AVERAGE STRUCTURES ASSOCIATED WITH A FINSLER SPACE

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Abstract. Given a Finsler space (M, F) on a manifold M and an *averaging procedure*, the averaging method associates to *Finslerian geometric objects affine geometric objects* living on M. In particular, a Riemannian metric is associated to the fundamental tensor g and an affine, torsion free connection is associated to the Chern connection, for instance. The *average curvature endomorphisms* are obtained by averaging on suitable submanifolds of each tangent space T_xM the curvature endomorphisms of the given Finslerian connection. We discuss the generalization of Riemannian results to the Finslerian connection the average geometric objects. As an illustration of the technique, a generalization of the Gauss-Bonnet theorem to Berwald surfaces using the average metric is presented. The parallel transport and curvature endomorphisms of the average connection are obtained, together with new affine, local isometric invariants of the original Finsler metric. The relation between symmetric Finsler spaces and completeness is explained. The work concludes with the introduction of the notion of convex invariance and how this notion articulates the structure of the Finslerian category.

1. Introduction

1.1. **Average equivalence classes and the structure of the Finsler category.** It is notable that many important results can be generalized from the Riemannian to the Finslerian category by properly adapting the proofs, in some cases in a rather direct way, from the Riemannian category to the Finslerian category (see for instance [4] for an abundant collection of results of this kind). One of the aims of this work is to understand the underlying reasons that make possible such generalizations. In order to clarify that point, we have developed several *average equivalence relations* in the space of Finsler spaces on a given manifold M. These equivalence relations provide a conceptual explanation for the similarity of the Riemannian and Finslerian category.

Among such equivalence relations, a relevant one is the following. Fixed an *averaging procedure*, that is, a procedure for taking averages of Finslerian objects, we say that two Finsler spaces on the same manifold M are *average metric equivalent* iff the corresponding *average Riemannian metrics* are the same. However, this is not the only natural average equivalence relation that can be defined in the Finsler category. Once the Finslerian connection (Chern, Cartan or any other *Finslerian connection*) and the average procedure have been fixed, the following average equivalence relation is defined: two Finsler spaces (M, F_1) and (M, F_2) are *average connection equivalent* iff they have the same *average Finslerian connection.*

These equivalence relations can be used to generalize theorems from the Riemannian to the Finslerian category by the following method. If a *proposition* in the Finsler category is formulated in terms of *invariant notions*, that is, independent of the particular representative of the equivalence class, Riemannian or affine geometry

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results can be generalized to the whole equivalence class. The strategy to perform the generalization is as follows. One choose the convenient representatives of the average equivalence classes (a Riemannian representative in the case of the average metric equivalence or the average connection representative for connections). Fixed an averaging procedure, each equivalence class has an unique representative (a Riemannian metric in the case of equivalence relation defined from averaging the fundamental tensor or an affine connection in the case of an equivalence relation for Finslerian connections). If the hypotheses of the *proposition* admit a formulation which do not depend on the representatives of the equivalence classes, the thesis is valid for all the elements of the equivalence class and the theorem can be proved using Riemannian geometry or affine geometry, in particular, by using the corresponding Riemannian results.

1.2. **Average of Finslerian geometric objects.** The way to choose a convenient representative of a Finslerian geometric object (fundamental tensor, Finslerian connections,curvatures, etc...) is by the *method of averaging.* For a generic Finsler space the average geometric objects provide unrelated geometric objects living on M. However, in the case of Berwald spaces, the different averages are related: the average connection of the Chern connection is the Levi-Civita connection of the average Riemannian metric and the Riemannian curvature tensor of the average connection is the Riemannian curvature tensor of the average metric. However, even in this case, there are details from the original Finsler space that are lost in the averaging process. For instance, given to metrics F_1 and F_2 with the same metric average, one of the metrics can be non-reversible and the other reversible. Thus, the reversible/non-reversible character of a Finsler metric is lost in the averaging.

The average metric has been used in the literature of Finsler geometry in the application to several problems [10, 17, 18, 1, 13, 14], just to mention few examples of different applications. The average connection has been also applied in different problems [11, 12, 1]. Also, an *average Finser-Laplace operator* has been considered in the literature [5]. In this paper, however, we describe the general theory of averaging in the Finsler category, mainly as an attempt to understand the category properly.

1.3. **Description of this work.** Let M be a n-dimensional, real, smooth manifold and let us consider the Finsler space (M,F). A Riemannian structure (M,h) is obtained by integrating at each point $x\in M$ the fundamental tensor g on an appropriate sub-manifold of each tangent space T_xM. Similar average geometric objects are defined for other geometric objects living on the sphere bundle SM. A relevant example of this is the averaging of Finslerian connections, providing a natural affine connection.

The structure of this paper is the following. In *section 2* we introduce several standard notions of Finsler geometry. In *section 3* we explain the measure used for the averaging. The averaging can be seen as an integration operation along fibers of the pull-back bundles $\hat{\pi}^*T^{(p,q)}M$ over the fibered manifold $TM\setminus\{0\}$. We obtain the Riemannian structure (M,h) from the initial Finsler space (M,F) by performing an average on the indicatrix I_x at each point $x\in M$ of the components of the fundamental tensor g. The quasi-metric distance function and the topology induced by the Finsler distance function are related with the analogous notions associated with the Riemannian metric h. We show that if a given average Riemannian structure is complete, then the Finsler structure is forward and backward complete. In *section 4* we obtain the average connection $\langle\nabla\rangle$ of a Finslerian connection ∇. We show that for Berwald spaces, the Riemann curvature tensor of the average metric h can be obtained by averaging the corresponding tensor of the Chern connection of g.

The average metric is used to prove several generalizations of Riemannian results to Berwald spaces. In particular, we have considered a generalization of the Gauss-Bonnet theorem for Berwald surfaces. In *section 5* the parallel transport operator and curvatures of the average connection of $\langle\nabla\rangle$ in terms of the corresponding parallel transport and curvatures of the Finslerian connection ∇. The average of the curvature tensor of the connection ∇ provides new isometric invariants for F, still to be fully exploited in applications. In *section 6*, motivated by these constructions, the notion of convex invariance is introduced. Then we use the notion of convex invariance to state what is the structure of the space M_F of Finsler metrics on M.

2. Basic notions of Finsler geometry

Let $(U,\bar{x})$ be a local coordinate chart of the n-dimensional manifold M, where a point $x \in U$ has local coordinates $(x^1,...,x^n)$ and $U \subset M$ is an open set. TM is the tangent bundle manifold of M. The slit tangent bundle is the bundle over M $\hat{\pi}: N \to M$, with $N = TM \setminus \{0\} \hookrightarrow TM$. A generic tangent vector at the point $x \in M$ is denoted in local coordinates by $y^i \frac{\partial}{\partial x^i} \in T_xM$, $y^i \in \mathbf{R}$ (in this work we use Einstein's convention for up and down equal indices, if anything else is not directly stated). The set of sections of the bundle TM is denoted by ΓTM. Fixed a local chart $(U,\mathbf{x})$ on the manifold M, a the point $x \in U$ has coordinates $(x^1,...,x^n)$ and the tangent vector $y \in T_xM$ at x with its components $y = (y^1,...,y^n)$. Thus, each local chart $(U,\mathbf{x})$ on M induces a local chart on M denoted by $(TU,\mathbf{x},\mathbf{y})$ such that $y = y^i \frac{\partial}{\partial x^i} \in T_xM$ has local natural coordinates $(x^1,...,x^n,y^1,...,y^n)$.

Definition 2.1. *A Finsler space on the manifold M is a pair (M,F) where F is a non-negative, real function $F : M \to [0,\infty[$ such that*

- *It is smooth in the slit tangent bundle N,*
- *Positive homogeneity holds: $F(x,\lambda y) = \lambda F(x,y)$ for every $\lambda > 0$,*
- *Strong convexity holds: the Hessian matrix*

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad i,j = 1,...,n \tag{2.1}$$

is positive definite on N.

The minimal smoothness requirement for the Finsler space (M,F) is that F must be a $\mathcal{C}^4$-smooth function on N. However, if Bianchi identities are required, then it is necessary that F is at least $\mathcal{C}^5$-smooth on N. The matrix $(g)_{ij} := g_{ij}(x,y)$ is the matrix components of the fundamental tensor g at $u \in N$.

Definition 2.2. *Let (M,F) be a Finsler space and $(TU,\mathbf{x},\mathbf{y})$ a local chart induced on N from the coordinate system $(U,\mathbf{x})$ of M. The components of the Cartan tensor are defined by the collection of functions*

$$A_{ijk} = \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}, \quad i,j,k = 1,...,n. \tag{2.2}$$

The component functions $\{A_{ijk},\, i,j,k = 1,...,n\}$ are homogeneous of degree zero in the coordinates $(y^1,...,y^n)$. In the Riemannian case the functions A_{ijk} are zero and this fact characterizes Riemannian geometry from other types of Finsler geometries.

2.1. **The non-linear connection.** There are several notions of connection in the literature. We adopt the following definition, which is equivalent to the notion of general connection as discussed in [16]. Let us consider the tangent bundle $\bar{\pi}: TN \to N$.

Definition 2.3. *A non-linear connection is a distribution $\mathcal{H} \subset TN$ supplementary to the canonical vertical distribution $\mathcal{V} = \ker d\bar{\pi}$.*

Therefore, for each $u \in N$ the splitting $T_uN = \mathcal{V}_u \oplus \mathcal{H}_u$ holds. Given a Finsler space (M, F), there is defined a non-linear connection in the manifold N. Let us consider the local chart $(TU, \mathbf{x}, \mathbf{y})$ of the manifold N. The local sections

$$\{\frac{\partial}{\partial y^1}|_u, ..., \frac{\partial}{\partial y^n}|_u,\, u \in \pi^{-1}(x),\, x \in U\}$$

determine a local frame for the vertical distribution $\mathcal{V}$. To obtain a supplementary distribution $\mathcal{H}$ we use a local construction. First, let us introduce the non-linear connection coefficients N^i_j by the expression

$$\frac{N^i_j}{F} = \gamma^i_{jk}\frac{y^k}{F} - A^i_{jk}\gamma^k_{rs}\frac{y^r}{F}\frac{y^s}{F}, \quad i, j, k, r, s = 1, ..., n,$$

where the *formal second kind Christoffel's symbols* $\gamma^i_{jk}(x, y)$ are defined by

$$\gamma^i_{jk} = \frac{1}{2}g^{is}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j}), \quad i, j, k = 1, ..., n.$$

$A^i_{jk} := g^{il}A_{ljk}$ and $g^{il}g_{lj} = \delta^i_j$. A tangent basis for T_uN is determined by the vectors

(2.3)
$$\{\frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u, F\frac{\partial}{\partial y^1}|_u, ..., F\frac{\partial}{\partial y^n}|_u\}, \quad \frac{\delta}{\delta x^j}|_u = \frac{\partial}{\partial x^j}|_u - N^i_j\frac{\partial}{\partial y^i}|_u, \quad i, j = 1, ..., n.$$

The local sections

$$\{\frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u,\, u \in \pi^{-1}(x),\, x \in U\}$$

determine a local frame for the *horizontal distribution* $\mathcal{H}$ [2, 4].

A local basis of the dual vector space T^*_uN is

(2.4)
$$\{dx^1|_u, ..., dx^n|_u, \frac{\delta y^1}{F}|_u, ..., \frac{\delta y^n}{F}|_u\}, \quad \frac{\delta y^i}{F}|_u = \frac{1}{F}(dy^i + N^i_j dx^j)|_u, \quad i, j = 1, ..., n.$$

The collection for each $u \in TU$ of such local basis determine a local dual frame of the local frame for T^*N.

The horizontal lift of vector fields is the homomorphism

$$\imath_u : \Gamma TM \to \Gamma TN, \quad X = X^i\frac{\partial}{\partial x^i}|_x \mapsto \imath_u(X) = X^i\frac{\delta}{\delta x^i}|_u,\, u \in \pi^{-1}(x). \tag{2.5}$$

Definition 2.4. *Let (M, F) be a Finsler space. The fundamental and the Cartan tensors are defined in the natural local coordinate system $(TU, \mathbf{x}, \mathbf{y})$ by the equations*

- *The fundamental tensor is*

$$g(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}\, dx^i \otimes dx^j, \tag{2.6}$$

- *The Cartan tensor is*

$$A(x, y) := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}\frac{\delta y^i}{F} \otimes dx^j \otimes dx^k = A_{ijk}\frac{\delta y^i}{F} \otimes dx^j \otimes dx^k. \tag{2.7}$$

Remark 2.5. *Note that our definition of the Cartan's tensor* (2.7) *differs from the usual definition* [4], *which is defined as the completely symmetric tensor $A = A_{ijk}\, dx^i \otimes dx^j \otimes dx^k$. However, such difference is only formal and not interference in the properties that one expects from the homogeneity of the Cartan tensor.*

2.2. **The Chern connection and other Finslerian notable connections.** Let us consider the cartesian product $N \times TM$ and the canonical projections

$$\hat{\pi}_1 : N \times TM \to N, \quad (u, \xi) \mapsto u, \qquad \hat{\pi}_2 : N \times TM \to TM, \quad (u, \xi) \mapsto \xi.$$

The pull-back $\pi_1 : \hat{\pi}^*TM \to N$ of the bundle $\pi : TM \to M$ by the map $\hat{\pi} : N \to M$ is the maximal sub-manifold of the cartesian product $N \times TM$ such that the following relation holds: for $u \in N$ and $(u, \xi) \in \hat{\pi}_1^{-1}(u)$, $(u, \xi) \in \hat{\pi}^*TM$ iff $\pi \circ \hat{\pi}_2(u, \xi) = \hat{\pi} \circ \hat{\pi}_1(u, \xi)$. From the definition of the pull-back bundle $\hat{\pi}^*TM$ it follows that the diagram

$$\begin{array}{ccc} \hat{\pi}^*TM & \xrightarrow{\pi_2} & TM \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\hat{\pi}} & M \end{array}$$

commutes. Here, $\pi_1 : \hat{\pi}^*TM \to N$ and $\pi_2 : \hat{\pi}^*TM \to TM$ are the restrictions of the natural projections $\hat{\pi}_1 : N \times TM \to N$ and $\hat{\pi}_2 : N \times TM \to TM$ to $\hat{\pi}^*TM$. $\pi_1 : \hat{\pi}^*TM \to N$ is a real vector bundle, with fiber over $u = (x, Z) \in N$ isomorphic to T_xM. The fiber dimension of $\hat{\pi}^*TM$ is equal to $n = dim(M)$, while the dimension of the base manifold N is $2n$. Each vector field $Z \in T_xM$ can be pulled-back $\hat{\pi}^*Z$ uniquely by the conditions

- $\pi_1(\hat{\pi}^*Z) = (x, Z) \in N_x$,
- $\pi_2(\hat{\pi}^*Z) = Z$.

The pull-back of a smooth function $f \in \mathcal{F}(M)$ is a smooth function $\hat{\pi}^*f \in \mathcal{F}(\hat{\pi}^*TM)$ such that $\hat{\pi}^*f(u) = f(\hat{\pi}(u))$ for every $u \in N$. Analogously, a pull-back tensor bundle $\hat{\pi}^*T^{(p,q)}M$ can be obtained from each tensor bundle $T^{(p,q)}M$ over M.

Definition 2.6. *A Finslerian connection is a linear connection on the pull-back bundle $\pi_1 : \hat{\pi}^*TM \to N$.*

The associated covariant derivative is an operator $\nabla : \Gamma\,\hat{\pi}^*TM \times \Gamma\,TN \to \Gamma\,\hat{\pi}^*TM$ such that

- For every $X \in \Gamma TN$, $S_1, S_2 \in \Gamma\,\hat{\pi}^*TM$ and $f \in \mathcal{F}(M)$ it holds that

$$\nabla_X(\hat{\pi}^*fS_1 + S_2) = (X \cdot \hat{\pi}^*f)S_1 + f\nabla_X S_1 + \nabla_X S_2. \tag{2.8}$$

- For every $X_1, X_2 \in \Gamma\,TN$, $S \in \Gamma\,\hat{\pi}^*TM$ and $\lambda \in \mathcal{F}(N)$ it holds that

$$\nabla_{fX_1 + X_2} S = f\nabla_{X_1} S + \nabla_{X_2} S. \tag{2.9}$$

The Chern connection ${}^{ch}\nabla$ is a Finslerian connection on $\hat{\pi}^*TM$. As any linear connection, it is determined by the structure equations for the connection 1-forms $\{\,{}^{ch}\omega^i_j,\ i, j = 1, ..., n\}$ as given by the following theorem [4],

Theorem 2.7. *Let (M, F) be a Finsler space. The vector bundle $\hat{\pi}^*TM$ admits a unique linear connection characterized by the collection of connection 1-forms $\{\,{}^{ch}\omega^i_j,\ i, j = 1, ..., n\}$ such that the following structure equations hold:*

- *"Torsion free" condition,*

$$d(dx^i) - dx^j \wedge {}^{ch}w^i_j = 0, \quad i, j = 1, ..., n. \tag{2.10}$$

- *Almost g-compatibility condition,*

$$dg_{ij} - g_{kj}\,{}^{ch}w^k_i - g_{ik}\,{}^{ch}w^k_j = 2A_{ijk}\frac{\delta y^k}{F}, \quad i, j, k = 1, ..., n. \tag{2.11}$$

The torsion free condition is equivalent to the absence of terms containing dy^i in the connection 1-forms ${}^{ch}\omega^i_j$ and also implies the symmetry of the connection coefficients $\Gamma^i_{jk}(x,y)$,

$$(2.12)\qquad {}^{ch}\omega^i_j(x,y) = \Gamma^i_{jk}(x,y)\,dx^k, \quad \Gamma^i_{jk}(x,y) = \Gamma^i_{kj}(x,y), \quad i,j,k = 1,...,n.$$

The torsion free condition and almost g-compatibility determine the expression of the connection coefficients of the Chern connection in terms of the Cartan and fundamental tensors. We can describe the Chern connection by using the associated covariant derivative operator along directions $\tilde{X} \in \Gamma TN$. We denote this covariant derivative operator by ${}^{ch}\nabla_{\tilde{X}}$, $\tilde{X} \in T_xN$. Let us denote by $V(X)$ the vertical and by $H(X)$ the horizontal components of a tangent vector $X \in T_uN$ defined by the non-linear connection on N. Then the *generalized torsion tensor* is given by the expression

(2.13)

$$T_{{}^{ch}\nabla} : \Gamma TM \times \Gamma TM \to \Gamma TM, \quad (X,Y) \mapsto \pi_2\big({}^{ch}\nabla_{\tilde{X}}\hat{\pi}^*Y\big) - \pi_2\big({}^{ch}\nabla_{\tilde{Y}}\hat{\pi}^*X\big) - [X,Y].$$

Then the following corollaries are direct consequences of *Theorem* 2.7.

Corollary 2.8. *Let (M,F) be a Finsler space. The torsion free condition* (2.10) *is equivalent to the following conditions:*

(1) *Null vertical covariant derivative of sections of $\hat{\pi}^*TM$. For any $\tilde{X} \in TN$ and $Y \in M$, the following relations hold,*

$$(2.14)\qquad {}^{ch}\nabla_{V(\tilde{X})}\hat{\pi}^*Y = 0.$$

(2) *Let us consider $X, Y \in \Gamma TM$ and the horizontal lifts $\tilde{X} = X^i\frac{\delta}{\delta x^i}$ and $\tilde{Y} = Y^i\frac{\delta}{\delta x^i}$. Then the following relation holds,*

$$(2.15)\qquad T_{{}^{ch}\nabla}(X,Y) = 0.$$

Proof. In order to prove the first condition, we make the following calculation. Let us consider a local frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ on U and $X = e_i = \frac{\partial}{\partial x^i}$, $Y = e_j = \frac{\partial}{\partial x^j}$. Then we have

$${}^{ch}\nabla_{V(\tilde{X})}\hat{\pi}^*Y := \hat{\pi}^*e_k\, w^k_j(V(e_i) = \hat{\pi}^*e_k\Gamma^k_{aj}dx^a(V(e_i)) = \hat{\pi}^*e_k\Gamma^k_{aj}dx^a(\frac{\partial}{\partial y^i}) = 0.$$

This is extended by linearity and the Leibnitz rule to arbitrary sections $X, Y \in \Gamma TM$.

To prove the relation (2.15), let us consider the torsion condition in local coordinates where the local frame $\{e_j\}$ of ΓTM commutes, $[e_i, e_j] = 0$. Then from the symmetry in the connection coefficients and the definition of the condition (2.15) one obtains

$$\pi_2({}^{ch}\nabla_{\tilde{e_i}}\hat{\pi}^*e_j) - \pi_2({}^{ch}\nabla_{\tilde{e_j}}\hat{\pi}^*e_i) - [e_i,e_j] = \pi_2({}^{ch}\nabla_{\tilde{e}_i}\hat{\pi}^*e_j) - \pi_2({}^{ch}\nabla_{\tilde{e}_j}\hat{\pi}^*e_i) = (\Gamma^k_{ij} - \Gamma^k_{ji})\,\hat{\pi}^*e_k = 0.$$

This is extended by linearity to arbitrary vectors $X, Y \in \Gamma TM$, since by construction $T_{{}^{ch}\nabla}$ is an $\mathcal{F}M$-linear homomorphism, $T_{{}^{ch}\nabla} \in Hom(\Gamma TM \times \Gamma TM,\ \Gamma\hat{\pi}^*TM)$. □

Corollary 2.9. *Let (M,F) be a Finsler space and $\tilde{X} \in \Gamma TN$. The almost g-compatibility condition* (2.11) *is equivalent to the conditions:*

- *${}^{ch}\nabla$ is metric compatible in the horizontal directions,*

$$(2.16)\qquad {}^{ch}\nabla_{H(\tilde{X})}g = 0.$$

- *${}^{ch}\nabla$ is almost-metric compatible in the vertical directions in the sense that*

$$(2.17)\qquad {}^{ch}\nabla_{V(\tilde{X})}g = 2A(\tilde{X},\cdot,\cdot)$$

holds good.

Proof. Using local natural coordinates and reading from 2.11, it follows that

$$ {}^{ch}\nabla g = (dg_{ij} - g_{kj}\,{}^{ch}w^k_i - g_{ik}\,{}^{ch}w^k_j)\,\hat{\pi}^*e^i \otimes \hat{\pi}^*e^j \;=\; 2A_{ijk}\frac{\delta y^k}{F} \otimes \hat{\pi}^*e^i \otimes \hat{\pi}^*e^j, $$

that corresponds to (2.16). By the definition of covariant derivative along a direction and since $2A_{ijk}\frac{\delta y^k}{F}$ is vertical, one obtains

$$ {}^{ch}\nabla_{(\tilde{X})}(g) := \; 2A_{ijk}\frac{\delta y^k}{F}(\tilde{X})\,\hat{\pi}^*e^i \otimes \hat{\pi}^*e^j,\; \forall \tilde{X} \in \Gamma TN. $$

From this expression follows the condition (2.17). □

The curvature endomorphisms associated with a Finslerian connection ∇ are defined by the covariant exterior differential of the connection 1-forms $\{\omega^i_j\, i,j = 1,...,n\}$,

$$ \Omega^i_j := dw^i_j - w^k_j \wedge w^i_k, \quad i,j,k = 1,...,n. \tag{2.18} $$

In local coordinates, the curvature endomorphisms are decomposed as

$$ \Omega^i_j = \frac{1}{2}R^i_{jkl}dx^k \wedge dx^l + P^i_{jkl}dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2}Q^i_{jkl}\frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}. \tag{2.19} $$

The quantities R^i_{jkl}, P^i_{jkl} and Q^i_{jkl} are the hh, hv, and vv-curvature tensor components. For the Chern connection the Q-component is zero, for any Finsler space [4]. However, for other linear connections in $\hat{\pi}^*TM$ the three curvature types could be non-zero.

2.3. **Other Finslerian connections.** Other Finslerian connection relevant in Finsler geometry is Cartan's connection. Cartan's connection is metric compatible, but has torsion [4]. The relation between the Cartan connection 1-forms $({}^c\omega)^k_i$ and the Chern connection 1-forms ${}^{ch}\omega^k_i$ is given by

$$ ({}^c\omega)^k_i = \,{}^{ch}\omega^k_i \,+\, A^k_{ij}\frac{\delta y^j}{F}, \quad i,j,k = 1,...,n. \tag{2.20} $$

There is also the Berwald (linear) connection, given by the 1-forms [2],

$$ ({}^b\omega)^k_i = \,{}^{ch}\omega^k_i \,+\, \dot{A}^k_{ij}\frac{\delta y^j}{F} \tag{2.21} $$

$\{\dot{A}^k_{ij}\}$ are determined by the components of the Landsberg tensor [2]. We remark that the results of *section 3* the main results of *section 4* and *section 5* are valid for any linear Finslerian connection on $\pi_1 : \hat{\pi}^*TM \to N$.

3. Riemannian average metrics from a Finsler space

3.1. **Definition of the averaging procedure for Finsler metrics.** Let $(U, \mathbf{x})$ be a local chart on the base manifold M and $(TU, \mathbf{x}, \mathbf{y})$ the induced local chart on the tangent bundle TM. The n-form d^ny is defined in local coordinates by the expression

$$ d^ny = \sqrt{det\, g(x,y)}\, dy^1 \wedge \cdots \wedge dy^n, \tag{3.1} $$

where $det\, g(x,y)$ is the determinant of the fundamental tensor $(g)_{ij} = g_{ij}(x,y)$. The n-form d^ny is invariant by local diffeomorphisms on TM. Let us denote the metric on T_xM $g_x(y) := g(x,y)$. Then the pair $(T_xM \setminus \{0\}, g_x)$ is a Riemannian structure and the n-form (3.1) induces a diffeomorphic invariant volume form on $T_xM \setminus \{0\}$. The volume form for the manifold $T_xM \setminus \{0\}$ is the n-form d^ny restricted to N.

Definition 3.1. *Let (M, F) be a Finsler space. The submanifold*

$$I_x := \{y \in T_xM \mid F(x, y) = 1\} \subset T_xM$$

is the indicatrix over the point $x \in M$. $\mathcal{I}$ is the fibered manifold $\pi_{\mathcal{I}} : \mathcal{I} \to M$ with $\pi_{\mathcal{I}}^{-1}(x) = I_x$ and the base manifold is M.

The indicatrix I_x is a compact, strictly convex submanifold of T_xM [4].

Let $(T_xM \setminus \{0\}, g_x)$ be the standard Riemannian metric induced on T_xM [4] and $\tilde{g}$ be the Riemannian metric on I_x induced from $(T_xM \setminus \{0\}, g_x)$ by isometric embedding. Then the pair $(I_x, \tilde{g})$ is a Riemannian manifold. Moreover, for each embedding $i_x : I_x \hookrightarrow N_x$ one consider the volume form on I_x given by

$$dvol_x := i_x^* d^n y. \tag{3.2}$$

Then the volume function $vol(I_x)$ is defined by

$$vol : M \to R^+, \quad x \mapsto vol(I_x) = \int_{I_x} dvol_x, \tag{3.3}$$

where the weight factor $f : TM \setminus \{0\}$ is a homogenous of degree zero in y, positive, smooth function on the tangent bundle. With these volume forms one can perform the following integration of $(n-1)$-forms to define averages of functions $\psi(x, y) \in \mathcal{F}(\mathcal{I})$,

$$\langle \psi(x, y) \rangle_f(x) = \frac{1}{vol(I_x)} \int_{I_x} \psi(x, y)\, f(x, y)\, dvol_x$$

Let (M, F) be a Finsler function as let us define the matrix components by

$$h_{ij}(f, x) := \langle g_{ij}(x, y) \rangle_f,\ i, j = 1, ..., n, \tag{3.4}$$

for each point $x \in M$.

Proposition 3.2. *Let (M, F) be a Finsler space. Then the coefficients*

$$h_{ij}(f, x) = \langle g_{ij}(x, y) \rangle_f, \quad i, j = 1, ..., n$$

are the components of a Riemannian metric defined on M such that in the local coordinate system $(U, \mathbf{x})$ the metric is

$$h_f(x) = h_{ij}(f, x)\, dx^i \otimes dx^j, \quad i, j = 1, ..., n. \tag{3.5}$$

Proof. The average (3.4) of a positive defined, real and symmetric $n \times n$ matrix is also a positive, real, symmetric matrix. To show this, let us consider

$$g_x(y)(\tilde{y}, \tilde{y}) = g_{ij}(x, y)\tilde{y}^i \tilde{y}^j,\ y, \tilde{y} \in T_xM.$$

This is because $g_x(y)$ for each fixed $y \in T_xM$ is a positive defined scalar product in T_xM. Therefore, the product $g_x(y)(\tilde{y}, \tilde{y})$ is positive definite for any $\tilde{y} \in T_xM$. In addition, we need to show the tensorial character of the expression (3.5). But it is direct from the expression (3.4), from the transformation rule of g and the invariance of the weight function f by coordinates transformations of the form

$$x \mapsto \tilde{x}(x),\ y^i \mapsto \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\, y^j$$

we have that

$$F(x, y) = \tilde{F}(\tilde{x}(x), \tilde{y}(y)), \quad I_x = \tilde{I}_{\tilde{x}}, \quad \tilde{g}_{ij}(\tilde{x}, \tilde{y}) = \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial x^k}{\partial \tilde{x}^i} g_{lk}(x, y),$$

being $\tilde{I}_{\tilde{x}} = \{y \in T_xM \mid \tilde{F}(\tilde{x}, \tilde{y}) = 1\}$. These relations imply that the transformation rule for the components $h_{ij}(f, x)$ is tensorial. □

The fact that there is not uniqueness on the definition of the averaging procedure motivates the following notion,

Definition 3.3. *An averaging procedure is a triplet* $(\Upsilon,\, \{dvol_\Upsilon(x)\}_{x\in M},\, f)$*, where*

- *The fiber bundle* $\pi_\Upsilon : \Upsilon \to M$ *is a sub-bundle of* $\hat{\pi} : N \to M$ *with fiber codimension* 1 *and whose fibers* $\pi_\Upsilon^{-1}(x)$ *are smooth, start domain manifolds of* N_x*, with center at the origin* $0 \in T_xM$*,*
- $\{dvol_\Upsilon(x)\}_{x\in M}$ *is a family of volume forms on each fiber* $\pi_\Upsilon^{-1}(x)$*,*
- $f : \Upsilon \to \mathbf{R}^+$ *is a homogenous of degree zero in* y*-coordinates, positive and smooth weight function.*

This definition allows to consider several averages. In particular, it is clear that the following result holds,

Theorem 3.4. *Let* (M, F) *be a Finsler space. Then for each triplet* $(\Upsilon,\, \{dvol_\Upsilon(x)\}_{x\in M},\, f)$ *there is an averaged metric given by the expression*

(3.6)
$$h_\gamma(x) = (h_\Upsilon)_{ij}\, dx^i \otimes dx^j, \quad (h_\Upsilon)_{ij} = \frac{1}{vol(\Upsilon(x))} \int_{\Upsilon(x)} g_{ij}(x,y)\, f(x,y)\, dvol_{\Upsilon(x)},$$

with

$$vol(\Upsilon(x)) = \int_{\Upsilon(x)} dvol_{\Upsilon(x)}. \tag{3.7}$$

The proof of this *Theorem* is analogous to the proof of *Proposition* 3.2 and we will not repeat it. Also, we will fix the averaging procedure as

$$\{(\Upsilon,\, \{dvol_\Upsilon(x)\}_{x\in M},\, f)\} = \{(\mathcal{I}, \{dvol_x\}_{x\in M}, f = 1)\},$$

if anything else is not stated.

Definition 3.5. *Two Finsler spaces* (M, F_1)*,* (M, F_2) *are said to be* h*-equivalent if the average metrics associated to* F_1 *and* F_2 *are the same.*

It can be shown that this definition provides an equivalence relation on the space M_F of Finsler spaces on M. The equivalence class containing g, the fundamental tensor of (M, F), is $[g]$. Let us investigate the structure of the equivalence classes

$$\{[g],\, g = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial^j},\, F \in M_F\}.$$

We first note that there is an unique average metric h as in (3.5) on each equivalence class. Therefore, for each equivalence class $[h = \langle g\rangle]$ there is defined a bundle $\pi_\Sigma : \Sigma \to M$, whose fibers are

$$\Sigma_x = \pi^{-1}(x) := \{y \in T_xM\, s.t.\, h_x(y) = 1\}. \tag{3.8}$$

It is direct that $\{(\Sigma_x,\, dvol_{\Sigma_x}, f = 1)\}$ defines an averaging procedure, where $dvol_{\Sigma_x}$ is the volume form on Σ_x and the Riemannian volume form $dvol_{\Sigma_x}$ is induced from the volume form of the corresponding Riemannian metric $\hat{h}$ defined on T_xM. Therefore, for each equivalence class $[h]$ one can define another average metric $\tilde{h}$ whose components are given by

$$\tilde{h}_{ij} = \frac{1}{vol(\Sigma_x)} \int_{\Sigma_x} g_{ij}(x,y)\, dvol_{\Sigma_x}, \tag{3.9}$$

We call the metric $\tilde{h} = (\tilde{h}_\Sigma)_{ij}\, dx^i \otimes dx^j$ the Σ-average metric.

Definition 3.6. *Let us consider (M, F) be a fixed Finsler space and let $\tilde{h}$ be the Σ-averaged metric. Two Finsler spaces (M, F_1), (M, F_2) are said to be Σ-equivalent if the Σ-average metrics associated to F_1 and F_2 are the same.*

This relation is also an equivalence relation in the space of Finsler spaces M_F. The equivalence class containing the Σ-averaged metric $\tilde{h}$ is $[\tilde{h}]_\Sigma$.

Proposition 3.7. *The equivalence class $[\tilde{h}]_\Sigma$ of the Finsler metrics having the same Σ-average metric $\tilde{h}$ is a convex set.*

Proof. Let (M, F_1) and (M, F_2) be two Finsler spaces with the same Σ-average metric $\tilde{h}$. Then one considers the one-parameter family of fundamental tensors

$$^t g = t\, g_1 + (1-t)\, g_2, \quad t \in [0,1],$$

with g_i the fundamental tensor of F_i. Each of these tensors has Σ-average metric $\tilde{h}$ and the Finsler function F_t is given by the combination

$$F_t = \sqrt{tF_1^2 + (1-t)F_2^2}, \quad t \in [0,1].$$

□

Proposition 3.8. *If two Finsler spaces (M, F_1) and (M, F_2) have the same average metric h, then they have the same Σ-average metric $\tilde{h}$.*

Proof. If $\langle g_1 \rangle = \langle g_2 \rangle$, then the corresponding bundles Σ_1, $\Sigma_2 \to M$ are the same, and the volume forms $dvol_{1,\Sigma_x}, dvol_{2,\Sigma_x}$ are also the same. There is a dilatation map $\Sigma_{x1} \to \Sigma_{x2}$, $y \mapsto \lambda(y)\, y$, $\lambda(y) \in \mathbf{R}^+$. Since the components $g_{ij}(x,y)$ are 0-homogeneous in y, one has that $\tilde{h}_{ij}(1,x) = \tilde{h}_{ij}(2,x)$. □

Theorem 3.9. *h-equivalence and Σ-equivalence are the same equivalence relation in M_F.*

Proof. Let us consider to average metrics h_1 and h_2 in the same equivalence class $[\tilde{h}]_\Sigma$. Then, we have

$$\tilde{h} = \langle (h_1)_{ij} \rangle_\Sigma = (h_1)_{ij}, \quad \tilde{h} = \langle (h_2)_{ij} \rangle_\Sigma = (h_2)_{ij}. \tag{3.10}$$

Therefore, $h_1 = h_2$. This means that in each equivalence class $[\tilde{h}]_\Sigma$ there is only one average metric h and therefore, there is only one equivalence class $[h]$. □

3.2. **A coordinate-free formula for h.** A geometric formula for the *isotropic* metric h is presented. It will be expressed in terms of the canonical projections on the bundle $\Gamma\, \hat{\pi}^* T^{(0,2)} M$. Let us consider the fiber metric

$$\bar{g} = g_{ij}(x,y) \hat{\pi}^* dx^i \otimes \hat{\pi}^* dx^j, \quad \xi \in \pi_2^{-1}(u),\, u \in \hat{\pi}^{-1}(x).$$

Then we have the following

Proposition 3.10. *Let (M, F) be a Finsler space. Then the following equation holds,*

$$h(x)(X,X) = \frac{1}{vol(I_x)} \Big(\int_{I_x} dvol_x\ \bar{g}\, (\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X)) \Big) \tag{3.11}$$

for each vector field $X \in \Gamma TM$.

Proof. For an arbitrary point $x \in M$, $h(x) = \langle g_{ij}\rangle(x)\, dx^i|_x \otimes dx^j|_x$, $u \in I_x$ and $X \in \Gamma TM$ it follows that

$$\begin{aligned} h(x)(X,X) &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\; g_{ij}(u)\Big) X^i\, X^j \\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\; \bar{g}_{ij}(u)\Big) X^i\, X^j \\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\; \bar{g}\,(\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X))\Big). \end{aligned}$$

□

Remark 3.11. *The proof of 3.10 is valid for any metric h_f obtained from averaging g with a positive, smooth, weight factor function $f : \mathcal{I} \to \mathbf{R}^+$.*

3.3. **Applications of the average metric in Finsler geometry.** We can relate the quasi-metric determined by (M, F) with the metric (M, h) as follows. Given a regular curve $\gamma : [a, b] \to M$, the length of the curve using the metric F is given by

$$L_F[\gamma] := \int_a^b F(\gamma(t), \dot{\gamma}(t))dt, \tag{3.12}$$

and calculated using the metric h is given by the expression

$$L_h[\gamma] := \int_a^b \sqrt{h_{ij}(\gamma(t))\dot{\gamma}^i\dot{\gamma}^j}\; dt. \tag{3.13}$$

Since the metric F is a Finsler metric, for each tangent vector $v \in T_{\gamma(t)}M$ the fundamental tensor $g_{ij}(\gamma(t), v)$ determines a positive defined scalar product on the vector space $T_{\gamma(t)}M$. Therefore, we have that for each $t \in [a, b]$

$$\begin{aligned} &\frac{1}{vol(I_{\gamma(t)})}\, g_{ij}(\gamma(t), \dot{\gamma}(t))\dot{\gamma}^i(t)\,\dot{\gamma}^j(t) \leq \frac{1}{vol(I_{\gamma(t)})}\Big(\int_{I_{\gamma(t)}} g_{ij}(\gamma(t), v)\, dvol_x(v)\Big)\dot{\gamma}^i(t)\dot{\gamma}^j(t) \\ &= h_{ij}(\gamma(t))\dot{\gamma}^i(t)\,\dot{\gamma}^j(t). \end{aligned}$$

This implies the relation

$$L_F[\gamma] \leq \Big(\sup_{t\in\,[a,b]}\,\{\sqrt{vol(I_{\gamma(t)})}\}\Big)\, L_h[\gamma] \tag{3.14}$$

for any smooth curve $\gamma : [a, b] \to M$.

Given a Finsler space (M, F), the metric distance between two points $p, q \in M$ in the same connected component of M is defined in the following way. Let $\Omega_1(p, q)$ be the space of smooth normalized curves such that $F(\gamma(t), \dot{\gamma}(t)) = 1$ starting at p and ending at q. If $\Omega_1(p, q) \neq \emptyset$, the *Finslerian distance* between p and q is

$$d_F(p, q) = \inf\{L_F[\gamma],\ \gamma \in \Omega_1(p, q)\}. \tag{3.15}$$

Similarly the *Riemannian distance* $d_h(p, q)$ between p and q is associated to the Riemannian metric h. Then for any curve $\gamma \in \Omega_1(p, q)$ we have the relation

$$d_F(p, q) \leq \text{Ł}_F[\gamma] \leq \sqrt{vol(I_{\gamma(t)})}\, L_h[\gamma]. \tag{3.16}$$

Proposition 3.12. *Let (M, F) be a Finsler space and $vol : M \to R^+,\ x \mapsto vol(I_x)$ be the Finslerian volume function. Then for any pair of path-wise connected points $p, q \in M$ the following relation holds,*

$$\frac{1}{\sup_{x\in M}\{\sqrt{vol(I_x)}\}}\, d_F(p, q) \leq\, d_h(p, q). \tag{3.17}$$

Proof. Let $\tilde{\gamma}$ be a curve realizing the Riemannian distance $d_h(p,q)$. Then we have

$$d_F(p,q) \leq L_F[\tilde{\gamma}] \leq \Big(\sup_{t\in [a,b]} \big\{ \sqrt{vol(I_{\tilde{\gamma}(t)})} \big\} \Big) L_h[\tilde{\gamma}]$$
$$\leq \Big(\sup_{x\in M} \big\{ \sqrt{vol(I_x)} \big\} \Big) L_h[\tilde{\gamma}] = \Big(\sup_{x\in M} \big\{ \sqrt{vol(I_x)} \big\} \Big) d_h(p,q).$$

□

As direct consequence, if the volume function is bounded by $vol(I_{x_0})$ for some $x_0 \in M$, the following relation holds,

$$d_F(p,q) \leq \sqrt{vol(I_{x_0})}\, d_h(p,q). \tag{3.18}$$

Corollary 3.13. *Let (M,F) be a Finsler space such that the volume function $vol : M \to \mathbf{R}^+$ is locally bounded on M. Then the metric topology induced by d_F is finer than the topology induced by d_h.*

Proof. As a consequence of the relation (3.18), given an open ball $B_h(x,\delta)$ with the topology of h, there is a ball $B_F(x,\delta')$ in the topology of F, with $\delta' < (\sqrt{vol(I_{x_0})})^{-1}\,\delta$ and such that $B_F(x,\delta') \subset B_h(x,\delta)$. □

For the notion of forward and backward metric completeness and geodesically completeness, see [4]. For any Cauchy's sequence $\{x_i\}_{i=1}^{+\infty}$ of h, we have by (3.18) that given an $\epsilon > 0$ and for some $i_0 < i < j$,

$$d_F(x_i,x_j) \leq vol(I_{x_0})\, d_h(x_i,x_j) < vol(I_{x_0})\,\epsilon, \quad i<j. \tag{3.19}$$

Then $\{x_i\}_{i=1}^{+\infty}$ is a forward Cauchy sequence of F (and similarly, also a backward sequence of F). Similarly, a convergent sequence in d_h is also convergent in the metric d_F. Then, if (M,h) is complete, any sequence $\{x_i\}$ that is Cauchy in d_h must be Cauchy in d_F. Indeed, we have as consequence of applying the relation (3.18) to Cauchy sequences and convergent sequences that

Proposition 3.14. *If the average metric (M,h) is complete and the volume function is bounded on M, then (M,F) is forward and backward complete.*

Examples of Finsler spaces with bounded volume function includes Riemannian spaces and Berwald spaces (since in the last case, Berwald spaces are characterized by the fact that each (T_xM, g_x) are all isometric Riemannian structures [4]). However, the requirement of bounded volume is artificial. To see this, let (M,F) be a Finsler space and let us consider the average metric $\hat{h}$ given by

$$\hat{h}_{ij}(x) := \int_{I_x} g_{ij}(x,y)\, dvol_x, \tag{3.20}$$

where the weight factor has been chosen to be $f : \mathcal{I} \to \mathbf{R}^+$, $u \mapsto vol(I_{\hat{\pi}(u)})$. Then there is an analogous relation to (3.18), given by

$$d_F(p,q) \leq d_h(p,q),\ \forall\, p,q \in M. \tag{3.21}$$

Proposition 3.15. *If $(M,\hat{h})$ is complete, then (M,F) is forward and backward complete.*

Proof. If $(M,\hat{h})$ is complete, then for a F-forward Cauchy sequence $\{x_i\}_{i=1}^{+\infty}$, we have that given an $\epsilon > 0$ and for some $i_0 < i < j$,

$$d_F(x_i,x_j) \leq d_h(x_i,x_j) < \epsilon, \quad i_0 < i < j,$$

implies the existence of a limit point $x \in M$ such that

$$d_F(x_i,x) \leq d_h(x_i,x) < \epsilon, \quad i_0 < i.$$

Similarly, it follows that (M,F) is backward complete. □

Corollary 3.16. *Let (M,F) be a connected Finsler space with bounded volume function. Then*

- *For each pair of points $p,q \in M$ it holds good the relation*
$$\lim_{p\to q} \frac{d_h(p,q)}{d_F(p,q)} > 0,$$
- *Reciprocally, it holds good the relation*
$$\lim_{p\to q} \frac{d_F(p,q)}{d_h(p,q)} < +\infty.$$

Proof. Let us consider a sequence of points in M $\{x_k\}_{k=1}^{+\infty}$ with $x_1 = p$ and with $\lim_{k\to+\infty} x_k = q$. From the equality (3.18) it follows that for any $x_k \in \{x_i\}_{i=1}^{+\infty}$ we have that
$$\frac{1}{vol(I_{x_k})} \leq \frac{d_h(x_k,q)}{d_F(x_k,q)}.$$
Thus, taking limits,
$$\lim_{k\to\infty} \frac{d_h(x_k,q)}{d_F(x_k,q)} \geq \lim_{k\to+\infty} \frac{1}{vol(I_{x_k})} \geq \frac{1}{vol(I_0)} > 0,$$
since $0 < vol(I_{x_k}) < +\infty$.

In a similar way, one has the following boundness condition,
$$\frac{d_F(x_k,q)}{d_h(x_k,q)} < vol(I_{x_k}).$$
Thus, we have
$$\lim_{k\to\infty} \frac{d_F(x_k,q)}{d_h(x_k,q)} \leq \lim_{k\to\infty} vol(I_{x_k}) = vol(I_q) < +\infty.$$
□

4. Average of linear connections in $\hat{\pi}^*TM$ and applications

4.1. Average of a family of tensor spaces automorphism. For each tensor $S_z \in T_z^{(p,q)}M$ with $v \in \hat{\pi}^{-1}(z)$, $z \in U \subset M$ the following isomorphism is defined,
$$\hat{\pi}^*|_v : \hat{\pi}^{-1}(x) \to \pi_2^{-1}(v), \quad S_z \mapsto \hat{\pi}_v^* S_z.$$
Consider a family of automorphism
$$A := \big\{A_w : \hat{\pi}_w^* T^{(p,q)}M \to \hat{\pi}_w^* T^{(p,q)}M,\, w \in N_x,\, x \in M\big\}.$$
Then one can define the vector valued integrals,
$$\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\Big) \cdot S_x := \int_{N_x} \big(\pi_2(\, A_u\, \hat{\pi}_u^*\, S_x)\, dvol_x\big), \quad S_x, \in T_x^{(p,q)}M, \tag{4.1}$$
where $u = (x,y)$ is a point on the fiber N_x. This integral operation is a fiber integration on a sub-manifold N_x of T_xM and with values of $T_x^{(p,q)}M$, for each $x \in M$. The tensor S_x is pulled-back $\{\hat{\pi}_u^* S_x,\, u \in N_x \subset N\}$ such that the following diagram commutes,
$$\begin{array}{ccc} \hat{\pi}_u^* T^{(p,q)}M & \xrightarrow{\pi_2} & S_x \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi} \\ N_x & \xrightarrow{\hat{\pi}} & x. \end{array}$$

One can perform a fiber integration with values on a vector space. The chain of compositions defining such integration is the following,

(4.2)

$$x \mapsto S_x \mapsto \{\hat{\pi}_u^* S_x\} \mapsto \{A_u(\hat{\pi}_u^* S_x)\} \mapsto \{\pi_2(A_u(\hat{\pi}_u^* S_x))\} \mapsto \int_{N_x} \pi_2(A_u(\hat{\pi}_u^* S_x))\, dvol_x.$$

Definition 4.1. *The average operator of the family of automorphism A is the automorphism*

$$\langle A\rangle_x : T_x^{(p,q)}M \to T_x^{(p,q)}M$$
$$S_x \mapsto \frac{1}{vol(N_x)}\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\Big) \cdot S_x, \quad u \in \pi_1^{-1}(x),\ \forall\, S_x \in T_x^{(p,q)}M.$$

Proposition 4.2. *The average operator associated with a family of a linear of operators A is a covariant operation, independent of the local coordinate system on M.*

Proof. Let us consider first sections of the bundle $\hat{\pi}^*TM$ and two local basis $\{e_i,\, i = 1, ..., n\}$ and $\{\tilde{e}_i,\, i = 1, ..., n\}$ of T_xM. Then we have

$$\begin{aligned}
\langle A\rangle(S(x)) &= \frac{1}{vol(N_x)}\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\Big) \cdot S(x)\\
&= \frac{1}{vol(N_x)}\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\Big) \cdot S^i(x) e_i(x)\\
&= \frac{1}{vol(N_x)}\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\, \tilde{S}^k(x)\, \tilde{e}_k(x)\, dvol_x\Big)\\
&= \frac{1}{vol(N_x)}\Big(\int_{N_x} \pi_2|_u A_u \hat{\pi}_u^*\Big) \cdot \tilde{S}^k(x) \tilde{e}_k(x).
\end{aligned}$$

One can extend this calculation to other tensor bundles. □

It is possible to extend the averaging operation to families of local operators acting on sections of tensor bundles $T^{(p,q)}M \to M$ by applying the construction (4.2) in the definition of 4.1 to the evaluation of sections $\hat{\pi}_v^*(S) := (\hat{\pi}^* S)(v)$, where S is a section of the corresponding bundle and $v \in U_u$ is a point in an open set $U_u \ni u$. In particular, we can average Finslerian connections,

Theorem 4.3. *Let ∇ be a linear connection of the vector bundle $\hat{\pi}^*TM \to N$. Then there is defined an affine connection of M determined by the covariant derivative of each section $Y \in \Gamma TM$ along each direction $X \in T_xM$,*

$$\langle\nabla\rangle_X Y := \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^* Y\, \rangle,\ \forall\ v \in U_u, \tag{4.3}$$

for each $X \in T_xM$ and $Y \in \Gamma TM$, where U_u is an open neighborhood of u.

Proof. We check that the properties for a linear covariant derivative hold for $\langle\nabla\rangle$:

(1) Using the linearity of the original covariant derivative and the linearity of the averaging operation,

$$\begin{aligned}
\langle\nabla\rangle_X(Y_1 + Y_2) &= \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^*(Y_1 + Y_2)\rangle \quad = \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^* Y_1\rangle + \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^* Y_2\rangle\\
&= \langle\nabla\rangle_X Y_1 + \langle\nabla\rangle_X Y_2,
\end{aligned}$$

$\forall\ Y_1, Y_2$. For the second condition of linearity we have

$$\langle\nabla\rangle_X(\lambda Y) = \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^*(\lambda Y)\rangle_v = \lambda \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^*\rangle = \lambda \langle\nabla\rangle_X Y$$

$\forall\ Y \in \Gamma TM, \lambda \in\ R,\ X \in T_xM.$

(2) $\langle\nabla\rangle_X Y$ is a $\mathcal{F}$-linear respect X, that is,

$$\langle\nabla\rangle_{X_1+X_2}Y = \langle\nabla\rangle_{X_1}Y + \langle\nabla\rangle_{X_2}Y, \qquad \langle\nabla\rangle_{fX}(Y) = f\,\langle\nabla\rangle_X Y, \tag{4.4}$$

holds good for each $Y \in \Gamma TM, v \in \pi^{-1}(z)$, $X, X_1, X_2 \in T_xM$ and $f \in \mathcal{F}M$. To prove the first equation it is enough the following calculation,

$$\begin{aligned}\langle\nabla\rangle_{X_1+X_2}Y &= \langle \pi_2|_u(\nabla_{\iota_u(X_1+X_2)})\hat{\pi}^*_v Y\rangle_v = \langle \pi_2|_u\nabla_{\iota_u(X_1)}\hat{\pi}^*_v Y\rangle_v + \langle \pi_2|_u\nabla_{\iota_u(X_2)}\hat{\pi}^*_v Y\rangle\\ &= \langle\nabla\rangle_{X_1}Y + \langle\nabla\rangle_{X_2}Y.\end{aligned}$$

For the second condition the proof is similar.

(3) The Leibnitz rule holds:

$$\langle\nabla\rangle_X(\varphi Y) = \,(d\varphi(X))Y + \varphi\langle\nabla\rangle_X Y, \quad \forall\, Y \in \,\Gamma TM, \quad \varphi \in \mathcal{F}(M), \quad X \in T_xM, \tag{4.5}$$

where $d\varphi(X)$ is the action of the 1-form $d\varphi \in \Lambda^1M$ evaluated at $x \in M$ on the tangent vector $X \in T_xM$. In order to prove (4.5) we use the following property,

$$\hat{\pi}^*_v(\varphi Y) = \,\hat{\pi}^*_v\varphi\hat{\pi}^*_v Y, \quad \forall\; Y \in TM,\, \varphi \in \mathcal{F}(M).$$

Then one obtains the following expressions,

$$\begin{aligned}\langle\nabla\rangle_X(\varphi Y) &= \langle \pi_2|_u\nabla_{\iota_u(X)}\hat{\pi}^*_v(\varphi Y)\rangle = \langle \pi_2|_u\nabla_{\iota_u(X)}\hat{\pi}^*_v(\varphi)\hat{\pi}^*_v Y\rangle\\ &= \,\langle \pi_2|_u(\nabla_{\iota_u(X)}(\hat{\pi}^*_v\varphi))\hat{\pi}^*_v(Y)\rangle + \langle \pi_2|_u(\hat{\pi}^*_u f)\nabla_{\iota_u(X)}\hat{\pi}^*_v(Y)\rangle\\ &= \,\langle \pi_2|_u(\iota_u(X)\cdot(\hat{\pi}^*_v\varphi))\hat{\pi}^*_v(Y)\rangle + \varphi\,\langle \pi_2|_u\nabla_{\iota_u(X)}\hat{\pi}^*_v(Y)\rangle\\ &= \,\langle (X\cdot\varphi)\pi_2|_u\hat{\pi}^*_u(Y)\rangle + \varphi\,\langle \pi_2|_v\nabla_{\iota_u(X)}\hat{\pi}^*_v(Y)\rangle.\end{aligned}$$

For the first term we perform the following simplification,

$$\begin{aligned}\langle (X\cdot\varphi)\pi_2|_u\hat{\pi}^*_u(Y)\rangle &= \,(X\cdot\varphi)\langle \pi_2|_u\hat{\pi}^*_u(Y)\rangle = \,(X\cdot\varphi)(\langle \pi_2|_u\hat{\pi}^*_u\rangle)Y\\ &= \,(X\cdot\varphi)Y = \,(d\varphi(X))Y.\end{aligned}$$

Finally we obtain that

$$\langle\nabla\rangle_X(\varphi Y) = \,(\langle\nabla\rangle_{X_x}\varphi)Y + \varphi\langle\nabla\rangle_X Y = \,(d\varphi(X))Y + \varphi\langle\nabla\rangle_X Y.$$

□

The average covariant derivative commutes with contractions. For each $\alpha \in \Lambda^1M$ and $X \in \Gamma TM$,

$$\langle\nabla\rangle_X[\alpha(Z)] = \,(\langle\nabla\rangle_X\alpha)\cdot Z + \,\alpha\cdot(\langle\nabla\rangle_X Z).$$

Therefore, the extension of the covariant derivative $\langle\nabla\rangle_X$ acting on sections of the tensor bundle $T^{(p,q)}M \to M$ is defined by the rule

$$\begin{aligned}\langle\nabla\rangle_X\, K(X_1,...,X_s,\alpha^1,...,\alpha^r) &= \langle\nabla\rangle_X\, K(X_1,...,X_s,\alpha^1,...,\alpha^r)\\ &\quad - \sum_{i=1}^s K(X_1,...,\langle\nabla\rangle_X X_i,...,X_s,\alpha^1,...,\alpha^r)\\ &\quad + \sum_{j=1}^r K(X_1,...,X_s,\alpha^1,...,\langle\nabla\rangle_X\alpha^j,...,\alpha^r).\end{aligned}$$

We denote the affine connection associated with the above covariant derivative by $\langle\nabla\rangle$. Thus, for each section $Y \in \Gamma\, TM$, $\langle\nabla\rangle Y \in \Gamma\, T^{(1,1)}M$ is given by the action on pairs $X, Y \in \Gamma\, TM$,

$$\langle\nabla\rangle(X,Y) := \langle\nabla\rangle_X Y. \tag{4.6}$$

Proposition 4.4. *Let (M, F) be a Finsler space and ∇ a Finslerian connection. Then it holds*

$$T_{\langle\nabla\rangle} = \langle T_\nabla\rangle. \tag{4.7}$$

Proof. We can calculate the torsion of the connection $\langle\nabla\rangle$,

$$\begin{aligned} T_{\langle\nabla\rangle}(X,Y) &= \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v\rangle Y - \langle \pi_2|_u \nabla_{\iota_u(Y)}\hat{\pi}^*_v\rangle X - [X,Y] \\ &= \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v\rangle Y - \langle \pi_2|_u \nabla_{\iota_u(Y)}\hat{\pi}^*_v\rangle X - \langle \pi_2|_u \hat{\pi}^*_u[X,Y]\rangle \\ &= \langle \pi_2|_u \big(\nabla_{\iota_u(X)}\hat{\pi}^* Y - \nabla_{\iota_u(Y)}\hat{\pi}^* X - \hat{\pi}^*|_u[X,Y]\big)\rangle \\ &= \langle T_\nabla(X,Y)\rangle. \end{aligned}$$

□

Corollary 4.5. *Let (M, F) be a Finsler space with average Chern connection $\langle {}^{ch}\nabla\rangle$. Then the torsion $T_{\langle {}^{ch}\nabla\rangle}$ is zero.*

Convex invariance of the averaged connection. Let us consider the averaged metric h (3.5) of the Finsler space (M, F). Let (M, F) and $g_t = (1-t)g + th$, $t \in [0,1]$. Each fundamental tensor g_t defines a Finsler space in M. The associated Chern connection is denoted by ${}^{ch}\nabla_t$. Then we have the following proposition, which is proof by direct calculation,

Proposition 4.6. *The Σ-average connections*

$$\langle {}^{ch}\nabla_t\rangle_\Sigma : \Gamma TM \times \Gamma TM \to \Gamma TM, \quad (\langle {}^{ch}\nabla_t\rangle_\Sigma)_X\, Y = \frac{1}{vol(\Sigma_x)} \int_{\Sigma_x} \pi_2|_u({}^{ch}\nabla_t)_X \hat{\pi}^*_u Y, \tag{4.8}$$

for each $X, Y \in \Gamma TM$ are affine, torsion free connections on M and they are the same connection for each $t \in [0,1]$.

In the space M_F of Finsler spaces on M, there are relations defined as follows. Let us consider first a fixed Finsler space (M, F_0) and the associated fibre bundle $\pi_\Sigma : \Sigma \to M$.

Definition 4.7. *Two Finsler spaces (M, F_1) and (M, F_2) are Γ-related iff the corresponding Σ-averaging connections are the same.*

This relation is an equivalence. Equivalence classes are denoted as $[g]_\Gamma$. Indeed, he equivalence classes $[g]_\Gamma$ are convex subsets in the space M_F.

Corollary 4.8. *The Σ-equivalence is contained in the Γ-equivalence relations on M_F.*

Proof. Let F_1, F_2 be two Finsler metrics of the same Σ-equivalence class $[\tilde{h}]$. Then by *Proposition* 4.6, then they are in the same Γ-equivalence class. □

The converse statement is not true, since in general the Levi-Civita connection does not determine the metric on M.

4.2. **Applications to Berwald spaces.**

Definition 4.9. *A Berwald space is a Finsler space such that its Chern connection also defines an affine connection on M.*

In this case, the connection coefficients ${}^{ch}\Gamma^i_{jk}(x, y)$ depend on $x \in M$ only. Thus, we have the following,

Theorem 4.10. *For a Berwald structure (M, F)*

- *The average of the Chern connection coincides with the Chern connection in the sense that*

$$\hat{\pi}^*\langle\, {}^{ch}\nabla\rangle_X \hat{\pi}^*\, S = {}^{ch}\nabla_{\iota_u(X)}\hat{\pi}^* S. \tag{4.9}$$

- *The average of the Chern connection coincides with the Levi-Civita connection of the average metric,*

$$\langle\, {}^{ch}\nabla\rangle \,=\, {}^{h}\nabla. \tag{4.10}$$

Proof. The relation (4.9) is direct from the definition of average connection. A detailed proof can be found in [11]. For the proof of relation (4.10), let us note first calculate the covariant derivative of the metric h for $\langle {}^{ch}\nabla\rangle$ as follows,

$$\begin{aligned}\langle\, {}^{ch}\nabla\rangle_{X=\frac{\partial}{\partial x^i}}\, h &= \Big(\frac{\partial h_{jk}}{\partial x^i} - \langle\, {}^{ch}\nabla\rangle^l{}_{ik} h_{jl} - \langle\, {}^{ch}\nabla\rangle^l{}_{ij} h_{lk}\Big) dx^j\otimes dx^k\\ &= \Big(\frac{\partial h_{jk}}{\partial x^i} - {}^{ch}\Gamma^l{}_{ik} h_{jl} - {}^{ch}\Gamma^l{}_{ij} h_{lk}\Big) dx^j\otimes dx^k\\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x}\big(\frac{\partial g_{jk}}{\partial x^i} - {}^{ch}\Gamma^l{}_{ik} h_{jl} - {}^{ch}\Gamma^l{}_{ij} h_{lk}\big)\, dvol_x\Big) dx^j\otimes dx^k\\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x}\big({}^{ch}\nabla_{\frac{\partial}{\partial \tilde{x}^i}}\, g)\big)_{jk}\, dvol_x\Big) dx^j\otimes dx^k.\end{aligned}$$

Because of the horizontal metric compatibility of the Chern connection (*equation* (2.17)), the integrand is zero, since the covariant derivative of the metric g is on the horizontal direction $\frac{\partial}{\partial \tilde{x}^i}$. Therefore,

$$\langle\, {}^{ch}\nabla\rangle_{X=\frac{\partial}{\partial x^i}}\, h = 0, \quad i = 1, ..., n.$$

Moreover, by *Corollary* 4.5 $\langle\, {}^{ch}\nabla\rangle$ is torsion free. Therefore, $\langle\, {}^{ch}\nabla\rangle$ must be the Levi-Civita connection of h. □

A direct consequence of this property is that for a Berwald space, the parameterized geodesics of the average connection (parameterized by the proper parameter of h) and the parameterized geodesics of the original metric F (parameterized by the proper parameter of F) coincide. Thus, for instance,

Corollary 4.11. *For a Berwald space (M, F), geodesically forward completeness implies backward completeness and viceversa.*

Proof. The relation (4.10) implies that if (M, F) is forward geodesic complete, then the metric h is geodesically forward completed. But since the metric h is Riemannian, it is also backward complete. Finally, applying the relation (4.9), it is clear that (M, F) must be backward complete. □

Let (M, F) be a Berwald space with fundamental tensor g. It is known that in this case the exponential map is a smooth diffeomorphism and that normal coordinates can be defined [4]. Moreover, by the relation (4.10), the normal coordinates of g are also normal coordinates of h. This contrasts with the case of a generic Finsler space, where the exponential map is only $\mathcal{C}^1$ and there not exist smooth normal coordinates systems [7].

Let (M, F) be a Berwald manifold, the Riemann tensor of g is a $(0, 4)$-tensor along the map $\hat{\pi} : N \to M$ whose components are given in normal coordinates of g by the expression

$${}^{g}R_{ijkl} := g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}, \tag{4.11}$$

where $g_{ij,kl}$ stands for $\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}$, etc... Then we have,

Proposition 4.12. *Let (M, F) be a Berwald space and consider the isometric average metric h_{ij}. Then the following relation holds,*

$$({4.12}) \qquad {}^{h}R_{ijkl}(x) = \langle\, {}^{g}R_{ijkl}(x,y)\rangle.$$

Proof. In normal coordinates for h, the Riemann tensor ${}^{h}R_{ijkl}$ is linear on the second derivatives of the components of the curvature tensor of h and can be expressed as

$${}^{h}R_{ijkl} = h_{il,jk} - h_{ik,jl} + h_{jk,il} - h_{jl,ik},$$

where for instance $h_{il,jk} = \frac{\partial^2 h_{il}(x)}{\partial x^j \partial x^k}$. From the definition of h it follows that

$${}^{h}R_{ijkl} = \langle\, g_{il}\rangle_{,jk} - \langle g_{ik}\rangle_{,jl} + \langle g_{jk}\rangle_{,il} - \langle h_{jl}\rangle_{,ik}$$

holds in the normal coordinate chard of h. Since the weight factor is $f = 1$ and the volume function $vol(I_x)$ is constant for a Berwald space [3], the partial derivatives can be introduced in the integrals,

$${}^{\langle g\rangle}R_{ijkl}(x) = {}^{h}R_{ijkl} = \langle\, g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}\rangle = \langle\, {}^{g}R_{ijkl}(x,y)\rangle,$$

where in the last equality we use that normal coordinates for h are also normal coordinates for g. This formulae has been proved in normal coordinates, but since it is an identity between tensor components, it holds in any coordinate system. □

The proof of *Proposition* 4.12 fails for general Finsler spaces since in this case there are no smooth normal coordinates. If the flag curvatures ${}^{g}K(X_1, X_2)$ are all negative, non-positive or non-negative, the metric h is of negative, non-positive or non-negative sectional curvature respectively. This open the possibility to generalize rigidity results from positive or non-negative curvature to the Berwald category. The following example clarifies the idea.

Remark 4.13. *A simple calculation shows that the average of the curvature endomorphism does not coincide with the curvature endomorphism of the average metric,*

$$\begin{aligned} {}^{h}R^{i}{}_{jkl} &= h^{im}\,{}^{h}R_{mjkl} = h^{im}\,\langle\, {}^{g}R_{mjkl}\rangle = \langle\, h^{im}\,{}^{g}R_{mjkl}\rangle \\ &= \langle\, h^{im} g_{ms} g^{sa}\,{}^{g}R_{ajkl}\rangle = \langle\, \vartheta^{i}{}_{s}\,{}^{g}R^{s}{}_{jkl}\rangle, \end{aligned}$$

where the condition $\vartheta^{i}{}_{s} := h^{im} g_{ms} \neq \delta^{i}_{s}$ measures the departure of the fundamental tensor g of being Riemannian.

Gauss-Bonnet theorem for Berwald surfaces. The construction of the average metric h as an average of the fundamental tensor over the indicatrix opens the possibility to generalize results from Riemannian to Berwald geometry, using directly the Riemannian results. We consider here two additional examples of this technique: a weak version of the Gauss-Bonnet theorem to arbitrary Berwald surfaces. Let I be the fibered manifold whose fibers are indicatrix over M. Then

Theorem 4.14. *Let (M, F) be a compact Berwald surface with average metric h and Gaussian curvature ${}^{h}K = -\,{}^{h}R_{1212}$ in some orthonormal basis of h. Then the following formula holds,*

$$({4.13}) \qquad \frac{1}{vol(I_x)} \int_I {}^{g}R_{1212}(x,y)\, dvol_x \wedge d\mu(x) = -2\pi\, \chi(M),$$

where $\chi(M)$ is the Euler's characteristic of M and $d\mu$ is the Riemannian volume form of h on M.

Proof. For the Riemannian metric h one can make use of the standard Gauss-Bonnet theorem for compact surfaces M. Thus, the relation

$$\int_M {}^hR_{1212}(x)\,d\mu = -2\pi\,\chi(M) \tag{4.14}$$

holds. Fixed the integration measure by a given function $\psi(x,y)$ as in *Proposition* 4.12 an using an orthonormal frame, ${}^hK(x) = -{}^hR_{1212}(x) = -\langle {}^gR_{1212}\rangle$, from which follows formula (4.13). □

P. Dazord proved this result for Landsberg surfaces [8]. Berwald surfaces are Riemannian surfaces or Minkowski surfaces (by the rigidity theorem from Szabó [19]). Therefore, our example is exclusively academic, although serves to illustrate the method of averaging in generalizing results. Furthermore, note that a main difficulty to extend the Chern-Gauss-Bonnet theorem to Finsler spaces of higher order dimensions by the averaging method is to understand the properties of the *total curvature* under averaging.

5. The parallel transport of the average connection

The parallel transport of a linear connection ∇ along a path $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = z$ is defined as the linear homomorphism

$$p_{xz}(\gamma) : T^{(p,q)}_x M \to T^{(p,q)}_z M, \quad S_x \mapsto S_z$$

such that the section $S(t)$ along γ is a solution of the following linear differential equation

$$\nabla_{\dot\gamma(0)} p(\gamma)(S)(t) = 0, \quad p(\gamma)(S)(0) = S(0). \tag{5.1}$$

In this *section* we will obtain the parallel transport of $\langle\nabla\rangle$ when ∇ is a linear Finslerian connection in $\hat\pi^*TM$. We first consider a formal solution for the parallel transport equation (5.1) along a curve $\gamma : [a,b] \to M$. Let (M,F) be a Finsler space. A *polygonal approximation* $\bar\gamma$ of γ is determined by a set of points $\{\gamma(0) = x, ..., ..., \gamma(t_{A-1}), \gamma(t_A) = z, \gamma(t_i) \in \gamma([a,b])\, \gamma(t_i) \in \gamma([a,b])\}$ joined by geodesic segments $\bar\gamma_{k,k+1}$ of F, with initial and ending points $\gamma(t_k)$ and $\gamma(t_{k+1})$ respectively. One can also consider the case when $t_k - t_{k-1} = \epsilon$. Then the parallel transport operator along $\bar\gamma$ is given by the composition of *elementary parallel transports*

$$p(\bar\gamma) := \prod_{k=1}^{A} \circ\, p_{t_k,t_{k-1}},$$

where the composition of elementary parallel transport $p_{t_k,t_{k-1}}$ is along the geodesic $\gamma(k-1)$ of F and is given by the endomorphism

$$p_{t_k,t_{k-1}} : T_{\gamma(t_{k-1})}M \to T_{\gamma(t_k)}M, \quad X^i e_i \mapsto \big(\delta^{j_k}_{i_{k-1}} X^{i_{k-1}} - \epsilon\,\Gamma^{j_k}_{i_{k-1}l_{k-1}} X^{i_{k-1}}\dot\gamma^{l_{k-1}}\big)e_{j_k}.$$

$\dot\gamma^{l_{k-1}}$ is the tangent vector at the point $\gamma(k-1)$. The double limit $A \to +\infty$ and $\epsilon = t_k - t_{k-1} \to 0$ is taken in this parallel transport operation, under the constraint

$$\lim_{A\to+\infty,\,\epsilon\to 0} A\epsilon = b - a.$$

We take as a definition of the parallel transport $\epsilon = \frac{b-a}{A}$. Therefore, the parallel transport of $X \in T_{\gamma(0)}M$ along γ between the point $x = \gamma(a)$ and $z = \gamma(b)$ is given by

(5.2)
$$(p_{xz}X)^j = \lim_{A\to+\infty,\epsilon\to 0} \Big((\delta^{j_A}_{i_{A-1}} - \epsilon\,\Gamma^{j_A}_{i_{A-1}l_{A-1}}\,\dot\gamma^{l_{A-1}})\Big)(p_{x\gamma(t_{A-1})}X)^{i_{A-1}},\ j = 1,...,n,$$

with $(p_{xz}X)^{j_0} = X^{j_0}$, $\lim_{A\to+\infty}\gamma(t_{A-1}) = \gamma(b)$ and $j_A = j$. This expression is equivalent to the infinite product

$$(5.3)\qquad (p_{xz}X)^j = \lim_{A\to+\infty,\epsilon\to 0}\prod_{k=1}^{A}\sum_{i_{k-1}=1}^{n}\Big((\delta^{j_k}_{i_{k-1}} - \epsilon\,\Gamma^{j_k}_{i_{k-1}l_k}\,\dot\gamma^{l_k})\Big)X^{i_0}.$$

One has the finite difference expression

$$\begin{aligned}(p_{xgamma(t+\epsilon)}X)^j - (p_{xgamma(t)}X)^j &= \lim_{A\to+\infty,\epsilon\to 0}\Big((\delta^{j_A}_{i_{A-1}} - \epsilon\,\Gamma^{j_A}{}_{i_{A-1}l_{A-1}}\,\dot\gamma^{l_{A-1}})\Big)(p_{x\gamma(t)}X)^{i_{A-1}}\\ &\quad - (p_{x\gamma(t)}X)^j\\ &= -\,\epsilon\,(p_{x\gamma(t)}\Gamma^j_{ik}(\gamma(t))\,\dot\gamma^i(t)(p_{x\gamma(t)}X)^k.\end{aligned}$$

For smooth vector fields and connections, one can take the limit

$$\lim_{\epsilon\to 0}\frac{(p_{x\gamma(t+\epsilon)}X)^j - (p_{x\gamma(t)}X)^j}{\epsilon} = -(p_{x\gamma(t)}\Gamma^j_{ik}(\gamma(t))\,\dot\gamma^i(t)(p_{x\gamma(t)}X)^k,$$

showing that the expression (5.2) is the solution of the parallel transport equation (5.1).

Formula (5.3) applies to the parallel transport of any linear connection. In particular, it can be applied to the average connection. Then it follows

Proposition 5.1. *Let $\langle\Gamma^i_{jk}\rangle$ be the connection coefficients of the average connection $\langle\nabla\rangle$. Then the parallel transport operation is*

(5.4)

$$(p_{xz}X)^j = \lim_{A\to\infty,\,\epsilon\to 0}\prod_{k=1}^{A}\Big((\delta^{j_k}{}_{i_{k-1}} - \epsilon\,\langle\Gamma^{j_k}_{i_{k-1}l_k}\rangle(\gamma(t_k))\,\dot\gamma^{l_k}\Big)X^{i_0},\qquad j = 1,...,n.$$

For a weight factor $f : N \to \mathbf{R}^+$ in each integration one obtains the expression for the parallel transport, one obtains

$$\begin{aligned}(5.5)\qquad (p_{xz}X)^j &= \lim_{A\to+\infty,\epsilon\to 0}\prod_{k=1}^{A}\Big(\frac{1}{vol(I_{\gamma(t_{k-1})})}\,(\int_{I_{\gamma(t_{k-1})}} dvol_x(\gamma(t_{k-1}))\\ &f(\gamma(t_k), y_{\gamma(t_k)})\,(\delta^{j_k}{}_{i_{k-1}} - \epsilon\,\Gamma^{j_k}{}_{i_{k-1}l_{k-1}}(\gamma(t_k), y_{\gamma(t_k)})\,\dot\gamma^{l_{k-1}}(\gamma(t_{k-1}))\Big)X^{i_0}.\end{aligned}$$

Let us remark that one can also consider the *average of the parallel transport operation*. However, in general this is not the parallel transport of the average connection: The average of the parallel transport of ∇ involves only one fiber integration, while the parallel transport of the average connection involves a formal infinite number of integrations along each fiber $\hat\pi^{-1}(\gamma(t))$, $t \in [a, b]$.

5.1. **Curvature of the average connection.** Let us consider the averaging procedure $(\Sigma, \{dvol_{\Sigma_x}\}_{x,\in M}, f = 1)$ and consider the curvature endomorphisms for the Σ-average connection $\langle\nabla\rangle$,

$$R^{\langle\nabla\rangle_\Sigma}_x(X_1, X_2)Z = \big(\langle\nabla\rangle_{\Sigma X_1}\langle\nabla\rangle_{\Sigma X_2} - \langle\nabla\rangle_{\Sigma X_2}\langle\nabla\rangle_{\Sigma X_1} - \langle\nabla\rangle_{\Sigma[X_1,X_2]}\big)Z.$$

Developing this expression in terms of the original connection ∇ one obtains

$$\begin{aligned}R^{\langle\nabla\rangle_\Sigma}_x(X_1, X_2)(Z) &= \frac{1}{vol^2(\Sigma_x)}\int_{\Sigma_x}\int_{\Sigma_x} dvol_{\Sigma_x}(v)dvol_{\Sigma_x}(u)\,\pi_2(v)\\ &\Big(\nabla_{\iota_v(X_1)}\hat\pi^*_v\,\pi_2(u)\,\nabla_{\iota_u(X_2)}\hat\pi^*_u\,Z - \nabla_{\iota_v(X_2)}\hat\pi^*_v\,\pi_2(u)\,\nabla_{\iota_u(X_1)}\hat\pi^*_u\,Z\\ &- \nabla_{\iota_u([X_1,X_2])}\,\hat\pi^*_u\,Z\Big).\end{aligned}$$

There is an analogous construction for the average connection with the averaging procedure $(\mathcal{I}, \{dvol_x\}_{x\in M}, f = 1)$. The reason why we have considered Σ-average connection is because in this case one has nicely that,

Proposition 5.2. *Given two metrics with the same Σ-average metric $\tilde{h}$ and the same Σ-average connection $\langle\nabla\rangle_\Sigma$, then all the metrics $(1-t)g_1 + tg_2, t \in [0,1]$ have the same curvature endormorphisms.*

Proof. Note that $vol(\Sigma_x)$ is a constant function on M, since it is the volume function of Riemannian spheres. Then the result follows from the invariance of $\langle\nabla\rangle_{\Sigma_t}$ calculated for each metric $F_t^2(x,y) = (g_t)_{ij}(x,y)y^i y^j$. □

It is interesting to note that this curvature is not equal to the *average curvature* of the Finslerian connection ∇. For instance, the averaged hh-curvature is

$$\begin{aligned}\langle R^\nabla(\iota_u(X_1), \iota_u(X_2))\rangle Z := & \frac{1}{vol(\Sigma_x)}\int_{\Sigma_x} dvol_{\Sigma_x}(u)\,\pi_2(u)\,\nabla_{\iota_u(X_1)}\nabla_{\iota_u(X_2)}\hat{\pi}_u^* Z \\ & - \frac{1}{vol(\Sigma_x)}\int_{\Sigma_x} \pi_2(u)\nabla_{\iota_u(X_2)}\,\nabla_{\iota_u(X_1)}\hat{\pi}_u^* Z \\ & - \frac{1}{vol(\Sigma_x)}\int_{\Sigma_x} du\,\pi_2(u)\,\nabla_{\iota_u([X_1,X_2])}\,\hat{\pi}_u^* Z \\ & = \frac{1}{vol(\Sigma_x)}\int_{\Sigma_x} dvol_{\Sigma_x}(u)\,\pi_2(u)\,\Big(\nabla_{\iota_u(X_1)}\nabla_{\iota_u(X_2)} \\ & - \nabla_{\iota_u(X_2)}\nabla_{\iota_u(X_1)} - \nabla_{\iota_u([X_1,X_2])}\Big)\hat{\pi}_u^* Z.\end{aligned}$$

Thus, given a Finslerian connection ∇, there are two notions of *average curvature endomorphisms*, $R_x^{\langle\nabla\rangle}(X_1, X_2)$ and $\langle R^\nabla(\iota_u(X_1), \iota_u(X_2))\rangle$. In general, the tensors $R_x^{\langle\nabla\rangle}(X_1, X_2)$ and $\langle R^\nabla(\iota_u(X_1), \iota_u(X_2))\rangle$ do not coincide because the covariant derivative $\nabla_{\iota_u(Y)}$ depends on $u \in N$ for a general Finsler space. However, for Berwald spaces the Chern connection lives on the manifold M and the curvature of the average connection coincide with the average of the curvature of the original Finslerian connection. An interesting problem is to obtain the general criteria that allows this phenomenon to happen.

5.2. **Average affine isometric invariants.**

Notions of isometry in Finsler geometry. There are several notions of Finsler isometry. A particularly suitable definition for us is the following,

Definition 5.3. *Given two Finsler spaces (M_1, F_1) and (M_2, F_2), a base manifold Finsler isometry (or simply a Finsler isometry) is a diffeomorphism $\Phi : M_1 \to M_2$ such that preserves the Finsler function,*

$$F_2(\Phi(x), d\Phi(y)) = F_1(x,y). \tag{5.6}$$

Moreover, every Finsler isometry is an isometry of the average structure,

Proposition 5.4. *Every isometry of (N_{1x}, g_{1x}) to (N_{2x}, g_{2x}) is an isometry between the corresponding average metrics (M, h_1) and (M, h_2).*

Proof. In local coordinates the condition of being $\tilde{\phi} : N_1 \to N_2$ an isometry of g_x is

$$(g_1)_{ij}(\tilde{x}(x), \tilde{y}(x,y)) = \frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^k}{\partial \tilde{x}^j}(g_2)_{lk}(x,y).$$

Therefore, this linear condition is translated to the corresponding average metrics. □

As an example of the use of the average metric we have that [13],

Proposition 5.5. *The group of isometries of (M, F) is contained as a closed subgroup of the isometries of (M, h) (in the compact-open topology).*

It follows that the group of isometries of (M, F) is a Lie group [9] and it is a subgroup of the group of isometries of the average metric (M, h).

Definition 5.6. *A Finsler space (M, F) is symmetric if for each point $x \in M$ there is a Finsler isometry $\varphi_x : M \to M$ such that*

- $\varphi_x(x) = x$,
- $(d\varphi_x)|_x = -Id|_{T_x M}$.

It follows that if (M, F) is a symmetric space, then (M, h) is symmetric.

Theorem 5.7. *If (M, F) is symmetric, then it is forward and backward complete.*

Proof. Let us consider instead of h, the average metric $\hat{h}$ defined by the expression (3.20). If $(M.F)$ is symmetric, then $(M, \hat{h})$ is symmetric. Then $(M, \hat{h})$ is homogeneous [6], p. 192. If $(M, \hat{h}$ is homogeneous, it is complete [15], p.176. By *Proposition* 3.15, then (M, F) is forward and backward complete. □

Other two related definitions of isometry in Finsler space-times are the following,

Definition 5.8. *Given two Finsler spaces (M_1, F_1) and (M_2, F_2), they are fiber isometric iff there is a bundle morphism*

$$\begin{array}{ccc} \hat{\pi}^* TM_1 & \xrightarrow{\tilde{\phi}} & \hat{\pi}^* TM_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ N_1 & \xrightarrow{\phi} & N_2 \end{array}$$

that preserves the fiber metric and such that

$$\tilde{\phi}^* \bar{g}_2 = \bar{g}_1. \tag{5.7}$$

It is easily recognized that a fiber isometry determines a Finsler isometry as in 5.3.

Definition 5.9. *Let (M_1, F_1) and (M_2, F_2) be two Finsler spaces and $(\tilde{\phi}, \phi)$ a bundle morphism such that the diagram*

$$\begin{array}{ccc} N_1 & \xrightarrow{\tilde{\phi}} & N_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

commutes. Then $(\tilde{\phi}, \phi)$ is an isometry iff

$$\tilde{\phi}^* g_2 = g_1, \tag{5.8}$$

where g_1 and g_2 are the fiber Riemannian metrics induced on N_{x_1} and N_{x_2} respectively.

Curvature average isometric invariants. Let us consider the averaging procedure $(\mathcal{I}, \{dvol_x\}_{x\in M}, f = 1)$. We can consider the average of all curvature endomorphisms of the Finslerian connection ∇. The first of these average operators is the *average hh-curvature endomorphisms*, defined as the endomorphism

(5.9)
$$\langle R\rangle_x(X_1, X_2) : T_xM \to T_xM, \quad Y \mapsto \langle R\rangle_x(X_1, X_2)\, Y := \langle R^{\nabla}(\,\iota_u(X_1), \iota_u(X_2))\rangle Y.$$

Let us denote the vertical lift of $X = X^i\frac{\partial}{\partial x^i} \in T_xM$ by $\kappa(X) = X^i\frac{\partial}{\partial y^i} \in \mathcal{V}_u$ with $u \in \hat{\pi}^{-1}(x)$. The average hv-curvature in the directions X_1 and X_2 is the endomorphism

(5.10)
$$\langle P\rangle_x(X_1, X_2) : T_xM \to T_xM, \quad Y \mapsto \langle P\rangle_x(X_1, X_2)\, Y := \langle \pi_2 P_u(\iota_u(X_1), \kappa_u(X_2))\hat{\pi}^*_u Y\rangle,$$

with $u \in I_x \subset \hat{\pi}^{-1}(x) \subset N$. Similarly, for the vv-curvature in the case of an arbitrary linear connection on $\hat{\pi}^*TM$, we define the average homomorphisms,

(5.11)
$$\langle Q\rangle_x(X_1, X_2) : T_xM \to T_xM, \quad Y \mapsto \langle Q\rangle_x(X_1, X_2) := \langle \pi_2 Q_u(\kappa_u(X_1), \kappa_u(X_2))\hat{\pi}^*_v Y\rangle,$$

with $u \in I_x \subset \hat{\pi}^{-1}(x) \subset N$. The average endomorphisms (5.9)-(5.11) are living on the manifold M. In the case of a Riemannian metric, the average curvatures $\langle P\rangle_x(X_1, X_2)$ and $\langle Q\rangle_x(X_1, X_2)$ are both zero, for any $X_1, X_2 \in \Gamma TM$.

The Cartan and Chern connections are invariant under fiber isometries, since the connections are defined in terms of the Finsler function F and the fundamental tensor g (that determines the fiber isometries). Therefore, if the measure used in the definition of the averaging operation is invariant under fiber isometries, the endomorphisms $\langle P\rangle(X_1, X_2)$ and $\langle Q\rangle(X_1, X_2)$ are also invariant under the fiber isometries of F. This fact can be used to define global affine invariants as integrals of the type

$$Inv = \int_M d\mu\, \mathcal{F}(\langle Q\rangle, \langle P\rangle, \langle R\rangle, h), \tag{5.12}$$

where $\mathcal{F}(\langle Q\rangle, \langle P\rangle, h)$ is a scalar function and the volume form $d\mu$ is the volume form associated to the average Riemannian metric $\langle g\rangle$. Thus, the integral (5.12) is invariant under fiber isometries. Furthermore, if we consider the averaging procedure $(\Sigma, \{dvol_{\Sigma_x}\}_{x,\in M}, f = 1)$, the analogous invariant associated to (5.12) is also convex invariant.

6. On the notion of convex invariance in Finsler geometry

The Σ-averaging operation is not injective. Each convex sum $(1-t)g + t\tilde{h}$, $t \in [0, 1]$ determines a fundamental tensors g_t with the same Σ-average metric $\tilde{h}$. Also, any Riemannian structure $(M, \hat{g})$ is invariant under averaging, except by a conformal factor depending on the weight factor f used, $\langle\hat{g}\rangle = \lambda(x)\hat{g}$.

These invariance properties can be extended to other geometric objects, specially to the invariance of the connection coefficients $\langle\Gamma^i_{jk}(x, y)\rangle$ of the average connection $\langle\nabla\rangle$: any convex combination $t_1\nabla_1 + t_2\nabla_2$ with $t_1 + t_2 = 1$ of Finslerian connections ∇_1, ∇_2 with the same Σ-average connection has the same Σ-average connection.

These examples motivate the following notion,

Definition 6.1. *Let $[M_F] := M_F/\sim$ be the quotient space by the Σ-equivalence relation in the space M_F of Finsler spaces over M. A geometric property or object is called convex invariant if it is well defined in $[M_F]$.*

From the examples discussed in this paper, we give a short list of properties that are convex invariant,

- The Σ-average metric is convex invariant.
- The Σ-average connection is convex invariant.
- For compact, Berwald surfaces, the Euler characteristic is convex invariant.
- The quasi-metric topology induced by a Finsler function is convex invariant.
- Metric forward and backward completeness is convex invariant, since a complete average metric implies that the original Finsler metric is both forward and backward metric complete.
- The Σ-average torsion of a Finslerian connection $\langle T_{\nabla}(X,Y)\rangle$ is convex invariant.
- The curvature of the Σ-average connection is in some specific cases (see *Proposition* 5.2.
- The notion of symmetric space is convex invariant.

There are properties that are not convex invariant. We have seen some of them,

- There are examples of Finsler metrics that are forward but not backward geodesic complete (see for instance the Poincaré Finslerian disc [4]).
- Non-reversibility of the Finsler space (M,F) is not convex invariant. For instance, the average of a Randers metric, which is not reversible, is a Riemannian metric, which is reversible.

The notion of convex invariance in Finsler geometry can be turned on an elegant method in the study of Finsler geometry, suggesting two different research strategies in Finsler geometry,

- Re-formulation of Riemannian results in a convex invariant form. Then the results are valid in the Finsler category directly. This method provides direct generalizations or *weak generalizations* of Riemannian results.
- The differences between Finsler and Riemannian geometry using the notion of convex invariance are associated with non-convex invariant properties. Thus, we need to go further from the averaging, since relevant properties (like non-reversibility) are lost in the averaging.

The notion of convex invariance can be applied naturally to the average of fundamental tensors, linear connections, torsion tensors and the curvature of the average metric. On the other hand, it cannot be applied directly with complete generality to the average of curvatures and parallel transports homomorphisms, since they require multiple fiber integrations.

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