THE EULER OBSTRUCTION OF A FUNCTION ON A DETERMINANTAL VARIETY AND ON A CURVE

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ABSTRACT. Given an analytic function germ $f : (X, 0) \to \mathbb{C}$ on an isolated determinantal singularity or on a reduced curve, we present formulas relating the local Euler obstruction of f to the vanishing Euler characteristic of the fiber $X \cap f^{-1}(0)$ and to the Milnor number of f. Restricting ourselves to the case where X is a complete intersection, we obtain an easy way to calculate the local Euler obstruction of f as the difference between the dimension of two algebras.

Keywords: Determinantal singularity, Euler obstruction, Milnor number, vanishing Euler characteristic

Mathematical subject classification: 32S10, 32S30, 32S05, 14B05.

1. INTRODUCTION

The Euler obstruction was introduced by MacPherson ([Ma]) for the construction of characteristic classes of singular complex algebraic varieties.

Several authors have studied the Euler obstruction (see for instance [BS], [Go], [LT]). In particular, the following formula was proved by Brasselet, Lê and Seade ([BLS]).

Theorem 1.1. [BLS] *Let $(X, 0)$ be an equidimensional complex analytic singularity germ with a Whitney stratification $\{V_i\}$ and let $p : U \to \mathbb{C}$ be a general complex linear form, in an open set U containing 0 in $\mathbb{C}^N$. Then,*

$$Eu_X(0) = \sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap p^{-1}(t_0)).Eu_X(V_i),$$

where $\mathbb{B}_\varepsilon$ is a small closed ball around 0 in $\mathbb{C}^N$, $t_0 \in \mathbb{C}\backslash\{0\}$ is sufficiently near $\{0\}$ and $Eu_X(V_i)$ is the Euler obstruction of X at any point of the stratum V_i.

The previous formula says that the Euler obstruction, as a constructible function on X satisfies the Euler condition relatively to a generic linear form. The Euler obstruction of a function was defined by Brasselet, Massey, Parameswaran and Seade ([BMPS]). It is the obstacle for the Euler obstruction to satisfy the Euler condition relatively to analytic functions with isolated singularity.

Theorem 1.2. [BMPS] *Let $(X, 0)$ be an equidimensional complex analytic singularity germ with a Whitney stratification $\{V_i\}$ and let $f : (X, 0) \to \mathbb{C}$ be an analytic function with an isolated singularity. Then,*

$$Eu_X(0) = \left(\sum \chi(V_i \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)).Eu_X(V_i)\right) + Eu_{f,X}(0),$$

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where $\mathbb{B}_\varepsilon$ *is a small closed ball around* 0 *in* $\mathbb{C}^N$, $t_0 \in \mathbb{C}\backslash\{0\}$ *is sufficiently near* $\{0\}$ *and* $Eu_X(V_i)$ *is the Euler obstruction of* X *at any point of the stratum* V_i.

In this work, we just consider $(X, 0)$ with isolated singularity. In this case, it follows directly from the theorems 1.1 and 1.2 and from the fact that the Euler obstruction of the regular part of X is equal to one, that

$$Eu_X(0) = \chi(X \cap p^{-1}(t_0)) \tag{1}$$

and

$$Eu_{f,X}(0) = Eu_X(0) - \chi(X \cap f^{-1}(t_0)). \tag{2}$$

Let $f : (X, 0) \to \mathbb{C}$ be an analytic function germ on an isolated complete intersection singularity (ICIS), $(X, 0)$. If f has an isolated singularity, then the Milnor fiber of the f has the homotopy type of a bouquet of spheres ([Ha], [Lê2]). The number of spheres in this bouquet is called the Milnor-Lê number of f and it is denoted by $\mu_L(f)$. Since we have two invariants related to a function germ on an analytic variety $f : (X, 0) \to \mathbb{C}$, it is natural to look for some relation between them. In this direction Seade, Tibar and Verjovsky, in [STV], show the following result about those invariants in the case, where $(X, 0)$ is an ICIS.

Theorem 1.3. [STV] *Let* $(X, 0)$ *be an ICIS, let* $f : (X, 0) \to \mathbb{C}$ *be an analytic function germ with isolated singularity and let* p *be a generic linear form. Then,*

$$Eu_{f,X}(0) = (-1)^{dim_{\mathbb{C}} X}[\mu_L(f) - \mu_L(p)].$$

Our goal in this work is to prove results like the previous theorem for a kind of analytic variety more general that ICIS. Namely, we want to obtain an analogous result for the case where $(X, 0)$ is an isolated determinantal singularity.

The determinantal varieties are those defined as the zero set of a given size minor of a matrix which elements are analytic function germs with the right assumptions on the codimension. These varieties frequently appear on the study of singularity theory, for instance, the singular locus of a map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is the zero set of the ideal generated by the maximal minors of the Jacobian matrix of g. Many authors studied those varieties ([DP, EG, FN, GaR, NOT, PR]).

The Euler obstruction of a germ $f : (X, 0) \to \mathbb{C}$ with isolated singularity is defined for any analytic germ $(X, 0)$. Nuño-Ballesteros, Oréfice-Okamoto and Tomazella ([NOT]) define an analogous for the Milnor-Lê number of a germ $f : (X, 0) \to \mathbb{C}$ when $(X, 0)$ is an isolated determinantal singularity, which the authors call vanishing Euler characteristic of the fiber $X \cap f^{-1}(0)$ and denote $\nu(X \cap f^{-1}(0), 0)$. In this paper we show that

$$Eu_{f,X}(0) = (-1)^d(\nu(X \cap f^{-1}(0), 0) - \nu(X \cap p^{-1}(0), 0)).$$

Also, in [NOT], we have a definition of kind of a Milnor number of the germ $f : (X, 0) \to \mathbb{C}$, which we call the determinantal Milnor number and denote $\mu_D(f)$. Still in [NOT], there is kind of a Lê-Greuel formula for $\mu_D(f)$. With this result we can relate the Euler obstruction of a function to the determinantal Milnor number in the following way

$$Eu_{f,X}(0) = (-1)^{\dim_{\mathbb{C}} X}(\mu_D(f) - \mu_D(p)).$$

If $(X, 0)$ is an ICIS defined by the map $(\phi_1, \cdots, \phi_k) : (\mathbb{C}^N, 0) \to (\mathbb{C}^k, 0)$ then

$$\mu_D(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi_1, \cdots, \phi_k \rangle + J(f, \phi_1, \cdots, \phi_k)},$$

where $\mathcal{O}_N$ denotes the ring of function germs from $(\mathbb{C}^N, 0)$ to $\mathbb{C}$ and $J(f, \phi_1, \cdots, \phi_k)$ denotes the ideal in $\mathcal{O}_N$ generated by the maximal minors of the Jacobian matrix of $(f, \phi_1, \cdots, \phi_k)$. Therefore, we can easily calculate the Euler obstruction of a function on an ICIS as a difference between dimensions of algebras.

We also present a relation between the Euler obstruction of a function germ and the Milnor number of a finite function germ $f : (X, 0) \to \mathbb{C}$ defined on a reduced curve.

2. Determinantal singularities

In this section, we show the definition of isolated determinantal singularity and some results about it. See [NOT] for more detailed information.

Let $0 < s \leq m \leq n$ be integer numbers. We denote by $M_{m,n}$ the set of complex matrices of size $m \times n$, by $M^s_{m,n}$ the subset consisting of the matrices with rank less than s and by Σ^s the subset consisting of those with rank equal to s.

The set $M^s_{m,n}$ is an irreducible algebraic subvariety of $M_{m,n}$ with codimension equal to $(m-s+1)(n-s+1)$ and it is called the *generic determinantal variety* of type $(m, n; s)$. The singular set of $M^s_{m,n}$ is $M^{s-1}_{m,n}$ and the family $\mathcal{S} = \{\Sigma^i\}_{0 \leq i \leq s-1}$ provides a Whitney stratification of $M^s_{m,n}$ (see [ACGH]).

Let $F : (\mathbb{C}^N, 0) \to (M_{m,n}, 0)$ be the map germ defined by $F(x) = (f_{ij}(x))$ with $f_{ij} \in \mathcal{O}_N$, the ring of holomorphic function germs from $(\mathbb{C}^N, 0)$ to $\mathbb{C}$, for $0 \leq i \leq m$, $0 \leq j \leq n$.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be the analytic germ defined by $(X, 0) = (F^{-1}(M^s_{m,n}), 0)$. We say that $(X, 0)$ is a *determinantal singularity* of type $(m, n; s)$ in $(\mathbb{C}^N, 0)$ if the dimension of $(X, 0)$ is equal to

$$N - (m-s+1)(n-s+1).$$

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type $(m, n; s)$ defined by $(X, 0) = F^{-1}(M^s_{m,n})$, satisfying condition

$$\text{either } s = 1 \text{ or } N < (m-s+2)(n-s+2). \qquad (*)$$

We say that such an $(X, 0)$ is an *isolated determinantal singularity* (IDS) if X is smooth at x and $\operatorname{rank} F(x) = s-1$, for all $x \neq 0$ in a neighbourhood of the origin.

We remark that if $s = 1$, then condition (*) is automatically satisfied and in this case $(X, 0)$ is an IDS if and only if it is an ICIS.

Let $(X, 0)$ be the IDS defined by $F^{-1}(M^s_{m,n})$ for an analytic map germ $F : (\mathbb{C}^N, 0) \to (M_{m,n}, 0)$. For a matrix $A \in M_{m,n}$, let $F_A : (\mathbb{C}^N, 0) \to M_{m,n}$ the map defined by $F_A(x) = F(x) + A$, we denoted $X_A := F_A^{-1}(M^s_{m,n})$.

In [NOT], it is defined the vanishing Euler characteristic of $(X, 0)$ by

$$\nu(X, 0) := (-1)^{\dim X}(\chi(X_A) - 1),$$

where A is such that X_A is smooth and $\operatorname{rank}(F_A(x)) = s-1$ for all $x \in X_A$.

Given $f : (X, 0) \to (\mathbb{C}, 0)$ a function germ with an isolated singularity, the *determinantal Milnor number* of f was defined in [NOT] by

$$\mu_D(f) := \#\Sigma\left(f_a|_{X_A}\right),$$

with $A \in M_{m,n}$ and $a = (a_1, \ldots, a_N) \in \mathbb{C}^N$ generic such that $f_a|_{X_A} : X_A \to \mathbb{C}$ is a Morse function, where $f_a(x_1, \ldots, x_N) = f(x_1, \ldots, x_N) + a_1 x_1 + \cdots + a_N x_N$.

The *vanishing Euler characteristic* of the fibre $(X \cap f^{-1}(0), 0)$ was defined in [NOT] by

$$\nu(X \cap f^{-1}(0), 0) := (-1)^{\dim X - 1}(\chi(X_A \cap f_a^{-1}(c)) - 1),$$

with $(a, A, c) \in \mathbb{C}^N \times M_{m,n} \times \mathbb{C}$ generic (such that, X_A is smooth and $\text{rank}(F_A(x)) = s-1$ for all $x \in X_A$; f_a is a Morse function and c is a regular value of $f_a|_{X_A}$).

We remark that if the IDS $(X,0)$ is an ICIS, then $\nu(X,0)$ and $\nu(X \cap f^{-1}(0),0)$ are equal to the Milnor numbers $\mu(X,0)$ and $\mu(X\cap f^{-1}(0),0)$, respectively, as defined by Hamm ([Ha]). The Lê-Greuel formula ([Lê1, BrG]) is a well known result about Milnor number of an ICIS. It says that if $(X,0)$ is the ICIS defined by the map $(\phi_1,\cdots,\phi_k) : (\mathbb{C}^N,0) \to (\mathbb{C}^k,0)$ and $f : (\mathbb{C}^N,0) \to (\mathbb{C},0)$ is a function germ such that $(X\cap f^{-1}(0),0)$ is an ICIS, then

$$\mu(X,0) + \mu(X \cap f^{-1}(0),0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi_1,\cdots,\phi_k\rangle + J(\phi_1,\cdots,\phi_k,f)}.$$

If we interpret

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi_1,\cdots,\phi_k\rangle + J(\phi_1,\cdots,\phi_k,f)}$$

as the Milnor number of the function germ $f|_X : (X,0) \to (\mathbb{C},0)$, we can rewrite the Lê-Greuel formula by

$$\mu(X,0) + \mu(X\cap f^{-1}(0),0) = \mu(f|_X).$$

In [NOT] there is a similar Lê-Greuel type formula for the case when $(X,0)$ is an IDS.

Theorem 2.1. [NOT] *Given a function $f : (X,0) \to (\mathbb{C},0)$ with isolated singularity on an IDS $(X,0)$, we have:*

$$\mu_D(f) = \nu(X,0) + \nu(X\cap f^{-1}(0),0).$$

By the theorem 2.1 and the Lê-Greuel formula, we have that if $(X,0)$ is an ICIS defined by the map $(\phi_1,\cdots,\phi_k) : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$, then

$$\mu_D(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi_1,\cdots,\phi_k\rangle + J(\phi_1,\cdots,\phi_k,f)}.$$

In [Ga], Gaffney defines the dth polar multiplicity of $(X,0)$ (where d is the dimension of X) as

$$m_d(X,0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi_1,\cdots,\phi_k\rangle + J(\phi_1,\cdots,\phi_k,p)},$$

where $(X,0) = ((\phi_1,\cdots,\phi_k)^{-1}(0),0) \subset (\mathbb{C}^N,0)$ and $p : (\mathbb{C}^N,0) \to \mathbb{C}$ is a generic linear projection. Following this, in [NOT], it is defined the dth polar multiplicity of an IDS by

$$m_d(X,0) := \#\Sigma(p|_{X_A}),$$

that is $m_d(X,0) = \mu_D(p)$, where $p : (X,0) \to \mathbb{C}$ is a generic linear map.

As we mentioned before, our main goal is to relate the Euler obstruction of a function $f : (X,0) \to (\mathbb{C},0)$ to the vanishing Euler characteristic of the fiber $X\cap f^{-1}(0)$. We remember, then, that the paper [NOT] shows a relation between the Euler obstruction and the vanishing Euler characteristic of an IDS.

Theorem 2.2. [NOT] *Let $(X,0)$ be an IDS of dimension d. Then,*

$$Eu_X(0) + (-1)^d m_d(X,0) = 1 + (-1)^d \nu(X,0).$$

3. Relating the Milnor number to the Euler obstruction of a function

In this section, we will relate the invariants defined in the previous section to the Euler obstruction of a function on an IDS.

By [STV], we know that if $(X,0)$ is an ICIS and $f:(X,0)\to(\mathbb{C},0)$ is a function germ with isolated singularity, then

$$Eu_{f,X}(0)=(-1)^{dim_{\mathbb{C}}X}[\mu_L(f)-\mu_L(p)].$$

where $p:(\mathbb{C}^N,0)\to\mathbb{C}$ is a generic linear form. Here we will obtain an analogous result when $(X,0)$ is an IDS. We will show that

$$Eu_{f,X}(0)=(-1)^d(\nu(X\cap f^{-1}(0),0)-\nu(X\cap p^{-1}(0),0)).$$

In order to do this we need the following lemmas.

Lemma 3.1. *Let $(X,0)=(F^{-1}(M^s_{m,n}),0)\subset(\mathbb{C}^N,0)$ be an IDS and let $f:(X,0)\to(\mathbb{C},0)$ be the germ of an analytic function with isolated singularity. Then, there exists a nonempty Zariski open subset $W\subset\mathbb{C}^N\times M_{m,n}\times\mathbb{C}$ such that for all $(a,A,c)\in W$, $X_A\cap f_a^{-1}(c)$ is smooth and $rank(F_A(x))=s-1$, for all $x\in X_A\cap f_a^{-1}(c)$.*

Proof. We take $B\subset\mathbb{C}^N$ in such a way that X is smooth at x, $rank(F(x))=s-1$ and f is regular at x, for all $x\in B\backslash\{0\}$. We denote

$$\begin{aligned}\tilde{C}=\ &\{(a,A,c,x)\in\mathbb{C}^N\times M_{m,n}\times\mathbb{C}\times\mathbb{C}^N : x\in X_A\cap f_a^{-1}(c)\text{ and either } x \text{ is a}\\ &\text{singular point of } X_A\cap f_a^{-1}(c)\text{ or } rank(F_A(x))<s-1\}.\end{aligned}$$

Note that $\tilde{C}$ is an analytic subset of $\mathbb{C}^N\times M_{m,n}\times\mathbb{C}\times\mathbb{C}^N$. In fact,

$$\tilde{C}=v(I_{\mathrm{codim}X+1}(J_{a,A,c}),g_{1A},\cdots,g_{kA},h)\cup v(I_{s-1}(F_A(x)),g_{1A},\cdots,g_{kA},h),$$

where $g_{1A},\cdots,g_{kA}\in\mathcal{O}_{N+mn}$ denotes the $s\times s$-minors de $F+A$, $I_r(M)$ denotes the ideal generated by $r\times r$-minors of a matrix M, $h:\mathbb{C}^N\times\mathbb{C}\times\mathbb{C}^N\to\mathbb{C}$ defined by $h(a,c,x)=f_a(x)-c$ and

$$J_{a,A,c}=\begin{pmatrix}\frac{\partial g_{1A}}{\partial x_1} & \cdots & \frac{\partial g_{1A}}{\partial x_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial g_{kA}}{\partial x_1} & \cdots & \frac{\partial g_{kA}}{\partial x_N}\\ \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_N}\end{pmatrix}.$$

We also denote $C=\pi(\tilde{C})$ where

$$\begin{array}{cccc}\pi: & (\tilde{C},0) & \to & (\mathbb{C}^N\times M_{m,n}\times\mathbb{C},0)\\ & (a,A,c,x) & \mapsto & (a,A,c)\end{array}$$

We have X with isolated singularity at 0, $f(0)=0$ and $S(X\cap f^{-1}(0))\subset S(f)$ where $S(X\cap f^{-1}(0))$ and $S(f)$ denote the sets containing the singular points of $X\cap f^{-1}(0)$ and of f, respectively, then $\pi^{-1}(0)=\{0\}$. By shrinking the neighbourhood B if necessary, we may assume that π is a finite map and its image $C=\pi(\tilde{C})$ is an analytic subset of $\mathbb{C}^N\times M_{m,n}\times\mathbb{C}$ (see, for instance, [GrR, Theorem 2 on page 53]).

Then, we put $W=\mathbb{C}^N\times M_{m,n}\times\mathbb{C}\setminus C$. Hence, W is Zariski open.

We consider now $(0,0,c)\in\mathbb{C}^N\times M_{m,n}\times\mathbb{C}$, with c a regular value of $f:(X,0)\to\mathbb{C}$ sufficiently near $\{0\}$. Since $S(X\cap f^{-1}(c))\subset S(f)$, $X\cap f^{-1}(c)$ does not contain $\{0\}$ and f have isolated singularity at $\{0\}$, then we have $X\cap f^{-1}(c)$ smooth. We also have $rank(F(x))=s-1$, for all $x\in X\cap f^{-1}(c)$, because X is an IDS.

Therefore $(0,0,c)$ is in W and hence W is a nonempty set. □

Lemma 3.2. *Let* $\phi : \mathbb{C}^N \times M_{m,n} \times \mathbb{C} \times \mathbb{C}^N \to M_{m,n} \times \mathbb{C}$ *be the map defined by* $\phi(a,A,c,x) = (F(x)+A, f_a(x)-c)$. *We denote* $\phi_{a,A,c} : \mathbb{C}_N \to M_{m,n} \times \mathbb{C}$ *defined by* $\phi_{a,A,c}(x) = \phi(a,A,c,x)$. *If* (a,A,c) *is in* W *the nonempty Zariski open set given by lemma 3.1, then* $\phi_{a,A,c}$ *is transverse to* $\Sigma^{s-1} \times \{0\}$.

Proof. First, we consider $M_{m,n} \times \mathbb{C} \equiv \mathbb{C}^{mn} \times \mathbb{C}$ with coordinate system $((x_{ij})_{m\times n}, z)$. Let $(B,0) \in \phi_{a,A,c}(\mathbb{C}^N) \cap (\Sigma^{s-1} \times \{0\})$. We have $\Sigma^{s-1} \times \{0\}$ is an open subset of $M^s_{m,n} \times \{0\}$, then the local ring $\frac{\mathcal{O}_{(mn+1,(B,0))}}{i(\Sigma^{s-1}\times\{0\})}$ is equal to local ring $\frac{\mathcal{O}_{(mn+1,(B,0))}}{J}$, where

$$J = I_s((x_{ij})_{m\times n}) + < z >.$$

Thus, $\frac{\mathcal{O}_{(mn+1,(B,0))}}{J}$ is a regular ring and $\frac{\mathcal{O}_N}{I}$ is regular, where $I = \phi^*_{a,A,c}(J)$, because $X_A \cap f_a^{-1}(c)$ is smooth and $rank(F_A(x)) = s-1$, for all $x \in X_A \cap f_a^{-1}(c)$. Therefore, $\phi_{a,A,c}$ is transverse to $\Sigma^{s-1} \times \{0\}$ by [BN, Lemma 4.2]. □

In order to relate the determinantal Milnor number to the Euler obstruction of a function, we need to show that the vanishing Euler characteristic of the fibre can be seen as

$$\nu(X \cap f^{-1}(0), 0) = (-1)^{d-1}(\chi(X \cap f^{-1}(c)) - 1).$$

For this we need to prove that $\chi(X_A \cap f_a^{-1}(c))$ is independent of (a,A,c) in the nonempty Zariski open set given by lemma 3.1.

Proposition 3.3. *Let* $(X,0) = (F^{-1}(M^s_{m,n}),0) \subset (\mathbb{C}^N,0)$ *be an IDS and let* $f : (X,0) \to (\mathbb{C},0)$ *be a germ of an analytic function with isolated singularity. Then,* $\chi(X_A \cap f_a^{-1}(c))$ *is independent of* (a,A,c) *in* W *the nonempty Zariski open set given by lemma 3.1.*

Proof. Let W the nonempty Zariski open set given by lemma 3.1. We consider the map

$$\begin{array}{rccc} \pi : & \phi^{-1}(\Sigma^{s-1} \times \{0\}) & \to & W \\ & (a,A,c,x) & \mapsto & (a,A,c) \end{array}$$

Then, π is a submersion. In fact, given $(a,A,c) \in W$, we have $\phi_{a,A,c}$ is transverse to $\Sigma^{s-1} \times \{0\}$, by lemma 3.2. Hence, for each $x \in \phi^{-1}_{a,A,c}(\Sigma^{s-1} \times \{0\})$,

$$d\phi_{a,A,c}|_x(T_x\mathbb{C}^N) + T_{\phi_{a,A,c}(x)}(\Sigma^{s-1} \times \{0\}) = T_{\phi_{a,A,c}(x)}(M_{m,n} \times \mathbb{C}).$$

Then,

$$d\phi|_{(a,A,c,x)}(0 \times 0 \times 0 \times T_x\mathbb{C}^N) + T_{\phi_{(a,A,c,x)}}(\Sigma^{s-1} \times \{0\}) = T_{\phi(a,A,c,x)}(M_{m,n} \times \mathbb{C}).$$

We have,

$$d\pi|_{(a,A,c,x)} : \ T_{(a,A,c,x)}\phi^{-1}(\Sigma^{s-1} \times \{0\}) \ \to \ T_{(a,A,c)}(\mathbb{C}^N \times M_{m,n} \times \mathbb{C}).$$

Thus,

$$\begin{aligned} T_{(a,A,c,x)}(\mathbb{C}^N \times M_{m,n} \times \mathbb{C} \times \mathbb{C}^N) &= (d\phi|_{(a,A,c,x)})^{-1}(T_{\phi(a,A,c,x)}\mathbb{C}^N \times M_{m,n} \times \mathbb{C}) = \\ &= 0 \times 0 \times 0 \times T_x\mathbb{C}^N + (d\phi|_{(a,A,c,x)})^{-1}(T_{\phi(a,A,c,x)}\Sigma^{s-1} \times \{0\}) = \\ &= 0 \times 0 \times 0 \times T_x\mathbb{C}^N + T_{(a,A,c,x)}\phi^{-1}(\Sigma^{s-1} \times \{0\}). \end{aligned}$$

Therefore,

$$d\pi|_{(a,A,c,x)}(T_{(a,A,c,x)}\phi^{-1}(\Sigma^{s-1} \times \{0\})) = T_{(a,A,c)}(\mathbb{C}^N \times M_{m,n} \times \mathbb{C}).$$

Thus, (a, A, c) is a regular value of π. Therefore, π is a submersion and hence a fibration over the connected set W. We have,

$$\pi^{-1}(a,A,c) = \{(a,A,c,x); (a,A,c,x) \in \phi^{-1}(\Sigma^{s-1}\times\{0\})\} =$$
$$= \{(a,A,c,x); \phi(a,A,c,x) \in \Sigma^{s-1}\times\{0\}\} = \{(a,A,c,x); \phi_{a,A,c}(x) \in \Sigma^{s-1}\times\{0\}\} =$$
$$= \{(a,A,c,x); x \in X_A \cap f_a^{-1}(c)\}) = \{(a,A,c)\times X_A\cap f_a^{-1}(c)\}),$$

then,

$$\chi(X_A\cap f_a^{-1}(c)) = \chi(\{(a,A,c)\times X_A\cap f_a^{-1}(c)\}) = \chi(\pi^{-1}(a,A,c)).$$

Therefore, $\chi(X_A \cap f_a^{-1}(c))$ does not depend on (a, A, c) in W. □

Corollary 3.4. *Let* $(X,0) = (F^{-1}(M^s_{m,n}),0) \subset (\mathbb{C}^N,0)$ *be an IDS and let* $f : (X,0) \to (\mathbb{C},0)$ *be a germ of an analytic function with isolated singularity. Given a regular value c of* f *sufficiently near* $\{0\}$*, then the vanishing Euler characteristic of the fibre* $(X \cap f^{-1}(0),0)$ *is*

$$\nu(X\cap f^{-1}(0),0) = (-1)^{d-1}(\chi(X\cap f^{-1}(c)) - 1).$$

Proof. By definition, the vanishing Euler characteristic of the fibre $(X \cap f^{-1}(0), 0)$ is

$$\nu(X\cap f^{-1}(0),0) = (-1)^{d-1}(\chi(X_A\cap f_a^{-1}(c)) - 1),$$

where $(a, A, c) \in \mathbb{C}^N \times M_{m,n} \times \mathbb{C}$ are generic values such that X_A is smooth, $f_a|_{X_A}$ is a Morse function and c is a regular value of $f_a|_{X_A}$. Then, (a, A, c) is in the nonempty Zariski open set given by lemma 3.1.

Thus, by proposition 3.3 , the vanishing Euler characteristic of the fibre $(X\cap f^{-1}(0), 0)$ is independent of (a, A, c) in W.

Given $(0, 0, c) \in \mathbb{C}^N \times M_{m,n} \times \mathbb{C}$, with c a regular value of $f : (X, 0) \to \mathbb{C}$, then $(0, 0, c)$ in W, therefore

$$\nu(X\cap f^{-1}(0),0) = (-1)^{d-1}(\chi(X\cap f^{-1}(c)) - 1).$$

□

We can now state our main result.

Theorem 3.5. *Let* $(X,0) = (F^{-1}(M^s_{m,n}),0) \subset (\mathbb{C}^N,0)$ *be an IDS and let* $f : (X,0) \to (\mathbb{C},0)$ *be a germ of an analytic function with isolated singularity. Then,*

$$Eu_{f,X}(0) = (-1)^d(\nu(X\cap f^{-1}(0),0) - \nu(X\cap p^{-1}(0),0)).$$

Proof. By the equations 1 and 2, we have

$$Eu_{f,X}(0) = \chi(X\cap p^{-1}(t_0)) - \chi(X\cap f^{-1}(t_0)).$$

By the corollary 3.4, $\nu(X \cap f^{-1}(0), 0) = (-1)^{d-1}(\chi(X \cap f^{-1}(t_0)) - 1)$ and $\nu(X \cap p^{-1}(0), 0) = (-1)^{d-1}(\chi(X \cap p^{-1}(t_0)) - 1)$, therefore,

$$Eu_{f,X}(0) = (-1)^d(\nu(X\cap f^{-1}(0),0) - \nu(X\cap p^{-1}(0),0)).$$

□

Now, applying the Lê-Greuel type formula to the formula on the previous theorem, we get

$$\begin{aligned} Eu_{f,X}(0) &= (-1)^d(\nu(X\cap f^{-1}(0),0) - \nu(X\cap p^{-1}(0),0)) \\ &= (-1)^d(\nu(X\cap f^{-1}(0),0) + \nu(X,0) - \nu(X,0) - \nu(X\cap p^{-1}(0),0)) \\ &= (-1)^d(\mu_D(f) - m_d(X,0)). \end{aligned}$$

Therefore, we have

Corollary 3.6. *Let* $(X,0)=(F^{-1}(M^s_{m,n}),0)\subset(\mathbb{C}^N,0)$ *be an IDS and let* $f:(X,0)\to(\mathbb{C},0)$ *be a germ of an analytic function with isolated singularity. Then,*

$$Eu_{f,X}(0)=(-1)^d(\mu_D(f)-m_d(X,0)).$$

For germs $(X,0)\subset(\mathbb{C}^N,0)$ and $f:(X,0)\to\mathbb{C}$, we have the Brasselet number defined by Dutertre and Grulha Jr ([DG]). In the particular case when f has an isolated singularity, the Brasselet number is equal to

$$B_{f,X}(0)=Eu_X(0)-Eu_{f,X}(0).$$

By the equations 1 and 2, we know that

$$Eu_X(0)-Eu_{f,X}(0)=\chi(X\cap f^{-1}(t_0)),$$

for a generic complex number t_0. Moreover, by the corollary 3.4, we have

$$\chi(X\cap f^{-1}(t_0))=(-1)^{d-1}\nu(X\cap f^{-1}(0),0)+1.$$

Therefore, we have the following proposition.

Proposition 3.7. *Let* $(X,0)=(F^{-1}(M^s_{m,n}),0)\subset(\mathbb{C}^N,0)$ *be an IDS and let* $f:(X,0)\to(\mathbb{C},0)$ *be a germ of an analytic function with isolated singularity. Then,*

$$B_{f,X}(0)=(-1)^{d-1}\nu(X\cap f^{-1}(0))+1.$$

We can also relate the Euler obstruction of a function $Eu_{f,X}(0)$ and the Milnor number $\mu(f)$, for a finite function germ $f:(X,0)\to(\mathbb{C},0)$, where X is a reduced curve and $\mu(f)$ is the Milnor number as defined by Goryunov ([Go]) and Mond, van Straten ([MS]). By the equations 1 and 2, we have

$$Eu_{f,X}(0)=\chi(X\cap p^{-1}(c))-\chi(X\cap f^{-1}(c)),$$

where p is a generic linear map and c is a generic complex number. We remember that $\chi(X\cap p^{-1}(c))$ is equal to the multiplicity of $(X,0)$, $m_0(X,0)$. Also, $\chi(X\cap f^{-1}(c))$ is the degree of f, $\deg f$. Nuño-Ballesteros and Tomazella ([NT]) prove that

$$\deg f=\mu(f)-\mu(X,0)+1,$$

where $\mu(X,0)$ is the Milnor number of the curve as defined by Buchweitz and Greuel ([BuG]). Besides, we know (see [NT]) that

$$m_0(X,0)=1+m_1(X,0)-\mu(X,0),$$

where $m_1(X,0)$ is the first polar multiplicity as defined in [NT]. Therefore,

$$\begin{aligned}Eu_{f,X}(0)&=m_0(X,0)-degf\\&=1+m_1(X,0)-\mu(X,0)-\mu(f)+\mu(X,0)-1\\&=m_1(X,0)-\mu(f)\\&=(-1)[\mu(f)-m_1(X,0)].\end{aligned}$$

Hence, we have

Proposition 3.8. *Let* $f:(X,0)\to(\mathbb{C},0)$ *be a finite function germ with isolated singularity on a reduced curve* $(X,0)$*. Then,*

$$Eu_{f,X}(0)=(-1)[\mu(f)-m_1(X,0)]\ \textit{and}\ B_{f,X}(0)=\deg(f).$$

4. Some examples and applications

Example 4.1. Let $F : (\mathbb{C}^4, 0) \to M_{2,3}$ be the map germ given by

$$F(x, y, z, w) = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,3})$. Let also $f : (X, 0) \to \mathbb{C}$ be defined by $f(x, y, z, w) = x^2 + y^2 + zw$.

Take

$$A = \frac{1}{100} \begin{pmatrix} 6 & -8 & 5 \\ 1 & 8 & 7 \end{pmatrix}$$

and $p = 2x - 3y + 4z - w$. Making computations with Mathematica, we see that

$$m_2(X, 0) = \#\Sigma(p|_{X_A}) = 3 \text{ and } \mu_D(f) = 10.$$

Therefore,

$$Eu_{f,X}(0) = 10 - 3 = 7.$$

Example 4.2. Let $F : (\mathbb{C}^4, 0) \to M_{2,3}$ be the map germ given by

$$F(x, y, z, w) = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,3})$. Let also $f : (X, 0) \to \mathbb{C}$ be defined by $f(x, y, z, w) = x^4 + y^2 + w$.

As in the example 4.1, we have that

$$m_2(X, 0) = 3 \text{ and } \mu_D(f) = 9.$$

Therefore,

$$Eu_{f,X}(0) = 9 - 3 = 6.$$

Remark 4.3. In the previous example $(X, 0)$ is a toric surface. Dalbelo, Grulha Jr. and Pereira ([DGP]) prove a formula to calculate $m_2(X, 0)$. Also, according to [DGP, Example 5.1], the Euler obstruction of f, in the example 4.2, can be calculated using the formula $Eu_{f,X}(0) = n^2 - n$, with $n = 3$.

Example 4.4. Let $F : (\mathbb{C}^5, 0) \to M_{2,4}$ be the map germ given by

$$F(x, y, z, w, v) = \begin{pmatrix} x & y & z & w \\ y & z & w & v \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,4})$. Let also $f : (X, 0) \to \mathbb{C}$ be defined by $f(x, y, z, w, v) = x^3 + v + yz$.

Take

$$A = \frac{1}{100} \begin{pmatrix} -1 & 9 & -7 & -4 \\ -4 & 7 & -8 & -5 \end{pmatrix}$$

and $p = 4x + 6y - 5z + 8w - 8v$. Making computations with Mathematica, we see that

$$m_2(X, 0) = \#\Sigma(p|_{X_A}) = 4 \text{ and } \mu_D(f) = 12.$$

Therefore,

$$Eu_{f,X}(0) = 12 - 4 = 8.$$

Example 4.5. Let $F : (\mathbb{C}^5, 0) \to M_{2,3}$ be the map germ given by

$$F(x, y, z, w, v) = \begin{pmatrix} x & y & z \\ z & w & v \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,3})$. Let also $f : (X, 0) \to \mathbb{C}$ be defined by $f(x, y, z, w, v) = x^2 + y^2 + z^2 + w^2 + v^2$.

Making computations with Mathematica, we have that

$$m_3(X, 0) = 0 \text{ and } \mu_D(f) = 10.$$

Then,

$$Eu_{f,X}(0) = -10.$$

We can apply the corollary 3.6 for the case where $(X, 0)$ is an ICIS to get an easy way to calculate the Euler obstruction of an isolated singularity function germ in an ICIS.

Theorem 4.6. *Let $(X, 0) = (\phi^{-1}(0), 0) \subset (\mathbb{C}^N, 0)$ be an ICIS and let $f : (X, 0) \to (\mathbb{C}, 0)$ be an isolated singularity function germ. Then*

$$Eu_{f,X}(0) = (-1)^d \left(\dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi \rangle + J(f, \phi)} - \dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi \rangle + J(p, \phi)} \right),$$

where $p : (X, 0) \to \mathbb{C}$ is a generic linear map.

Example 4.7. Let $(X, 0) \subset \mathbb{C}^3$ be the germ defined as the zeros of the function $\phi(x, y, z) = x^2 + y^3 - z^5$ and let $f : (X, 0) \to \mathbb{C}$ be the function germ $f(x, y, z) = x^2 + y^2 + z^{10}$. We will calculate the local Euler obstruction of f using the formula from the previous theorem. We take a generic linear function, for instance, $p(x, y, z) = 2x + 3y - 2$. With the help of SINGULAR, we calculate the dimensions of the algebras

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle \phi \rangle + J(f, \phi)} = 25 \text{ and } \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle \phi \rangle + J(p, \phi)} = 10.$$

Therefore,

$$Eu_{f,X}(0) = (-1)^2(25 - 10) = 15.$$

Example 4.8. Let $(X, 0) \subset \mathbb{C}^4$ be the germ defined as the zeros of the function $\phi(x, y, z, w) = x^2 + y^3 - z^5 + w^2$ and let $f : (X, 0) \to (\mathbb{C}, 0)$ be the function germ $f(x, y, z) = x^2 + y^3 + z^2 - wx$.

We take a generic linear function, for instance, $p(x, y, z) = x + y + 7z - 5w$. With the help of SINGULAR, we calculate the dimensions of the algebras

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\langle \phi \rangle + J(f, \phi)} = 21 \text{ and } \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\langle \phi \rangle + J(p, \phi)} = 10.$$

Therefore,

$$Eu_{f,X}(0) = (-1)^3(21 - 10) = -11.$$

Also, we can apply the result

$$Eu_{f,X}(0) = (-1)^d(\mu_D(f) - m_d(X, 0)).$$

to show that the determinantal Milnor number be constant for the family is equivalent to the Euler obstruction of a function be constant for the family.

Proposition 4.9. *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an IDS and let $G : (\mathbb{C}^N \times \mathbb{C}^r, 0) \to \mathbb{C}$ be a family of functions with isolated singularity, we denote $G(x, u) = g_u(x)$. Then, the following are equivalent:*

(1) $Eu_{g_u,X}(0)$ *is constant for the family;*
(2) $\mu_D(g_u)$ *is constant for the family.*

We remark that in the case of an ICIS the previous result was proved by Grulha Jr. ([Gr]).

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