n}$, i.e. in the coefficient of each $\mu_{q,n}^{j=2k-1}$, the two terms (coming from $\hat{\phi}_{q,j,\pm}(\lambda)$)

$$\pm \frac{\alpha_q}{\sqrt{1-\lambda}}U_{j-1}(\lambda) + \frac{(-1)^{j+1}}{j}(\pm\alpha_q)^j,$$

cancel each other, while the term (coming from $\phi_{q,j}(\lambda)$)

$$2U_k(\lambda),$$

will cancel out by the two terms (coming from $\hat{\phi}_{q,j,\pm}(\lambda)$)

$$-2V_k(\lambda).$$

This prove the first formula in the last statement. The proof of the second one is similar. \square

Lemma 7.2. *For $j > 0$, the following functions of $\Phi_{q,j}$ and $\hat{\Phi}_{q,j,\pm}$ are regular at $s = 0$:*

$$\begin{aligned} \operatorname{Res}_{s=0} \left(2\Phi_{q,2k-1}(s) - \hat{\Phi}_{q,2k-1,+}(s) - \hat{\Phi}_{q,2k-1,-}(s) \right) &= 0, \\ \operatorname{Res}_{s=0} \left(\hat{\Phi}_{q,2k,-}(s) - \hat{\Phi}_{q,2k,+}(s) \right) &= 0. \end{aligned}$$

Proof. The argument is the same for the functions $\hat{\Phi}$ and the function Φ , so just consider the last ones. By Lemma 7.1

$$\phi_{q,j}(\lambda) = \sum_{k=j}^{3j} c_k \frac{1}{(1-\lambda)^{\frac{k}{2}}},$$

and since (see [20])

$$\int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \frac{1}{(1-\lambda)^{\frac{k}{2}}} d\lambda dt = \begin{cases} 0, & k = 0, \\ \frac{\Gamma(s+k)}{\Gamma(k)s}, & k \neq 0, \end{cases}$$

we have that

$$\Phi_{q,j}(s) = \sum_{k=1}^{3j} c_k \frac{\Gamma(s+k)}{\Gamma(k)s},$$

and this means that

$$\begin{aligned} \operatorname{Res}_{s=0} \Phi_{q,2h-1}(s) &= \sum_{k=j}^{3(2h-1)} c_k = \phi_{q,2h-1}(0), \\ \operatorname{Res}_{s=0} \Phi_{q,2h}(s) &= \sum_{k=j}^{6h} c_k = \phi_{q,2h}(0). \end{aligned}$$

□

Applying the definition in equation (5.5), we see that all relevant $b_{\sigma_h,0,0/1}$ and $\hat{b}_{\sigma_h,0,0/1}$ vanishes. For $j = \sigma_h = m - h$, with $h = 0, 2, 4, \dots$, $h \neq m$; and therefore when $j = 0$, $b_{0,0,0/1}$ and $\hat{b}_{0,0,0/1}$ do not appears in the sums, while when $j > 0$ there are neither logarithmic terms nor constant terms in the expansion for large λ of the functions $\phi_{q,j}(\lambda)$ and $\hat{\phi}_{q,j,\pm}(\lambda)$ by the previous lemmas. Thus,

$$(7.7) \quad \begin{aligned} A_{0,0,q}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} a_{0,0,n,q} \mu_{q,n}^{-2s}, & A_{0,1,q}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} a_{0,1,n,q} \mu_{q,n}^{-2s}, \\ \hat{A}_{0,0,q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \hat{a}_{0,0,n,q,\pm} \mu_{q,n}^{-2s}, & \hat{A}_{0,1,q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \hat{a}_{0,1,n,q,\pm} \mu_{q,n}^{-2s}. \end{aligned}$$

where $a_{0,0,n,q}$ and $a_{0,1,n,q}$ are the constant and the logarithmic term in the expansion in equation (7.5), and $\hat{a}_{0,0,n,q,\pm}$ and $\hat{a}_{0,1,n,q,\pm}$ are the constant and the logarithmic term in the expansion in equation (7.6).

Third, concerning the singular term, the functions defined in equation (5.3), are

$$(7.8) \quad \begin{aligned} \Phi_{q,j}(s) &= \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \phi_{q,j}(\lambda) d\lambda dt, \\ \hat{\Phi}_{q,j,\pm}(s) &= \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \hat{\phi}_{q,j,\pm}(\lambda) d\lambda dt, \end{aligned}$$

where the ϕ and the $\hat{\phi}$ are given in equations (7.2) and (7.4), respectively.

7.2. The frustum with mixed BC. Consider the double series

$$\hat{D}_{q,\pm}(s; l_1, l_2) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{f}_{\mu_{q,n},\pm\alpha_q,k}^{-2s}(l_1, l_2),$$

associated to the double sequence $\hat{\Theta}_{q,\pm}(l_1, l_2) = \{m_{\text{cex},q,n} : \hat{f}_{\mu_{q,n},\pm\alpha_q,k}^{-2s}(l_1, l_2)\}$. It is easy to see that this sequence as power $\kappa = 2$. By Definition 5.1, the relevant spectral function associated to $\hat{\Theta}_{q,\pm}(l_1, l_2)$ is the function (compare with equation (5.1)):

$$\begin{aligned} \log(-\lambda; \hat{\Theta}_{q,\pm}(l_1, l_2)) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{-\lambda}{\hat{f}_{\mu_{q,n},\pm\alpha_q,k}^2(l_1, l_2)} \right) \\ &= -\log \hat{G}_{\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda}; l_1, l_2) \\ &\quad + \log \frac{1}{\pi} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right), \end{aligned}$$

where (for $-\pi < \arg(z) \leq \frac{\pi}{2}$)

$$\hat{G}_{\mu,c}(z; l_1, l_2) = \hat{F}_{\mu,c}(iz; l_1, l_2).$$

We need (uniform) asymptotic expansions of the function $\log \Gamma$ for large n and for large λ . Such expansions can be obtained from those of the Bessel function. A long calculation gives the following results. For large n , uniformly in λ , (compare with equation (5.2))

$$\begin{aligned} \log(-\lambda; \hat{\Theta}_{q,\pm}(l_1, l_2)/\mu_{q,n}^2) &= -\mu_{q,n} \left(\sqrt{1 - l_2^2 \lambda} - \sqrt{1 - l_1^2 \lambda} \right) - \mu_{q,n} \log \frac{l_2(1 + \sqrt{1 - l_1^2 \lambda})}{l_1(1 + \sqrt{1 - l_2^2 \lambda})} \\ &\quad - \frac{1}{4} \log \frac{(1 - l_1^2 \lambda)}{(1 - l_2^2 \lambda)} - \sum_{j=1}^m \frac{\hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2)}{\mu_{q,n}^j} \\ &\quad + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) + O(\mu_{q,n}^{1-m}), \end{aligned}$$

where $\hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2)$ is the coefficient of $\mu_{q,n}^{-j}$ in the asymptotic expansion for large $\mu_{q,n}$, i.e. of

$$\log \left(1 + \sum_{k=1}^{\infty} \Psi_{q,k,\pm}(l_1, l_2) \mu_{q,n}^{-k} \right),$$

where

$$\begin{aligned} \Psi_{q,k,\pm}(l_1, l_2) &= \text{sgn}(l_1 - l_2)^k U_k(l_1 \sqrt{-\lambda}) + \text{sgn}(l_2 - l_1)^k W_{\pm \text{sgn}(l_2 - l_1) \alpha_q, k}(l_2 \sqrt{-\lambda}) \\ &\quad + \sum_{h=1}^{k-1} \text{sgn}(l_1 - l_2)^h \text{sgn}(l_2 - l_1)^{k-h} U_h(l_1 \sqrt{-\lambda}) W_{\pm \text{sgn}(l_2 - l_1) \alpha_q, k-h}(l_2 \sqrt{-\lambda}). \end{aligned}$$

In other words,

$$\log \left(1 + \sum_{k=1}^{\infty} \Psi_{q,k,\pm}(l_1, l_2) \mu_{q,n}^{-k} \right) = \sum_{j=1}^{\infty} \hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2) \mu_{q,n}^{-j},$$

therefore the complete coefficient of $\mu_{q,n}^{-j}$ in $\log \Gamma(-\lambda, \hat{\Theta}_{q,\pm}(l_1, l_2) / \mu_{q,n}^2)$ is

$$(7.9) \quad \hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2) = \hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2) + \frac{(-1)^{j+1}}{j} (\operatorname{sgn}(l_2 - l_1) \alpha_q)^j.$$

For large λ , (compare with equation (5.4))

$$(7.10) \quad \begin{aligned} \log \Gamma(-\lambda, \hat{S}_{\mu_{q,n}, \pm \alpha_q, k}(l_1, l_2) / \mu_{q,n}^2) &= -\mu_{q,n}(l_2 - l_1) \sqrt{-\lambda} - \frac{1}{2} \log \frac{l_2}{l_1} \\ &+ \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) \\ &+ O \left(\frac{1}{\sqrt{-\lambda}} \right) \end{aligned}$$

Next consider the functions $\hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2)$. By the following lemma we see that all the results given in Lemmas 7.1 and 7.2 hold for these functions.

Lemma 7.3. *The following relation holds:*

$$\hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2) = \operatorname{sgn}(l_1 - l_2)^j \phi_{q,j}(l_1^2 \lambda) + \operatorname{sgn}(l_2 - l_1)^j \hat{\phi}_{q,j,\pm \operatorname{sgn}(l_2 - l_1)}(l_2^2 \lambda).$$

Proof. By definition in equations (7.2) and (7.4)

$$\log \left(1 + \sum_{k=1}^{\infty} U_k(\sqrt{-\lambda}) \mu_{q,n}^{-k} \right) = \sum_{j=1}^{\infty} \phi_{q,j}(\lambda) \mu_{q,n}^{-j}$$

and

$$\hat{\phi}_{q,j,\pm}(\lambda) = \hat{\psi}_{q,j,\pm}(\lambda) \pm \frac{(-1)^{j+1}}{j} \alpha_q,$$

where

$$\log \left(1 + \sum_{k=1}^{\infty} W_{\pm \alpha_q, k}(\sqrt{-\lambda}) \mu_{q,n}^{-k} \right) = \sum_{j=1}^{\infty} \hat{\psi}_{q,j,\pm}(\lambda) \mu_{q,n}^{-j},$$

while by equation (7.9)

$$\hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2) = \hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2) + \frac{(-1)^{j+1}}{j} (\operatorname{sgn}(l_2 - l_1) \alpha_q)^j,$$

where

$$\log \left(1 + \sum_{k=1}^{\infty} \Psi_{q,k,\pm}(l_1, l_2) \mu_{q,n}^{-k} \right) = \sum_{j=1}^{\infty} \hat{\psi}_{q,j,\pm}(\lambda; l_1, l_2) \mu_{q,n}^{-j}$$

with

$$\begin{aligned} \Psi_{q,k,\pm}(l_1, l_2) &= \operatorname{sgn}(l_1 - l_2)^k U_k(l_1 \sqrt{-\lambda}) + \operatorname{sgn}(l_2 - l_1)^k W_{\pm \operatorname{sgn}(l_2 - l_1) \alpha_q, k}(l_2 \sqrt{-\lambda}) \\ &+ \sum_{h=1}^{k-1} \operatorname{sgn}(l_1 - l_2)^h \operatorname{sgn}(l_2 - l_1)^{k-h} U_h(l_1 \sqrt{-\lambda}) W_{\pm \operatorname{sgn}(l_2 - l_1) \alpha_q, k-h}(l_2 \sqrt{-\lambda}). \end{aligned}$$

Now,

$$\begin{aligned} \log \left(1 + \sum_{k=1}^{\infty} \Psi_{q,k,\pm}(l_1, l_2) \mu_{q,n}^{-k} \right) &= \log \left(1 + \sum_{k=1}^{\infty} \operatorname{sgn}(l_1 - l_2)^k U_k(l_1 \sqrt{-\lambda}) \mu_{q,n}^{-k} \right) \\ &\quad + \log \left(1 + \sum_{k=1}^{\infty} \operatorname{sgn}(l_2 - l_1)^k W_{\pm \operatorname{sgn}(l_2 - l_1) \alpha_q, k}(l_2 \sqrt{-\lambda}) \mu_{q,n}^{-k} \right), \end{aligned}$$

and hence the thesis follows by the very definition. \square

This means that the $b_{\sigma_h, j, k}$ in equation (5.5) are all disappearing, and hence

$$\begin{aligned} A_{0,0,\pm}(s; l_1, l_2) &= \sum_{n=1}^{\infty} m_{\text{cex}, q, n} a_{0,0,n,\pm}(l_1, l_2) \mu_{q,n}^{-2s}, \\ A_{0,1,\pm}(s; l_1, l_2) &= \sum_{n=1}^{\infty} m_{\text{cex}, q, n} a_{0,1,n,\pm}(l_1, l_2) \mu_{q,n}^{-2s}, \end{aligned}$$

where $a_{0,0,n,\pm}(l_1, l_2)$ and $a_{0,1,n,\pm}(l_1, l_2)$ are the constant and the logarithmic term in the expansion in equation (7.10).

Eventually, concerning the singular term, the functions defined in equation (5.3) are

$$(7.11) \quad \hat{\Phi}_{q,j,\pm}(s; l_1, l_2) = \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2) d\lambda dt,$$

where the $\hat{\phi}$ are given in equation (7.9).

8. CALCULATION II: APPLICATION OF THEOREM 5.1

We now apply Theorem 5.1 and its corollary to compute the derivative at $s = 0$ of the functions Z , D , and E appearing in Lemmas 4.1, 4.2, and 4.3. According to Remark 5.1 we split the result in the regular and singular parts, denoted by the obvious subscript. In order to improve readability we will use in this section the simplified notation A and B for the two terms $A_{0,0}$ and $A_{0,1}$ defined in equation (5.5). We observe that, in all cases, when computing the regular part, all the terms that are equal in $B_q(0)$ and $\hat{B}_{q,\pm}(0)$ and in $A_q(0)$ and $\hat{A}_{q,\pm}(0)$ respectively, cancel in the final formula for $t_{0,\text{reg}}^{(m)}(0)$, and similarly for the others. We thus introduce some regularized terms, denoted by a slashed letter, where only the relevant parts appear.

8.1. The cone. We split the calculation into three parts: the regular contribution, the singular contribution and the contribution of the harmonics. We give all details for the cone.

8.1.1. The contribution of the regular part. By Theorem 5.1, we have

$$\begin{aligned} Z_{q,\text{reg}}(0) &= -B_q(0), & Z'_{q,\text{reg}}(0) &= -A_q(0) - B'_q(0), \\ \hat{Z}_{q,\pm,\text{reg}}(0) &= -\hat{B}_{q,\pm}(0), & \hat{Z}'_{q,\pm,\text{reg}}(0) &= -\hat{A}_{q,\pm}(0) - \hat{B}'_{q,\pm}(0), \end{aligned}$$

where, by equation (7.7),

$$\begin{aligned}
A_q(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) \right) \mu_{q,n}^{-2s}, \\
B_q(s) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} + \frac{1}{2} \right) \mu_{q,n}^{-2s}, \\
B'_q(s) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} + \frac{1}{2} \right) \mu_{q,n}^{-2s} \log \mu_{q,n}, \\
\hat{A}_{q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right) \right) \mu_{q,n}^{-2s}, \\
\hat{B}_{q,\pm}(s) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} - \frac{1}{2} \right) \mu_{q,n}^{-2s}, \\
\hat{B}'_{q,\pm}(s) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} - \frac{1}{2} \right) \mu_{q,n}^{-2s} \log \mu_{q,n}.
\end{aligned}$$

This gives (in the following formulas we are taking the finite part)

$$\begin{aligned}
A_q(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) \right), \\
B_q(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} + \frac{1}{2} \right), \\
B'_q(0) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} + \frac{1}{2} \right) \log \mu_{q,n}, \\
\hat{A}_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right) \right), \\
\hat{B}_{q,\pm}(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} - \frac{1}{2} \right), \\
\hat{B}'_{q,\pm}(0) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n}.
\end{aligned}$$

Thus, considering first $t_{0,\text{reg}}^{(m)}$, by Lemma 4.1, we have

$$\begin{aligned}
t_{0,\text{reg}}^{(m)'}(0) &= \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((Z_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0)) + (-1)^{m-1} (Z_{q,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0)) \right) \log l, \\
&\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((Z'_{q,\text{reg}}(0) - \hat{Z}'_{q,+,\text{reg}}(0)) + (-1)^{m-1} (Z'_{q,\text{reg}}(0) - \hat{Z}'_{q,-,\text{reg}}(0)) \right).
\end{aligned}$$

and hence

$$\begin{aligned}
t_{0,\text{reg}}^{(m)'}(0) &= - \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((B_q(0) - \hat{B}_{q,+}(0)) + (-1)^{m-1} (B_q(0) - \hat{B}_{q,-}(0)) \right) \log l \\
&\quad - \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(A_q(0) + B'_q(0) - \hat{A}_{q,+}(0) - \hat{B}'_{q,+}(0) \right) \\
&\quad - (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(A_q(0) + B'_q(0) - \hat{A}_{q,-}(0) - \hat{B}'_{q,-}(0) \right).
\end{aligned}$$

We observe that all the terms that are equal in $B_q(0)$ and $\hat{B}_{q,\pm}(0)$ and in $A_q(0)$ and $\hat{A}_{q,\pm}(0)$ respectively, cancel in the final formula for $t_{0,\text{reg}}^{(m)}(0)$. This is because the final formulas are

$$\begin{aligned}
(Z'_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0)) + (Z'_{q,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0)) &= 2Z'_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0), \\
(Z_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0)) + (Z_{q,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0)) &= 2Z_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0),
\end{aligned}$$

if $m = 2p - 1$ is odd, and

$$\begin{aligned}
(Z'_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0)) - (Z'_{q,\text{reg}}(0) - \hat{Z}'_{q,-,\text{reg}}(0)) &= -\hat{Z}'_{q,+,\text{reg}}(0) + \hat{Z}'_{q,-,\text{reg}}(0), \\
(Z_{q,\text{reg}}(0) - \hat{Z}_{q,+,\text{reg}}(0)) - (Z_{q,\text{reg}}(0) - \hat{Z}_{q,-,\text{reg}}(0)) &= -\hat{Z}_{q,+,\text{reg}}(0) + \hat{Z}_{q,-,\text{reg}}(0),
\end{aligned}$$

if $m = 2p$ is even. Therefore, we can rewrite

$$\begin{aligned}
t_{0,\text{reg}}^{(m)'}(0) &= - \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\mathbb{B}_q(0) - \hat{\mathbb{B}}_{q,+}(0)) + (-1)^{m-1} (\mathbb{B}_q(0) - \hat{\mathbb{B}}_{q,-}(0)) \right) \log l \\
&\quad - \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathbb{A}_q(0) + \mathbb{B}'_q(0) - \hat{\mathbb{A}}_{q,+}(0) - \hat{\mathbb{B}}'_{q,+}(0) \right) \\
&\quad - (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathbb{A}_q(0) + \mathbb{B}'_q(0) - \hat{\mathbb{A}}_{q,-}(0) - \hat{\mathbb{B}}'_{q,-}(0) \right).
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{A}_q(0) &= 0, \\
\mathbb{B}_q(0) &= \frac{1}{4} \sum_{n=1}^{\infty} m_{\text{cex},q,n} = \frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2) = \frac{1}{4} \zeta(0, \tilde{\Delta}^{(q)}), \\
\mathbb{B}'_q(0) &= -\frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \\
\hat{\mathbb{A}}_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right), \\
\hat{\mathbb{B}}_{q,\pm}(0) &= -\frac{1}{4} \sum_{n=1}^{\infty} m_{\text{cex},q,n} = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2) = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}), \\
\hat{\mathbb{B}}'_{q,\pm}(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n},
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} m_{\text{cex},q,n} = \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2), \quad -2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} = \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2).$$

Thus,

$$\begin{aligned}
(8.1) \quad t_{0,\text{reg}}^{(m)'}(0) &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + (-1)^{m-1} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \right) \log l \\
&\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 + \frac{\alpha_q}{\mu_{q,n}} \right) + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right) \\
&\quad + (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 - \frac{\alpha_q}{\mu_{q,n}} \right) + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right).
\end{aligned}$$

Next, consider $t_{1,\text{reg}}^{(2p-1)}$. By Lemma 4.1, we have

$$t_{1,\text{reg}}^{(2p-1)'}(0) = -(-1)^{p-1} \left(B_{p-1}(0) - \hat{B}_{p-1,0}(0) \right) \log l - \frac{1}{2} \left(A_{p-1}(0) + B'_{p-1}(0) - \hat{A}_{p-1,0}(0) - \hat{B}'_{p-1,0}(0) \right).$$

As above, all the terms that are equal in $B_q(0)$ and $\hat{B}_{q,0}(0)$ and in $A_q(0)$ and $\hat{A}_{q,0}(0)$ respectively, cancel in the final formula for $t_{1,\text{reg}}^{(2p-1)}(0)$. Therefore, we can rewrite

$$t_{1,\text{reg}}^{(2p-1)'}(0) = -(-1)^{p-1} \left(\mathbb{B}_{p-1}(0) - \hat{\mathbb{B}}_{p-1,0}(0) \right) \log l - \frac{1}{2} \left(\mathbb{A}_{p-1}(0) + \mathbb{B}'_{p-1}(0) - \hat{\mathbb{A}}_{p-1,0}(0) - \hat{\mathbb{B}}'_{p-1,0}(0) \right).$$

where

$$\begin{aligned}\hat{A}_{p-1,0}(0) &= 0, \\ \hat{B}_{p-1,0}(0) &= -\frac{1}{4} \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)} + \alpha_{p-1}^2) = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}), \\ \hat{B}'_{p-1,0}(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \log \mu_{p-1,n}.\end{aligned}$$

Observing that $\alpha_{p-1} = \frac{1}{2}(1 + 2p - 2 - 2p + 1) = 0$, $\mu_{p-1,n} = \lambda_{p-1,n}$, and hence

$$\zeta_{\text{cex}}(s, \tilde{\Delta}^{(p-1)} + \alpha_{p-1}^2) = \zeta_{\text{cex}}(s, \tilde{\Delta}^{(p-1)}) = \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \lambda_{p-1,n}^{-2s},$$

and

$$\hat{B}'_{p-1,0}(0) = \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \log \lambda_{p-1,n} = -\frac{1}{4} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).$$

Thus,

$$(8.2) \quad t_{1,\text{reg}}^{(2p-1)'}(0) = -(-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \log l - \frac{1}{4} (-1)^{p-1} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).$$

8.1.2. *The contribution of the singular part.* By Lemma 7.2, the functions $\Phi_{q,2j-1}$ and $\Phi_{q,2j-1,\pm}$ are regular at $s = 0$ when m is odd, and the functions $\Phi_{q,2j}$ and $\Phi_{q,2j,\pm}$ are regular at $s = 0$ when m is even. By the location of the poles of $\zeta(s, U)$ given at the beginning of Section 7, it follows that the formula for the singular part in Theorem 5.1 reduces to the single sums (recall $\kappa = 2$ and the zeta of U is regular at $s = 0$)

$$\begin{aligned}Z_{q,\text{sing}}(0) &= 0, & \hat{Z}_{q,\pm,\text{sing}}(0) &= 0, \\ Z'_{q,\text{sing}}(0) &= \frac{1}{2} \sum_{j=1}^m \Phi_{q,j}(0) \text{Res}_1 \zeta(s, U), & \hat{Z}'_{q,\pm,\text{sing}}(0) &= \frac{1}{2} \sum_{j=1}^m \hat{\Phi}_{q,j,\pm}(0) \text{Res}_1 \zeta(s, U).\end{aligned}$$

where the Φ are defined in equation (7.8). Thus, considering first $t_{0,\text{sing}}^{(m)}$, by Lemma 4.1, we have

$$t_{0,\text{sing}}^{(m)'}(0) = \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((Z'_{q,\text{sing}}(0) - \hat{Z}'_{q,+,\text{sing}}(0)) + (-1)^{m-1} (Z'_{q,\text{sing}}(0) - \hat{Z}'_{q,-,\text{sing}}(0)) \right),$$

and hence

$$t_{0,\text{sing}}^{(m)'}(0) = \frac{1}{4} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \sum_{j=1}^m \left((\Phi_{q,j}(0) - \hat{\Phi}_{q,j,+}(0)) + (-1)^{m-1} (\Phi_{q,j}(0) - \hat{\Phi}_{q,j,-}(0)) \right) \text{Res}_1 \zeta(s, U),$$

Second, consider $t_{1,\text{sing}}^{(2p-1)'}$ as given in Lemma 4.1. We have

$$t_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{2} \left(Z'_{p-1,\text{sing}}(0) - \hat{Z}'_{p-1,0,\text{sing}}(0) \right),$$

and hence

$$t_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{4} \sum_{j=1}^m \left(\Phi_{p-1,j}(0) - \hat{\Phi}_{p-1,j,0}(0) \right) \text{Res}_1 \zeta(s, U).$$

8.1.3. *The contribution of the harmonics.* The contribution of the harmonics can be computed using the formula in equation (5.7) at the end of Section 5. By the definition in Lemma 4.1, since

$$z_{q,\pm}(s) = z(s, \pm\alpha_q, 0, 1),$$

we have

$$t_2^{(m)}(s) = \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (z(s, -\alpha_{q-1}, 0, 1) + (-1)^m z(s, -\alpha_q, 0, 1));$$

by equations (5.7) and (5.6),

$$\begin{aligned} t_2^{(m)'}(0) &= \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (z(0, -\alpha_{q-1}, 0, 1) + (-1)^m z(0, -\alpha_q, 0, 1)) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (z'(0, -\alpha_{q-1}, 0, 1) + (-1)^m z'(0, -\alpha_q, 0, 1)) \\ (8.3) \quad &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} \left(\left(-\alpha_{q-1} + \frac{1}{2} \right) + (-1)^m \left(-\alpha_q + \frac{1}{2} \right) \right) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} \left(\log \frac{2^{-\alpha_{q-1}-\frac{1}{2}} \Gamma(-\alpha_{q-1} + 1)}{\sqrt{\pi}} + (-1)^m \log \frac{2^{-\alpha_q-\frac{1}{2}} \Gamma(-\alpha_q + 1)}{\sqrt{\pi}} \right). \end{aligned}$$

On the other side,

$$t_3^{(2p)}(s) = (-1)^{p+1} \frac{l^{2s}}{4} m_{\text{har},p} (z(s, \alpha_p, 0, 1) + z(s, -\alpha_p, 0, 1)).$$

Whence

$$\begin{aligned} t_3^{(2p)'}(0) &= (-1)^{p+1} m_{\text{har},p} \frac{1}{2} (z(0, \alpha_p, 0, 1) + z(0, -\alpha_p, 0, 1)) \log l \\ &\quad + (-1)^{p+1} m_{\text{har},p} \frac{1}{4} (z'(s, \alpha_p, 0, 1) + z'(s, -\alpha_p, 0, 1)) \\ &= -(-1)^{p+1} \frac{1}{4} m_{\text{har},p} \left(\left(\alpha_p + \frac{1}{2} \right) + \left(-\alpha_p + \frac{1}{2} \right) \right) \log l \\ &\quad + (-1)^{p+1} \frac{1}{4} m_{\text{har},p} \left(\log \frac{2^{\alpha_p-\frac{1}{2}} \Gamma(\alpha_p + 1)}{\sqrt{\pi}} + \log \frac{2^{-\alpha_p-\frac{1}{2}} \Gamma(-\alpha_p + 1)}{\sqrt{\pi}} \right) \\ &= (-1)^p \frac{1}{4} m_{\text{har},p} \log l + (-1)^{p+1} \frac{1}{4} m_{\text{har},p} \log \frac{\alpha_p}{2 \sin(\pi \alpha_p)}. \end{aligned}$$

Since $\alpha_p = \frac{1}{2}(1 + 2p - 2p) = \frac{1}{2}$, this gives

$$(8.4) \quad t_3^{(2p)'}(0) = (-1)^p \frac{1}{4} m_{\text{har},p} \log l + (-1)^p \frac{1}{2} m_{\text{har},p} \log 2.$$

8.2. **The frustum with mixed BC.** We omit the details, since the calculations are similar to the one performed for the cone. As above, we split into three parts.

8.2.1. *The contribution of the regular part.* According to Theorem 5.1. we have

$$\hat{D}'_{q,\pm,\text{reg}}(0; l_1, l_2) = -\hat{A}_{q,\pm}(0; l_1, l_2) - \hat{B}'_{q,\pm}(0; l_1, l_2),$$

where, according to equation (7.10)

$$\hat{A}_{\pm}(s; l_1, l_2) = \sum_{n=1}^{\infty} m_{q,\text{cex}} \left(-\frac{1}{2} \log \frac{l_2}{l_1} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) \right) \mu_{q,n}^{-2s},$$

$$\hat{B}_{\pm}(s; l_1, l_2) = 0,$$

and hence

$$\hat{A}_{\pm}(0; l_1, l_2) = \sum_{n=1}^{\infty} m_{q,\text{cex}} \left(-\frac{1}{2} \log \frac{l_2}{l_1} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) \right),$$

$$\hat{B}_{\pm}(0; l_1, l_2) = 0,$$

Now, note that

$$\hat{A}_-(0; l_2, l_1) - \hat{A}_+(0; l_1, l_2) = \sum_{n=1}^{\infty} m_{q,\text{cex}} \log \frac{l_2}{l_1} = \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \log \frac{l_2}{l_1},$$

and hence, by Lemma 4.2,

$$\begin{aligned} w_{0,\text{reg}}^{(m)'}(0) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\hat{D}'_{q,-,\text{reg}}(0; l_2, l_1) - \hat{D}'_{q,+,\text{reg}}(0; l_1, l_2)) \right. \\ &\quad \left. + (-1)^{m-1} (\hat{D}'_{q,+,\text{reg}}(0; l_2, l_1) - \hat{D}'_{q,-,\text{reg}}(0; l_1, l_2)) \right) \\ (8.5) \quad &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\hat{A}_{q,-}(0; l_2, l_1) - \hat{A}_{q,+}(0; l_1, l_2)) \right. \\ &\quad \left. + (-1)^{m-1} (\hat{A}_{q,+}(0; l_2, l_1) - \hat{A}_{q,-}(0; l_1, l_2)) \right) \\ &= -\frac{1}{2} \left(\log \frac{l_2}{l_1} \right) \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q (\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) + (-1)^{m+1} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)})), \end{aligned}$$

and

$$\begin{aligned} (8.6) \quad w_{1,\text{reg}}^{(2p-1)'}(0) &= (-1)^{p-1} \frac{1}{2} \left(\hat{D}'_{p-1,0,\text{reg}}(0; l_2, l_1) - \hat{D}'_{p-1,0,\text{reg}}(0; l_1, l_2) \right) \\ &= (-1)^p \frac{1}{2} \log \frac{l_2}{l_1} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}). \end{aligned}$$

8.2.2. *The contribution of the singular part.* Since the functions $\Phi_{q,2j-1}$ and $\Phi_{q,2j-1,\pm}$ are regular at $s = 0$, and by the location of the poles of $\zeta(s, U)$ given at the beginning of Section 7, it follows that the formula for the singular part in Theorem 5.1 reduces to the single sums (recall $\kappa = 2$ and the zeta of U is regular at $s = 0$)

$$\hat{D}_{q,\pm,\text{sing}}(0; l_1, l_2) = 0, \quad \hat{D}'_{q,\pm,\text{sing}}(0; l_1, l_2) = \frac{1}{2} \sum_{j=1}^m \hat{\Phi}_{q,j,\pm}(0; l_1, l_2) \text{Res}_{s=j} \zeta(s, U).$$

where the Φ are defined in equation (7.11). Thus, considering first $w_{0,\text{sing}}^{(m)}$, by Lemma 4.2, we have

$$w_{0,\text{sing}}^{(m)'}(0) = \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\hat{D}'_{q,-,\text{sing}}(0; l_2, l_1) - \hat{D}'_{q,+,\text{sing}}(0; l_1, l_2)) \right. \\ \left. + (-1)^{m-1} (\hat{D}'_{q,+,\text{sing}}(0; l_2, l_1) - \hat{D}'_{q,-,\text{sing}}(0; l_1, l_2)) \right),$$

and hence

$$(8.7) \quad w_{0,\text{sing}}^{(m)'}(0) = \frac{1}{4} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \sum_{j=1}^m \left((\hat{\Phi}_{q,j,-}(0; l_2, l_1) - \hat{\Phi}_{q,j,+}(0; l_1, l_2)) \right. \\ \left. + (-1)^{m-1} (\hat{\Phi}_{q,j,+}(0; l_2, l_1) - \hat{\Phi}_{q,j,-}(0; l_1, l_2)) \right) \text{Res}_{s=j} \zeta(s, U),$$

Second, for $w_{1,\text{sing}}^{(2p-1)'}$ (0), we have

$$w_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{2} \left(\hat{D}'_{p-1,0,\text{sing}}(0; l_2, l_1) - \hat{D}'_{p-1,0,\text{sing}}(0; l_1, l_2) \right),$$

and hence

$$(8.8) \quad w_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{4} \sum_{j=1}^{2p-1} \left(\hat{\Phi}_{p-1,j,0}(0; l_2, l_1) - \hat{\Phi}_{p-1,j,0}(0; l_1, l_2) \right) \text{Res}_{s=j} \zeta(s, U).$$

8.2.3. The contribution of the harmonics. In order to determinate the contribution of the harmonics, we use the technique described in Section 2 of [22] to deal with (simple) sequences of spectral type (in fact this was already in [21]). Recall that, by Lemma 4.2, we need $d'_q(0; l_1, l_2)$, where by definition

$$d_q(s; l_1, l_2) = m_{\text{har},q} \sum_{k=1}^{\infty} f_{-\alpha_{q-1,k}}^{-2s}(l_1, l_2),$$

and the $f_{\nu,k}^{-2s}(l_1, l_2)$ are the positive zeros of the function

$$F_\nu(z; l_1, l_2) = J_\nu(l_1 z) Y_{\nu-1}(l_2 z) - Y_\nu(l_1 z) J_{\nu-1}(l_2 z).$$

Recalling the series definition of the Bessel functions, we obtain that near $z = 0$,

$$F_\nu(z) = \frac{2}{\pi z} \frac{l_2^{\nu-1}}{l_1^\nu}.$$

This means that the function $F_\nu(z)$ is an even function of z . By the Hadamard factorization Theorem, we have the product expansion

$$F_\nu(z) = \frac{2}{\pi z} \frac{l_2^{\nu-1}}{l_1^\nu} \prod_{k=-\infty}^{+\infty} \left(1 - \frac{z}{f_{\nu,k}(l_1, l_2)} \right),$$

and therefore

$$F_\nu(z) = \frac{2}{\pi z} \frac{l_2^{\nu-1}}{l_1^\nu} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{f_{\nu,k}^2(l_1, l_2)} \right),$$

Defining, for $-\pi < \arg(z) < \frac{\pi}{2}$,

$$G_\nu(z) = -F_\nu(iz),$$

we have

$$G_\nu(z) = \frac{2}{\pi z} \frac{l_2^{\nu-1}}{l_1^\nu} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{f_{\nu,k}^2(l_1, l_2)} \right).$$

By equation (5.1), the logarithmic Gamma function associated to the simple sequence $\Theta_\nu(l_1, l_2) = \{f_{\nu,k}^2(l_1, l_2)\}$ is

$$\begin{aligned} \log \Gamma(-\lambda, \Theta_\nu(l_1, l_2)) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{f_{\nu,k}^2} \right) \\ &= -\log G_\nu(\sqrt{-\lambda}) - \log \frac{\pi}{2} - \log l_2 \sqrt{\lambda} + \nu \log \frac{l_2}{l_1}. \end{aligned}$$

Combining the asymptotic expansions of the Bessel functions, we obtain for large λ :

$$\log \Gamma(-\lambda, \Theta_\nu(l_1, l_2)) = \left(\frac{1}{2} - \nu \right) \log \frac{l_1}{l_2} + \log 2 - (l_2 - l_1) \sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Therefore, by Theorem 2.11 of [22],

$$d'_q(0; l_1, l_2) = -m_{\text{har},q} \left(\left(\frac{1}{2} + \alpha_{q-1} \right) \log \frac{l_1}{l_2} + \log 2 \right),$$

and hence, by Lemma 4.2,

$$\begin{aligned} (8.9) \quad w_2^{(m)'}(0) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{q+1} (d'_q(0; l_1, l_2) + (-1)^m d'_q(0; l_2, l_1)) \\ &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^q m_{\text{har},q} \left(\left(\frac{1}{2} + \alpha_{q-1} \right) \log \frac{l_1}{l_2} + \log 2 \right. \\ &\quad \left. + (-1)^m \left(\left(\frac{1}{2} + \alpha_{q-1} \right) \log \frac{l_2}{l_1} + \log 2 \right) \right). \end{aligned}$$

$$\begin{aligned} (8.10) \quad w_3^{(2p)'}(0) &= (-1)^{p+1} \frac{1}{2} d'_p(0; l_1, l_2) = (-1)^p \frac{1}{2} m_{\text{har},p} \left(\left(\frac{1}{2} + \alpha_{p-1} \right) \log \frac{l_1}{l_2} + \log 2 \right) \\ &= (-1)^p \frac{1}{2} m_{\text{har},p} \log 2. \end{aligned}$$

8.3. The furstum with absolute BC.

8.3.1. *The contribution of the regular part.* Proceeding as in Section 8, by Theorem 5.1, we have

$$E'_{q,\text{reg}}(0) = -A_q(0) - B'_q(0), \quad \hat{E}'_{q,\pm,\text{reg}}(0) = -\hat{A}_{q,\pm}(0) - \hat{B}'_{q,\pm}(0),$$

where,

$$\begin{aligned}
A_q(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\log \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) + \frac{1}{2} \log l_1 l_2 \right) \mu_{q,n}^{-2s}, \\
B_q(s) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \mu_{q,n}^{-2s}, \\
B'_q(s) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \mu_{q,n}^{-2s} \log \mu_{q,n}, \\
\hat{A}_{q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(-\frac{1}{2} \log l_1 l_2 + \log \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) + \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \right) \mu_{q,n}^{-2s}, \\
\hat{B}_{q,\pm}(s) &= -\frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \mu_{q,n}^{-2s}, \\
\hat{B}'_{q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \mu_{q,n}^{-2s} \log \mu_{q,n}.
\end{aligned}$$

This gives (in the following formulas we are taking the finite part)

$$\begin{aligned}
A_q(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\log \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) + \frac{1}{2} \log l_1 l_2 \right), \\
B_q(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n}, \\
B'_q(0) &= - \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \\
\hat{A}_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(-\frac{1}{2} \log l_1 l_2 + \log \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) + \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \right), \\
\hat{B}_{q,\pm}(0) &= -\frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n}, \\
\hat{B}'_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}.
\end{aligned}$$

Thus, considering first $y_{0,\text{reg}}^{(m)}$, we have

$$y_{0,\text{reg}}^{(m)'}(0) = \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((E'_{q,\text{reg}}(0) - \hat{E}'_{q,+,\text{reg}}(0)) + (-1)^{m-1} (E'_{q,\text{reg}}(0) - \hat{E}'_{q,-,\text{reg}}(0)) \right).$$

and hence

$$\begin{aligned} y_{0,\text{reg}}^{(m)'}(0) &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(A_q(0) + B'_q(0) - \hat{A}_{q,+}(0) - \hat{B}'_{q,+}(0) \right) \\ &\quad - (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(A_q(0) + B'_q(0) - \hat{A}_{q,-}(0) - \hat{B}'_{q,-}(0) \right). \end{aligned}$$

We observe that all the terms that are equal in $B_q(0)$ and $\hat{B}_{q,\pm}(0)$ and in $A_q(0)$ and $\hat{A}_{q,\pm}(0)$ respectively cancel in the final formula for $t_{0,\text{reg}}^{(m)}(0)$. This is because the final formulas are

$$(E'_{q,\text{reg}}(0) - \hat{E}_{q,+,\text{reg}}(0)) + (E'_{q,\text{reg}}(0) - \hat{E}_{q,-,\text{reg}}(0)) = 2E'_{q,\text{reg}}(0) - \hat{E}_{q,+,\text{reg}}(0) - \hat{E}_{q,-,\text{reg}}(0),$$

if $m = 2p - 1$ is odd, and

$$(E'_{q,\text{reg}}(0) - \hat{E}_{q,+,\text{reg}}(0)) - (E'_{q,\text{reg}}(0) - \hat{E}_{q,-,\text{reg}}(0)) = -\hat{E}'_{q,+,\text{reg}}(0) + \hat{E}'_{q,-,\text{reg}}(0),$$

if $m = 2p$ is even. Therefore, we can rewrite

$$\begin{aligned} y_{0,\text{reg}}^{(m)'}(0) &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathbb{A}_q(0) + \mathbb{B}'_q(0) - \hat{\mathbb{A}}_{q,+}(0) - \hat{\mathbb{B}}'_{q,+}(0) \right) \\ &\quad - (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathbb{A}_q(0) + \mathbb{B}'_q(0) - \hat{\mathbb{A}}_{q,-}(0) - \hat{\mathbb{B}}'_{q,-}(0) \right). \end{aligned}$$

where

$$\begin{aligned} \mathbb{A}_q(0) &= \frac{1}{2} \log l_1 l_2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} = \frac{1}{2} \zeta(0, \tilde{\Delta}^{(q)}) \log l_1 l_2, \\ \mathbb{B}'_q(0) &= -\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \\ \hat{\mathbb{A}}_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(-\frac{1}{2} \log l_1 l_2 + \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \right) \\ &= -\frac{1}{2} \zeta(0, \tilde{\Delta}^{(q)}) \log l_1 l_2 + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right), \\ \hat{\mathbb{B}}'_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} m_{\text{cex},q,n} = \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2), \quad -2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} = \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2).$$

Thus,
(8.11)

$$\begin{aligned} y_{0,\text{reg}}^{(m)'}(0) &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\zeta(0, \tilde{\Delta}^{(q)}) + (-1)^{m-1} \zeta(0, \tilde{\Delta}^{(q)}) \right) \log l_1 l_2 \\ &\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(-\frac{\alpha_q^2}{\mu_{q,n}^2} \right) + 2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right) \\ &\quad + (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) + 2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right). \end{aligned}$$

Next, consider $y_{1,\text{reg}}^{(2p-1)}$. By Lemma 4.1, we have

$$y_{1,\text{reg}}^{(2p-1)'}(0) = -\frac{1}{2} \left(A_{p-1}(0) + B'_{p-1}(0) - \hat{A}_{p-1,0}(0) - \hat{B}'_{p-1,0}(0) \right).$$

As above, all the terms that are equal in $B_q(0)$ and $\hat{B}_{q,0}(0)$ and in $A_q(0)$ and $\hat{A}_{q,0}(0)$ respectively cancel in the final formula for $t_{1,\text{reg}}^{(2p-1)}(0)$. Therefore, we can write

$$y_{1,\text{reg}}^{(2p-1)'}(0) = -\frac{1}{2} \left(\hat{A}_{p-1}(0) + \hat{B}'_{p-1}(0) - \hat{A}_{p-1,0}(0) - \hat{B}'_{p-1,0}(0) \right),$$

where

$$\begin{aligned} \hat{A}_{p-1}(0) &= \frac{1}{2} \zeta(0, \tilde{\Delta}^{(p-1)}) \log l_1 l_2, & \hat{B}'_{p-1}(0) &= -\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \\ \hat{A}_{q,\pm}(0) &= -\frac{1}{2} \log l_1 l_2 \zeta(0, \tilde{\Delta}^{(q)}), & \hat{B}'_{q,\pm}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}. \end{aligned}$$

Observing that $\alpha_{p-1} = \frac{1}{2}(1 + 2p - 2 - 2p + 1) = 0$, and $\mu_{p-1,n} = \lambda_{p-1,n}$, we have

$$\zeta_{\text{cex}}(s, \tilde{\Delta}^{(p-1)} + \alpha_{p-1}^2) = \zeta_{\text{cex}}(s, \tilde{\Delta}^{(p-1)}) = \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \lambda_{p-1,n}^{-2s},$$

and

$$\hat{B}'_{p-1,0}(0) = \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \log \lambda_{p-1,n} = -\frac{1}{2} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).$$

Thus,

$$(8.12) \quad y_{1,\text{reg}}^{(2p-1)'}(0) = -(-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \log l_1 l_2 - \frac{1}{2} (-1)^{p-1} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).$$

8.3.2. *The contribution of the harmonics.* The contribution of the harmonics can be computed directly, proceeding as in Section 8.1.3. We obtain (this formula holds for all cases in which $\alpha_q \neq 0$, the particular case only appears when m is odd and is described below)

$$e'_{q,-}(0) = \log(-\alpha_q) - \log \left(\frac{l_2^{-\alpha_q}}{l_1^{-\alpha_q}} - \frac{l_1^{-\alpha_q}}{l_2^{-\alpha_q}} \right) - \frac{1}{2} \log l_1 l_2,$$

and hence

$$\begin{aligned}
y_2^{(m)'}(0) &= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (e'_{q-1,-}(0) + (-1)^m e'_{q,-}(0)) \\
&= \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} \log \left(\frac{m-2q+1}{2} \right) \left(\frac{m-2q-1}{2} \right)^{(-1)^m} \\
&\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^q m_{\text{har},q} \log \left(\frac{l_2^{m-2q+1} - l_1^{m-2q+1}}{(l_1 l_2)^{\frac{m-2q+1}{2}}} \right) \left(\frac{l_2^{m-2q-1} - l_1^{m-2q-1}}{(l_1 l_2)^{\frac{m-2q-1}{2}}} \right)^{(-1)^m} \\
&\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^q m_{\text{har},q} \frac{1 + (-1)^m}{2} \log l_1 l_2.
\end{aligned}$$

It is convenient to distinguish odd and even cases, as in Section 9.1. When, $m = 2p - 1$, we need to isolate the case $q = p - 1$, when the correct formula is

$$e'_{p-1,-}(0) = -\log \log \frac{l_2}{l_1} - \log 2 - \log \sqrt{l_1 l_2}.$$

After some calculations, we obtain

$$\begin{aligned}
y_2^{(2p-1)'}(0) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} - \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q-2} - l_1^{2p-2q-2}}{2p-2q-2} \\
&\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \log l_1 l_2 + \frac{1}{2} (-1)^p m_{\text{har},p} \log \log \frac{l_2}{l_1}.
\end{aligned}$$

By duality on the section:

$$\sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{l_2^{2p-2-2q} - l_1^{2p-2-2q}}{2p-2-2q} = \sum_{q=p+1}^{2p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} - \sum_{q=0}^{p-2} (-1)^q r_q \log (l_1 l_2)^{2p-2-2q},$$

and hence

$$\begin{aligned}
(8.13) \quad y_2^{(2p-1)'}(0) &= \frac{1}{2} \sum_{q=0, q \neq p}^{2p-1} (-1)^{q+1} m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} \\
&\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} (2p-1-2q) m_{\text{har},q} \log l_1 l_2 + \frac{1}{2} (-1)^p m_{\text{har},p} \log \log \frac{l_2}{l_1}.
\end{aligned}$$

When $m = 2p$, we obtain

$$\begin{aligned}
y_2^{(2p)'}(0) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} (\log(2p-2q+1) + \log(2p-2q-1) - 2 \log 2) \\
&\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \left(m_{\text{har},q} \log \left(l_2^{2p-2q+1} - l_1^{2p-2q+1} \right) + \log \left(l_2^{2p-2q-1} - l_1^{2p-2q-1} \right) - (2p-2q) \log l_1 l_2 \right) \\
&\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log l_1 l_2.
\end{aligned}$$

The last contribution is

$$\begin{aligned}
y_3^{(2p)'}(0) &= (-1)^{p+1} \frac{1}{2} m_{\text{har},p} e'_{p,-}(0) \\
&= (-1)^{p+1} \frac{1}{2} m_{\text{har},p} \left(-\log 2 - \log \left(\frac{l_2^{\frac{1}{2}}}{l_1^{\frac{1}{2}}} - \frac{l_1^{\frac{1}{2}}}{l_2^{\frac{1}{2}}} \right) - \frac{1}{2} \log l_1 l_2 \right) \\
&= (-1)^p \frac{1}{2} m_{\text{har},p} (\log(l_2 - l_1) + \log 2).
\end{aligned}$$

Collecting

$$\begin{aligned}
y_2^{(2p)'}(0) + y_3^{(2p)'}(0) &= \sum_{q=0}^{p-1} (-1)^q \log 2 + \frac{1}{2} (-1)^p m_{\text{har},p} \log 2 \\
&\quad + \frac{1}{2} \sum_{q=0}^p (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q-1} - l_1^{2p-2q-1}}{2p-2q-1} \\
&\quad + \frac{1}{2} \log l_1 l_2 \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} (2p-2q-1).
\end{aligned}$$

Using duality on the section

$$\begin{aligned}
\sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q-1} - l_1^{2p-2q-1}}{2p-2q-1} &= \sum_{q=p+1}^{2p} (-1)^q r_q \log \frac{-1}{2p-2q+1} \left(\frac{1}{l_2^{2p-2q+1}} - \frac{1}{l_1^{2p-2q+1}} \right) \\
&= \sum_{q=p+1}^{2p} (-1)^q r_q \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} \\
&\quad + \sum_{q=0}^{p-1} (-1)^{q+1} r_q (2p-2q-1) \log l_1 l_2,
\end{aligned}$$

and hence

$$(8.14) \quad y_2^{(2p)'}(0) + y_3^{(2p)'}(0) = \frac{1}{2} \chi(W) \log 2 + \frac{1}{2} \sum_{q=0}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1}.$$

8.3.3. *The contribution of the singular part.* The calculations show that the functions Φ appearing in the singular part are the same appearing in the singular part for the cone and the frustum with mixed BC, as described in Sections 7 and 8. We obtain the following result

$$\begin{aligned} E_{q,\text{sing}}(0) &= 0, & E'_{q,\text{sing}}(0) &= \frac{1}{2} \sum_{j=1}^m \Phi_{q,j}^{\text{Frustum,abs}}(0) \text{Res}_1 \zeta(s, U), \\ \hat{E}_{q,\pm,\text{sing}}(0) &= 0, & \hat{E}'_{q,\pm,\text{sing}}(0) &= \frac{1}{2} \sum_{j=1}^m \hat{\Phi}_{q,j,\pm}^{\text{Frustum,abs}}(0) \text{Res}_1 \zeta(s, U). \end{aligned}$$

Since,

$$y_{0,\text{sing}}^{(m)'}(0) = \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((E'_{q,\text{sing}}(0) - \hat{E}'_{q,+,\text{sing}}(0)) + (-1)^{m-1} (E'_{q,\text{sing}}(0) - \hat{E}'_{q,-,\text{sing}}(0)) \right),$$

we obtain

$$(8.15) \quad \begin{aligned} y_{0,\text{sing}}^{(m)'}(0) &= \frac{1}{4} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \sum_{j=1}^m \left((\Phi_{q,j}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(0)) \right. \\ &\quad \left. + (-1)^{m-1} (\Phi_{q,j}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(0)) \right) \text{Res}_1 \zeta(s, U), \end{aligned}$$

If $m = 2p - 1$:

$$y_{0,\text{sing}}^{(2p-1)'}(0) = \frac{1}{4} \sum_{q=0}^{p-2} (-1)^q \sum_{j=1}^{2p-1} \left(2\Phi_{q,j}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(0) \right) \text{Res}_1 \zeta(s, U),$$

if $m = 2p$:

$$y_{0,\text{sing}}^{(2p)'}(0) = -\frac{1}{4} \sum_{q=0}^{p-1} (-1)^q \sum_{j=1}^{2p} \left(\hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(0) \right) \text{Res}_1 \zeta(s, U),$$

For $y_{1,\text{sing}}^{(2p-1)'}(0)$, we have

$$y_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{2} \left(E'_{p-1,\text{sing}}(0) - \hat{E}'_{p-1,0,\text{sing}}(0) \right),$$

and hence

$$(8.16) \quad y_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{4} \sum_{j=1}^m \left(\Phi_{p-1,j}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{p-1,j,0}^{\text{Frustum,abs}}(0) \right) \text{Res}_1 \zeta(s, U).$$

In particular, we have a result analogous to Lemma 7.3, that leads to the following formula

$$\begin{aligned} &\Phi_{q,j}^{\text{Frustum,abs}}(s) - \hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(s) + (-1)^{m-1} (\Phi_{q,j}^{\text{Frustum,abs}}(s) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(s)) = \\ &= (l_2^{2s} + (-1)^j l_1^{2s}) \left(\Phi_{q,j}(s) - \hat{\Phi}_{q,j,+}(s) + (-1)^{m-1} \left(\Phi_{q,j}(s) - \hat{\Phi}_{q,j,-}(s) \right) \right), \end{aligned}$$

and therefore, if $m = 2p - 1$,

$$2\Phi_{q,j}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(0) = (1 + (-1)^j) \left(2\Phi_{q,j}(0) - \hat{\Phi}_{q,j,+}(0) - \hat{\Phi}_{q,j,-}(0) \right),$$

if $m = 2p$,

$$\hat{\Phi}_{q,j,+}^{\text{Frustum,abs}}(0) - \hat{\Phi}_{q,j,-}^{\text{Frustum,abs}}(0) = (1 + (-1)^j) \left(\hat{\Phi}_{q,j,+}(0) - \hat{\Phi}_{q,j,-}(0) \right).$$

Since in the odd case the relevant terms are those with odd index j , we have that

$$y_{0,\text{sing}}^{(2p-1)'}(0) + y_{1,\text{sing}}^{(2p-1)'}(0) = 0.$$

In the even case, we obtain

$$y_{0,\text{sing}}^{(2p)'}(0) = 2t_{0,\text{sing}}^{(2p-1)'}(0).$$

9. THE REGULAR PART OF THE TORSION

We decompose the torsion in two parts, the regular part and the anomaly boundary term, according to the formulas (6.1), (6.2), and (6.3) of Section 6. In this section we give the formulas for the regular part, for the cone and for the frustum. As a consequence, we have the proof of the formulas in equations (6.1), and (6.2) namely that, in the case of the frustum, the regular part of the torsion coincides with the Reidemeister torsion plus the Euler part of the boundary contribution, and the singular part is precisely the anomaly boundary contribution.

9.1. The regular part of the torsion for the cone. We distinguish odd and even cases.

9.1.1. *Odd case:* $m = 2p - 1$. $\lfloor \frac{m}{2} \rfloor - 1 = p - 2$, and hence, using equation (8.1),

$$\begin{aligned} t_{0,\text{reg}}^{(2p-1)'}(0) &= - \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}^2 \right). \end{aligned}$$

Since $\mu_{q,n}^2 - \alpha_q^2 = \lambda_{q,n}^2$, we obtain

$$\begin{aligned} t_{0,\text{reg}}^{(2p-1)'}(0) &= - \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \log l + \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \lambda_{q,n} \right) \\ &= - \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \log l - \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)}). \end{aligned}$$

Next, from equation (8.2)

$$t_{1,\text{reg}}^{(2p-1)'}(0) = -(-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \log l - (-1)^{p-1} \frac{1}{4} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).$$

Eventually, the contribution of the harmonics is given in equation (8.3)

$$\begin{aligned} t_2^{(2p-1)'}(0) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} (\alpha_{q-1} - \alpha_q) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \left(\log \frac{2^{-\alpha_{q-1}} \Gamma(-\alpha_{q-1} + 1)}{2^{-\alpha_q} \Gamma(-\alpha_q + 1)} \right). \end{aligned}$$

Since, by definition $\alpha_q = \frac{1}{2}(1 + 2q - 2p + 1)$ and $\alpha_{q-1} = \frac{1}{2}(1 + 2q - 2 - 2p + 1)$, we obtain

$$t_2^{(2p-1)'}(0) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log l + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \log(2(p-q)).$$

Summing up, as in equation (6.3), we obtain

$$\begin{aligned} \log T_{\text{global}}(C_l(W)) &= t_{0,\text{reg}}^{(2p-1)'}(0) + t_{1,\text{reg}}^{(2p-1)'}(0) + t_2^{(2p-1)'}(0) + t_3^{(2p-1)'}(0) \\ &= \left(\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} - \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) - (-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \right) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-2} (-1)^{q+1} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)}) + \frac{1}{4} (-1)^p \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \\ &\quad - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log(2(p-q)). \end{aligned}$$

Using equations (A.4) and (A.1), this formula can be rewritten in terms of harmonics:

$$\log T_{\text{global}}(C_l(W)) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l^{2p-2q}}{2p-2q} + \frac{1}{2} \log T(W, g).$$

9.1.2. *Even case:* $m = 2p$. $\lfloor \frac{m}{2} \rfloor - 1 = p - 1$, and hence, using equation (8.1),

$$t_{0,\text{reg}}^{(2p)'}(0) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\log \left(1 + \frac{\alpha_q}{\mu_{q,n}} \right) - \log \left(1 - \frac{\alpha_q}{\mu_{q,n}} \right) \right).$$

Next, the contribution of the harmonics is given in equation (8.3)

$$\begin{aligned} t_2^{(2p)'}(0) &= -\frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} (-\alpha_{q-1} - \alpha_q + 1) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \log \frac{2^{-\alpha_{q-1} - \alpha_q - 1} \Gamma(-\alpha_{q-1} + 1) \Gamma(-\alpha_q + 1)}{\pi}. \end{aligned}$$

Since, by definition $\alpha_q = \frac{1}{2}(1 + 2q - 2p)$, and $\alpha_{q-1} = \frac{1}{2}(1 + 2q - 2 - 2p)$, we obtain

$$\begin{aligned} t_2^{(2p)'}(0) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} (2p - 2q + 1) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \log(2^{-2} ((2p - 2q - 1)!)^2 (2p - 2q + 1)). \end{aligned}$$

Eventually, from equation (8.4)

$$t_3^{(2p)'}(0) = (-1)^p \frac{1}{4} m_{\text{har},p} \log l + (-1)^p \frac{1}{2} m_{\text{har},p} \log 2.$$

Summing up, as in equation (6.3), we obtain

$$\begin{aligned} \log T_{\text{global}}(C_l(W)) + \frac{1}{4}\chi(\partial C_l(W)) \log 2 &= t_{0,\text{reg}}^{(2p)'}(0) + t_{1,\text{reg}}^{(2p)'}(0) + t_2^{(2p)'}(0) + t_3^{(2p)'}(0) \\ &= \left(\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} (2p - 2q + 1) + (-1)^p \frac{1}{4} m_{\text{har},p} \right) \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \log(2p - 2q + 1) ((2p - 2q - 1)!)^2 \\ &\quad + \frac{1}{2} \chi(W) \log 2 + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \frac{1 + \frac{\alpha_q}{\mu_{q,n}}}{1 - \frac{\alpha_q}{\mu_{q,n}}}. \end{aligned}$$

9.2. The regular part of the torsion for the frustum with mixed BC. By equation (8.5),

$$w_{0,\text{reg}}^{(m)'}(0) = -\frac{1}{2} \left(\log \frac{l_2}{l_1} \right)^{\left[\frac{m}{2} \right] - 1} \sum_{q=0}^{\left[\frac{m}{2} \right] - 1} (-1)^q (\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) + (-1)^{m+1} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)})),$$

whence

$$w_{0,\text{reg}}^{(2p-1)'}(0) = -\log \left(\frac{l_2}{l_1} \right) \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}), \quad w_{0,\text{reg}}^{(2p)'}(0) = 0;$$

by equation (8.6),

$$w_{1,\text{reg}}^{(2p-1)'}(0) = (-1)^p \frac{1}{2} \log \left(\frac{l_2}{l_1} \right) \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)});$$

by equation (8.9)

$$w_2^{(m)'}(0) = \frac{1}{2} \sum_{q=0}^{\left[\frac{m}{2} \right]} (-1)^q m_{\text{har},q} \left(\left(\frac{1}{2} + \alpha_{q-1} \right) \log \frac{l_1}{l_2} + \log 2 + (-1)^m \left(\left(\frac{1}{2} + \alpha_{q-1} \right) \log \frac{l_2}{l_1} + \log 2 \right) \right),$$

and hence

$$w_2^{(2p-1)'}(0) = \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q} \left(p - q - \frac{1}{2} \right) \log \frac{l_2}{l_1}, \quad w_2^{(2p)'}(0) = \sum_{q=0}^p (-1)^q m_{\text{har},q} \log 2;$$

By equation (8.10),

$$w_3^{(2p)'}(0) = (-1)^p \frac{1}{2} m_{\text{har},p} \log 2.$$

Thus, when $m = 2p$,

$$w_{0,\text{reg}}^{(2p)'}(0) + w_2^{(2p)'}(0) + w_3^{(2p)'}(0) = \sum_{q=0}^p (-1)^q m_{\text{har},q} \log 2 + (-1)^p \frac{1}{2} m_{\text{har},p} \log 2 = \frac{1}{2} \chi(W) \log 2.$$

When $m = 2p - 1$, recalling equations (A.2) and (A.3),

$$\zeta_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) = (-1)^q \sum_{k=0}^q (-1)^k \zeta(0, \tilde{\Delta}^{(k)}) = -(-1)^q \sum_{k=0}^q (-1)^k m_{\text{har},k},$$

since the dimension of W is odd; whence, after some calculation we obtain

$$w_{0,\text{reg}}^{(2p-1)'}(0) + w_2^{(2p-1)'}(0) = \frac{1}{2} \log \left(\frac{l_2}{l_1} \right) \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q},$$

$$w_{1,\text{reg}}^{(2p-1)'}(0) = \frac{1}{2} \log \left(\frac{l_2}{l_1} \right) \sum_{k=0}^{p-1} (-1)^k m_{\text{har},k},$$

and hence

$$w_{0,\text{reg}}^{(2p-1)'}(0) + w_{1,\text{reg}}^{(2p-1)'}(0) + w_2^{(m)'}(0) = 0.$$

Therefore, for any parity of m , the regular part of the torsion is

$$w_{0,\text{reg}}^{(m)'}(0) + w_{1,\text{reg}}^{(m)'}(0) + w_2^{(m)'}(0) + w_3^{(m)'}(0) = \frac{1}{2} \chi(W) \log 2,$$

and this is equal to the Reidemeister torsion plus the Euler part of the boundary contribution, i.e.

$$\log \tau(C_{[l_1, l_2]}(W), \{l_1\} \times W) + \frac{1}{4} \chi(\partial C_{[l_1, l_2]}(W)) \log 2 = w_{0,\text{reg}}^{(m)'}(0) + w_{1,\text{reg}}^{(m)'}(0) + w_2^{(m)'}(0) + w_3^{(m)'}(0),$$

and hence

$$A_{\text{BM},\text{mix}}(\partial C_{[l_1, l_2]}(W)) = w_{0,\text{sing}}^{(m)'}(0) + w_{1,\text{sing}}^{(m)'}(0),$$

and this concludes the proof of the formulas in equation (6.1).

9.3. The regular part of the torsion for the frustum with absolute BC. By equations (8.11) and (8.12),

$$y_{0,\text{reg}}^{(2p)'}(0) = 0,$$

$$y_{0,\text{reg}}^{(2p-1)'}(0) = - \sum_{q=0}^{p-2} (-1)^q \zeta(0, \tilde{\Delta}^{(q)}) \log l_1 l_2 - \sum_{q=0}^{p-2} (-1)^q \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)}),$$

$$y_{1,\text{reg}}^{(2p-1)'}(0) = - (-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \log l_1 l_2 - \frac{1}{2} (-1)^{p-1} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}),$$

and, using equation (A.4),

$$y_{0,\text{reg}}^{(2p-1)'}(0) + y_{1,\text{reg}}^{(2p-1)'}(0) = \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} (2p-1-2q) \log l_1 l_2 + \log T(W, g).$$

By equation (8.13)

$$y_{0,\text{reg}}^{(2p-1)'}(0) + y_{1,\text{reg}}^{(2p-1)'}(0) + y_2^{(2p-1)'}(0) = \log T(W, g) + \frac{1}{2} \sum_{q=0, q \neq p}^{2p-1} (-1)^{q+1} m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q}$$

$$+ \frac{1}{2} (-1)^p m_{\text{har},p} \log \log \frac{l_2}{l_1},$$

and by equation (8.14)

$$y_{0,\text{reg}}^{(2p)'}(0) + y_2^{(2p)'}(0) + y_3^{(2p)'}(0) = \frac{1}{2} \chi(W) \log 2 + \frac{1}{2} \sum_{q=0}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1}.$$

Comparison with Proposition 3.3 of [11], proves the following formulas:

$$y_{0,\text{reg}}^{(m)'}(0) + y_{1,\text{reg}}^{(m)'}(0) + y_2^{(m)'}(0) + y_3^{(m)'}(0) = \log \tau_{\mathbb{R}}(C_{[l_1, l_2]}(W^{(m)})) + \frac{1}{2} \chi(W^{(m)}) \log 2,$$

and

$$y_{0,\text{sing}}^{(m)'}(0) + y_{1,\text{sing}}^{(m)'}(0) = A_{\text{BM,abs}}(\partial C_{[l_1, l_2]}(W)),$$

and this concludes the proof of the formulas in equation (6.2).

10. THE ANOMALY BOUNDARY TERM

We show in this section that the anomaly boundary term of the frustum with mixed BC is twice the one of the cone. This concludes the proof of Theorem 1.1.

By equations (8.7) and (8.8),

$$\begin{aligned} w_{0,\text{sing}}^{(m)'}(0) &= \frac{1}{4} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \sum_{j=1}^m \left((\hat{\Phi}_{q,j,-}(0; l_2, l_1) - \hat{\Phi}_{q,j,+}(0; l_1, l_2)) \right. \\ &\quad \left. + (-1)^{m+1} (\hat{\Phi}_{q,j,+}(0; l_2, l_1) - \hat{\Phi}_{q,j,-}(0; l_1, l_2)) \right) \text{Res}_1 \zeta(s, U), \\ w_{1,\text{sing}}^{(2p-1)'}(0) &= (-1)^{p-1} \frac{1}{4} \sum_{j=1}^{2p-1} \left(\hat{\Phi}_{p-1,j,0}(0; l_2, l_1) - \hat{\Phi}_{p-1,j,0}(0; l_1, l_2) \right) \text{Res}_1 \zeta(s, U), \end{aligned}$$

where (see equation (7.11))

$$\hat{\Phi}_{q,j,\pm}(s; l_1, l_2) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \hat{\phi}_{q,j,\pm}(\lambda; l_1, l_2) d\lambda dt,$$

and the $\hat{\phi}$ are given in equation (7.9). Let introduce the linear operator

$$\mathcal{T}(f) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} f(\lambda) d\lambda dt.$$

Note that

$$\mathcal{T}(f(a_-))(-) = a^s \mathcal{T}(f(-))(-),$$

and hence by Lemma 7.3

$$\hat{\Phi}_{q,j,\pm}(s; l_1, l_2) = \text{sgn}(l_1 - l_2)^j l_1^{2s} \Phi_{q,j}(s) + \text{sgn}(l_2 - l_1)^j l_2^{2s} \hat{\Phi}_{q,j,\pm \text{sgn}(l_2 - l_1)}(s).$$

Therefore,

$$\begin{aligned} &(\hat{\Phi}_{q,j,-}(0; l_2, l_1) - \hat{\Phi}_{q,j,+}(0; l_1, l_2)) + (-1)^{m+1} (\hat{\Phi}_{q,j,+}(0; l_2, l_1) - \hat{\Phi}_{q,j,-}(0; l_1, l_2)) \\ &= (1 - (-1)^j) \Phi_{q,j}(0) - \hat{\Phi}_{q,j,+}(0) + (-1)^j \hat{\Phi}_{q,j,-}(0) \\ &\quad + (-1)^{m+1} \left((1 - (-1)^j) \Phi_{q,j}(0) - \hat{\Phi}_{q,j,-}(0) + (-1)^j \hat{\Phi}_{q,j,+}(0) \right); \end{aligned}$$

and

$$\hat{\Phi}_{p-1,j,0}(0; l_2, l_1) - \hat{\Phi}_{p-1,j,0}(0; l_1, l_2) = (1 - (-1)^j) \Phi_{p-1,j}(0) - (1 - (-1)^j) \hat{\Phi}_{p-1,j,0}(0).$$

It is now convenient to distinguish odd and even cases. If $m = 2p - 1$ is odd, then the relevant values of the index j are the odd ones, since the zeta function $\zeta(s, U)$ only has poles at odd integers

s , while if $m = 2p$ is even, the relevant values for j are the even ones, for similar reason. Thus, if $m = 2p - 1$,

$$\begin{aligned} & (\hat{\Phi}_{q,2k-1,-}(0; l_2, l_1) - \hat{\Phi}_{q,2k-1,+}(0; l_1, l_2)) + (\hat{\Phi}_{q,2k-1,+}(0; l_2, l_1) - \hat{\Phi}_{q,2k-1,-}(0; l_1, l_2)) \\ & = 4\hat{\Phi}_{q,2k-1}(0) - 2\hat{\Phi}_{q,2k-1,+}(0) - 2\hat{\Phi}_{q,2k-1,-}(0), \end{aligned}$$

and

$$\hat{\Phi}_{p-1,2k-1,0}(0; l_2, l_1) - \hat{\Phi}_{p-1,2k-1,0}(0; l_1, l_2) = 2\hat{\Phi}_{p-1,j}(0) - 2\hat{\Phi}_{p-1,j,0}(0),$$

if $m = 2p$ is even,

$$\begin{aligned} & (\hat{\Phi}_{q,2k,-}(0; l_2, l_1) - \hat{\Phi}_{q,2k,+}(0; l_1, l_2)) + (\hat{\Phi}_{q,2k,+}(0; l_2, l_1) - \hat{\Phi}_{q,2k,-}(0; l_1, l_2)) \\ & = 2\hat{\Phi}_{q,2k-1,-}(0) - 2\hat{\Phi}_{q,2k-1,+}(0). \end{aligned}$$

Now, recalling the singular contribution for the cone given in Section 8.1.2

$$\begin{aligned} t_{0,\text{sing}}^{(2p-1)'}(0) &= \frac{1}{4} \sum_{q=0}^{p-2} (-1)^q \sum_{j=1}^{p-2} (2\hat{\Phi}_{q,2j-1}(0) - \hat{\Phi}_{q,2j-1,+}(0) - \hat{\Phi}_{q,2j-1,-}(0)) \text{Res}_{s=2j-1} \zeta(s, U), \\ t_{0,\text{sing}}^{(2p)'}(0) &= \frac{1}{4} \sum_{q=0}^{p-1} (-1)^q \sum_{j=1}^{p-1} (\hat{\Phi}_{q,2j,-}(0) - \hat{\Phi}_{q,2j,+}(0)) \text{Res}_{s=2j} \zeta(s, U), \end{aligned}$$

and

$$t_{1,\text{sing}}^{(2p-1)'}(0) = (-1)^{p-1} \frac{1}{4} \sum_{j=1}^m (\hat{\Phi}_{p-1,j}(0) - \hat{\Phi}_{p-1,j,0}(0)) \text{Res}_{s=j} \zeta(s, U),$$

and hence we have proved that

$$\begin{aligned} w_{0,\text{sing}}^{(m)'}(0) &= 2t_{0,\text{sing}}^{(m)'}(0), \\ w_{1,\text{sing}}^{(2p-1)'}(0) &= 2t_{1,\text{sing}}^{(2p-1)'}(0). \end{aligned}$$

Now, by the final formula in the previous section, and by Lemma 4.1 of [11]

$$w_{0,\text{sing}}^{(m)'}(0) + w_{1,\text{sing}}^{(m)'}(0) = A_{\text{BM,mix}}(\partial C_{[l_1, l_2]}(W)) = \int_W B,$$

where B is defined in Section 4.3 of [11] (based on [1]). Therefore,

$$t_{0,\text{sing}}^{(m)'}(0) + t_{1,\text{sing}}^{(m)'}(0) = \frac{1}{2} \int_W B,$$

and this is exactly the term that we would obtain applying the formula of [1] [2] on the cone, as if it were a smooth manifold. This concludes the proof of Theorem 1.1.

11. THE LIMITING CASE

We address the following question: is there any relationship between the analytic torsion of the frustum and that of the cone? In [11] we proved that, in the odd case $m = 2p - 1$, regularising the analytic torsion of the frustum with absolute BC (taking the quotient by the suitable factor) and taking the limit for $l_1 \rightarrow 0^+$ we obtained the torsion of the cone. The results of the present paper permit a more explicit unified analysis for all dimensions, and a possible interpretation for the regularising factor.

The idea is to consider the set of the formal eigenfunctions of the cone that are not square integrable. If we proceed formally, applying the boundary conditions, these eigenfunctions gives a new set of eigenvalues for the formal operator, that for simplicity we call the negative part of the spectrum. We verify that the negative part of the spectrum can be treated by the same method used for the positive part of the spectrum, up to some technical points, that we describe in details. As a result, we obtain a new term, that we call the negative part of the analytic torsion, and that we denote by $\log T_-(C_l(W))$. We give a formula for this term in Proposition 11.1, that shows clearly the analogies and the differences with the regular torsion. Next, in Proposition 11.2 we give the expansion for the logarithm of the ratio of the analytic torsion of the cone to the negative negative torsion of the cone (with $l = l_1$), and eventually we show in Theorem 11.1 that the finite part of this ratio coincides with the analytic torsion of the cone (with $l = l_2$) up to a classical boundary term (see Remark 2.1).

11.1. The “negative” analytic torsion of the cone. We proceed with the notation of Section 3. The set $\mathcal{S}^{(q)}$ of the eigenvalues of the equation $\Delta^{(q)}u = \lambda^2 u$, with absolute BC and $\lambda \neq 0$, is $\mathcal{S}^{(q)} = \mathcal{S}_+^{(q)} \cup \mathcal{S}_-^{(q)}$, where $\mathcal{S}_+^{(q)} = \text{Sp}_+ \Delta_{\text{abs}}^{(q)}$, and $\mathcal{S}_-^{(q)}$ is the same set with $\mu_{x,n}$ replaced by $-\mu_{x,n}$, $x = q, q-1, q-2$, namely: if $m = \dim W = 2p-1$, $p \geq 1$:

$$\begin{aligned} \mathcal{S}_-^{(q)} = & \left\{ m_{\text{cex},q,n} : \hat{j}_{-\mu_{q,n},\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{-\mu_{q-1,n},\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},q-1,n} : j_{-\mu_{q-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-2,n} : j_{-\mu_{q-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},q} : \hat{j}_{-|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1} : \hat{j}_{-|\alpha_{q-1}|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty ; \end{aligned}$$

if $m = \dim W = 2p$, $p \geq 1$:

$$\begin{aligned} \mathcal{S}_-^{(q \neq p, p+1)} = & \left\{ m_{\text{cex},q,n} : \hat{j}_{-\mu_{q,n},\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{-\mu_{q-1,n},\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},q-1,n} : j_{-\mu_{q-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-2,n} : j_{-\mu_{q-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},q} : \hat{j}_{-|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1} : \hat{j}_{-|\alpha_{q-1}|,\alpha_{q-1},k}^2 / l^2 \right\}_{k=1}^\infty , \end{aligned}$$

$$\begin{aligned} \mathcal{S}_-^{(p)} = & \left\{ m_{\text{cex},p,n} : \hat{j}_{-\mu_{p,n},\alpha_p,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p-1,n} : \hat{j}_{-\mu_{p-1,n},\alpha_{p-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},p-1,n} : j_{-\mu_{p-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p-2,n} : j_{-\mu_{p-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ \frac{1}{2} m_{\text{har},p} : j_{\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ \frac{1}{2} m_{\text{har},p} : j_{-\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},p-1} : \hat{j}_{-|\alpha_{p-1}|,\alpha_{p-1},k}^2 / l^2 \right\}_{k=1}^\infty , \end{aligned}$$

$$\begin{aligned} \mathcal{S}_-^{(p+1)} = & \left\{ m_{\text{cex},p+1,n} : \hat{j}_{-\mu_{p+1,n},\alpha_{p+1},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p,n} : \hat{j}_{-\mu_{p,n},\alpha_p,k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},p,n} : j_{-\mu_{p,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},p-1,n} : j_{-\mu_{p-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},p+1} : \hat{j}_{-|\alpha_{p+1}|,\alpha_{p+1},k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ \frac{1}{2} m_{\text{har},p} : j_{-\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ \frac{1}{2} m_{\text{har},p} : j_{\frac{1}{2}}^2 / l^2 \right\}_{k=1}^\infty , \end{aligned}$$

where the $j_{\mu,k}$ are the positive zeros of the Bessel function $J_\mu(x)$, the $\hat{j}_{\mu,c,k}$ are the positive zeros of the function $\hat{J}_{\mu,c}(x) = cJ_\mu(x) + xJ'_\mu(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q,n}$ are defined in Lemma 3.1.

Remark 11.1. *The above description of the negative spectrum is always valid except that for the eigenvalues associated to the harmonics of the section in the odd case $m = 2p - 1$. In such a case, the eigenfunctions associated to these eigenvalues are not the Bessel function themselves, as observed at the end of Lemma 3.1. We will take care of this difference explicitly when we treat the term of the analytic torsion associated to these eigenvalues, in Subsection 11.1.3 below.*

Note that the set $\mathcal{S}^{(q)}$ satisfy all the same properties satisfied by the set $\mathcal{S}_+^{(q)}$ and used in the previous sections in order to define and analyse the associated spectral functions. Following this idea, and proceeding as in Section 6, we consider the functions

$$Z_q^-(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{-\mu_{q,n},k}^{-2s}, \quad \hat{Z}_{q,\pm}^-(s) = \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{-\mu_{q,n},\pm\alpha_q,k}^{-2s}, \quad z_{q,\pm}(s) = \sum_{k=1}^{\infty} j_{\pm\alpha_q,k}^{-2s},$$

and

$$t_{\text{Cone},-}^{(m)}(s) = t_{0,-}^{(m)}(s) + t_{1,-}^{(m)}(s) + t_{2,-}^{(m)}(s) + t_{3,-}^{(m)}(s),$$

where

$$\begin{aligned} t_{0,-}^{(m)}(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((Z_q^-(s) - \hat{Z}_{q,+}^-(s)) + (-1)^{m-1} (Z_q^-(s) - \hat{Z}_{q,-}^-(s)) \right), \\ t_{1,-}^{(2p-1)}(s) &= (-1)^{p-1} \frac{l^{2s}}{2} \left(Z_{p-1}^-(s) - \hat{Z}_{p-1,0}^-(s) \right), & t_{1,-}^{(2p)}(s) &= 0 \\ t_{2,-}^{(m)}(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{q+1} m_{\text{har},q} (z_{q-1,+}(s) + (-1)^m z_{q,+}(s)), \\ t_{3,-}^{(2p)}(s) &= (-1)^{p+1} m_{\text{har},p} \frac{l^{2s}}{4} (z_{p,+}(s) + z_{p,-}(s)), & t_{3,-}^{(2p-1)}(s) &= 0, \end{aligned}$$

The aim of this section is to determine the quantity

$$\log T_-(C_l(W^{(m)})) = t_{\text{Cone},-}^{(m)}(0).$$

11.1.1. *The contribution of the regular part.* The functions $t_{0,-}$ and $t_{1,-}$ are double series as t_0 and t_1 , and, up to solving some little technical problems, can be analysed by the same method used in Section 7. The relevant sequences are now $S_q^- = \{m_{\text{cex},q,n} : j_{-\mu_{q,n},k}^2\}$, and $\hat{S}_{q,\pm}^- = \{m_{\text{cex},q,n} : j_{-\mu_{q,n},\pm\alpha_q,k}^2\}$. We obtain the following representation for associated logarithmic Gamma functions:

$$\begin{aligned} \log \Gamma(-\lambda, S_q^-) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{j_{-\mu_{q,n},k}^2} \right) \\ &= -\log I_{-\mu_{q,n}}(\sqrt{-\lambda}) - \mu_{q,n} \log \sqrt{-\lambda} + \mu_{q,n} \log 2 - \log \Gamma(1 - \mu_{q,n}). \\ \log \Gamma(-\lambda, \hat{S}_{q,\pm}^-) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{j}_{-\mu_{q,n},\pm\alpha_q,k}^2} \right) \\ &= -\log \hat{I}_{-\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda}) - \mu_{q,n} \log \sqrt{-\lambda} + \mu_{q,n} \log 2 \\ &\quad - \log \Gamma(-\mu_{q,n}) + \log \left(1 \mp \frac{\alpha_q}{\mu_{q,n}} \right). \end{aligned}$$

Recalling

$$I_{-\nu}(z) = \frac{2}{\pi} \sin \nu \pi K_{\nu}(z) + I_{\nu}(z),$$

substitution in the representations above of the logarithmic Gamma functions, and using the known asymptotic expansions for the Bessel functions, shows that the asymptotic expansions for the negative case may be deduced from the ones computed for the positive case. We give here the relevant results, using the same notation as in Section 8, with an added minus index.

$$\begin{aligned} t_{0,\text{reg},-}^{(m)'}(0) &= - \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left((\mathfrak{B}_{q,-}(0) - \hat{\mathfrak{B}}_{q,+,-}(0)) + (-1)^{m-1} (\mathfrak{B}_{q,-}(0) - \hat{\mathfrak{B}}_{q,-,-}(0)) \right) \log l \\ &\quad - \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathfrak{A}_{q,-}(0) + \mathfrak{B}'_{q,-}(0) - \hat{\mathfrak{A}}_{q,+,-}(0) - \hat{\mathfrak{B}}'_{q,+,-}(0) \right) \\ &\quad - (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\mathfrak{A}_{q,-}(0) + \mathfrak{B}'_{q,-}(0) - \hat{\mathfrak{A}}_{q,-,-}(0) - \hat{\mathfrak{B}}'_{q,-,-}(0) \right). \end{aligned}$$

where

$$\begin{aligned} \mathfrak{A}_{q,-}(0) &= 0, \\ \mathfrak{B}_{q,-}(0) &= \frac{1}{4} \sum_{n=1}^{\infty} m_{\text{cex},q,n} = \frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2) = \frac{1}{4} \zeta(0, \tilde{\Delta}^{(q)}), \\ \mathfrak{B}'_{q,-}(0) &= -\frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \\ \hat{\mathfrak{A}}_{q,\pm,-}(0) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right), \\ \hat{\mathfrak{B}}_{q,\pm,-}(0) &= -\frac{1}{4} \sum_{n=1}^{\infty} m_{\text{cex},q,n} = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2) = -\frac{1}{4} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}), \\ \hat{\mathfrak{B}}'_{q,\pm,-}(0) &= \frac{1}{2} \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n}, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} m_{\text{cex},q,n} = \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2), \quad -2 \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} = \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)} + \alpha_q^2).$$

Thus,

(11.1)

$$\begin{aligned}
t_{0,\text{reg},-}^{(m)'}(0) &= -\frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) + (-1)^{m-1} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}) \right) \log l \\
&\quad + \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 - \frac{\alpha_q}{\mu_{q,n}} \right) + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right) \\
&\quad + (-1)^{m-1} \frac{1}{2} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^q \left(\sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \left(1 + \frac{\alpha_q}{\mu_{q,n}} \right) + \sum_{n=1}^{\infty} m_{\text{cex},q,n} \log \mu_{q,n} \right) \\
&= (-1)^{m+1} t_{0,\text{reg},-}^{(m)'}(0)
\end{aligned}$$

Similar analysis gives

$$(11.2) \quad t_{1,\text{reg},-}^{(2p-1)'}(0) = -(-1)^{p-1} \frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) \log l - \frac{1}{4} (-1)^{p-1} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}) = t_{1,\text{reg}}^{(2p-1)'}(0).$$

11.1.2. *The contribution of the singular part.* It is easy to realise that the singular part coincides exactly with the singular part in the positive case, namely:

$$(11.3) \quad \begin{aligned} t_{0,\text{sing},-}^{(m)'}(0) &= t_{0,\text{sing},-}^{(m)'}(0), \\ t_{1,\text{sing},-}^{(2p-1)'}(0) &= t_{1,\text{sing},-}^{(2p-1)'}(0). \end{aligned}$$

11.1.3. *The contribution of the harmonics.* This is the contribution coming from the simple series $z_{q,+}(s)$. Respect with the harmonics for the positive torsion, where we studied the functions $z_{q,-}(s)$, there is now a technical problem, since the values of α_q appearing in the $z_{q,-}(s)$ are never negative integers, while the values of the α_q appearing in the $z_{q,+}(s)$ maybe negative integers. This problem appears only when $m = 2p - 1$ is odd, that is treated below in details. In the even case $m = 2p$, we can use the formulas in equations (5.6) and (5.7) as in Section 8.1.3. For $0 \leq q < p$, $\alpha_q = \frac{1}{2} + q - p$, and we obtain:

$$\begin{aligned}
z_{q,+}(0) &= -\frac{1}{2} \left(\alpha_q + \frac{1}{2} \right) = -\frac{1}{2} (1 + q - p) = -z_{q,-}(0) - \frac{1}{2}, \\
z'_{q,+}(0) &= \log \frac{\sqrt{\pi}}{2^{\alpha_q - \frac{1}{2}} \Gamma(\alpha_q + 1)} = -\log 2 - \log(2(p - q - 1) - 1)!! \\
&= -z'_{q,-}(0) + \log(2p - 2q - 1) - 2 \log 2.
\end{aligned}$$

This gives:

$$(11.4) \quad \begin{aligned} t_{2,-}^{(2p)'}(0) &= -t_2^{(2p)'}(0) - \sum_{q=0}^{p-1} (-1)^{q+1} r_q \log l \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} r_q \log(2p - 2q - 1)(2p - 2q + 1) - 2 \sum_{q=0}^{p-1} (-1)^{q+1} r_q \log 2, \\ t_{3,-}^{(2p)'}(0) &= t_3^{(2p)'}(0). \end{aligned}$$

In the odd case, $m = 2p - 1$, since $\alpha_q = 1 + q - p < 0$ for $0 \leq q \leq p - 1$, as observed in Remark 11.1, the eigenfunctions of type E and O are not the Bessel function J_{α_q} of negative index, but the functions Y_{α_q} . Whence, function under study is the function is

$$z_{q,+}(s) = \sum_{k=1}^{\infty} y_{\alpha_q,k}^{-2s},$$

where $S = \{y_{\alpha_q,k}\}$ is the sequence of the zeros of the Bessel function $Y_{-\alpha_q}$. We proceed as follows. Consider the series representation of the Bessel function Y_n ($n = -\alpha_q$),

$$Y_n(z) = \frac{2}{\pi} J_n(z) \left(\log \frac{z}{2} + C \right) + \sum_{k=0}^{n-1} \frac{(n-k-1)!}{\pi k!} \left(\frac{z}{2} \right)^{2k-n} - \left(\frac{z}{2} \right)^n \frac{1}{n!} \sum_{k=1}^n \frac{1}{\pi k} - z^n \sum_{k=1}^{\infty} a_{n,k}.$$

Since

$$\lim_{z \rightarrow 0} z^n Y_n(z) = \frac{1}{\pi} \frac{(n-1)!}{2^{-n}} = \frac{2^n (n-1)!}{\pi},$$

we have the product expansion is

$$G_n(z) = z^n Y_n(z) = \frac{2^n (n-1)!}{\pi} \prod_{k=-\infty, k \neq 0}^{+\infty} \left(1 - \frac{z}{y_{n,k}} \right).$$

Define

$$\Upsilon_n(z) = (niz)^n \left(J_n(iz) - H_n^{(1)}(iz) \right) = z^n \left(I_n(z) - \frac{2}{\pi i} K_n(z) \right),$$

then

$$\Upsilon_n(z) = \frac{2^n (n-1)!}{\pi} \prod_{k=1}^{+\infty} \left(1 + \frac{z^2}{y_{n,k}^2} \right),$$

and

$$\log(-\lambda, S) = -\log \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{y_{n,k}^2} \right) = -\log \Upsilon_n(\sqrt{-\lambda}) + n \log 2 + \log(n-1)! - \log \pi.$$

Using the classical expansion for large z of $I_n(z)$ and $K_n(z)$,

$$\log \left(I_n(z) - \frac{2}{\pi i} K_n(z) \right) \sim \log I_n(z) + O(e^{-z}).$$

This implies that

$$\log \Upsilon_n(\sqrt{-\lambda}) = n \log \sqrt{-\lambda} + \sqrt{-\lambda} - \frac{1}{2} \log 2\pi - \frac{1}{2} \log \sqrt{-\lambda} = \frac{2n-1}{2} \log \sqrt{-\lambda} + \sqrt{-\lambda} - \frac{1}{2} \log 2\pi,$$

and

$$\log(-\lambda, S) = -\frac{2n-1}{2} \log \sqrt{-\lambda} - \sqrt{-\lambda} + \frac{2n+1}{2} \log 2 + \log(n-1)! - \frac{1}{2} \log \pi.$$

Whence, for $q \neq p-1$,

$$z_{q,+}(0) = \frac{1}{2} \left(p - q - 1 - \frac{1}{2} \right) = \frac{1}{2} \left(-\alpha_q + \frac{1}{2} \right) - \frac{1}{2} = -z_{q,-}(0) - \frac{1}{2},$$

$$z'_{q,+}(0) = -\log \frac{2^{(-\frac{1}{2}-q+p)}(p-q-2)!}{\pi^{\frac{1}{2}}} = -\log \frac{2^{-\alpha_q + \frac{1}{2}} \Gamma(-\alpha_q)}{\pi^{\frac{1}{2}}} = -z'_{q,-}(0) - \log 2 + \log(-\alpha_q)$$

$$z_{p-1,+}(0) = z_{p-1,-}(0)$$

$$z'_{p-1,+}(0) = -z_{p-1,-}(0) - \log 2,$$

and

$$(11.5) \quad t_{2,-}^{(2p-1)'}(0) = -t_2^{(2p-1)'}(0) + \sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{p-q}{p-q-1}.$$

11.2. Formula for the “negative” torsion and the limiting case $l_1 = 0$. Collecting the results of the previous subsections we have:

$$\begin{aligned} \log T_-(C_l(W^{(2p-1)})) &= t_{0,\text{reg},-}^{(2p-1)'}(0) + t_{2,-}^{(2p-1)'}(0) + t_{3,-}^{(2p-1)'}(0) + t_{0,\text{sing},-}^{(2p-1)'}(0) \\ &= -\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log \frac{l^{2p-2q}}{2p-2q} + \frac{1}{2} \log T(W, g) + t_{0,\text{sing}}^{(2p-1)'}(0) \\ &\quad + \sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{p-q}{p-q-1}. \end{aligned}$$

$$\begin{aligned} \log T_-(C_l(W^{(2p)})) &= t_{0,\text{reg},-}^{(2p)'}(0) + t_{2,-}^{(2p)'}(0) + t_{3,-}^{(2p)'}(0) + t_{0,\text{sing},-}^{(2p)'}(0) \\ &= -t_{0,\text{reg}}^{(2p)'}(0) - t_2^{(2p)'}(0) + t_3^{(2p)'}(0) + t_{0,\text{sing}}^{(2p)'}(0) \\ &\quad + \sum_{q=0}^{p-1} (-1)^q r_q \log l - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p-2q-1)(2p-2q+1) + 2 \sum_{q=0}^{p-1} (-1)^q r_q \log 2 \\ &= -\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l^{2p-2q+1}}{2p-2q+1} - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p-2q-1)(2p-2q+1) \\ &\quad + (-1)^p \frac{1}{4} m_{\text{har},p} \log l + \sum_{q=0}^{p-1} (-1)^q r_q \log l + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log((2p-2q-1)!!)^2 \\ &\quad - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(\log \left(1 + \frac{\alpha_q}{\mu_{q,n}} \right) - \log \left(1 - \frac{\alpha_q}{\mu_{q,n}} \right) \right) \\ &\quad + \sum_{q=0}^{p-1} (-1)^q r_q \log 2 + (-1)^p \frac{1}{2} m_{\text{har},p} \log 2 + t_{0,\text{sing}}^{(2p)'}(0). \end{aligned}$$

The previous formulas in the notation of Theorem 8.4 read:

Proposition 11.1.

$$\begin{aligned}
\log T_-(C_l(W^{(2p-1)})) &= \frac{1}{2} \log T(W, g) - \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_l}}{\text{Det}_0^{p-1} \alpha} + A_{\text{BS,abs}}(\partial C_l(W)) \\
&\quad - \sum_{q=0}^{p-2} (-1)^q r_q \log \frac{p-q}{p-q-1}; \\
\log T_-(C_l(W^{(2p)})) &= -\log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_l}}{\text{Det}_0^{p-1} \alpha} + \frac{1}{4} \chi(W) \log 2 + A_{\text{BS,abs}}(\partial C_l(W)) - B_1^{(2p)}(C_l(W)) + B_2^{(2p)}(C_l(W)) \\
&\quad + \sum_{q=0}^{p-1} (-1)^q r_q \log l - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p-2q-1)(2p-2q+1).
\end{aligned}$$

We are now able to study the constant part of the limit for $l_1 \rightarrow 0^+$ of the difference between the logarithms of the torsion of the frustum and the one of the negative torsion with $l = l_1$. For we need a few lemmas and a remark.

Lemma 11.1. *For $l_1 \rightarrow 0^+$, we have the expansions:*

$$\begin{aligned}
\log \frac{\text{Det} \ddot{\alpha}_F}{\text{Det} \alpha} &= \frac{(-1)^p}{2} r_p \log \log \frac{1}{l_1} + \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q r_q \log l_1^{2p-2-2q} \\
&\quad + \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_2}}}{\text{Det}_0^{p-1} \alpha} + \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q r_q \log(2p-2-2q) + O(l_1),
\end{aligned}$$

if $m = 2p - 1$, and

$$\begin{aligned}
\log \frac{\text{Det} \ddot{\alpha}_F}{\text{Det} \alpha} &= -\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log l_1^{2p-2q-1} \\
&\quad + \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_2}}}{\text{Det}_0^{p-1} \alpha} - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p-2q-1) + \frac{1}{4} (-1)^p r_p \log l_2 + O(l_1),
\end{aligned}$$

if $m = 2p$.

Proof. When $m = 2p - 1$,

$$\begin{aligned}
\sum_{q=p+1}^{2p-1} (-1)^q r_q \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} &= \sum_{q=p+1}^{2p-1} (-1)^q r_q \log \frac{l_2^{2q-2p} - l_1^{2q-2p}}{(2q-2p)(l_1 l_2)^{2q-2p}} \\
&= \sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{l_2^{2p-2-2q} - l_1^{2p-2-2q}}{(2p-2-2q)(l_1 l_2)^{2p-2-2q}} \\
&= \sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{1 - \left(\frac{l_1}{l_2}\right)^{2p-2-2q}}{2p-2-2q} + \sum_{q=0}^{p-2} (-1)^q r_q \log l_1^{2p-2-2q}.
\end{aligned}$$

Whence,

$$\begin{aligned}
\log \prod_{q=0}^{2p-1} \Gamma_q &= \sum_{q=0, q \neq p}^{2p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} + \frac{(-1)^p}{2} r_p \log \log \frac{l_2}{l_1} \\
&= \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{2p-2q} + \frac{(-1)^p}{2} r_p \log \log \frac{l_2}{l_1} \\
&\quad + \sum_{q=0}^{p-2} (-1)^{q+1} r_q \log \frac{1 - \left(\frac{l_1}{l_2}\right)^{2p-2-2q}}{2p-2-2q} + \sum_{q=0}^{p-2} (-1)^q r_q \log l_1^{2p-2-2q}.
\end{aligned}$$

When $m = 2p$,

$$\begin{aligned}
\log \prod_{q=0}^{2p} \Gamma_q &= \sum_{q=0}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} \\
&= \sum_{q=0}^p (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} + \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} \\
&= \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} + \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} \\
&\quad + (-1)^p r_p \log(l_2 - l_1).
\end{aligned}$$

$$\begin{aligned}
\sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \frac{l_2^{2p-2q+1} - l_1^{2p-2q+1}}{2p-2q+1} &= \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \left(l_1^{2p-2q+1} - l_2^{2p-2q+1} \right) \\
&\quad - \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log(2q-1-2p).
\end{aligned}$$

$$\sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log(2q-1-2p) = \sum_{q=0}^{p-1} (-1)^q m_{\text{har},q} \log(2p-1-2q).$$

$$\begin{aligned}
\sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \left(l_1^{2p-2q+1} - l_2^{2p-2q+1} \right) &= \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log l_1^{2p-2q+1} \\
&\quad + \sum_{q=p+1}^{2p} (-1)^q m_{\text{har},q} \log \left(1 - \left(\frac{l_1}{l_2} \right)^{2q-2p-1} \right).
\end{aligned}$$

□

Remark 11.2. Observe that the anomaly boundary term of the frustum is the sum of two equal terms each defined as an integral over on of the two boundaries, so we can write

$$A_{\text{BS,abs}}(\partial C_{[l_1, l_2]}(W)) = A_{\text{BS,abs}}(\partial C_{l_2}(W)) + A_{\text{BS,abs}}(\partial C_{l_1}(W)),$$

and this is true in all dimensions, see Lemma 4.1 of [11].

Proposition 11.2. *For small l_1 , we have the expansions:*

$$\begin{aligned} \log T_{\text{abs}}(C_{[l_1, l_2]}(W^{(2p-1)})) - \log T_-(C_{l_1}(W^{(2p-1)})) &= \frac{(-1)^p}{2} r_p \log \log \frac{1}{l_1} + \sum_{q=0}^{p-1} (-1)^q r_q \log l_1^{2p-1-2q} \\ &\quad + \log T_{\text{abs}}(C_{l_2}(W)) + O(l_1), \\ \log T_{\text{abs}}(C_{[l_1, l_2]}(W^{(2p)})) - \log T_-(C_{l_1}(W^{(2p)})) &= \log T_{\text{abs,ideal}}(C_{l_2}(W)) - \frac{1}{2} \chi(W) \log 2 + O(l_1). \end{aligned}$$

Proof. Using the expansion in the previous lemma and the formulas in Theorem 2.1 we compute, in the odd case $m = 2p - 1$,

$$\begin{aligned} \log T_{\text{abs}}(C_{[l_1, l_2]}(W)) - \log T_-(C_{l_1}(W)) &= \log T(W, g) + \log \frac{\text{Det} \ddot{\alpha}_F}{\text{Det} \alpha} + A_{\text{BS,abs}}(\partial C_{[l_1, l_2]}(W)) \\ &\quad - \frac{1}{2} \log T(W, g) + \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_1}}}{\text{Det}_0^{p-1} \alpha} - A_{\text{BS,abs}}(\partial C_{l_1}(W)) \\ &\quad + \sum_{q=0}^{p-2} (-1)^q r_q \log \frac{p-q}{p-q-1} \\ &= \frac{(-1)^p}{2} r_p \log \log \frac{1}{l_1} + \sum_{q=0}^{p-1} (-1)^q r_q \log l_1^{2p-1-2q} \\ &\quad + \frac{1}{2} \log T(W, g) + \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_2}}}{\text{Det}_0^{p-1} \alpha} \\ &\quad + A_{\text{BS,abs}}(\partial C_{l_2}(W)) + O(l_1). \end{aligned}$$

In the even case $m = 2p$,

$$\begin{aligned} \log T_{\text{abs}}(C_{[l_1, l_2]}(W)) - \log T_-(C_{l_1}(W)) &= \log \frac{\text{Det} \ddot{\alpha}_F}{\text{Det} \alpha} + \frac{1}{2} \chi(W) \log 2 + A_{\text{BS,abs}}(\partial C_{[l_1, l_2]}(W)) \\ &\quad + \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_1}}}{\text{Det}_0^{p-1} \alpha} - \frac{1}{4} \chi(W) \log 2 - A_{\text{BS,abs}}(\partial C_{l_1}(W)) \\ &\quad + B_1^{(2p)}(C_{l_1}(W)) - B_2^{(2p)}(C_{l_1}(W)) - \sum_{q=0}^{p-1} (-1)^q r_q \log l_1 \\ &\quad + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(2p-2q-1)(2p-2q+1) \\ &= \log \frac{\text{Det}_0^{p-1} \ddot{\alpha}_{C_{l_2}}}{\text{Det}_0^{p-1} \alpha} + \frac{1}{4} (-1)^p r_p \log l_2 \\ &\quad + A_{\text{BS,abs}}(\partial C_{[l_1, l_2]}(W)) - A_{\text{BS,abs}}(\partial C_{l_1}(W)) \\ &\quad + B_1^{(2p)}(C_{l_2}(W)) + O(l_1). \end{aligned}$$

□

Theorem 11.1.

$$\operatorname{Res}_0 \left(\log T_{\text{abs}}(C_{[l_1, l_2]}(W)) - \log T_-(C_{l_1}(W)) \right) = \log T_{\text{abs,ideal}}(C_{l_2}(W)) - \frac{1}{2} \chi(W) \log 2.$$

As announced in the introduction, we obtained the formula for the torsion written in Theorem 1.1 (or better 2.1) as a limit of a regularisation of the torsion of the frustum (extending a result obtained in [11] for the odd case $m = 2p - 1$). Beside the intrinsic interest of this result, this also shows that the anomaly term B_1 appearing in the formula for the analytic torsion of the cone (compare Theorem 2.1) is due to the fact that in this approach to the problem of extending the definition of Ray and Singer of analytic torsion to spaces with conical singularities, a set of eigenfunctions of the Laplace Beltrami operator are missed (those that are not square integrable near the tip of the cone). As a consequence, the spectrum change, and the lost part of the spectrum is exactly the one that produce in computing the analytic torsion a counter term to the anomaly term B_1 (namely what we called the negative part of the torsion). Due to the symmetry of the problem, that depends on a sing spending on the dimension, this anomaly term cancels in odd dimension, but does not cancel in even dimension. This emerges by the formulas given in this section for the different terms composing the negative torsion, and by the comparison with the same terms in the case of the regular (positive) torsion given in the previous sections.

APPENDIX A. FORMULAS FOR THE ZETA FUNCTION OF THE HODGE-LAPLACE OPERATOR

Decomposing the zeta function of the Hodge-Laplace operator Δ on an m -dimensional oriented compact connected Riemannian manifold (W, g) , we have that

$$\zeta(s, \Delta^{(q)}) = \zeta_{\text{ex}}(s, \Delta^{(q)}) + \zeta_{\text{cex}}(s, \Delta^{(q)}).$$

When $m = 2p - 1$ is odd, using duality, this gives

$$(A.1) \quad \log T(W, g) = \sum_{q=0}^{p-2} (-1)^{q+1} \zeta'_{\text{cex}}(0, \Delta^{(q)}) + \frac{1}{2} (-1)^p \zeta'_{\text{cex}}(0, \Delta^{(p-1)}).$$

Moreover, observing that

$$(A.2) \quad \zeta_{\text{cex}}(s, \Delta^{(q)}) = (-1)^q \sum_{k=0}^q (-1)^k \zeta(s, \Delta^{(k)}),$$

and that

$$(A.3) \quad \zeta(0, \Delta^{(q)}) = -\dim \ker \Delta^{(q)},$$

using duality we have

$$(A.4) \quad \sum_{q=0}^{p-2} (-1)^q \zeta_{\text{cex}}(0, \Delta^{(q)}) + \frac{1}{2} (-1)^{p-1} \zeta_{\text{cex}}(0, \Delta^{(p-1)}) = -\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q (2p - 1 - 2q) \operatorname{rk} H_q(W).$$

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